Ordinal Type Theory

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Abstract

Higher-order logic, with its type-theoretic apparatus known as the *simple theory of types* (STT), has increasingly come to be employed in theorizing about properties, relations, and states of affairs—or 'intensional entities' for short. This paper argues against this employment of STT and offers an alternative: *ordinal type theory* (OTT). Very roughly, STT and OTT can be regarded as complementary simplifications of the 'ramified theory of types' outlined in the Introduction to *Principia Mathematica* (on a realist reading). While STT, understood as a theory of intensional entities, retains the Fregean division of properties and relations into a multiplicity of categories according to their adicities and 'input types' and discards the division of intensional entities into different 'orders', OTT takes the opposite approach: it *retains* the hierarchy of orders (though with some modifications) and *discards* the categorization of properties and relations according to their adicities and input types. In contrast to STT, this latter approach avoids intensional counterparts of the Epimenides and related paradoxes. *Fundamental* intensional entities lie at the base of the proposed hierarchy and are also given a prominent part to play in the individuation of non-fundamental intensional entities.

Keywords: Properties; relations; states of affairs; type theory; higher-order metaphysics; fundamentality

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> [O]ne can maintain in effect an impredicative point of view for extensions, while maintaining the predicative (ramified) point of view for intensions.
>
> – Kripke, 'A Puzzle about Time and Thought'

1 Introduction

How should one go about describing the richness of the world, taking 'richness' in the broadest possible sense? It will not in general suffice to say that there are so-and-so many *F*s and so-and-so many *G*s: if every *F* is a *G* and *vice versa*, one will also have to say (if it is true) that to be *F* is not the same as to be *G*. More generally, one will have to count the ways there are for things to be, as well as the ways there are for two or more things to be related to each other. To have a formal grasp of such ways, philosophers have often resorted to the 'simple theory of types' (STT), which in various ways goes back to Frege (1884: §53, 1891/2008), Russell (1908), Chwistek (1921; 1922), and Ramsey (1925/1931). In recent years, STT has been the standard framework of 'higher-order metaphysics'.[1]

When employed in theorizing about the "ways there are for things to be", or the "ways there are for two or more things to be related to each other", the types of STT are naturally understood in a *non*-extensional manner: the semantic values of formulas are taken to be states of affairs (or 'propositions') rather than truth-values, and the semantic values of predicates are taken to be properties and relations rather than sets—though a proponent of STT will be quick to point out that this talk of 'states of affairs', 'properties', and 'relations' should be regarded as a merely heuristic gloss. Unfortunately, the non-extensionalistic interpretation of formulas and predicates, if carried out in accordance with STT, leads to Epimenidean paradoxes, among other difficulties. To some theorists, such as Thomason (1989), Church (1993), and Martino (2001), this has suggested that it would be better to work instead with some version of the 'ramified' theory of types employed in *Principia Mathematica*, from which STT was derived by omitting certain complications. My principal aim in this paper is to motivate and develop a radically different approach, which consists not in adding back onto STT the original complications of ramified type theory, but rather in *giving up* STT in favor of a hierarchy of 'orders' more or less akin to that employed by the ramifiers. Accordingly, the type-theoretic apparatus that results from this alternative approach might be best referred to as *ordinal* type theory (OTT).

A key aspect of the to-be-proposed theory is its reliance on a concept of *fundamentality* applicable to intensional entities—i.e., properties, relations, and states of affairs.[2] Instantiations of fundamental properties or relations roughly correspond to

[1] I mainly have in mind Williamson (2013), Dorr (2016), Fritz & Goodman (2016), Goodman (2017), Bacon (2019; 2020), Dorr, Hawthorne & Yli-Vakkuri (2021), and Bacon & Dorr (forthcoming). Major influences on this literature include Prior (1971) and Williamson (2003). Skiba (2021) offers a survey. Critical voices include Orilia & Landini (2019), Hofweber (2022), Whittle (2023: 1642–45), Menzel (forthcoming), and Sider (MS). Also cf. Florio (forthcoming).

[2] In the literature of higher-order metaphysics, the term 'state of affairs' is usually avoided in

the 'atomic propositions' of the *Principia*. While I will here not attempt any definition of 'fundamental', I will formulate four assumptions that should go some way towards narrowing down that concept. Very briefly, these are to the effect that: (i) the identity relation is fundamental; (ii) for every fundamental relation, any converse of that relation is also fundamental; (iii) everything is 'fully analyzable' in terms of particulars and fundamental intensional entities; and (iv) with certain exceptions, having to do with (i) and (ii), there are no 'necessary connections' among distinct particulars or fundamental intensional entities. This last assumption will play a crucial role in clarifying the individuation of intensional entities.

The mathematical bedrock of the theory will be ZFC with urelements. However, rather than to let a vast universe of sets sit next to a galaxy of intensional entities, I will follow Carnap (1947: §23) in identifying sets with properties (of a certain sort). A major philosophical advantage of this identification lies in the fact that it integrates the ontology of mathematics into the theory of intensional entities, thereby unifying the overall ontological picture.

I should warn the reader that there are (at least) two flies in the ointment. First, I have no proof that the theory proposed below is consistent. Second, the last of the mentioned assumptions about fundamentality is quite unattractively complicated. Much of its complexity stems from special provisions that have to be put in place to enable an adequate treatment of relations. I take this to be a strong indication that the present conception of relations is in some ways too naïve and simplistic: the term 'relation' is defined on the basis of a multigrade instantiation predicate, which is taken as primitive; and thus no attempt is made to understand what it might mean for a state of affairs (henceforth simply 'state') to be an 'instantiation' of a given relation by some entities in a given order. One might reasonably hope that a suitable account of instantiation, together with a conception of relations as consisting of conceptually simpler constituents, may help to reduce the complexity of that fourth assumption. This is left for future work.

The plan for the rest of this paper is as follows. Section 2 outlines some of the discontents that arise when STT is put to use in theorizing about intensional entities. Some of these criticisms also affect various generalizations of STT, such as the *calculus of constructions* (Coquand & Huet 1988), if these are interpreted as theories of intensional entities. Sections 3–5 introduce OTT. Section 3 specifies the syntax and semantics of a formal language whose formulas and predicates will serve to denote intensional entities, and outlines an ontology of such entities. Section 4 defines how entities are assigned their respective orders, motivates the resulting hierarchy, and describes how this approach avoids the intensional paradoxes. Section 5 contains the mentioned assumptions about fundamental intensional entities, and derives some of their consequences. A central result (stated in §5.1) will be the thesis that, for every ordinal α, there are proper-class many αth-order entities. Section 6 addresses two potential concerns, and Section 7 concludes.

favor of 'proposition'. Nonetheless, I shall here speak of states of affairs (or simply 'states'), as this term goes more naturally together with 'property' and 'relation', whereas 'proposition' connotes something more fine-grained and sentence-like. (Cf., e.g., Dorr [2016: 54n.].)

2 Simple Type Theory and Its Discontents

2.1 The Types of Simple Type Theory

In the familiar language of first-order predicate logic with identity (hereinafter LFO), we are used to distinguishing between predicates, individual constants, and individual variables. Predicates other than '=' are upper-case letters with adicity-indicating superscripts; individual constants are lower-case letters taken from the beginning of the Latin alphabet; and individual variables (which in LFO are the *only* variables) are lower-case letters taken from the end of the Latin alphabet.[3] In a well-formed formula, any occurrence of an n-adic predicate (for any $n > 0$) stands at predicate-position, i.e., at the head of an expression of the form '$P^n(t_1, \ldots, t_n)$', while any occurrence of a zeroadic predicate stands at sentence-position. In *second*-order logic, predicate variables are added to the mix, and occurrences of these can be bound by quantifier-occurrences just like occurrences of individual variables.

As we move beyond second-order to *higher*-order logic, the simple typographic distinction between lower and upper case becomes insufficient, for we now have 'super-predicates' that stand to other predicates in the same way as 'normal' predicates stand to individual constants. Accordingly, we need a more versatile device: type superscripts. On one possible implementation, each type superscript is either 'e' or a comma-delimited list, enclosed in angular brackets, of zero or more elements, where each element is in turn a type superscript.[4] Individual constants and variables are given the superscript 'e', whereas the superscript of any n-adic predicate or predicate variable (for any natural number n) is a list of length n. Predicates and predicate variables can now appear as elements of argument lists, rather than only at either predicate- or sentence-position. In this language of higher-order logic (LHO), predicates and their arguments are related by the following constraint:

(C) For any expressions E and F and any $n > 0$: if E appears, in a well-formed formula, as the nth element of an argument list immediately preceded by an occurrence of F, then F is a predicate or predicate variable whose superscript is a list that has E's superscript as its nth element.

Thus, while '$M^{\langle\rangle}$' and '$R^{\langle e,e\rangle}(a^e, b^e)$' are well-formed formulas, '$P^{\langle e\rangle}(P^{\langle e\rangle})$' isn't. Minor variations aside, it is this language—often augmented by modal operators and a variable-binding operator 'λ' as a device of predicate abstraction—that practitioners of higher-order metaphysics tend to employ when theorizing about intensional entities.

This gives rise to a number of concerns, not least of which is the worry that from a philosophical point of view the restrictions of LHO are under-motivated. To substantiate this worry, I will begin with a critical (albeit brief and unsystematic)

[3] A more detailed overview of LFO is given in Shapiro & Kouri Kissel (2018).

[4] This notation follows Williamson (2013: 221f.), whose notation is a variant of that used by Gallin (1975: 68).

discussion of several ways in which a defender of STT might take those restrictions to be justified.

2.2 Unclear Motivation

In arguing for the thesis that a propositional function cannot have itself as argument, Russell in the *Principia* appeals to the *vicious circle principle*, which is in turn motivated by the need to avoid certain kinds of paradox. In a similar vein, Hilbert and Ackermann (1950: 152) thought that "we must differentiate the predicates according to the kind of arguments they have" in order to avoid Russell's paradox of properties. However, as was soon pointed out by Behmann (1931), this paradox can be avoided by other means than differentiating the predicates according to their argument-types. Briefly put, Behmann's proposal was that no predicate should be regarded as predicable of a given object (or sequence of objects) unless it be possible for the predication to be rewritten using nothing but 'primitive vocabulary' (*Grundzeichen*). For example, if we introduce a property N (and the predicate 'N') by declaring that '$N(\varphi)$' is short for '$\neg\varphi(\varphi)$', and then try to get a Russellian paradox by considering '$N(N)$', we find that the symbol 'N' can never be eliminated: from '$N(N)$' we are led to '$\neg N(N)$', from here to '$\neg\neg N(N)$', and so on. Behmann took this to show that the expression '$N(N)$' is not a statement (*Aussage*) at all, and hence not a contradiction.

Unlike STT, Behmann's proposal did not take the form of a calculus with precisely specified rules of inference, and it has never caught on.[5] Nonetheless, insofar as it points to an alternative way of avoiding intensional paradox, it can be reasonably regarded as an indication that the need to avoid paradox is not enough by itself to justify the restrictions imposed by STT.

Another possible avenue towards motivating STT emphasizes the syntactic categories of certain natural languages. Arthur Prior took STT to be "at bottom little more than a matter of being sensible about syntax" (1971: 40).[6] Just as English syntax distinguishes between nouns and verbs (and does not allow a sentence to be formed by putting a verb after a verb), and just as LFO distinguishes between individual constants and predicates (and does not allow a predicate to appear as an element of an argument list), so LHO distinguishes between "predicates, which form sentences out of names, and higher-order 'functions' which form sentences out of predicates,

[5]Thiel (2002) gives a brief overview and helpful discussion of Behmann's correspondence, relating to his proposal, with Gödel and Dubislav. Also cf. Feferman (1984). In a letter to Behmann, Ramsey wrote that "the hierarchy of functions and arguments is the one part [of the Theory of Types] which I feel can hardly be questioned" (Mancosu 2020: 24). He motivates this assessment by pointing out that some functions are defined only on functions and some only on functions of functions (*op. cit.*, p. 23). But the existence of such functions—or of properties that have instantiations only by properties—can be granted without having to embrace STT.

[6]Cf. also Tarski (1935: §4), who proposes a form of STT based on considerations about the "semantical categories" of natural-language expressions. In his postscript, however, he abandons his earlier standpoint and takes quite seriously the possibility that STT may need to be given up in the interest of expressive power.

and so on up this hierarchy" (*ibid.*). These "higher-order 'functions'" are simply the aforementioned super-predicates, or what one might call predicates of 'higher' types. But this trajectory of generalization—from a language like LFO, whose predicates take as arguments only individual constants, to one like LHO, where we also have those super-predicates—is not inevitable, for one can easily imagine as one's starting-point an alternative to LFO that *does* allow predicates (and formulas) to appear as elements of argument lists. For example, Menzel (1986; 1993) allows "predicate terms", which can take the form of constants, variables, and lambda-expressions, to appear both as elements of argument lists and at predicate-position.

"But hasn't Frege taught us", the defender of STT might retort, "that there is a world of a difference between, e.g., the predicate 'is a horse' (as in 'Secretariat is a horse') and 'is instantiated' (as in 'Horsehood is instantiated')? It just *doesn't make any sense* to say that Secretariat is instantiated!" And so it might be thought that STT is exactly what we need, since it allows us to say that 'is a horse' is a first-level predicate (superscript: '$\langle e \rangle$') and that 'is instantiated' is a second-level predicate (superscript: '$\langle\langle e \rangle\rangle$'). However, as an argument for STT this would be too quick. From the observation that it "doesn't make any sense" to say that Secretariat is instantiated, little more follows than that the predicate 'is instantiated' cannot be meaningfully applied to such things as horses. While it would not be unreasonable to conclude that some properties have instantiations only by properties, it hardly follows that no property has instantiations by properties as well as by particulars. An argument for having *some* kind of type theory is not automatically an argument for STT.[7]

Nominalists might be drawn to STT in the hope that adopting this kind of system will make it impossible to formulate certain awkward questions. For example, the question of where in the world the virtues are located turns out to be ill-posed, at least if we assume that virtues and location-properties are equally of type $\langle e \rangle$. This might be thought to dissolve the dispute between those who take properties to be 'immanent' and those who regard them as 'transcendental'.[8] But one could wonder whether STT gives nominalists what they actually want, or *should* want. On the one hand, what they want is (mainly) to avoid ontological commitment to properties and relations, and typically also states. On the other hand, it is hard to see why one could not fairly describe STT as proposing a division of ontology into different branches: apart from e-ontology (i.e., the study of what there is in the sense of '$\exists x^e$'), there are also $\langle\rangle$-ontology, $\langle e \rangle$-ontology, $\langle\langle e, e \rangle, \langle\rangle\rangle$-ontology, and so on. Under the interpreta-

[7]A recent defense of STT that draws explicitly on Fregean ideas can be found in Button & Trueman (2022), which in turn invokes Trueman's (2021) defense of 'Fregean realism'. Trueman's central thesis is that "it is *nonsense* to suppose that a property might be an object" (p. 2n.). Here an *object* is understood to be "anything which can be referred to with a singular term", while a *property* is "anything which can be referred to with a predicate" (pp. 1f.). Later in the book the relevant concept of reference undergoes a bifurcation when Trueman distinguishes between 'term-reference' and 'predicate-reference'. At a crucial juncture, Trueman's argument appeals to the requirement that "the notion of reference appropriate for predicates must allow us to disquote predicates" (p. 52). But it is not very clear why this requirement should be accepted.

[8]Cf. Jones (2018: §4.2). The rest of this paragraph is largely in line with §3.3 of the same paper.

tion of STT that is currently under consideration, the entities that fall under these other types (i.e., other than e) may be aptly described as properties, relations, and states; and thus anyone who holds that STT gives nominalists what they want will have to claim that the only kind of ontological commitment of concern to the nominalist is e-ontological commitment. But if this is correct, then so much the worse for nominalism, for it is unclear why the other kinds of commitment should not matter just as much in assessing the 'ontological costs' of a theory. Moreover, the maneuver of rendering certain awkward questions about intensional entities unformalizable through the expedient of type-restrictions leaves many other questions about intensional entities both formalizable and unresolved, so that the philosophical headache caused by these entities is not all that much lessened by distributing them over a roster of different types. For example, it might be asked whether every state $s^{\langle\rangle}$ is identical with its 'double negation' $\neg\neg s^{\langle\rangle}$, or how many fundamental entities of type $\langle\langle\rangle, e, \langle\langle e\rangle\rangle\rangle$ there are.[9]

Finally, in view of the fact that much work in recent metaphysics bases itself on STT, it might be suspected that there is a 'broadly abductive' argument in the offing. Very roughly, the idea is that STT might be motivated by its theoretical utility. To give just one example: Bacon (2019; 2020) has relied on STT in a project that sheds valuable light on, among other things, the concept of fundamentality. It might thus be thought that, by giving up STT, we would stand to lose valuable insights—or at least potentially fruitful ideas—about fundamentality, just as, by giving up set theory, we would stand to lose valuable insights about cardinality and types of infinity.

