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Article

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CONTEXT-INDEXED COUNTERFACTUALS

SUMMARY: It is commonly believed that the role of context cannot be ignored in the analysis of conditionals, and counterfactuals in particular. On truth conditional accounts involving possible worlds semantics, conditionals have been analysed as expressions of relative necessity: "If *A*, then *B*" is true at some world *w* if *B* is true at all the *A*-worlds deemed relevant to the evaluation of the conditional at *w*. A drawback of this approach is that for the evaluation of conditionals with the same antecedents at some world, the same worlds are deemed as relevant for all occasions of utterance. But surely this is inadequate, if shifts of contexts between occasions are to be accounted for. Both the linguistic and logical implications of this defect are discussed, and in order to overcome it a modification of David Lewis' ordering semantics for counterfactuals is developed for a modified language. I follow Lewis by letting contexts determine comparative similarity assignments, and show that the addition of syntactic context parameters (context indices) to the language gives the freedom required to switch between sets of relevant antecedent worlds from occasion to occasion by choosing the corresponding similarity assignment accordingly. Thus an account that extends Lewis' analysis of a language containing a single counterfactual connective > to a language containing infinitely many counterfactual connectives $>_c$, each indexed by a different context name *c*, overcomes the limitations of traditional analyses. Finally it is also shown that these traditional accounts can be recovered from the modified account if certain contextual restrictions are in place.

KEYWORDS: ordering semantics, counterfactuals, comparative similarity, context, contextual information.

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Introduction

On many possible world semantics for conditional logics, which famously include Stalnaker-Lewis truth conditional accounts, only the world of evaluation and the antecedent are considered in selecting worlds that are deemed relevant to determining the truth value of a conditional.[1] But that results in the underlying context being fixed for all occasions—even when contextual considerations underlying the evaluation of the uttered counterfactuals on various occasions may vary.[2] Alternative approaches go some of the way toward resolving this inadequacy by appealing to a difference in the consequents associated with counterfactuals with the same antecedent, but nevertheless such approaches are still limited to evaluating any conditional with a fixed truth value on any occasion. In this article I propose an analysis of a language that makes appropriate explicit access to the intended context available by introducing explicit names for contexts that index the counterfactual connective. That is, I give an account of a contextualized counterfactual of the form "In context *C*: If it were the case that…, then it would be the case that…". Although the proposal is largely based on David Lewis' analyses of counterfactuals, it does not require that any particular logic of conditionals should serve as its basis—rather, it is intended as a general prescription for contextualizing a conditional language. The contextualization can be applied to the weakest of conditional logics. That is, the method in the manner described is generalizable (extendable) to the weakest of conditional logics, e.g., the system **CE** (Chellas, 1975, p. 138; Nute, 1980, p. 53; Weiss, 2018, p. 15). The advantage of working with stronger logics and ordering semantics stems from existing results, due to Lewis (1981), concerning the properties of ordering frames that facilitate fashioning and implementing a notion of contextual information preservation, which is central to the semantics of the proposed account.

There are three key results concerning the account proposed in this article, which can only be described informally at this point. The first result is at the level of Lewis' ordering semantics for counterfactuals, and it concerns semantic (truth preserving) properties of a certain class of ordering frames (ordering frame

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[1] The tradition of analysis contested in this article refers mainly to Stalnaker, Lewis, and Gabbay all of whom offer truth conditional accounts of conditionals involving possible worlds semantics (Nute, 1981, Section 4). More generally, this concerns analyses that take only the semantic content provided by the world of evaluation and the antecedent (and the consequent, in Gabbay's case) in order to evaluate the conditional. The most general of those include—using Nute's (1981, Chapter 3) classification terminology—conditional logics characterized by world selection function models (WS-models), systems of spheres models (SOS-models), relational models (R-models), class selection function models (CS-models), and neighbourhood models (N-models). The most well-known of those include analyses given in (Chellas, 1975; Lewis, 1973; Montague, 1970; Scott, 1970; Stalnaker, 1968).

[2] To clarify the terminology, an *occasion* of utterance (consideration, or evaluation) of an expression is the *time and place* of such an utterance (consideration, or evaluation). What should be clear is that in any given possible world there are numerous occasions.

refinements), and its importance stems from the role it plays in establishing key results of the modified (contextualized) account. The two subsequent results concern the modified account, which is developed as an analysis of a language containing context-indexed conditionals (contextualized language). Informally, the first states that if discourse is restricted to a single context, then the model theory of the modified account reduces (as it would be expected) exactly to the Stalnaker-Lewis' analysis of counterfactuals, in particular, extensions of **VC**. In this sense, the modified account is really just an extension of Stalnaker-Lewis type of analyses—it is *equivalent* to those accounts when dealing with sets of formulae that contain counterfactual connectives ranging over a single context index, but it extends those accounts by offering a model theory that can handle evaluating, and making inferences over sets of formulae containing counterfactual connectives whose context indices vary. The second, and more general result concerns the recovery of Stalnaker-Lewis analysis on the modified account, if certain contextual information preservation conditions are satisfied. Namely, part of the logic given by the **VC** semantic consequence relation can be preserved on the proposed account for those inferences (ranging over the contextualized language) where the context index of the conclusion is said to preserve some of the mutual contextual information of the context indices over which the premises range. The second result is applied in fashioning a logic of contextualized counterfactuals, offered in the form of a semantic consequence. It is intended as a logic that is sensitive to explicit contextual content. Contextual validity is strengthened by adding the requirement of contextual information preservation to the standard requirement of truth preservation at all possible worlds.