[9]In §3 of his 'A Case for Higher-Order Metaphysics' (forthcoming), Andrew Bacon argues for the virtues of STT over "property theory" in part by maintaining that such questions as 'Is wisdom located?' or 'Is wisdom concrete or abstract?' do not have "straightforward higher-order analogues". It is true that, in an STT-based setting, *it is not obvious* that these questions have higher-order analogues. But this just means that we are faced with the difficult question of whether they *have* such analogues; and here of course it is not enough to verify that they have no higher-order analogues *in English*. For instance, there may (assuming STT) be a fundamental entity of type $\langle\langle e\rangle, \langle e\rangle\rangle$ that behaves just like a metaphysically primitive co-location relation among entities of type $\langle e\rangle$. Although considerations of ontological parsimony give us a reason to think that there isn't, this does not mean that the question does not arise.

A second way in which Bacon regards higher-order metaphysics as separate from "property theory" rests on the idea, which goes back at least to Prior (1971: ch. 3), that quantification into predicate-position is a purely logical affair. For instance, the existential generalization 'Socrates somethings' is supposed to follow unproblematically from 'Socrates is wise'; and since one can accept 'Socrates is wise' without having to believe in properties, 'Socrates somethings' should be similarly free of ontological commitment to properties. (Cf. also Liggins [2021: 10028].) But notice that, if this argument works at all, it cuts both ways: one can accept 'Socrates is wise' without having to believe in STT or any of the 'higher-order entities' posited by higher-order metaphysicians. Hence, if 'Socrates somethings' does indeed follow logically from 'Socrates is wise', then that sentence should not carry any commitment to such entities, either.

A final point worth noting is that Bacon uses as his foil a rather specific form of "property theory", whereas there are of course various possible ways to develop a theory of properties (and relations). As far as I can see, nothing hinders our regarding STT as one of these ways; though obviously this is not to say that STT, thus understood, will be unproblematic.

This line of reasoning has undeniable appeal.[10] Still, it does not constitute any compelling argument for STT *unless* it can be shown that no analogous insights or ideas could be developed within an at least equally viable alternative to STT. Specifically with regard to fundamentality, we will see in Sections 5.3 and 5.4 how counterparts of Bacon's (2020) principles of 'Fundamental Completeness' and 'Quantified Logical Necessity' (the latter of which links the concept of fundamentality to that of logical necessity) can be formulated within the context of *ordinal* type theory.

This much may suffice as an at least preliminary articulation of our first worry. While nothing that has been said here is meant to call into question STT's usefulness for a variety of purposes (such as the formalization of mathematical reasoning), it remains unclear why STT should be adopted as a theory of, or framework for theorizing about, intensional entities.

2.3 Ontological Profligacy and Procrusteanism

A second problem concerns the treatment of quantifiers, which function in STT as names of higher-order entities. For instance, the symbol '$\forall_{\langle\rangle}$' (or something similar) is effectively treated as the name of a 'property of properties of states' (i.e., of an entity of type $\langle\langle\langle\rangle\rangle\rangle$) that a given 'property of states' instantiates just in case it is instantiated by every state.[11] For every type, STT provides in this way a corresponding quantifier and posits a corresponding property. And each such property is ostensibly treated as fundamental, since no analysis in interestingly different terms is either given or sought. As a result, if ontological parsimony is measured by the number of fundamental entities posited by a given theory, then STT appears to fare rather poorly in this regard.

Related to the topic of quantification, there is also a third worry. Fundamental properties are traditionally regarded as *intrinsic*. Yet intuitively, a property such as that of *being instantiated by every state* is quite clearly *ex*trinsic.[12] So it is at least questionable that such properties should be treated as fundamental.

A fourth problem, analogous to the second, arises from the consideration of identity predicates. In the context of STT, talk of 'identity', as applied to the inhabitants of a given type, is often understood in the sense of 'Leibniz equivalence'. Thus the symbol '$=_{\langle\rangle}$' might be explained by saying that it is the case that $p^{\langle\rangle} =_{\langle\rangle} q^{\langle\rangle}$ if and only if, for all $x^{\langle\langle\rangle\rangle}$: $x^{\langle\langle\rangle\rangle}(p^{\langle\rangle})$ obtains just in case $x^{\langle\langle\rangle\rangle}(q^{\langle\rangle})$ obtains. But Leibniz equivalence will amount to bona fide *numerical* identity only if, for each inhabitant of the respective type, there exists a property that only that inhabitant has. Since it would be extremely unparsimonious to say that for each entity there exists a *fundamental* property that only that entity has, the better course would be to let each of these properties be 'constructed' from an identity relation and the entity in question,

[10] Thanks to an anonymous referee for raising this issue.

[11] The subscript-less symbols '$\forall$' and '$\exists$' that are used throughout this section should be read as abbreviatory devices. Cf., e.g., Church (1940: 58).

[12] On the intrinsic/extrinsic distinction, see, e.g., Plate (2018) and references therein.

along the lines of $\lambda x\,(x = y)$, where y is the respective entity. But now, given that STT prohibits relations that can take as arguments (in the same argument-place) entities of different types, the STTler will still need to posit infinitely many identity relations, one for each type.

A fifth worry, which has already been alluded to in the previous sentence, has to do with the fact that STT requires every property and every relation (every *attribute* for short) to have a tightly restricted 'range of application'. For instance, if R is a dyadic relation that has an instantiation by some entity of type e and another entity of type $\langle\rangle$ (in this order), then *every* instantiation of R will be of this sort. The type of R will be $\langle e, \langle\rangle\rangle$, and there will simply not exist an instantiation of R by, say, a pair of entities of types $\langle\rangle$ and $\langle e\rangle$ (respectively). Under STT, there is consequently no place for a relation of *loving* or *thinking of* that can hold not only among persons—usually considered to be of type e—but can also be borne by a person to an entity of some other type. This has often been perceived as counter-intuitive and Procrustean.[13]

2.4 Paradox and Indeterminacy

A sixth worry arises from the fact that STT gives rise to paradoxical or at least implausible consequences. For example, let $P^{\langle\langle\rangle\rangle}$ be a property that is instantiated by exactly one state, namely $\forall x^{\langle\rangle}\,(P^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \to \neg x^{\langle\rangle})$. (For the sake of concreteness, let us think of $P^{\langle\langle\rangle\rangle}$ as the property of *being Frege's favorite state*.) By a straightforward argument, we are led to conclude that this state obtains just in case it doesn't: a contradiction.

This paradox—an intensional analogue of the Epimenides—hardly deserves to be called a 'paradox' if we think of entities of type $\langle\rangle$ as nothing other than truth-values and of entities of type $\langle\langle\rangle\rangle$ as nothing other than functions from truth-values to truth-values. For under this *extensionalistic* conception, when we say 'Let $x^{\langle\rangle}$ be $\forall x^{\langle\rangle}\,(P^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \to \neg x^{\langle\rangle})$', we are *not* telling our audience in any straightforward way what entity they are supposed to be considering. (To do that, we would have to tell them *either* 'Let $x^{\langle\rangle}$ be the True' *or* 'Let $x^{\langle\rangle}$ be the False'.) Rather, we are telling them to let $x^{\langle\rangle}$ be the True if everything$^{\langle\rangle}$ that $P^{\langle\langle\rangle\rangle}$ maps to the True is *not* the True, and the False otherwise. Under this extensionalistic interpretation, the 'paradox' amounts to nothing more than a proof by *reductio* of an unremarkable logico-mathematical fact about functions: namely that there is no function f such that, for some x, it's both the case (i) that f maps only x to the True and (ii) that x itself *is* the True iff everything that f maps to the True is not the True.[14]

However, once we adopt a *non*-extensionalistic interpretation, under which a formula such as '$\forall x^{\langle\rangle}\,(P^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \to \neg x^{\langle\rangle})$' does not denote a truth-value but a *state*, we

[13]Cf., e.g., Chierchia (1982), Menzel (1986: 3f.; 1993: 64f.), Bealer & Mönnich (2003: §10).

[14]Similar remarks apply if one thinks of entities of type $\langle\rangle$ as sets of possible worlds. Indeed it might be wondered whether we should not, following Lewis (1986), conceive of intensional entities along these lines; but here I shall be taking for granted that the answer is 'no'. (For relevant critical discussion, see, e.g., Schnieder [2004: 72f.].)

have a very different situation. For now, when we say 'Let $x^{\langle\rangle}$ be $\forall x^{\langle\rangle}\,(P^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \rightarrow \neg x^{\langle\rangle})$', our audience may rightly take the formula '$\forall x^{\langle\rangle}\,(P^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \rightarrow \neg x^{\langle\rangle})$' to be straightforwardly descriptive of what entity $x^{\langle\rangle}$ is supposed to be: namely, the state that every state that instantiates $P^{\langle\langle\rangle\rangle}$ fails to obtain. Accordingly they will want to know how it is that this particular state cannot be the only state that instantiates $P^{\langle\langle\rangle\rangle}$. The paradox would seem to show that it just can't. But in the absence of a plausible mechanism, this is deeply unsatisfactory.[15]

A seventh and final worry has to do with the fact that STT leads to awkward indeterminacies akin to the truth-teller paradox. For example, suppose that some property $Q^{\langle\langle\rangle\rangle}$ is instantiated by the state $\exists x^{\langle\rangle}\,(Q^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \wedge x^{\langle\rangle})$ and by nothing else, and consider the claim that the state just mentioned obtains. Is this claim true? Except in certain special cases, there appears to be nothing that could determine the issue.[16] Arthur Prior, in discussing the same difficulty, in effect concludes that "it can never be" that $\exists x^{\langle\rangle}\,(Q^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \wedge x^{\langle\rangle})$ is the only thing$^{\langle\rangle}$ that instantiates $Q^{\langle\langle\rangle\rangle}$ (1971: 91f.). But this is a hard doctrine to maintain. Short of resolving the indeterminacy by hoping that the puzzling case can never arise, the STTler appears to be faced with a *prima facie* unappealing choice between dialetheism and epistemicism: *dialetheism* if it is held that, while the claim in question "fails to be true and fails to be false, nevertheless it is either true or false",[17] and *epistemicism* if it is held that the claim is either true or false but there is no telling which.[18]

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[15]Prior (1961) and Bacon, Hawthorne & Uzquiano (2016: 532) describe essentially the same conundrum. By similar reasoning (also within STT), already Chwistek reached the puzzling conclusion that "I cannot [...] consider as interesting those and only those propositions which I wish so to consider" (1921: 344f.). Related problems include the Russell–Myhill paradox and intensional analogues of, e.g., Grelling's paradox and the 'infinite liar' paradox devised by Yablo (1993). A form of the Russell–Myhill paradox will be briefly discussed in footnote 39 below. For relevant discussion of Grelling's paradox, see Church (1976).

[16]One of the "special cases" just alluded to is the case in which $Q^{\langle\langle\rangle\rangle}$ is the property of *being identical with s*, for some state s. (For example, let s be the state that snow is white.) The question of whether $\exists x^{\langle\rangle}\,(Q^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \wedge x^{\langle\rangle})$ obtains will then come down to whether s obtains. Moreover, the stipulation that $Q^{\langle\langle\rangle\rangle}$ should itself be instantiated by $\exists x^{\langle\rangle}\,(Q^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \wedge x^{\langle\rangle})$ "and by nothing else" will be satisfied as long as states are individuated in a sufficiently coarse-grained manner. In particular, s has to be identical with $\exists x^{\langle\rangle}\,(Q^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \wedge x^{\langle\rangle})$—i.e., with the state *that some state identical with s obtains*. This requires a somewhat coarse-grained conception of states, but the required level of coarse-grainedness is arguably far from implausible. So here we have a case in which, under not-implausible assumptions, the question of whether $\exists x^{\langle\rangle}\,(Q^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \wedge x^{\langle\rangle})$ obtains is *not* left indeterminate. (Thanks to Andrew Bacon and Jeremy Goodman for alerting me to this point.) Even so, it is clearly a rather special case. If $Q^{\langle\langle\rangle\rangle}$ were instead, say, the property of *being Quine's favorite state*, there would be no such easy path to determinacy.

[17]The quotation is from Mortensen & Priest (1981: 385), who discuss a simpler truth-teller paradox.

[18]In this connection, cases of 'reciprocal negation'—pairs of sentences, propositions, or states of which each 'says' of the other that it is false—also deserve consideration. (For discussion, see, e.g., Sorensen [2001: ch. 11; 2018: §9], Armour-Garb & Woodbridge [2006], Greenough [2011]. As Sorensen notes, the problem goes back to Buridan.) Let $P^{\langle\langle\rangle\rangle}$ and $Q^{\langle\langle\rangle\rangle}$ be the properties of, respectively, *being Prior's favorite state* and *being Quine's favorite state*, let $p^{\langle\rangle}$ and $q^{\langle\rangle}$ be, respectively, the states $\forall x^{\langle\rangle}\,(P^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \rightarrow \neg x^{\langle\rangle})$ and $\forall x^{\langle\rangle}\,(Q^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \rightarrow \neg x^{\langle\rangle})$, and suppose that $p^{\langle\rangle}$ is the only thing that

3 Ordinal Type Theory: Language and Ontology

As we have seen, the simple theory of types, when interpreted as a theory of intensional entities, faces various problems. The purpose of this and the next two sections, and the main purpose of this paper, is to provide an alternative theory. This calls for some expectation management. *Qua* theory of intensional entities, STT must count as a piece of metaphysics, and metaphysics rather than logic will be our primary concern in the following. Accordingly, although I will be introducing a formal language, I will *not* be formulating any inference rules or logical axioms. Any reasoning will be done in the meta-language, using classical logic.

The formal language in question will be simply called '$\mathcal{L}$'. In the rest of this section I will outline the syntax and semantics of $\mathcal{L}$ as well as an ontology of intensional entities. We have to begin with set-theoretic foundations and the conception of sets as complex properties. Section 3.2 then specifies the syntax of $\mathcal{L}$, while Section 3.3 will introduce a number of ontological assumptions. These will provide the basis for the semantics of $\mathcal{L}$, which will be laid out in Section 3.4.

3.1 Sets

Although sets are commonly thought of as *particulars*, i.e., entities that are neither attributes nor states, this conception is by no means forced upon us. Carnap (1947: §23) suggests a "reduction of extensions to intensions" under which sets are identified with properties of a certain special sort. Alternatively, one might take an *eliminativist* approach and let the theoretical work of sets be done by suitable properties, as proposed by Bealer (1982: 116–19). The choice between these two approaches will make little difference in practice. (Some suggest that it makes no difference at all: see Oliver [1996: 19n.].) Either option achieves the unification, alluded to in the Introduction, of the overall ontological picture: the first by assimilating sets, the second by eliminating them. Here I will take the first route and, following Carnap, identify sets with properties. More particularly, for any entities $x_1, x_2, \ldots$, the set $\{x_1, x_2, \ldots\}$ will be conceived of as the property of *being identical with* x_1 *or* x_2 *or* $\ldots$, while the empty set $\varnothing$ will be taken to be the uninstantiable property of *not being identical with anything*. Using vocabulary and notation that will be more formally introduced further below (although for the moment an intuitive grasp should suffice), this identification can be expressed by a pair of definitions:

Definition 1. Something is a *set* iff it is *either* the property $\lambda x\, \neg\exists y\, (x = y)$ *or*, for some (one or more) entities $x_1, x_2, \ldots$, the property $\lambda x\, ((x = x_1) \vee (x = x_2) \vee \ldots)$.

Definition 2. An entity x is a *member of* a set y iff x instantiates y.

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instantiates $Q^{\langle\langle\rangle\rangle}$ while $q^{\langle\rangle}$ is the only thing that instantiates $P^{\langle\langle\rangle\rangle}$. Question: Which of $p^{\langle\rangle}$ and $q^{\langle\rangle}$ obtains? The symmetry of the situation strongly suggests that they either both obtain or both fail to obtain, but either of these options leads to a contradiction.

We need both of these definitions, because Definition 1 by itself does not tell us, e.g., that $\lambda x \, \neg\exists y \, (x = y)$ is an *empty* set: a set without members. Nor does it tell us that, for any y, the property $\lambda x \, (x = y)$ has y as a member.

The syntax and semantics of lambda-expressions, which are relied on in Definition 1, will below be specified under heavy use of set-theoretic resources. Since Definition 1 tells us what it is to be a set, this might at first look circular. However, for the purposes of specifying the syntax and semantics of lambda-expressions, *what a set is* will matter much less than *what sets there are* (e.g., whether there is an infinite set). With regard to this latter question, we will adopt the usual axioms of Zermelo–Fraenkel set theory (including Choice), with quantifiers restricted to sets where appropriate,[19] and with the schemata of Separation and Replacement understood in such a way that their instances may contain vocabulary from other parts of the theory, such as 'state' and 'lambda-expression'.

3.2 Syntax

3.2.1 Preliminaries

In Section 3.3.3 below, we will adopt an existence assumption for attributes that 'generates' attributes from terms of $\mathcal{L}$. As a result, the more restricted $\mathcal{L}$'s syntax, the more impoverished the resulting ontology of attributes. But our ontology has to be at least rich enough to make it possible to regard all sets as properties, in accordance with Definition 1; and, more generally, a more plenitudinous ontology of attributes can be expected to be more useful as a background theory for metaphysical theorizing. We thus have excellent reason to allow the expressions of $\mathcal{L}$ to reach any set-sized length—to allow, for instance, that there is a lambda-expression $\ulcorner \lambda x \, ((x = v_1) \vee (x = v_2) \vee \ldots) \urcorner$ for any sequence of variables $v_1, v_2, \ldots$ of a length greater than zero. The natural way to do so is to let the expressions of $\mathcal{L}$ themselves be sequences, i.e., functions, conceived of as sets of ordered pairs, defined on initial segments of the ordinals. For a simple example, an expression of length 2 will be a set $\{(0, x), (1, y)\}$, for some entities x and y. Ordinals, in turn, will be conceived of *à la* von Neumann (1923/1967), meaning that every ordinal is the set of all its predecessors. Thus every ordinal is *itself* an initial segment of the ordinals. If a function has as its domain a given ordinal, then that ordinal will be called the 'length' of that function (*qua* sequence). Since there are transfinite ordinals, it follows that sequences can be of transfinite length.

Written representations of the expressions of $\mathcal{L}$ will form part of the meta-language, where they will function semantically in the same way in which the expressions of $\mathcal{L}$ function in $\mathcal{L}$. Quotation marks will be used ambiguously, serving to construct names of expressions of $\mathcal{L}$ as well as names of their written representations, with disambiguation being left to context.[20] These representations will be constructed in

[19]For example, the extensionality axiom will be to the effect that no two *sets* have all their members in common.

[20]For example, when below it is stipulated that '$\top$' abbreviates '&()', the quotation marks should

accordance with conventions detailed in Section 3.2.2.

A few general features of $\mathcal{L}$'s syntax are worth flagging in advance:

- Formulas and lambda-expressions are *terms* and can appear as elements of argument lists.
- Formulas can appear at predicate-position, and lambda-expressions can appear at sentence-position. (So, e.g., '$\neg\lambda x\, F(x)$' is a well-formed formula. However, the semantics of $\mathcal{L}$ will ensure that this formula has a denotation *only* relative to a variable-assignment relative to which '$\lambda x\, F(x)$' denotes a state. As a result this formula has a denotation relative to any given variable-assignment only if some property is also a state; which is ruled out by an assumption adopted in Section 3.3.3.)
- We distinguish between *typed* and *untyped* variables. Typed variables will be said to have positive (i.e., non-zero) ordinals as 'types'.

So as to leave no questions about the relationship between typed variables and their respective types, the nature of $\mathcal{L}$'s atomic terms will be specified in a little more detail than strictly necessary.

3.2.2 Atomic terms

From what has been said above, it follows that we take any $\mathcal{L}$-expression of length 1 to be a set whose only member is an ordered pair $(0, x)$ (i.e., a set $\{\{0\}, \{0, x\}\}$), for some entity x. Let us now stipulate that, for any positive ordinal α, a set $\{(i, x_i) \mid i < \alpha\}$ is an *expression of* $\mathcal{L}$ iff each x_i satisfies one of the following three conditions:

(a) It is one of the ordinals $0, \ldots, 6$.

(b) It is an ordered pair $(0, \alpha)$ or $(1, \alpha)$, for some ordinal α.

(c) It is an ordered pair (α, β), for some ordinals $\alpha > 1$ and $\beta > 0$.

While in principle every (finitely long) expression of $\mathcal{L}$ could be written using standard set-theoretic notation, this would not be very convenient to do. The following list provides some conventions for an alternative way of writing expressions of $\mathcal{L}$; this also offers an opportunity to introduce some terminology that will become relevant later on:

- If x is one of the ordinals $0, \ldots, 6$, then the expression $\{(0, x)\}$ will be written as '$\neg$', '&', '$\exists$', 'λ', ',', '(', and ')', respectively.
- If x is the ordered pair $(0, 0)$, then $\{(0, x)\}$ will be written as '$\mathbf{I}$' and will be called a *logical constant*.

be understood as serving to construct names of *written representations* of $\mathcal{L}$-expressions. Using them instead to form names of $\mathcal{L}$*-expressions*, we would have to say that '$\top$' is *identical* with '&()'. Quinean corner-quotes will be used in a similarly ambiguous way.

- If x is an ordered pair $(0, \alpha)$, for some ordinal $\alpha > 0$, then $\{(0, x)\}$ will be written as an unitalicized string and will be called a *non-logical constant*.
- If x is an ordered pair $(1, \alpha)$, for some ordinal α, then $\{(0, x)\}$ will be written as an italicized letter, possibly with sub- but without superscript, and will be called an *untyped variable*.[21]
- If x is an ordered pair (α, β), for some ordinals $\alpha > 1$ and $\beta > 0$, then $\{(0, x)\}$ will be written as an italicized letter (possibly with subscript) that carries as superscript a numeral or other expression representing β, and will be called a *typed variable*.
- Except for the use of abbreviatory devices (such as ellipses), expressions of any length greater than 1 will be written by concatenating representations of expressions of length 1.

Constants and variables together will be called *atomic terms*. An ordinal β will be said to be the *type* of a given variable $\{(0, x)\}$ iff, for some ordinal $\alpha > 1$, x is identical with (α, β). Note that the variables that have a type in this sense are exactly those that have above been labelled as 'typed'. In a sense to be made clearer in Section 3.4, *un*typed variables will be allowed to 'range' over everything there is,[22] whereas a variable of type α (for any $\alpha > 0$) will range only over all entities whose respective order is less than α.

3.2.3 Complex terms

There are in $\mathcal{L}$ two kinds of non-atomic (or *complex*) terms, namely formulas and lambda-expressions. Their syntax can be specified recursively:

(i) For any term t: $\ulcorner \neg t \urcorner$ is a formula.

(ii) For any (possibly empty) sequence of terms $t_1, t_2, \ldots$:[23] $\ulcorner \&(t_1, t_2, \ldots) \urcorner$ is a formula.

(iii) For any (one or more) pairwise distinct variables $v_1, v_2, \ldots$ and any term t: $\ulcorner \exists v_1, v_2, \ldots\ t \urcorner$ is a formula and $\ulcorner \lambda v_1, v_2, \ldots\ t \urcorner$ is a lambda-expression.

(iv) For any term t and any (one or more) terms $t_1, t_2, \ldots$: $\ulcorner (t)(t_1, t_2, \ldots) \urcorner$ is a formula.

(v) Nothing else is a formula or lambda-expression.

[21] As for *which* letter is to be used to represent a given variable, any convention at all would do, and so I shall leave this open. That there are far more ordinals than letters will not matter in practice.

[22] This formulation ("everything there is") might give rise to meta-ontological qualms, as it appears to presuppose an absolutely general (as opposed to 'indefinitely extensible') domain of entities. I acknowledge this presupposition, but will here not be trying to defend it.

[23] Throughout this paper, unless otherwise specified, sequences may be of any set-sized length.

Appropriate notions of predicate- and sentence-position, as well as of a bound or free variable-occurrence, may be defined more or less as one would expect. Thus, an occurrence of a term t will be said to stand *at predicate-position* iff it is the first occurrence of t in an occurrence of $\ulcorner(t)(t_1, t_2, \ldots)\urcorner$, for one or more terms $t_1, t_2, \ldots$; and it will be said to stand *at sentence-position* iff it is immediately preceded by (a) an occurrence of '$\neg$' or (b) an occurrence of '&(' or (c), for one ore more terms $t_1, t_2, \ldots$, an occurrence of $\ulcorner\&(t_1, t_2, \ldots,\urcorner$ or (d), for one or more variables $v_1, v_2, \ldots$, an occurrence of either $\ulcorner\exists v_1, v_2, \ldots\urcorner$ or $\ulcorner\lambda v_1, v_2, \ldots\urcorner$. For example, the variable 'x' has in '$\neg\&(x, \neg x)$' exactly two free occurrences at sentence-position.

3.2.4 Abbreviations

Besides ellipses, we will make use of several other abbreviatory devices. '$\top$' and '$\bot$' will respectively abbreviate '$\&()$' and '$\neg\&()$'. For any two terms t_1 and t_2, $\ulcorner(t_1 = t_2)\urcorner$ and $\ulcorner(t_1 \neq t_2)\urcorner$ will respectively abbreviate $\ulcorner(\mathbf{I})(t_1, t_2)\urcorner$ and $\ulcorner\neg(\mathbf{I})(t_1, t_2)\urcorner$, while $\ulcorner(t_1 \to t_2)\urcorner$ will abbreviate $\ulcorner\neg\&(t_1, \neg t_2)\urcorner$. Further, for any ordinal $\alpha > 1$ and any α-sequence of terms $t_1, t_2, \ldots$: $\ulcorner(t_1 \wedge t_2 \wedge \ldots)\urcorner$ and $\ulcorner(t_1 \vee t_2 \vee \ldots)\urcorner$ will respectively abbreviate $\ulcorner\&(t_1, t_2, \ldots)\urcorner$ and $\ulcorner\neg\&(\neg t_1, \neg t_2, \ldots)\urcorner$, while $\ulcorner(t_1 \leftrightarrow t_2 \leftrightarrow \ldots)\urcorner$ will abbreviate $\ulcorner\neg\&(\neg\&(t_1, t_2, \ldots), \neg\&(\neg t_1, \neg t_2, \ldots))\urcorner$. Lastly, for any (one or more) variables v_1, v_2, ... and any term t, $\ulcorner\forall v_1, v_2, \ldots\, t\urcorner$ will abbreviate $\ulcorner\neg\exists v_1, v_2, \ldots\, \neg t\urcorner$. Parentheses will often be omitted where no confusion is likely to arise, but will always be left in place around occurrences of complex terms at predicate-position.

3.3 Ontology

When it comes to the metaphysics of intensional entities, the literature offers at least six broad options. In no particular order we have, first, the *higher-orderist* approach, which takes properties, relations, and states to be *sui generis* entities inhabiting various logical types in accordance with STT. Second, there is the *possibilist* approach familiar from Montague (1969) and Lewis (1986: §1.5), under which properties are identified with, e.g., sets of *possibilia* or functions from possible worlds to extensions. A third approach, exemplified by Williams (1953), identifies properties with equivalence classes of 'tropes'. Fourth, there is the *broadly Aristotelian* approach that takes properties to be in some sense 'immanent' to their respective bearers. (Typically this approach is somewhat at odds with the existence of relations; cf., e.g., Lowe [2016].) Fifth, we have the *algebraic* approach endorsed by, e.g., Bealer (1982; 1998), Zalta (1983; 1988), and Menzel (1986; 1993), under which intensional entities are arranged in an algebra generated by (quasi-)logical operations. And sixth, there is what Bealer (1989: 187) calls the *semi-reductionistic* approach, which takes particulars and fundamental attributes as basic and identifies all other intensional entities with set-theoretic constructions over a space of such 'basic' entities. The ontology to be proposed below may be considered something of a middle ground between the higher-orderist and the algebraic approach: it deviates from the former in abandoning STT, and it differs from the latter in that it (i) eschews the highly fine-grained conception of intensional entities that the algebraic approach usually goes along with and (ii) relies on a meta-linguistic comprehension axiom, rather than algebraic principles, to generate ontological commitment to non-fundamental attributes.

3.3.1 Basic concepts

The following concepts will be taken as primitive: being a *state*; *obtaining*; being *a negation of* something; being *a conjunction of* some (or possibly no) things;[24] being *an existential quantification of* something; being *an instantiation of* something *by* some entities (in a given order); and the concept of the *identity relation*. The concepts of attribute and relation, among others, can be defined on the basis of those of state and instantiation, as follows:

Definition 3. An entity x is an *attribute* iff, for some non-empty sequence of entities $x_1, x_2, \ldots$, there exists an instantiation of x by $x_1, x_2, \ldots$, in this order.

Definition 4. Something is an *intensional entity* iff it is an attribute or state.

Definition 5. Something is a *particular* iff it is not an intensional entity.

Definition 6. An ordinal α is an *adicity* of an intensional entity x iff the following two conditions are satisfied:

(i) If $\alpha = 0$, then x is a state.

(ii) If $\alpha > 0$, then there exists an α-sequence (i.e., a sequence of length α) of entities $x_1, x_2, \ldots$ such that x has an instantiation by $x_1, x_2, \ldots$ (in this order).

Definition 7. A *property* is an attribute with an adicity of 1.

Definition 8. A *relation* is an attribute with an adicity of at least 2.

Definition 9. Some entities $x_1, x_2, \ldots$ (in this order) *instantiate* an attribute A iff there exists an obtaining instantiation of A by $x_1, x_2, \ldots$ (in this order).

These definitions call for a general hermeneutic remark. From the perspective of higher-order metaphysics, it may seem natural to understand the present talk of states and attributes as carrying a somewhat dubious commitment to certain special kinds of 'individuals' or 'first-order entities' (a.k.a. 'entities of type *e*') called 'states' and 'attributes',[25] coupled with a *failure* to recognize the plethora of entities *not* of type *e*. This interpretation should be resisted. Although talk of 'individuals' may *prima facie* sound perfectly innocent, it is unclear how it should be understood in

[24] The conjunction of "no" things is the *empty* conjunction, symbolized as '&()' or '$\top$'. It can be thought of as the state that obtains "no matter what".

[25] Cf., e.g., Goodman (forthcoming: §3.3).

the context of higher-order metaphysics other than in a way that effectively *presupposes* STT, as referring to the members of one of the latter's most basic categories.[26] Having rejected STT, the proponent of the present theory is consequently justified in suspecting that the higher-orderist's talk of 'individuals' is semantically defective, similar to an 18th-century chemist's talk of 'phlogiston'; and so it should strike her as equally problematic if her own claims about intensional entities are characterized as claims about certain kinds of 'individuals'.

But if the present theory's talk of states and attributes cannot be understood as talk about 'individuals', how else is the STTler to interpret this theory? This question will seem perplexing only to someone who thinks that it should somehow be possible to interpret the present theory *within* STT. But it isn't, and nor should it be: the present theory is a rival to STT, and must be understood as such.

3.3.2 States

With regard to states, we adopt, to begin with, the following five assumptions:

(S1) For any state s, there exists exactly one negation of s.

(S2) For any set of states S, there exists exactly one conjunction of the members of S.

(S3) For any attribute A, there exists exactly one existential quantification of A.

(S4) For any attribute A and any sequence of entities $x_1, x_2, \ldots$, there exists at most one instantiation of A by $x_1, x_2, \ldots$ (in this order).

(S5) For any entities x and y, there exists an instantiation of the identity relation by x and y (in this order).

It should further be understood that: anything that obtains is a state; only a state has a negation; the negation of a state s is a state that obtains iff s itself doesn't; only the members of a (possibly empty) set of states have a conjunction; the conjunction of the members of a set of states S is a state that obtains iff each member of S obtains; only an attribute has an existential quantification; the existential quantification of an attribute is a state that obtains iff that attribute has an obtaining instantiation; any

[26] It might be thought that talk of 'individuals' should be understood as referring to all and only those things that can be denoted by singular terms; but this is hardly convincing. Let x be some non-individual entity, for instance of type $\langle e, \langle\rangle\rangle$: on the face of it, nothing prevents us from introducing a singular term that denotes x. The STTler may point out that the relevant denotation relation would have to be of a different type than that which holds between singular terms and *individuals*. But this would be to presuppose, if not STT itself, then some similarly problematic division of entities into logical types. (Likewise if it is suggested that the individuals are just those things that are ranged over by 'our first-order quantifiers'.) Alternatively it might be supposed that an individual is simply anything that is neither an attribute nor a state. (Cf. Whitehead & Russell [1910: 53], where individuals are understood to be "objects which are neither propositions nor functions".) But someone who takes this route can of course no longer maintain that a commitment to the existence of attributes and states is a commitment to the existence of certain special kinds of 'individuals'.

instantiation is a state; and any entities x and y instantiate the identity relation iff x is numerically the same entity as y.

So far, very little has been said about the individuation of states. To remedy this, we first have to introduce a concept of 'analytic entailment', which rests on the semantics of $\mathcal{L}$ (to be specified in Section 3.4).

In general, a term of $\mathcal{L}$ will be said to have or lack a denotation only *relative to* (i) an interpretation of non-logical constants and (ii) an assignment of entities to variables. For brevity's sake, I will simply speak of *interpretations* and *variable-assignments*, meaning functions defined on sets of constants and sets of variables, respectively. Reference to either of these functions may be omitted if the function in question is just the empty set. (So, instead of saying that a certain term denotes an entity x relative to the empty interpretation and a variable-assignment g, I will simply say that it denotes x relative to g.) I shall write 'denotes$_{I,g}$' as shorthand for 'denotes relative to I and g'. The concept of analytic entailment can now be defined as follows:

Definition 10. A term t *analytically entails* a term t' iff (i) t denotes a state relative to some interpretation and variable-assignment, and (ii) for any interpretation I and variable-assignment g: if t denotes$_{I,g}$ a state, then so does t', and if t denotes$_{I,g}$ an *obtaining* state, then so does t'.