1. Counterfactuals and Context

Counterfactuals are expressions of the form "If it were the case that A, then it would be the case that B" (formally, $A > B$), where A and B are propositions. It is commonly believed that they are notoriously context sensitive. Take a well-known example:

1. If Caesar had been in command, he would have used the atom bomb.
2. If Caesar had been in command, he would have used catapults.[3]

Intuitively, the truth of each depends on contextual background assumptions. Clearly, for the first statement to be true, we require contexts where Caesar's knowledge of modern warfare is assumed to be in line with the military knowledge of a modern military general, whereas for the second to be true, no such contextual background assumption is required.[4] David Lewis (1973, pp. 66–67) approaches this contention by proposing a *rule of accommodation*,

[3] Quine (1960, p. 22) bases this example on similar ones given by Goodman (1954).

[4] Gabbay (1972, pp. 98–99) argues essentially along the same lines.

whereby the uttered counterfactual is taken as being asserted, and then context is called upon in resolving the vagueness of the comparative similarity in favor of the truth of the uttered counterfactual (for the purposes of the present discussion I will say that *a context justifies the assertion of a given conditional* to refer to the aforementioned role of context when employing the rule of accommodation).[5] However, the key drawback of this solution is that for any world of evaluation and explicit antecedent, a single context is called upon to justify the assertion of a counterfactual on any occasion. That is, a single context is fixed for all occasions. The formal semantics of Lewis' account and other, aforementioned analyses is clear in that regard. Donald Nute elucidates this fact as follows:

> SOS-models involve functions which take only possible worlds as arguments, while both SC-models and WS-models involve functions which take both possible worlds and sentences-qua-antecedents as arguments. [However] the evaluation of two conditionals with the same antecedent may require consideration of different sets of situations. Any semantics which takes into account only the antecedent of the conditional and the situation of the speaker in determining the situations to be considered in hypothetical deliberation does not explicitly recognize this fact. (1981, p. 73)[6]

Another way of seeing this major drawback is by highlighting a fundamental feature, pointed out by Chellas (1975, p. 138), that those possible world conditional analyses have in common—namely, of the conditionals being conceived of as expressions of relative necessity (for a detailed overview of such analyses, see Chellas, 1975; Priest, 2008, Sections 5.3 and 5.5; Weiss, 2018). This has the following consequence—when evaluating the truth of a counterfactual at some possible world *w*, the antecedent effectively acts as *restricted necessity operator*, making accessible only those possible worlds that have the features we take to be relevant to our deliberations in evaluating the conditional. But because only *w* and the explicit antecedent are employed in the determination of that restriction on those accounts, it is fixed for all occasions for conditionals with the same antecedent. But surely the features we take to be relevant to our deliberations in evaluating the conditional are not the same for all occasions, since contextual considerations underlying each occasion are bound to change.

In what follows I will argue why the aforementioned approaches (which in the current discussion shall be referred to as the *class of contested accounts*) are inadequate if we take the role of context seriously. The objection can be viewed as having two components. The first part is mainly linguistic, focusing on the inadequacy of analyzing sets of asserted conditionals accross various occasions

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[5] Lewis (1986, p. 251) maintains his approach and expresses this idea succinctly: "[t]here is a rule of accommodation: what you say makes itself true, if at all possible, by creating a context that selects the relevant features so as to make it true".

[6] Nute (1981, pp. 72, 76). Footnote 1 of the current paper disambiguates the acronyms used by Nute in the cited fragment.

in a manner that accounts for contextual differences underlying those occasions and consequently distinct justifications of the assertions. The second part of the objection focuses on the implications that such inadequacy has for the logic of counterfactuals, i.e., making inferences from sets of statements that include counterfactuals.

Accounts from the aforementioned, contested class fare fine when dealing with conditionals considered in isolation, however difficulties appear when we consider sets of conditionals, and in particular, inferences containing conditionals. It is clear from Lewis' formal semantics (1973; 1981) that asserting a set of conditionals across more than one occasion at any possible world is restricted to a single assertion-justifying context (modelled by a *single* similarity assignment to that world). However, given two conditionals with explicitly identical antecedents, we may wish to call upon different contexts (not just a single one) on distinct occasions to justify our assertions of either conditionals. For example, we may wish to have our assertion of (1) justified on one occasion by a context that does not justify the assertion of (2), and on another occasion have the assertion of (2) justified by a context that does not justify the assertion of (1).[7] To put it another way, on any two occasions we may wish to be free to assert conditionals with the same antecedent for different reasons (by recourse to different contexts that accordingly justify each assertion) or we may even wish to assert the same conditional for different reasons on two occasions, and as such not be restricted to relying on a single context in providing the corresponding justifications for those assertions.