On this basis we can next define a concept of ('analytic') necessitation:

Definition 11. A state s *necessitates* a state s' iff there exist an interpretation I, a variable-assignment g, and terms t and t' such that (i) t and t' respectively denote$_{I,g}$ s and s', and (ii) t analytically entails t'.

As a straightforward consequence of the semantics of $\mathcal{L}$, every entity is denoted by some variable relative to some variable-assignment. Since every term that denotes a state relative to some variable-assignment analytically entails itself, it follows that *every state necessitates itself*. This will become relevant in Section 5.5. For now, let us simply adopt the following assumption about the individuation of states:

(S6) No two states necessitate each other.

This assumption encapsulates the (highly plausible) idea that, if two states are to be distinct, there has to be some difference in what they require of the world—some difference in what the world has to be like in order for them to obtain. To illustrate this, consider a typical example of two terms that analytically entail each other: 'p' and '$\neg\neg p$'. For any states s and s', if there exists a variable-assignment g such that 'p' and '$\neg\neg p$' respectively denote$_{\emptyset,g}$ s and s', then s and s' necessitate each other, so that, by (S6), they are one and the same state. It follows that every state is identical with its double negation.

3.3.3 Attributes

We now have to formulate three assumptions that are more specifically concerned with attributes. The first of these reads as follows:

(A1) Every attribute has at most one adicity.

That every attribute has *at least* one adicity follows already from Definitions 3 and 6. Given (A1), we have that every attribute has *exactly* one adicity.[27] We can further note:

Corollary 1. No attribute is a state.

For, from Definitions 3 and 6, it follows that every attribute has a *positive* adicity and that every state has an adicity of zero. Hence, if a given attribute A were also a state, then A would have at least two adicities, contrary to (A1).

To formulate our next assumption, and also for the purposes of Section 5 below, we will first need a few definitions revolving around the notion of a *converse*:

Definition 12. For any ordinal $\alpha > 0$, any permutation f on α, and any α-adic attribute A: an entity x is an f*-converse* of A iff, for any α-sequence of entities $x_0, x_1, \ldots$ and any state s, the following holds: s is an instantiation of x by $x_{f(0)}, x_{f(1)}, \ldots$, in this order, iff s is an instantiation of A by $x_0, x_1, \ldots$, in this order.

Definition 13. An entity x is a *trivial* converse of an attribute A iff, for some ordinal α: A is α-adic, and x is an id_α-converse of A, where id_α is the identity permutation on α.

Definition 14. An entity x is a *non-trivial* converse of an attribute A iff, for some ordinal α and some non-identity permutation f on α: A is α-adic, and x is an f-converse of A.

Definition 15. Something is a *converse* of an attribute A iff it is a trivial or non-trivial converse of A.

It bears emphasis that a non-trivial converse of an attribute A is not simply a converse of A that fails to be a trivial converse of A. Nothing in these definitions rules out that an attribute may be both a trivial *and* a non-trivial converse of some other attribute (or even itself).

Equipped with the concept of a trivial converse, we can formulate the following assumption:

[27] An objector might argue that there are, or may well be, *multigrade* relations, which have more than one adicity. (Cf. MacBride [2005: §2].) To adjudicate this issue, we would have to clarify what, exactly, a relation should be taken to be. For now it may suffice to say that (A1) recommends itself at least as a simplifying assumption, and that its main purpose within the present theory lies in settling certain questions that would otherwise be left as glaring lacunae. For an example that will become relevant in Section 5.4, consider the question of whether the tautologous formula '$x = x$' analytically entails '$x = \lambda y\, x(y)$'. With the help of (A1), this can be answered in the negative. For, under the semantics of $\mathcal{L}$, '$x = x$' denotes a state relative to every variable-assignment that is defined on 'x', whereas '$x = \lambda y\, x(y)$' has a denotation relative to a given variable-assignment *only* if the latter maps 'x' to a property. Let now g be some assignment that maps 'x' to the identity relation. By (S5), this relation is dyadic, and hence, given (A1), *not* monadic: it is not a property. Consequently, relative to g, '$x = x$' denotes a state while '$x = \lambda y\, x(y)$' denotes nothing at all. So '$x = x$' does not analytically entail '$x = \lambda y\, x(y)$'.

(A2) Every attribute has at most one trivial converse.

It is easy to verify, based on the relevant definitions alone, that every attribute is a trivial converse of itself. With (A2), it follows that every attribute is its own *only* trivial converse. To appreciate the import of (A2), it may be helpful to unpack it using the above definitions:

(A2′) For any attribute A, there exists at most one attribute B such that, for some ordinal α, it is both the case that (i) A is α-adic and (ii) for any α-sequence of entities $x_0, x_1, \ldots$ and any state s, the following holds: s is an instantiation of B by $x_0, x_1, \ldots$, in this order, iff s is an instantiation of A by $x_0, x_1, \ldots$, in this order.

In the literature on higher-order metaphysics, an analogue of this thesis is known as 'Functionality'.[28] Given that instantiations are states, (A2) effectively ties the individuation of attributes to that of states: if states are individuated in a certain coarse-grained manner (on which see (S6) above), then (A2) will ensure that the individuation of attributes is similarly coarse-grained. It cannot happen, for instance, that triangularity and trilaterality are two distinct properties while, for any x, something is an instantiation of triangularity by x just in case it is an instantiation of trilaterality by x.

Finally, we come to the present theory's central existence assumption for attributes, which is essentially a comprehension axiom:

(A3) For any ordinal $\alpha > 0$, any α-sequence of pairwise distinct variables $v_0, v_1, \ldots$, any term t, any interpretation I, and any variable-assignment g: if t denotes$_{I,g}$ a state and, for each $i < \alpha$, the following two conditions are satisfied—

(i) v_i occurs free in t.

(ii) If v_i is identical with t or has in t a free occurrence at predicate- or sentence-position, then v_i is typed.

—then there exists at least one attribute A such that, for any α-sequence of entities $x_0, x_1, \ldots$ and any state s: s is an instantiation of A by $x_0, x_1, \ldots$ (in this order) iff, for some variable-assignment h that is just like g except that it maps each v_i to the corresponding x_i, s is denoted$_{I,h}$ by t.

For a simple example of how this works, consider the formula '$(\mathbf{I})(x, x)$', which (by what has been said in Section 3.2.4) may also be written as '$(x = x)$'. By the semantics to be specified below, this formula denotes, relative to any interpretation I and any variable-assignment g that maps 'x' to some entity x, the instantiation of the identity relation by x and x—provided that such an instantiation exists. By (S5) above, there does indeed exist such an instantiation. (For instance, let x be the

[28]Cf. Dorr (2016: 100). An earlier version can be found in Ramsey (1925/1931: 35). Anderson's (1986: §1) criticism can be answered by pointing out that the domain of quantification contains all sets and hence contains proper-class many entities.

identity relation.) Notice also that 'x' does not have in '$(x = x)$' any free occurrences at predicate- or sentence-position. So, if we let α be 1, let v_0 be 'x', let t be '$(x = x)$', and let I be any interpretation and g any variable-assignment that maps 'x' to some entity, then the antecedent of (A3)'s main conditional will be satisfied. From (A3), it then follows that there exists an attribute A such that, for any entity x_0 and any state s: s is an instantiation of A by x_0 iff s is denoted by '$(x = x)$' relative to I and a variable-assignment that maps 'x' to x_0 while otherwise being just like g. In other words, a state is an instantiation of A by an entity x_0 iff it is the instantiation of the identity relation by x_0 and x_0; and from (A2) it follows that there is only at most one such attribute. We can accordingly regard A as the property of *being self-identical*.

3.4 Semantics

To specify the semantics of $\mathcal{L}$, we have to specify what is denoted by a given term of $\mathcal{L}$ relative to a given interpretation and variable-assignment. To this end, let us stipulate that, for any interpretation I and variable-assignment g, the clauses (d1)–(d8) below are satisfied. These are largely as one would expect, except perhaps for the reference to orders in (d3):

(d1) '**I**' denotes$_{I,g}$ the identity relation.

(d2) For any non-logical constant c and any x: c denotes$_{I,g}$ x iff I maps c to x.

(d3) For any variable v and any x: v denotes$_{I,g}$ x iff (i) g maps v to x and (ii) for any ordinal α: if v is of type α, then x is of some order $\beta < \alpha$.[29]

(d4) For any term t and any x: $\ulcorner \neg t \urcorner$ denotes$_{I,g}$ x iff t denotes$_{I,g}$ a state of which x is the negation.

(d5) For any (possibly empty) sequence of terms $t_1, t_2, \ldots$ and any x: $\ulcorner \&(t_1, t_2, \ldots) \urcorner$ denotes$_{I,g}$ x iff there exists a set S of states such that: each t_i denotes$_{I,g}$ some $s \in S$; each $s \in S$ is denoted$_{I,g}$ by some t_i; and x is the conjunction of the members of S.

(d6) For any (one or more) pairwise distinct variables $v_1, v_2, \ldots$, any term t, and any x: $\ulcorner \exists v_1, v_2, \ldots\ t \urcorner$ denotes$_{I,g}$ x iff $\ulcorner \lambda v_1, v_2, \ldots\ t \urcorner$ denotes$_{I,g}$ some attribute of which x is the existential quantification.

(d7) For any term t, ordinal $\alpha > 0$, any α-sequence of terms $t_1, t_2, \ldots$, and any x: $\ulcorner (t)(t_1, t_2, \ldots) \urcorner$ denotes$_{I,g}$ x iff t denotes$_{I,g}$ an attribute A such that, for some α-sequence of entities $x_1, x_2, \ldots$: each t_i denotes$_{I,g}$ the corresponding x_i, and x is an instantiation of A by $x_1, x_2, \ldots$ (in this order).

[29]Note that, for any *un*typed variable v, this second conjunct is trivially satisfied. Hence, any untyped variable will simply denote$_{I,g}$ whatever entity it is mapped to by g, provided that g *does* map it to some entity. (Since there are proper-class many variables, no variable-assignment is defined on all of them.)

(d8) For any ordinal $\alpha > 0$, any α-sequence of pairwise distinct variables $v_1, v_2, \ldots$, any term t, and any x: $\ulcorner \lambda v_1, v_2, \ldots\ t \urcorner$ denotes$_{I,g}$ x iff x is an α-adic attribute such that, for any α-sequence of entities $x_1, x_2, \ldots$ and any state s: s is an instantiation of x by $x_1, x_2, \ldots$ (in this order) iff, for some variable-assignment h that is just like g except that it maps each v_i to the corresponding x_i, s is denoted$_{I,h}$ by t.

Not coincidentally, this last clause bears some similarity to the above assumption (A3). As we have seen, that assumption—together with (S5)—has the consequence that there exists a property whose instantiation by any given entity x is the instantiation of the identity relation by x and x. From the above semantics, it can now be inferred that that property is denoted (relative to any interpretation and variable-assignment) by the lambda-expression '$\lambda x\,(x = x)$'.

A crucial fact about the above semantics, which will be made use of without comment in the rest of this paper, is the following: if a term t contains a free occurrence of some term that fails to have a denotation relative to some interpretation I and variable-assignment g, then t itself does not denote$_{I,g}$ anything, either. This can be shown by induction over the complexity of terms.

4 The Order-Theoretic Hierarchy

4.1 The Assignment of Orders

Under the above semantics, no typed variable can denote (relative to a given interpretation and variable-assignment) an entity of an order that is greater than or equal to the variable's type. Accordingly, for any typed variable v, the class of all entities whose order is *less* than v's type may also be called the *range* of v; whereas the range of an *un*typed variable is simply the class of all entities.[30]

We now have to proceed to the definition of 'order of', the basic idea of which can be put as follows: an entity has as its order the least ordinal that puts it outside the range of any variable that has a bound occurrence at predicate- or sentence-position in a term denoting that same entity relative to some interpretation and variable-assignment. (So far, so straightforward. However, to avoid the trivializing result that *every* entity is zeroth-order, an additional condition has to be imposed on the term in question, to the effect that any atomic term that is either identical with it or has in it a free occurrence at predicate- or sentence-position denotes either a particular or a fundamental intensional entity.)

Definition 16. An ordinal α is an *order of* an entity x (alternatively: x is of order α) iff α is the least ordinal β such that, for some interpretation I, variable-assignment g, and term t, the following three conditions are satisfied:

[30] The concept of *class* serves here merely as an informal convenience; likewise the concept of a variable's range. I am introducing the latter only for the purpose of formulating the basic idea behind the definition of 'order of', which is given below.

(o1) t denotes$_{I,g}$ x.

(o2) Any atomic term that is either identical with t or has in t a free occurrence at predicate- or sentence-position denotes$_{I,g}$ either a particular or a fundamental intensional entity.

(o3) For any variable v: if v has in t a bound occurrence at predicate- or sentence-position, then v is typed, and its type is less than or equal to β.

We can immediately note two corollaries:

Corollary 2. The states $\top$ and $\bot$ are zeroth-order.[31]

Corollary 3. Any particular and any fundamental intensional entity is zeroth-order.[32]

With the help of (S6) ("No two states necessitate each other"), we can further infer:

Corollary 4. For any ordinal α and any state s: s is of order α iff $\neg s$ is of order α.[33] Analogously for attributes.

By contrast, it does not follow that any *conjunction* of two or more αth-order states is again of order α. For example, let N be the state that Napoleon has all the zeroth-order properties that make for a great general. Relative to a suitable interpretation, N can be denoted by '$\forall x^1$ (great-making$(x^1) \to x^1$(Napoleon))'. Suppose for the moment that great-making is fundamental. Then, under the above definition, N will plausibly be classified as first- rather than zeroth-order.[34] Yet, by (S6), the conjunction of N with its own negation is nothing other than $(N \neq N)$; and under the assumption (to be adopted in Section 5.1 below) that the identity relation is fundamental, $(N \neq N)$ is *zeroth*-order.

The semantics of $\mathcal{L}$ invokes the concept of an entity's order; and now we have seen that the definition of 'order of' relies heavily on the semantics of $\mathcal{L}$, in that

[31] To see this, note that '$\top$' and '$\bot$' respectively abbreviate '&()' and '¬&()'. (Cf. Section 3.2.4 above.) So $\top$ and $\bot$ are respectively denoted, relative to any interpretation and variable-assignment, by '&()' and '¬&()', which contain no occurrences of any atomic terms.

[32] Let x be any particular or fundamental intensional entity, let I be the empty set, let t be any untyped variable, and let g be any variable-assignment that maps t to x. From clause (d3) of the above semantics, it then follows that t denotes$_{I,g}$ x; so condition (o1) on the right-hand side of Definition 16 is satisfied. Condition (o2) is also satisfied. And since t contains no bound variable-occurrence, condition (o3) is satisfied for $\beta = 0$.

[33] Let s be any state and α any ordinal. To prove the left-to-right direction, suppose that s is of order α. It is then clear that $\neg s$ is of *at most* order α, since, for any interpretation I, variable-assignment g, and term t: if t denotes$_{I,g}$ s, then $\ulcorner \neg t \urcorner$ denotes$_{I,g}$ $\neg s$. Now let β be any ordinal such that $\neg s$ is of order β. By what has just been said, we have that $\beta \leq \alpha$. Further, by an argument analogous to the foregoing, it can be seen that $\neg\neg s$ is of at most order β. But, by (S6), $\neg\neg s$ is nothing other than s. This shows that $\alpha \leq \beta$, and so we have that $\alpha = \beta$, as required. The right-to-left direction can be shown in a similar way.

[34] Whether N will *really* be classified as first-order depends, roughly put, on what other formulas it can be denoted by, and thus depends on how finely states are individuated. We will return to this topic in Section 5.5.3.

it makes the question of whether an entity is of a given order depend (to put it very roughly) on what terms that entity can be denoted by. This may at first seem objectionably circular. More concretely, a critic might worry that, for at least one entity x and some ordinal α, when we ask whether x is of order α, the definition will not produce any definite answer, because the question of what terms x can be denoted by will depend on the original question of whether x is of order α—or perhaps there will be an infinite chain of such dependencies. On closer inspection, this possibility turns out to be quite remote; but for reasons of space I will have to leave that inspection to the reader.

The same goes for the examination of alternative definitions of 'order of'. The above definition has the benefit of being relatively simple, but other, more complex definitions, and the different hierarchies to which they give rise, may also be worth discussing; and there may even be something to be said for an ontology that contains multiple such hierarchies existing side-by-side. For now, however, I will stick with a single hierarchy.

4.2 How the Paradoxes Are Avoided

According to Chihara (1979: 591), "nothing should be called a solution" of the semantic paradoxes unless it successfully diagnoses the disease(s) of which the paradoxes are merely, in Tarski's (1969: 66) famous words, symptoms. In the case of the liar paradox, Tarski's own diagnosis was to the effect that the disease underlying *this* particular malady is the *semantic universality* of natural language, as manifested in the fact that "for every sentence formulated in the common language, we can form in the same language another sentence to the effect that the first sentence is true or that it is false" (p. 67). Although Tarski's diagnosis has not been universally accepted, I take it to be reasonable to suspect that he was essentially right.[35] It should then seem natural to venture an analogous diagnosis in the case of the *intensional* paradoxes. Tarski was looking at a pathological feature of natural language: a truth predicate applicable to all declarative sentences. An analogous feature in the theory of intensional entities would be the assumption of a property of *obtainment* that has an instantiation by every state. Another instance of the same general pattern would be the assumption that there exists a property of non-self-instantiation that has an instantiation (obtaining or not) by any property whatsoever, including itself—an assumption that would lead to Russell's paradox of properties.

Suppose we accept a general diagnosis of the intensional paradoxes along these lines. The natural way to avoid them will then lie in adjusting our ontology. STT constitutes an approach in this vein, albeit a rather drastic one, since it institutes a complete ban on self-instantiation by means of syntactic rules: already the expression $\ulcorner v(v) \urcorner$ is ill-formed (for any variable v), and so the same goes for $\ulcorner \neg v(v) \urcorner$ and $\ulcorner \lambda v \, \neg v(v) \urcorner$. Indeed, not only are properties of (non-)self-instantiation banned from

[35] For an overview of the main approaches to avoiding semantic paradox, see, e.g., Horsten (2015: 687ff.) and Beall, Glanzberg & Ripley (2018: ch. 5). For more detailed critical discussion of some recently popular approaches, see Murzi & Rossi (2020a;b). Also cf. Hansen (2021), Sher (2023).

our ontology, but the very thought of them is supposed to be incoherent. The present theory, by contrast, seeks to avoid intensional paradox not by way of syntactic rules but rather by the weakening of an ontological assumption. This weakening is effected by the inclusion of clause (ii) in the antecedent of (A3)'s main conditional. Let us first see how this bears on Russell's paradox, and then turn to Epimenidean paradoxes. Thanks to said clause, (A3) does *not* commit us to the existence of a property $\lambda x \, \neg x(x)$; that is, it does not commit us to the existence of a property P such that, for any x and any state s: s is an instantiation of P by x if and only if s is denoted by '$\neg x(x)$' relative to a variable-assignment that maps 'x' to x. For the term '$\neg x(x)$' contains a free occurrence of 'x' at predicate-position, and 'x' is untyped. The existence of such a property would lead to Russell's paradox *if* it could be assumed that it had an instantiation by itself.

For a slightly different example, consider the property $\lambda x^1 \, \neg x^1(x^1)$. Assuming that it exists (which is to say: assuming that the term '$\lambda x^1 \, \neg x^1(x^1)$' has a denotation relative to the empty interpretation and variable-assignment), it will have instantiations *only* by entities of order zero.[36] Hence, if this property is first- rather than zeroth-order, then it will not have any instantiation by itself, thereby blocking the paradox. For another example of this sort, let R be the relation $\lambda x^1, y \, x^1(y)$, and let P be the property $\lambda x \, \neg R(x, x)$. A critic might wonder whether this won't lead to a Russellian paradox as soon as we ask whether P instantiates itself. Now, for P to instantiate itself is for there to be an obtaining instantiation of P by itself, which would require that P *have* an instantiation by itself, which would require that R have an instantiation by P and P. But since $R = \lambda x^1, y \, x^1(y)$, this relation will have an instantiation by P and P only if P is zeroth-order. And of course, even assuming that P *is* zeroth-order, this will not guarantee that R has an instantiation by P and P.[37]

At the beginning of Section 2.4 we supposed, working in the context of STT, that some property $P^{\langle\langle\rangle\rangle}$ is instantiated by no other state than $\forall x^{\langle\rangle} \, (P^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \rightarrow \neg x^{\langle\rangle})$. It then turned out that this state would obtain just in case it doesn't, and so we had a contradiction. As a first stab at constructing an analogous problem within the present system, it might be supposed that, for any property P, there exists a state $\forall x \, (P(x) \rightarrow \neg x)$, or equivalently, $\neg \exists x \, (P(x) \wedge x)$. If, for some property P, the corresponding state is the *only* thing that instantiates P, then we can again derive a contradiction. However, in order for there to be a state $\forall x \, (P(x) \rightarrow \neg x)$, there has to be a property $\lambda x \, (P(x) \wedge x)$. After all, '$\forall x \, (P(x) \rightarrow \neg x)$' abbreviates '$\neg \exists x \, \neg \neg \& ((P)(x), \neg \neg x)$', wherefore $\forall x \, (P(x) \rightarrow \neg x)$, if it exists, will be the negation of $\exists x \, \neg \neg (P(x) \wedge \neg \neg x)$, which by (S6) will be nothing other than $\exists x \, (P(x) \wedge x)$, which will be the existential quantification of $\lambda x \, (P(x) \wedge x)$. But now, due to clause (ii) of (A3), the present theory—assuming that it is consistent—*is not actually committed* to there being a property $\lambda x \, (P(x) \wedge x)$, for any given P. (That is, it is not committed

[36] The reason for this lies ultimately in clause (d3) of the above semantics.

[37] Incidentally, the property P, if it exists, will be nothing other than $\lambda x^1 \, \neg x^1(x^1)$. This can be seen from the semantics of lambda-expressions—given that $P = \lambda x \, \neg R(x, x)$ and $R = \lambda x^1, y \, x^1(y)$—in combination with (A2). Regarding the question of whether P *does* exist, cf. Section 5.1 below. Regarding the question of whether it is zeroth-order, see Section 5.5.3.

to there being a property Q such that, for any x and any state s: s is an instantiation of Q by x iff s is denoted by '$P(x) \wedge x$' relative to a variable-assignment that maps 'P' to P and 'x' to x. For the term '$P(x) \wedge x$' contains a free occurrence of 'x' at sentence-position, and 'x' is untyped.) As a result it is also not committed to there being a state $\forall x\,(P(x) \rightarrow \neg x)$, and the prospective paradox does not arise.

Let us now try to construct an Epimenidean paradox in a slightly different way. Given any property P that has an instantiation by at least one zeroth-order state—an example would be $\lambda x\,(x = x)$, which is instantiated, among other things, by $\top$—the present theory is committed to there being a property $\lambda x^1\,(P(x^1) \wedge x^1)$ and hence also a state $\forall x^1\,(P(x^1) \rightarrow \neg x^1)$. Suppose now that P is instantiated *only* by $\forall x^1\,(P(x^1) \rightarrow \neg x^1)$, and call this state '$E$' (for 'Epimenides'). Since '$\forall x^1\,(P(x^1) \rightarrow \neg x^1)$' is short for '$\neg\exists x^1\,\neg\neg\&((P)(x^1), \neg\neg x^1)$', we have that

(1) E obtains iff the property $\lambda x^1\,\neg\neg(P(x^1) \wedge \neg\neg x^1)$—which, given (A2) and (S6), is just $\lambda x^1\,(P(x^1) \wedge x^1)$—fails to have an obtaining instantiation.[38]

Further, with the help of the clauses (d3), (d5), (d7), and (d8) of the above semantics, we can infer that:

(2) The property $\lambda x^1\,(P(x^1) \wedge x^1)$ has an obtaining instantiation iff some zeroth-order entity both instantiates P and is an obtaining state.

Putting (1) and (2) together, we have:

(3) E obtains iff no zeroth-order entity both instantiates P and is an obtaining state.

Given that, by hypothesis, only E instantiates P, it follows from (3) that, if E is zeroth-order, then E obtains iff it doesn't. In other words (since 'E obtains iff it doesn't' is a contradiction), it follows that E fails to be zeroth-order. And fortunately, nothing in this argument or in our theory—assuming it is consistent—commits us to holding that E *is* zeroth-order. So paradox is again averted.

In essentially the same way, the theory also avoids the Russell–Myhill paradox and intensional versions of, e.g., the truth-teller and the reciprocal-negation paradox (i.e., the 'no-no paradox' in Sorensen's [2001] terminology).[39] This uniformity in the avoidance of different yet related paradoxes constitutes one of the main attractions of ordinal type theory.

[38]To verify this, one has to consult clauses (d4) and (d6) of the above semantics, bearing in mind that, as stated in Section 3.3.2, the negation of a state s obtains iff s itself doesn't, and that the existential quantification of an attribute obtains iff that attribute has an obtaining instantiation.

[39]On the reciprocal-negation paradox, cf. footnote 18 above. For a simple version of the Russell–Myhill paradox as it arises in the context of STT, consider the property F of *being Frege's only favorite entity of type* $\langle\langle\rangle\rangle$. Surely, for any *distinct* entities x and y of that type, the world will have to be at least a *little* bit different if $F(x)$ obtains than if $F(y)$ obtains; and so $F(x)$ has to be distinct from $F(y)$. (I am here omitting superscripts for the sake of readability.) But now let M be a property of type $\langle\langle\rangle\rangle$, of *being an entity x of type $\langle\rangle$ that is identical with the state $F(y)$ for some entity y of type $\langle\langle\rangle\rangle$ such that $y(x)$ fails to obtain*, or in symbols: $M = \lambda x^{\langle\rangle}\,\exists y^{\langle\langle\rangle\rangle}\,(\neg y^{\langle\langle\rangle\rangle}(x^{\langle\rangle}) \wedge (x^{\langle\rangle} = F^{\langle\langle\langle\rangle\rangle\rangle}(y^{\langle\langle\rangle\rangle})))$. If we ask whether the state $F(M)$ instantiates M, we are led to a contradiction. The upshot would seem to be that the STTler has to reject the existence of such properties as *being Frege's only favorite entity of type* $\langle\langle\rangle\rangle$. But provided that one buys into STT and its type distinctions at all, this is certainly an awkward result.

4.3 How the Hierarchy Is Motivated

It might now be urged that any restrictions enacted with a view to avoiding paradox ought to "result naturally and inevitably from our positive doctrines", as Russell puts it (1908: 226). In contrast to Russell's own ramified type theory, the present approach does not strictly speaking involve restriction so much as ontological *abstention*, in that it does not commit itself (as long as it is consistent) to the existence of certain problematic entities. However, it also involves ontological *expansion*, in the form of the order-theoretic hierarchy. Both aspects, abstention and (especially) expansion, stand in need of motivation. The former, embodied by clause (ii) of (A3), can be straightforwardly motivated by the need to avoid paradox. If a more 'philosophical' reason is needed, it can in addition be motivated by the "positive doctrine" that the question of whether a given state obtains should never lead back to that state itself or into an infinite regress.[40] As for the order-theoretic hierarchy, it may be taken to be embodied by the consequent of the mentioned clause. To motivate it is to produce reasons as to why that clause should not simply be replaced with the negation of its antecedent (in which case it might read: 'v_i is not identical with t and does also not have in t any free occurrence at predicate- or sentence-position'). In the rest of this section I will suggest two ways in which this might be done.

First, it might be argued that, even if there is no 'generally applicable' property of obtainment (i.e., no property $\lambda x\, x$), there should still be *something like it*—call it an 'approximation'—that can in at least some contexts function as a semantic value of our predicate 'obtains' and thereby keep it from being semantically defective.[41] One such property, and indeed the first of a series, is $\lambda x^1\, x^1$. But in order to avoid Epimenidean paradox (and in keeping with the aforementioned "positive doctrine"), states like $\forall x^1\,(P(x^1) \rightarrow \neg x^1)$ will in some cases, as we have seen, be assigned an order that puts them outside the class of entities by which $\lambda x^1\, x^1$ has an instantiation. So we need another, more generally applicable approximation that can function as

To see how the OTTist avoids being saddled with an analogous result, let G be the property of *being Gödel's only favorite entity*. To construct an analogue of M, the best we can do is to let N be the property of *being an entity x that is identical with the state $G(y)$ for some zeroth-order entity y such that $y(x)$ fails to obtain*, or in symbols: $N = \lambda x\, \exists y^1\,(\neg y^1(x) \wedge (x = G(y^1)))$. From the type of '$y^1$', it can be inferred that N is at most first-order; yet nothing (on the face of it) commits us to holding that N is zeroth-order. But if it isn't, then it falls outside of the range of 'y^1', which is enough to block the paradox.

[40]This "doctrine" is loosely related to Russell's vicious circle principle. It is also not too distantly related to Behmann's (1931) proposal, already briefly mentioned in Section 2.2 above, that no predicate is predicable of a given object unless the predication can be rewritten in primitive vocabulary. Behmann has presented a more sophisticated version of his proposal in his (1959). However, neither this nor the original version offers an escape from Epimenidean paradoxes.

[41]It is tempting to envision a broadly similar approach to the semantics of *vague* predicates, such as 'is bald' or 'is a table'. The idea would be that, even though there are no properties of tablehood and baldness (i.e., no properties precisely corresponding to 'is a table' and 'is bald'), there *are* properties that may be said to be picked out by possible precisifications of these predicates, and this is enough to keep them from being semantically defective—or at least enough to render them *less* semantically defective than, say, 'is phlogiston' or 'orbits Vulcan'.

the semantic value of our predicate 'obtains' when we talk about such states. In this way we are led to introduce the property $\lambda x^2\, x^2$; and so on up the levels of our hierarchy.

Against this line of reasoning it may be objected that having a semantic value is not the only possible way for a predicate to be semantically non-defective. In particular one might interpret 'obtains' in a deflationistic spirit, as a *syncategorematic operator* that (roughly speaking) takes a name of a state and yields a sentence that has that same state as its truth-condition. Although this would be a somewhat unnatural interpretation for a predicate, it cannot be dismissed out of hand; and so it may be thought that we do not need any properties of obtainment, after all. However, there is another way to motivate the hierarchy, using a slightly more complex example.

Consider the triadic predicate 'is an instantiation of ... by ...'. On the one hand, if we committed ourselves to there being a corresponding 'generally applicable' triadic relation $\lambda x, y, z\, \exists w\, ((w = y) \wedge (x = w(z)))$, we would be faced with something of a paradox.[42] On the other hand it is not easy to see how that predicate might be interpreted as syncategorematic. Unless we can appeal to some reductive analysis of instantiation, the best we can do will (on the face of it) be to say that 'is an instantiation of ... by ...' takes three names of entities x, y, and z and yields a sentence that has as its truth-condition the state that x is an instantiation of y by z. In other words, we are simply *re-using* the instantiation predicate, or using a synonymous expression in a meta-language; and so one could still ask how it is that that predicate, or its synonym in the meta-language, is not semantically defective. And if, instead of using a synonym, we are employing the terms of some reductive analysis, the buck will similarly get passed on to those terms, since we may now ask in virtue of what *they* fail to be semantically defective. (Other 'non-denotationalist' approaches to semantics—such as success semantics, biosemantics, or the Wittgensteinian idea that 'meaning is use'—also seem to be of little help in this connection.)

The order-theoretic hierarchy offers a way out of this dilemma: while there is no relation $\lambda x, y, z\, \exists w\, ((w = y) \wedge (x = w(z)))$, there *is*, for any positive ordinal α,

[42] Let R be the mentioned (hypothetical) relation, i.e., $\lambda x, y, z\, \exists w\, ((w = y) \wedge (x = w(z)))$. The reason why we are here considering this relatively complicated relation instead of the simpler $\lambda x, y, z\, (x = y(z))$ lies in the fact that the latter relation would, if it were to exist, have an instantiation by entities x, y, and z, in this order, *only* if there actually existed an instantiation of y by z. It would thus not be a 'generally applicable' relation, whereas R, if it were to exist, would have an instantiation by x, y, and z, in this order, even if there were *no* instantiation of y by z.

To return to the paradox, let Q be the property $\lambda x\, \exists y\, (R(y, x, x) \wedge (y \neq \top))$, and consider whether the state $Q(Q)$ obtains. We can first note that $Q(Q)$ is the state $\exists y\, (R(y, Q, Q) \wedge (y \neq \top))$, which, given that $R = \lambda x, y, z\, \exists w\, ((w = y) \wedge (x = w(z)))$, and given (S6), is identical with

$$\exists y, w\, ((w = Q) \wedge (y = w(Q)) \wedge (y \neq \top)). \qquad (*)$$

From this it is easy to see that, if $Q(Q)$ obtains, then it is distinct from $\top$. However, R is denoted$_{\emptyset,\emptyset}$ by a term that contains no non-logical constants or free variables; and so the same goes for Q and $Q(Q)$. Consequently, if $Q(Q)$ obtains, then it and $\top$ necessitate each other, which by (S6) means that they are one and the same state. Since a state cannot be both distinct from and identical with $\top$, we can conclude that $Q(Q)$ does *not* obtain. So it must be distinct from $\top$ (since $\top$ trivially obtains). But from this it can be inferred that $(*)$, i.e., $Q(Q)$ itself, *does* obtain: contradiction.

a relation $\lambda x, y, z \, \exists w^\alpha \, ((w^\alpha = y) \wedge (x = w^\alpha(z)))$, and one of these may serve as the semantic value of our instantiation predicate when we say, in a given context, that something is an instantiation of some other entity by a third thing. (But, to be sure, this only gives the barest hint as to how one might approach the task of constructing a semantics for the language in which the present theory is formulated.[43])

One more brief remark before we move on. The stratification of states or propositions into different orders is sometimes thought to be undermined by the restrictions it imposes on our talk of such things (i.e., states or propositions). For example, if we say that *everything* Socrates said is true, then on the present view we cannot be sure that we thereby express a state. To stay within the bounds of the above ontology, we have to pick some positive ordinal α and then express the higher-order state $\forall x^\alpha \, (R(\text{Socrates}, x^\alpha) \rightarrow x^\alpha)$, for some suitable relation R.[44] This is admittedly inconvenient; but so what? We generally have little reason to think that the metaphysics of intensional entities is fully laid bare in our everyday habits of speech; and a little inconvenience is surely a small price to pay for the avoidance of paradox.

5 Fundamentality

Although the concept of *fundamentality* has, in the assumptions and definitions listed above, been invoked only in clause (o2) of Definition 16, the role it plays there makes it a crucial ingredient of the present theory, in that it helps to anchor the order-theoretic hierarchy. In this section I will formulate four assumptions that will both help to narrow down that concept and give it further work to do, most notably in fleshing out the individuation of intensional entities. While particulars (i.e., non-intensional entities) are treated in a way that is largely analogous to fundamental intensional entities, I shall remain officially neutral on whether particulars are fundamental.

5.1 The Fundamentality of Identity

We begin with a straightforward thesis about the identity relation:

(F1) The identity relation is fundamental.

It might be wondered how we could ever tell whether this assumption is correct. Here I propose to sidestep this worry by treating (F1) as a meaning postulate.

[43]An objector might argue that I have put the cart in front of the horse: rather than to justify the hierarchy by relying on the assumption that my talk of instantiation is semantically non-defective, I should have established the hierarchy *before* indulging in talk of instantiation. Arguably, however, one can legitimately proceed in the opposite direction if there is independent reason to think that talk of instantiation is non-defective. Such a reason is given by the theoretical usefulness of the concept of instantiation in drawing up a theory of intensional entities. (Cf. also Section 6.1 below.)

[44]To be sure, the existence of R does not follow from the above ontology. But let us ignore this for the sake of the example.

From (F1) it follows nearly immediately that, for any y, the property $\lambda x\,(x = y)$—a.k.a. the singleton of y—is zeroth-order. (More generally, we can infer that every *set* is zeroth-order.) Consequently, the formulas '$x = y$' and '$\forall z^1\,(z^1(x) \leftrightarrow z^1(y))$' analytically entail each other. By (S6), it now follows that the indiscernibility relation $\lambda x, y\,\forall z^1\,(z^1(x) \leftrightarrow z^1(y))$ is a trivial converse of the identity relation; and so, by (A2), the former is nothing other than the identity relation itself.

We can also infer the following generalization of Corollary 2:

Corollary 5. Any entity denoted$_{\varnothing,\varnothing}$ by a term that contains no bound variable-occurrence at predicate- or sentence-position is zeroth-order.[45]

Let us say that an entity is *purely logical* iff it is denoted$_{\varnothing,\varnothing}$ by some term of $\mathcal{L}$. Typical examples include the property $\lambda x\,(x = x)$ (of *being self-identical*) and the identity relation.

Another set of consequences of (F1) has to do with such properties as $\lambda x^1\,x^1(x^1)$, or *being a self-instantiating zeroth-order property*. In order for the present view to be committed, via (A3), to the existence of this property, there have to exist an interpretation I and a variable-assignment g relative to which the formula '$x^1(x^1)$' denotes a state. This in turn means that there has to exist a zeroth-order property that has an instantiation by itself. Since $\lambda x\,(x = x)$ has an instantiation by itself and is (as we have just seen) zeroth-order, it fits the bill; and so we can infer that there exists a property $\lambda x^1\,x^1(x^1)$—as well as the further properties $\lambda x^2\,x^2(x^2)$, $\lambda x^3\,x^3(x^3)$, and so on. The existence of the 'negations' of these properties (i.e., $\lambda x^1\,\neg x^1(x^1)$, $\lambda x^2\,\neg x^2(x^2)$, and so on) follows by a similar argument. And in just the same way it can also be shown, for each positive ordinal α, that there exists a property $\lambda x\,\neg\exists y^\alpha\,((y^\alpha = x) \wedge y^\alpha(y^\alpha))$, of *not being a self-instantiating property of some order less than α*.

Unlike $\lambda x^1\,\neg x^1(x^1)$ and its variants, these latter properties have instantiations by themselves. To see this, let P be the property $\lambda x\,\neg\exists y^1\,((y^1 = x) \wedge y^1(y^1))$, of *not being a self-instantiating zeroth-order property*. Then the self-instantiation of P is the state $\neg\exists y^1\,((y^1 = P) \wedge y^1(y^1))$, whose existence can be proved with the help of (S5) and (F1). On this basis it can now be shown that P is not zeroth- but *first*-order.[46] By

[45] Let x be any entity, and let t be any term that denotes$_{\varnothing,\varnothing}$ x and contains no bound variable-occurrence at predicate- or sentence-position. By the semantics of $\mathcal{L}$, it can be seen that any atomic term that occurs free in t is identical with '$\mathbf{I}$' and hence denotes$_{\varnothing,\varnothing}$ the identity relation. By (F1) (together with Definition 16), it follows that x is zeroth-order.

[46] Suppose for *reductio* that P is zeroth-order, and let Q be the property $\lambda y^1\,((y^1 = P) \wedge y^1(y^1))$. Since $P = \lambda x\,\neg\exists y^1\,((y^1 = x) \wedge y^1(y^1))$, the state $P(P)$ obtains iff Q lacks an obtaining instantiation. Given that P is zeroth-order, Q has an instantiation by P, namely $((P = P) \wedge P(P))$. Suppose now that $P(P)$ obtains. It then follows that Q lacks an obtaining instantiation, so that, in particular, $Q(P)$ does not obtain. But $Q(P)$ is the conjunction of $(P = P)$ and $P(P)$. Since $(P = P)$ trivially obtains, we thus have that $P(P)$ fails to obtain, contradicting the supposition. So we can conclude that $P(P)$ does not obtain. But clearly, Q does not have an obtaining instantiation by any entity distinct from P. Hence Q does not have an obtaining instantiation, which means that $P(P)$ obtains, after all: contradiction. This completes the *reductio*. We can thus infer that P is not zeroth-order. So it must be first-order, since it is denoted$_{\varnothing,\varnothing}$ by '$\lambda x\,\neg\exists y^1\,((y^1 = x) \wedge y^1(y^1))$'.

an analogous argument, it can be shown that $\lambda x\, \neg\exists y^2\,((y^2 = x) \wedge y^2(y^2))$ is *second-order*; and so on for the remaining properties in this series. We can accordingly conclude that every level of the order-theoretic hierarchy is populated. In particular, for any ordinal α, there exists at least one α-order property. This conclusion can be readily generalized; for in essentially the same way one can prove, for any ordinal $\beta > 1$, that there is an αth-order β-adic relation. We thus arrive at the following:

Proposition 1. For any ordinal α, there are proper-class many entities of order α.

5.2 The Equifundamentality of Converses

Our next assumption concerns fundamental relations and their converses:

(F2) Any converse of any fundamental relation is again fundamental.

Ideally, this assumption would fall out of an account of *what it means* for something to be a fundamental relation; but even in the absence of such an account, (F2) should seem a reasonable assumption to take on board. To deny it would be to hold that nature draws certain 'invidious distinctions', treating, e.g., the relation *longer-than* as fundamental while its converse, *shorter-than*, isn't. By all appearances, nature does not draw such distinctions, and that seems a tolerably good reason to think that in fact it doesn't.[47]

5.3 The Analyzability Assumption

We next assume that, roughly put, every entity can be 'fully analyzed' in terms of particulars and fundamental intensional entities:[48]

(F3) For any entity x, there exist a variable-assignment g and a term t that satisfy the following two conditions:

  (i) t denotes$_{\emptyset,g}$ x.

  (ii) Any intensional entity in the range of g is fundamental.

This assumption is certainly controversial, as one may well doubt that everything has an analysis that 'bottoms out' in particulars and fundamental intensional entities. We will return to this issue in Section 6.2.

The chief attraction of (F3), at least for present purposes, lies in the consequences that can be derived from it in conjunction with (F4) below, such as the thesis that no set is a fundamental property. (Cf. Section 5.5.2.) But in the meantime it would clearly be wrong to claim for (F3) any kind of certainty. It may be best described as a 'default assumption'. For in effect it says that $\mathcal{L}$ is sufficiently powerful as a medium of analysis that every entity can in it be analyzed in terms of fundamental

[47] For related discussion, see, e.g., Sider (2011: 219), Bacon (2019: 1020; 2020: 569).

[48] Bacon (2020: 566) has a roughly analogous principle of 'Fundamental Completeness'.

intensional entities and particulars; and, as formal languages go, $\mathcal{L}$ is really rather simple. From the viewpoint of ideological parsimony it is an attractive hypothesis that indeed nothing more complicated is needed, but even so we should consider it an open question how much more complicated $\mathcal{L}$ would need to be in order for (F3) to be true.

5.4 The Independence Assumption

Our final assumption, (F4), can be thought of as expressing the vaguely Humean thought that—with certain exceptions, having to do with (F1) and (F2)—there are no 'necessary connections' among any particulars or fundamental intensional entities. This will become clearer when we look at its consequences. Stripped to its absolute core, the assumption can be symbolized as follows:

$$s \text{ necessitates } s' \quad \Rightarrow \quad t \text{ analytically entails } t',$$

where, relative to the empty interpretation and some variable-assignment g, t denotes s and t' denotes s'.

For a fuller exposition, we first have to introduce the notion of a *symmetry statement*. This is simply a formula $\ulcorner u = L \urcorner$, for some atomic term u and some lambda-expression L such that, for some ordinal $\alpha > 1$, some α-sequence of pairwise distinct untyped variables $v_0, v_1, \ldots$, all distinct from u, and some permutation f on α: L is identical with $\ulcorner \lambda v_0, v_1, \ldots\, u(v_{f(0)}, v_{f(1)}, \ldots) \urcorner$. Further, we will say that a term is *ontologically conservative* relative to another iff any variable or non-logical constant that occurs free in the first also occurs free in the second. (For example, the formula '$R = \lambda x, y\, R(y, x)$' is a symmetry statement that is ontologically conservative relative to the variable 'R'.) And lastly, we will say that, for any interpretation I and variable-assignment g, a given term is *true* relative to I and g—or true$_{I,g}$ for short—iff it denotes$_{I,g}$ an obtaining state.

The reason why (F4) has to make reference to symmetry statements lies in the need to avoid the consequence that no fundamental relation (other than the identity relation) is its own non-trivial converse. For example, let E be the *equidistance* relation, as expressed by 'x is as far from y as z is from w'. Then E is plausibly identical with $\lambda x, y, z, w\, E(y, x, w, z)$, and is hence one of its own non-trivial converses. But now let g be a variable-assignment that maps 'E' to E, and let t and t' be (respectively) the formulas

$$\lambda x, y, z, w\, E(x, y, z, w) = \lambda x, y, z, w\, E(x, y, z, w)$$

and

$$\lambda x, y, z, w\, E(x, y, z, w) = \lambda x, y, z, w\, E(y, x, w, z).$$

In this case t and t' both denote$_{\varnothing,g}$ the same state, namely $(E = E)$. As we know, $(E = E)$ necessitates itself, whereas t does not analytically entail t'. In this way (F4) would threaten to rule out the fundamentality of relations like E. That would be undesirable, since it would render the above assumption (F3) significantly less plausible; for (F3) requires that the fundamental intensional entities be sufficiently abundant to allow everything else to be analyzed in terms of them (together with particulars). To avoid this outcome, our independence assumption has to be suitably weakened. In particular, rather than to require that t should analytically entail t' all by itself, we have to require only that the *conjunction* of t and zero or more 'admissible' symmetry statements should analytically entail t'. A symmetry statement is here counted as *admissible* iff it is both true$_{\varnothing,g}$ and ontologically conservative relative to t.[49]

With these modifications in place, one could describe (F4) as a thesis to the effect that, if t and t' are terms that respectively denote$_{\varnothing,g}$ states s and s' such that s necessitates s', then—provided certain further conditions (to be specified below) are met—there are zero or more admissible symmetry statements in conjunction with which t analytically entails t'. For instance, suppose that t and t' are, respectively, the formulas $\ulcorner \tau = \tau \urcorner$ and $\ulcorner \tau' = \tau' \urcorner$, for some terms τ and τ' that respectively denote$_{\varnothing,g}$ two entities x and y. What (F4) then in effect says is that, provided that the to-be-specified further conditions are met, the self-identity of x necessitates that of y *only* if $\ulcorner \tau = \tau \urcorner$, possibly in conjunction with one or more admissible symmetry statements, analytically entails $\ulcorner \tau' = \tau' \urcorner$. Accordingly, (F4) imposes a constraint on what pairs of entities x and y are such that the self-identity of x necessitates that of y.

The "further conditions" alluded to above are needed to prevent (F4) from having absurd or otherwise undesirable consequences. Thus it has to be required, in the first place, that any intensional entity in the range of g should be *fundamental*. If this requirement were omitted, (F4) would be incompatible with the abundant ontology of intensional entities outlined in Section 3.3. To see why, suppose that some state s and its negation $\neg s$ are respectively denoted$_{\varnothing,g}$ by two variables u and v. Then, since the self-identity of s necessitates that of its negation—as can be seen from the fact that '$s = s$' analytically entails '$\neg s = \neg s$'—it would follow from (F4), if the requirement in question were not in place, that $\ulcorner u = u \urcorner$, possibly in conjunction with one or more admissible symmetry statements, analytically entails $\ulcorner v = v \urcorner$. However, since u and v are distinct, it is *not* the case that $\ulcorner u = u \urcorner$, whether by itself or in conjunction with one ore more admissible symmetry statements, analytically entails $\ulcorner v = v \urcorner$. So we would have a contradiction. By contrast, with the mentioned requirement in place, the argument that leads to this contradiction requires the additional premise that s and $\neg s$ are both fundamental; and so the argument then merely amounts to a

[49]The point of requiring the symmetry statements in question to be true$_{\varnothing,g}$ is to avoid weakening (F4) in such a way that, for any fundamental dyadic relation R that is *distinct* from its non-trivial converse $\lambda x, y\, R(y,x)$, (F4) no longer entails, e.g., that the state $(R \neq \lambda x, y\, R(y,x))$ does not necessitate $\forall x, y\, R(x,y)$. This would leave an unwelcome lacuna. (On the other hand, if R were *identical* with $\lambda x, y\, R(y,x)$, then the state $(R \neq \lambda x, y\, R(y,x))$ *would* necessitate $\forall x, y\, R(x,y)$, as it would be nothing other than the 'impossible' state $(R \neq R)$.) Similarly, the point of requiring the symmetry statements to be ontologically conservative relative to t is to avoid weakening (F4) in such a way that it no longer entails, e.g., that for any particular or fundamental property x and any fundamental relation R (other than identity), the state $(x = x)$ does not necessitate $(R = R)$.

reductio of this premise.[50]

In the second place, it has to be required that no two variables should under g be mapped to the same entity. For let x be any particular or fundamental intensional entity, let u and v be any two variables, and suppose that g is the smallest variable-assignment that maps both u and v to x. Then $\ulcorner u = u \urcorner$ and $\ulcorner v = v \urcorner$ will both denote$_{\emptyset,g}$ the state $(x = x)$ (which, like any state, necessitates itself), yet $\ulcorner u = u \urcorner$, even in conjunction with one or more admissible symmetry statements, does not analytically entail $\ulcorner v = v \urcorner$.

Two further conditions are needed to prevent (F4) from becoming incompatible with (F1) and (F2). To ensure compatibility with (F1), we have to require that the range of g should not contain the identity relation. To ensure compatibility with (F2),[51] we have to require that no two variables should under g be mapped to relations that are converses of each other. And finally, we have to impose two at first blush relatively arcane requirements. The first of these is to the effect that any variable in the range of g should be of type 1. If this requirement were omitted, (F4) would have the consequence that there are no particulars or fundamental intensional entities other than the identity relation. For, by Corollary 3, any particular or fundamental intensional entity is zeroth-order. Hence, for any particular or fundamental intensional entity x, the self-identity of x can be denoted by '$x^1 = x^1$'. Now let h be some variable-assignment that maps 'x^1' to x, and note that the formulas '$x^1 = x^1$' and '$\exists y^1\, (y^1 = x^1)$' analytically entail each other. By (S6), it follows that the state denoted$_{\emptyset,h}$ by '$\exists y^1\, (y^1 = x^1)$' is nothing other than the self-identity of x. From this it can further be seen that, relative to a variable-assignment that maps 'x' to x, the self-identity of x will be denoted by both '$x = x$' and '$\exists y^1\, (y^1 = x)$'; yet, given that not every entity is zeroth-order, the first of these formulas, even in conjunction with admissible symmetry statements, does not analytically entail the second.

The remaining requirement is to the effect that, for any variable v that occurs free in t and is under g mapped to an intensional entity, at least one free occurrence of v in t should stand at predicate- or sentence-position. To see the need for this, note that, if x is a fundamental property, then its self-identity can be denoted not only by '$x^1 = x^1$' but also by '$x^1 = \lambda y\, x^1(y)$', whereas '$x^1 = x^1$' (even in conjunction with admissible symmetry statements) does *not* analytically entail '$x^1 = \lambda y\, x^1(y)$'. Analogously if x is a fundamental state; for then its self-identity can be denoted by both '$x^1 = x^1$' and '$x^1 = \&(x^1)$', whereas '$x^1 = x^1$' does not analytically entail '$x^1 = \&(x^1)$'. Hence, if we were to omit the requirement in question, (F4) would have

[50]See footnote 56 below for a version of this argument that relies on (F4) in its fully developed form.

[51]More precisely: to ensure compatibility with (F2) *in conjunction* with the claim (which should arguably not be ruled out *a priori*) that there exists at least one fundamental relation R with a distinct converse R'. To see the problem, let u and v be two variables that are under g mapped to (respectively) R and R'. Then the formulas $\ulcorner u = u \urcorner$ and $\ulcorner v = v \urcorner$ will respectively denote$_{\emptyset,g}$ the self-identity of R and the self-identity of R', which necessitate each other. Moreover, by (F2), R' is fundamental, given that R is. Yet $\ulcorner u = u \urcorner$, even in conjunction with admissible symmetry statements, does not analytically entail $\ulcorner v = v \urcorner$.

the unwelcome consequence that there are no fundamental states or properties.

Putting all of the above requirements together, we arrive at the following:

(F4) For any variable-assignment g, any terms t and t', and any states s and s' respectively denoted$_{\varnothing,g}$ by t and t': if the following five conditions are satisfied—

(i) s necessitates s'.

(ii) Any variable in the domain of g is of type 1.

(iii) Any intensional entity in the range of g is fundamental and distinct from the identity relation.

(iv) No two variables are under g mapped to the same entity or to relations that are converses of each other.

(v) For any variable v: if v occurs free in t and is under g mapped to an intensional entity, then at least one free occurrence of v in t stands at predicate- or sentence-position.

—then there exist zero or more symmetry statements $\sigma_1, \sigma_2, \ldots$, each of them true$_{\varnothing,g}$ and ontologically conservative relative to t, such that $\ulcorner t \wedge \sigma_1 \wedge \sigma_2 \wedge \ldots \urcorner$ analytically entails t'.

So much for the statement of (F4). The consequences of this assumption deserve their own subsection.[52]

5.5 Consequences of the Independence Assumption

We can divide the consequences of (F4)—or at least those that will interest us here—into three classes: *independence* results, *non-fundamentality* results, and *distinctness* results.

5.5.1 Independence results

Let us say that an entity is *independent* of another iff the self-identity of the first does not necessitate that of the second. Independence results may be obtained from (F4) by a method that has already been illustrated above: to prove that x is independent of y, simply find two terms t and t', together with a variable-assignment g, such that t and t' respectively denote$_{\varnothing,g}$ the states $(x = x)$ and $(y = y)$, clauses (ii)–(v) are satisfied, and t (even in conjunction with one or more admissible symmetry statements) does not analytically entail t'. In this way it can, e.g., be shown that, if x and y are two particulars or fundamental intensional entities, then x is independent of y, *unless* y is the identity relation or a converse of x.

[52] With some qualifications, (F4) can be considered a counterpart of Bacon's (2020: 547) principle of 'Quantified Logical Necessity'. An important difference (among several) lies in the fact that, where Bacon's principle invokes a monadic notion of logical necessity, (F4) relies instead on the *dyadic* notion of necessitation.

5.5.2 Non-fundamentality results

In order to prove that a given intensional entity x is non-fundamental, one can (in principle) proceed as follows.[53] First, for some variable v of type 1, let ρ be *either* the formula $\ulcorner\&(v)\urcorner$ (if x is a state) *or* (if x is an α-adic attribute) the lambda-expression $\ulcorner\lambda u_1, u_2, \ldots\ v(u_1, u_2, \ldots)\urcorner$, for some α-sequence of variables $u_1, u_2, \ldots$ distinct from v, and let t be the formula $\ulcorner\rho = \rho\urcorner$. Next, choose a variable-assignment g and a complex term τ such that the following holds: g maps v to x; τ denotes$_{\emptyset,g}$ x; t, even in conjunction with one or more admissible symmetry statements, does not analytically entail $\ulcorner\rho = \tau\urcorner$; and, at least under the supposition that x is fundamental, clauses (ii)–(v) of (F4) are satisfied. Next, suppose for *reductio* that x *is* fundamental. It then follows from Corollary 3 that x is zeroth-order, so that v denotes$_{\emptyset,g}$ x, so that t and $\ulcorner\rho = \tau\urcorner$ both denote$_{\emptyset,g}$ the state $(x = x)$. Finally, let t' be $\ulcorner\rho = \tau\urcorner$, and let s and s' both be $(x = x)$. The antecedent of (F4)'s main conditional will then be satisfied, but the consequent won't, and this will complete the *reductio*. The following are some general results that can be obtained by this method:

- No set is fundamental.[54]
- No instantiation of a fundamental attribute is fundamental.[55]
- No purely logical entity other than the identity relation is fundamental.
- The negation of a fundamental state is not fundamental.[56] Analogously for attributes.
- The conjunction of any two fundamental states is not fundamental. (Corollary: no fundamental state necessitates another.[57])

[53] I say "in principle" because there are cases in which the procedure cannot be carried out: after all, given (F1), not every intensional entity is non-fundamental.

[54] The proof of this assertion requires the use of (F3), which is needed to guarantee that there exists a term that denotes the respective set relative to a variable-assignment g that satisfies clause (iii) of (F4). (For a similar reason, (F3) is also needed to prove the next assertion.)

[55] By contrast, there may be fundamental instantiations of *non*-fundamental attributes. For example, suppose that there exists a fundamental state s. Then s is zeroth-order, so that the non-fundamental property $\lambda x^1\, x^1$ has an instantiation by s. But that instantiation is nothing other than s itself, and is therefore fundamental.

[56] Let s be any fundamental state (assuming that there is one), let u and v be two variables of type 1, let g be the smallest variable-assignment that maps u to s and v to $\neg s$, and let τ be the formula $\ulcorner\neg u\urcorner$. The *reductio* succeeds because $\ulcorner\&(v) = \&(v)\urcorner$ does not analytically entail $\ulcorner\&(v) = \neg u\urcorner$.

[57] Suppose for *reductio* that there are two fundamental states s and s' such that s necessitates s'. By the definition of 'necessitates', there then exist an interpretation I, a variable-assignment g, and terms t and t' such that (i) t and t' respectively denote$_{I,g}$ s and s' and (ii) t analytically entails t'. Then t also analytically entails $\ulcorner t \wedge t'\urcorner$. But $\ulcorner t \wedge t'\urcorner$ denotes$_{I,g}$ $(s \wedge s')$, which is therefore necessitated by s. Likewise, s is necessitated by $(s \wedge s')$. Hence, by (S6), s is identical with $(s \wedge s')$. So s is a conjunction of two fundamental states. From the result stated in the text (namely, that the conjunction of any two fundamental states is not fundamental), it now follows that s fails to be fundamental, contrary to hypothesis.

5.5.3 Distinctness results

While (S6), the assumption that no two states necessitate each other, puts a *lower* bound on the coarse-grainedness of states, (F4) imposes a complementary upper bound. This manifests itself in *distinctness results*, i.e., results to the effect that certain pairs of terms denote (relative to the same interpretation and variable-assignment) distinct entities.

Distinctness results can be obtained by a procedure similar to the one described above, only that now the term ρ would not need to be built around a typed variable but could be any suitable complex term. (For an application of this method, see footnote 59 below.) By a more complicated argument, but still relying on (F4), it can be shown that $\lambda x^1\, \neg x^1(x^1)$ is not zeroth-order.[58] From this inobvious result it follows, by Corollary 4, that $\lambda x^1\, x^1(x^1)$ is not zeroth-order, either. This is of some relevance for the avoidance of a 'truth-teller' version of Russell's paradox; for, from the fact that $\lambda x^1\, x^1(x^1)$ is not zeroth-order, we can infer almost immediately that this property does not have an instantiation by itself. We may also note the following:

Proposition 2. If there are no particulars or fundamental intensional entities other than the identity relation, then $\top$ and $\bot$ are the only two states; but if there is at least one particular or at least one fundamental attribute other than the identity relation, then there are proper-class many zeroth-order facts (i.e., obtaining states).

[58]Let P be this property, let ρ be '$\lambda x^1\, \neg x^1(x^1)$', let t be $\ulcorner \rho = \rho \urcorner$, and suppose for *reductio* that P is zeroth-order. By Definition 16, there then exist an interpretation I and a variable-assignment g relative to which P is denoted by a term τ that satisfies the following two conditions:

(1) Any atomic term that is either identical with τ or has in τ a free occurrence at predicate- or sentence-position denotes$_{\emptyset,g}$ either a particular or a fundamental intensional entity.

(2) No variable has in τ a bound occurrence at predicate- or sentence-position.

Without loss of generality, we may take I to be the empty set and suppose that g and t jointly satisfy clauses (ii)–(v) of (F4). Given that τ denotes$_{\emptyset,g}$ the property P, which is purely logical and hence (as shown above) non-fundamental, it follows from (1) that τ is non-atomic. Now assume the following holds:

(3) At least one atomic term has in τ a free occurrence at predicate-position.

Let u be any such term, and let t' be $\ulcorner \rho = \tau \urcorner$. Evidently u does not occur free in ρ. Hence, unless u is the constant '**I**', any conjunction of t with zero or more admissible symmetry statements will *not* analytically entail t'; for there will exist an interpretation and variable-assignment relative to which t has a denotation while t' doesn't. But, since ρ and τ denote$_{\emptyset,g}$ the same entity (viz., P), we have that t and t' denote$_{\emptyset,g}$ one and the same state s. From (F4), it now follows that t, possibly together with one or more admissible symmetry statements, analytically entails t'. By what has just been said, we can thus infer that u is the constant '**I**'. But u was *any* atomic term that has in τ a free occurrence at predicate-position. Hence, given (2), we have that no atomic term other than '**I**' occurs in τ at predicate-position. (At this point we can 'discharge' the above assumption (3).) Given (S5), it can now be seen, by induction over the complexity of τ, that any property denoted$_{\emptyset,g}$ by τ must have an instantiation by any zeroth-order entity. So, since τ denotes$_{\emptyset,g}$ P, which by hypothesis is zeroth-order, P has an instantiation by itself. But since P is also denoted$_{\emptyset,\emptyset}$ by '$\lambda x^1\, \neg x^1(x^1)$', any instantiation of P by itself obtains just in case it doesn't. This completes the *reductio*.

The first conjunct follows straightforwardly from (F3) and (S6), while the second follows from (F4).[59] The first conjunct describes a scenario that might be termed a 'purely mathematical' reality, whereas the second describes what is (at least from an intuitive point of view) more likely to be the *actual* reality, i.e., the world as it actually is.

6 Potential Concerns

In this section I will briefly discuss two worries to which OTT might be thought to give rise. The discussion will be largely programmatic; a main objective will be to round off the overall metaphysical picture.

[59]Let x be any particular or fundamental attribute other than the identity relation, let v be any variable of type 1, and let g be the smallest variable-assignment that maps v to x. Further, let α and β be any two ordinals, and let L and M be any two lambda-expressions that denote$_{\emptyset,\emptyset}$ α and β, respectively. We have to consider three cases.

First, suppose that x is a particular. We can then show, with the help of (F4), that the zeroth-order fact $(x \neq \alpha)$ is distinct from the zeroth-order fact $(x \neq \beta)$. For let ρ be $\ulcorner v \neq L \urcorner$, let τ be $\ulcorner v \neq M \urcorner$, and suppose for *reductio* that $(x \neq \alpha)$ is identical with $(x \neq \beta)$. We then have that ρ and τ denote$_{\emptyset,g}$ the same state, and so, with $t = \ulcorner \rho = \rho \urcorner$ and $t' = \ulcorner \rho = \tau \urcorner$, we also have that t and t' denote$_{\emptyset,g}$ the same state. Moreover, clauses (ii)–(v) of (F4) are satisfied. Yet t, even in conjunction with one or more symmetry statements, does not analytically entail t'. (To see this, let h be some variable-assignment that maps v to α. Then ρ denotes$_{\emptyset,h}$ a non-obtaining state while τ denotes$_{\emptyset,h}$ an *obtaining* state. As a result, t denotes$_{\emptyset,h}$ an obtaining state while t' doesn't.) This completes the *reductio*, and we can conclude that $(x \neq \alpha)$ is distinct from $(x \neq \beta)$. But α and β were any two ordinals. Hence there are as many zeroth-order facts as there are ordinals.

Next, suppose that x is a fundamental property. The argument proceeds exactly as before, only that, for some untyped variable u, we let ρ be the formula $\ulcorner \lambda u\, v(u) \neq L \urcorner$. Finally, suppose that x is a fundamental γ-adic relation, for some $\gamma > 1$. The argument proceeds again largely as before, only that now we are considering not $(x \neq \alpha)$ and $(x \neq \beta)$ but rather the somewhat different zeroth-order facts $(x \neq R_\alpha)$ and $(x \neq R_\beta)$, where, for any ordinal δ, R_δ is the γ-adic relation $\lambda x_0, x_1, \ldots \big(\delta(x_0) \wedge \bigwedge_{1 \leq i < \gamma} x_i = x_i\big)$. (The symbol '$\bigwedge$' should here be read as an abbreviatory device for conjunctions. Basically, R_δ is δ with one or more 'dummy' argument-places tacked on.) Correspondingly, for some γ-sequence of pairwise distinct untyped variables $u_0, u_1, \ldots$, we let ρ and τ be (respectively) the formulas φ_L and φ_M, where, for any term t, φ_t is the formula

$$\lambda u_0, u_1, \ldots\, v(u_0, u_1, \ldots) \neq \lambda u_0, u_1, \ldots \Big((t)(u_0) \wedge \bigwedge_{1 \leq i < \gamma} u_i = u_i\Big).$$

Suppose for *reductio* that $(x \neq R_\alpha)$ is identical with $(x \neq R_\beta)$. Then ρ and τ will denote$_{\emptyset,g}$ the same state, and so, with $t = \ulcorner \rho = \rho \urcorner$ and $t' = \ulcorner \rho = \tau \urcorner$, we also have that t and t' denote$_{\emptyset,g}$ the same state. Yet t, even conjoined with one or more symmetry statements, does not analytically entail t'. (To see this, consider a variable-assignment h that maps v to R_α.) This completes the *reductio*, and we can conclude that $(x \neq R_\alpha)$ is distinct from $(x \neq R_\beta)$. As before, it follows that there are as many zeroth-order facts as there are ordinals.

6.1 Ontological Doubts

The first worry is epistemological in character, very general, and very simple: in the absence of any pertinent empirical evidence, why should we accept the staggering existence assumptions of Section 3.3? To see how this question might be answered, it may help to start with an even simpler one: why believe in mathematical objects? A popular way to answer this is to point out the indispensability of mathematics for the natural sciences, but a quick thought experiment may be more immediately compelling. Suppose you have four apples in front of you. How many ways are there to group them into disjoint pairs?

I assume it has not taken you long to work out that there are exactly three. Let us take this as a datum, and try to make sense of it. Nominalistic qualms aside, it seems pointless to try to finesse away the ontological commitment to those ways of grouping. Whatever they are, they exist in *some* sense, and we should be able to quantify over them, since we can count them; that's what quantification is *for*. The interesting question is not whether those ways of grouping exist, but what they are, or better: what sort of thing we should nominate for the relevant theoretical role.[60] When we convince ourselves that there are exactly three ways of grouping four apples into disjoint pairs, we typically abstract away from any physical details of how such grouping can be effected (e.g., using bags or bowls). And that is clearly the way to go if we are going to pursue questions of mathematical interest: the term 'way of grouping' has to be understood in as abstract a way as possible. By this route we are led to think of a way of grouping four apples into disjoint pairs as just a *set* of disjoint pairs of apples (each pair being in turn a set $\{a, b\}$, for distinct a and b) whose union is the set of the original four. The sheer abstractness of the concept of set does, however, tend to impart on this concept a certain air of mystery.[61] An attractive way to dispel the mystery—at least if it is antecedently accepted that there are such things as properties and relations, something that I have not tried to argue in this paper—is to follow Carnap in thinking of sets as properties, as suggested in Section 3.1.[62]

[60] Cf., e.g., Quine (1960: 257f.) on 'ordered pair' and Lewis (1986: 55f.) on 'property'.

[61] See, e.g., Leśniewski (1992: 209ff.), Black (1971), Bealer (1982: ch. 5), Lewis (1991: §2.1). For relevant historical discussion, see Kanamori (2003).

[62] It might be suggested that the mystery could have been avoided by thinking of those "ways of grouping" not as sets of sets but rather as pluralities of pluralities. However, it is somewhat hard to see why pluralities should be any less mysterious than sets. The metaphor of the 'invisible plastic bag', used by Bealer to lampoon the notion of set, is no less applicable here. (Cf. also Florio & Linnebo [2021: ch. 8], where this becomes clearer.) Moreover, pluralities lead to trouble within an ontology that also admits non-fundamental intensional entities, such as instantiations of fundamental attributes. For example, if we think that there is a plurality of *the flowers in the field* as well as the state of John's loving Mary, then presumably we will also want there to be such a thing as the state of John's loving the flowers in the field. But as innocent as this kind of commitment may seem, it tends to lead to a form of the Russell–Myhill paradox. (Cf. McGee & Rayo [2000], Pruss & Rasmussen [2015].) We thus seem to be faced with a choice between renouncing pluralities on the one hand and renouncing non-fundamental intensional entities on the other. Given the enormous theoretical utility of the latter, it is arguably pluralities that have to go. (A third option, which is to make pluralities

Once we have reached this point, the remaining components of the theory fall into place almost by themselves. First-order ZFC provides the mathematical foundation, as a well-understood and deductively powerful theory of cardinalities and lengths of sequence, largely free of arbitrary restrictions (though some modifications are needed to allow for urelements). Further, to make sense of sets in the manner proposed in Section 3.1, we need an ontology of attributes that admits such properties as $\lambda x\, \neg\exists y\, (x = y)$ and $\lambda x\, ((x = y) \vee (x = z))$ (for any y and z). A natural way to obtain such properties is through lambda-abstraction; and this requires a sufficiently rich ontology of states that those properties can be abstracted *from*. We accordingly—at least on one possible approach—need there to be an identity relation, we need it to be 'transcendental' (having an instantiation by any pair of entities whatsoever), and we need negations of states, existential quantifications of attributes, and conjunctions of states.[63] To avoid arbitrary restrictions, but also to enable our ontology to embed the set-theoretic universe in the way described, we have to allow that, for any set of states, there exists a conjunction (and hence also, given that we have negation, a *dis*junction) of the set's members. Along with complex properties, complex *relations* are obtained by lambda-abstraction; and it would be arbitrary to limit their adicities to any particular ordinal. From here it is only one more step to the order-theoretic hierarchy, and that step has been motivated in Section 4.3.

6.2 Logically Non-Well-Founded Entities?

Our second worry has to do with OTT's reliance on the concept of a fundamental intensional entity. More specifically, an objector might argue that, for all we know, there may be 'logically non-well-founded' entities whose existence would be incompatible with the conjunction of (F3) and (F4). This calls for some preliminary clarifications.

First, let us say that an entity x is *logically analyzable in terms of* an entity y iff, for some term t of $\mathcal{L}$ and for some variable-assignment g: t denotes$_{\emptyset,g}$ x and contains a free occurrence of an atomic term that denotes$_{\emptyset,g}$ y. Next, let us say that an entity x is *fully* logically analyzable in terms of a set S (notation: $x \prec S$) iff the following two conditions are satisfied:[64]

(i) For any $y \in S$, x is logically analyzable in in terms of y.

(ii) There exist a variable-assignment g and a term t of $\mathcal{L}$ such that: t denotes$_{\emptyset,g}$ x, and every atomic term that occurs free in t denotes$_{\emptyset,g}$ some member of S.

behave more like sets, has recently been advanced by Florio & Linnebo *op. cit.*)

[63] This talk of what we "need" should be understood in the sense of what has to be put in place (at least on one possible approach) in order to construct a philosophically satisfying theory that accommodates sets, in particular in accordance with the proposed conception of sets as complex properties.

[64] This definition is, like the previous one, essentially borrowed from Plate (2016: 30f.). I have added the qualifier 'logically' in order to leave terminological room for different notions of analyzability.

These two concepts are quite liberal; for instance, every entity is logically analyzable in terms of itself and fully logically analyzable in terms of its singleton. (For brevity's sake, the 'logically' will usually be dropped in the following.) Further, as may be recalled from Section 5.1, we call an entity 'purely logical' iff it is denoted$_{\emptyset,\emptyset}$ by some term of $\mathcal{L}$. Equivalently, we may say that an entity is purely logical—or just 'logical' for short—iff it is fully logically analyzable in terms of either $\{\mathbf{I}\}$ (i.e., the singleton of the identity relation) or the empty set.

On the basis of the concept of full analyzability, we can next define the concept of a 'logically abyssal' entity:

Definition 17. An entity x is *logically abyssal* iff every set X with $x \prec X$ contains an entity y that is analyzable in terms of some non-logical entity that fails to be analyzable in terms of y.

Intuitively, an entity x is logically abyssal (or just 'abyssal') iff, no matter how far one 'drills down' in one's analysis of x, there is always at least one non-logical entity that is 'more basic' than some entity mentioned in that analysis. The 'non-logical' is needed to avoid the unwelcome result that every non-logical intensional entity is classified as abyssal. (For, given (S6), any intensional entity y is analyzable in terms of any *logical* entity, whereas, given (F3) and (F4), any logical entity will fail to be analyzable in terms of y unless y is a logical entity, too.) I explain in a footnote how the existence of abyssal entities would pose a problem for the present theory.[65]

On the subject of abyssal entities, three further points are worth noting. First, it is easy to verify that any abyssal entity x is analyzable in terms of some entity distinct from x. From this it follows that every abyssal entity is denotable by a complex term (a formula or lambda-expression) and is hence either an attribute or a state. Second, for any abyssal entity x, any set X with $x \prec X$ contains at least one abyssal entity.[66] And third, a result of Plate (2016) has the corollary that any abyssal entity is *logically complex*.[67] Consequently, for any abyssal entity x, any set X with $x \prec X$ contains at least one logically complex entity. This will become relevant shortly.

[65]Suppose there exists some abyssal entity x. By (F3) (and regardless of whether x is abyssal), there exists a set X such that $x \prec X$ and every member of X is either a particular or a fundamental intensional entity. But since x is abyssal, X contains an entity y that is analyzable in terms of some non-logical entity z that fails to be analyzable in terms of y. Again by (F3), there exists a set Z such that $z \prec Z$ and every member of Z is either a particular or a fundamental intensional entity. Since z is non-logical, Z has to contain some entity w distinct from $\mathbf{I}$. Further, since z fails to be analyzable in terms of y, so must w. Consequently, w is distinct not only from $\mathbf{I}$ but also from y, and is also not a converse of y. Moreover, since $y \in X$ and $w \in Z$, we have that each of y and w is either a particular or a fundamental intensional entity. So, by (F4), y is independent of w. (Cf. Section 5.5.1.) But this cannot be, since y is, via z, analyzable in terms of w.

[66]Let x be any abyssal entity, and let X be any set with $x \prec X$. Suppose for *reductio* that X contains *no* abyssal entity. For each $w \in X$, there then exists a set S_w such that $w \prec S_w$ and S_w does *not* contain any intensional entity y that is analyzable in terms of some non-logical entity that fails to be analyzable in terms of y. The union of the S_w, i.e., $\bigcup_{w \in X} S_w$, will then also not contain any such entity. At the same time it can be seen that $x \prec \bigcup_{w \in X} S_w$. So x fails to be abyssal, contrary to hypothesis.

[67]Plate provides an account of logical complexity in terms of 'reductions'. The details of that account need here not interest us, but it should be noted that, while Plate's main account is concerned

Abyssality constitutes only *one* form of logical non-well-foundedness that would spell trouble for our theory. An objector might suggest that there may also be entities that are *logically ouroboric*, roughly in the sense of having themselves as 'logical constituents'. A formal definition can be given as follows:

Definition 18. An entity x is *logically ouroboric* iff every set X with $x \prec X$ contains a logically complex entity that is analyzable in terms of x.

If there were, for instance, a property Q such that $Q = \lambda x\,(x = Q)$, then this would be a natural candidate for being a logically ouroboric (or just 'ouroboric') property. In addition, of course, one can imagine entities that are not ouroboric themselves but that instead have ouroboric constituents.

David Armstrong took quite seriously the theoretical possibility that some logically complex properties "dissolve *ad infinitum* into constituents which themselves lacked simple constituents" (1978: 67). Whether or not this talk of infinite dissolution is to be understood in terms of abyssality, it is conceivable that Armstrong and others would have taken equally seriously the possibility that some properties are abyssal or ouroboric.[68] Nonetheless, I think that the existence of abyssal or ouroboric entities is sufficiently implausible that we need not be overly worried by this prospect. In particular, from what has been said above it can be seen that the existence of such entities would run counter to the natural and attractive view that logically complex entities form no part of the world's 'ultimate furniture'. More precisely, their existence would be incompatible with the following metaphysical thesis:

(M) There exists a set A such that (i) every entity is fully analyzable in terms of some subset of A and (ii) no logically complex entity is a member of A.

That this view is incompatible with the existence of *abyssal* intensional entities follows from the aforementioned fact that, for any abyssal entity x, any set X with

with the concept of logical simplicity *as applied to attributes*, an analogous concept applicable to *states* is defined on pp. 34f. of the same paper. With minor changes, the result in question can be stated as follows (cf. *op. cit.*, p. 32):

(R) If an attribute A is analyzable in terms of some non-logical entity that is *not* fully analyzable in terms of $\{A, \mathbf{I}\}$, then A is logically complex (i.e., not logically simple).

This result is obtained on the basis of assumptions about the individuation of intensional entities that, in all relevant respects, also hold in the present system. An analogous result could be obtained for states. Meanwhile, from Definition 17 above it can—with the help of the transitivity of analyzability—be inferred that any abyssal entity x is analyzable in terms of some non-logical entity that is not fully analyzable in terms of $\{x, \mathbf{I}\}$. By (R) and its analogue for states, it now quickly follows (given that every abyssal entity is an attribute or state) that every abyssal entity is logically complex.

[68] For example, Russell (1918/19: 180) seems to have taken seriously an analogous possibility in the case of facts. Decades later, theories of non-well-founded sets have been studied by, e.g., Aczel (1988). Around the same time, Barwise (1989: 192ff.) has defended the existence of 'situations' that have themselves as constituents. His examples include announcements and mental acts that are in some sense 'about' themselves. It seems tolerably clear, however, that such cases need not be interpreted as *evidence* for the existence of logically non-well-founded entities.

$x \prec X$ contains at least one logically complex entity; and in the case of *ouroboric* entities, the incompatibility follows straightforwardly from Definition 18. Yet (M) appears to be intuitively plausible, at least insofar as logically complex entities are naturally thought of as 'merely derivative'.

There is, however, a residual worry that needs to be briefly addressed. A critic might argue against (M) as follows: under classical mereological assumptions, there exists, for every set x, a mereological fusion of x and, say, the Eiffel Tower.[69] Plausibly, these various fusions, assuming they exist, are pairwise distinct; and it is also plausible to think that they are particulars. So there will be as many particulars as there are sets, which means that they cannot all be members of the *same* set. But then, contrary to (M), there cannot be a set A such that every entity is fully analyzable in terms of some subset of A. For, given that no particular is (logically) analyzable in terms of anything other than itself, such a set would need to contain all of those particulars.

To address this objection, we could add to $\mathcal{L}$ a mereological fusion operator 'Σ' that functions in such a way that '$\Sigma(a, b, c)$', for instance, denotes the fusion of the entities respectively denoted by 'a', 'b', and 'c'. A fusion of the Eiffel Tower and a given set could then be denoted by $\ulcorner\Sigma(t, t')\urcorner$, with terms t and t' respectively denoting the Eiffel tower and the set in question. This would effectively defuse the objection by rendering particulars analyzable, though at the cost of complicating $\mathcal{L}$. But there is also another way to respond to the objection, which is more ideologically parsimonious. This is to reject the traditional 'topic-neutral' mereological notions on which the objection relies.

Arguments in this general direction can be found in the work of 'mereological nihilists'. Thus Sider (2013: 239) writes that mereological nihilism—the view that nothing has proper parts—"allows us to eliminate the extra-logical (or perhaps quasi-logical) notion of 'part' from our ideology", adding that "this kind of ideological simplification is an epistemic improvement". It should be noted, however, that the rejection of a topic-neutral notion of parthood does not obviously amount to the rejection of the mereological notions that we employ in our everyday talk of parts and wholes, for it is not obvious that the latter *are* topic-neutral. In particular it is far from clear that the notions of parthood and composition that we employ in our everyday talk about material objects (such as the Eiffel Tower) are also applicable to properties. Indeed there may not even be a fact of the matter: our linguistic usage may have left the question undecided. If so, it arguably falls to metaphysics to supply possible precisifications.

Though here is not the place to pursue the question in any detail, with regard to composite material objects it may be argued that these will be best thought of as *events*; and it may further be proposed that events are nothing but certain conjunctions of obtaining states.[70] Admittedly this view is somewhat revisionary. But suppose it is correct: then composition, at least as applied to material objects, will

[69]For discussion and references, see Cotnoir & Varzi (2021: §5.2.1).

[70]Cf. Plate (MS). Also see, e.g., Nolan (2012).

be best thought of as the logical (or logico-metaphysical) operation of *conjunction* (as applied to states). On the resulting interpretation of our talk of material objects and their fusions, there will be no such thing as the fusion of the Eiffel Tower and a set, given that no set is a state. (Cf. Corollary 1.) That there is no such thing as the fusion of the Eiffel Tower and a set should seem common-sensical enough. More important for present purposes, it also blocks the above objection against (M).

7 Conclusion

The theory proposed in this paper may be roughly characterized by the following seven features:

1. Sets are identified with complex properties, following Carnap (1947: §23).
2. The theory's assumptions regarding the existence of attributes (i.e., properties and relations) and the individuation of states rely on an infinitary formal language $\mathcal{L}$.
3. This language contains both *typed* and *un*typed variables. The type of a typed variable is given by an ordinal, and the ranges of typed variables are *cumulative*: any typed variable ranges over all and only those entities whose 'order' is exceeded by the variable's type.
4. The *order* of an entity x is determined by what terms t of $\mathcal{L}$ are such that, for some interpretation I and variable-assignment g: t denotes$_{I,g}$ x, and any atomic term that is either identical with t or has in t a free occurrence at predicate- or sentence-position denotes$_{I,g}$ either a fundamental intensional entity (i.e., a fundamental state or attribute) or a particular (i.e., something that is neither a state nor an attribute).
5. States, and consequently also attributes, are individuated in a moderately coarse-grained manner, relying on a concept of *analytic entailment* as well as on the concept of a fundamental intensional entity.
6. Particulars, fundamental intensional entities, and sets (among other things) are classified as zeroth-order.
7. The identity relation is stipulated to be fundamental.

The theory may be naturally extended by statements as to what large cardinals, particulars, or fundamental intensional entities there are.

To be sure, for some tastes the proposed ontology may already be too rich as it stands, especially in light of its infinite hierarchy of intensional entities. It is worth keeping in mind, however, that the theory arises quite naturally from the idea that there are sets—or at any rate entities capable of playing the theoretical role of sets—and that those entities are complex properties (cf. Section 6.1 above). Moreover, there are many theoretical uses not only for complex properties and relations but also, more specifically, for *higher-order* attributes (as well as states), both inside and outside of philosophy. For instance, the relation of *being similarly colored* is most naturally construed as higher-order, involving quantification over color properties. Putnam (1969: §VI) influentially suggested that *psychological* properties are higher-order in something like the present sense of the term. Later on, such variegated properties as *being fragile*, *being red*, and *being good* have all been claimed to be higher-order.[71] And outside of philosophy, we encounter higher-order properties in, e.g., the field of artificial intelligence. For instance, many pattern recognition tasks call for the representation of properties such as that of containing, for *some* property P (of a certain sort), exactly two objects instantiating P.[72]

While higher-order entities in something much like the present sense are also admitted by STT, the difficulties discussed in Section 2 suggest that we would be better off with a different approach. The theory proposed in this paper presents an alternative—and indeed something more than that. For, unlike STT, it provides identity conditions for intensional entities, and it also has something to say about fundamentality. But as yet it is early days. Certainly more could be said about the theory's formal characteristics. It would also be of interest to ask whether the present concept of fundamentality may be amenable to reductive analysis, or whether some of the theoretical tasks to which this concept has here been put—namely, in anchoring the order-theoretic hierarchy and constraining the individuation of intensional entities—should be handled differently. Finally, as noted in the Introduction, there may be something to be said for an account of *instantiation* that does not treat this latter concept as primitive.[73]

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[71]See, e.g., Prior, Pargetter & Jackson (1982), Prior (1985: ch. 7), Campbell & Pargetter (1986), Pargetter (1988), Lewis (1997).

[72]For context, see, e.g., Bongard (1970).

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