Presently I shall give examples that aim to illustrate the inadequacy of the aforementioned accounts when tasked with a treatment that is supposed to account for context sensitivity when dealing with *sets* of counterfactuals. Let us first consider the following pair of counteridenticals given by Goodman (1954). Here the antecedents are the same, but their consequents are contradictory, on the assumed identity.

3. If I was Julius Caesar, I would not be alive in the 21st century.
4. If I was Julius Caesar, he would be alive in the 21st century (Goodman, 1983, p. 6).

Imagine asserting (3) on one occasion and asserting (4) on another occasion, at the same possible world. Those assertions are justified by recourse to different contexts on those occasions—clearly for the truth of (3) I assume being alive in the 1st century BCE, whereas no such assumption is required for the truth of (4)—but on the traditional accounts only a single context is available for both of

[7] Berto (2017, Section 5) essentially agrees with this view, the interpretation difference being that when Lewis speaks of *assertions* Berto speaks of *acts of imagination*, and when Lewis speaks of *explicit antecedents*, Berto speaks of *explicit content of the imagination acts*.

those occasions, i.e., the same set of antecedent worlds is considered as relevant in the evaluation of both conditionals. This is clearly inadequate. It seems that both (3) and (4) can be asserted or at least they can both be heard as true, albeit *according to different contexts* (see also Priest, 2016, p. 4). However, on accounts in the contested class of analyses, such pairs cannot be both evaluated as true at the same world, since their formal model theories allow only a single context to underlie the evaluation of any counterfactual with the same antecedent on any occasion, which means that on possible world analyses at most one conditional in the above pair can be evaluated as true at the same world.[8] Another interesting class of examples similar to (3) and (4) comes from a widespread phenomenon of contentious pairs of indicative conditionals known in the literature as "Gibbardian Stand-Offs", whereby its seems clear that there are good reasons for the truth (or assertion) of two conditionals with identical antecedents yet contradictory consequents, albeit each in its own context. I argue in Section 3.2 that my proposal can also be applied in offering a solution to these phenomena burdening the indicative conditional.

Those limitations have direct implications for the logic of counterfactuals, as becomes evident from the inference forms that the presence of those limitations is responsible for validating. Let us consider the example given by Quine again:

1. If Caesar had been in command, he would have used the atom bomb.
2. If Caesar had been in command, he would have used catapults.

As it has been already said, the kind of relevant assumptions required for the truth of (1) are not the same as those required for the truth of (2). There may be good reasons to assert (1) in some contexts and (2) in others. Moreover we may wish to assert (or evaluate as true) both on a single occasion, yet with recourse to distinct contexts that justify the assertion of each. However, the truth of both (1) and (2) should not entail the truth of:

5. If Caesar had been in command, he would have used catapults and the atom bomb.

Sure, there may exist a strange context that accounts for such idiosyncratic decisions (after all, it is possible to use both nukes and catapults), but inferring (5) from (1) and (2) should not be an automatic entailment, because clearly that depends on what contexts have been employed in the justification of (1) and (2), i.e., presumably, not always a single strange context (see also Berto, 2014, p. 113). However, all accounts in the contested class, that evaluate both (1) and (2) as true at some world are committed to evaluating (5) as true at that world.

[8] On analyses that invalidate conditional excluded middle, i.e., $(A > B) \vee (A > \sim B)$, there may be the third possibility of both being evaluated as false, e.g., this is one of the differences between Stalnaker's and Lewis' accounts.

This stems from the fact that whenever sets of conditionals with the same antecedent are modelled as jointly true at some world *w*, the set of relevant antecedent worlds employed in the evaluation of those conditionals at *w* is the same for all those conditionals. This becomes even more starkly evident in the example from Goodman. As discussed earlier, we may wish to accept both (3) and (4) as true, on different occasions, but we would never accept the truth of:

> If I was Julius Caesar, I (Julius Caesar) would and would not be alive in the 21st century.

It is not surprising that such analyses validate the inference of $A > (B \wedge C)$ from both $A > B$ and $A > C$ (henceforth referred to as *Adjunction of Consequents*) or equivalently have $(A > B \wedge A > C) \supset A > (B \wedge C)$ as the corresponding axiom in their respective proof theories.[9] Note that since on the contested accounts (3) and (4) can never be both evaluated as true, the inference goes through vacuously.

Gabbay's (1972) analysis of conditionals has one apparent advantage over the analyses in the contested class as it offers a semantic counterpart for the fact, which we have observed, that the evaluation of two conditionals with the same antecedent may require consideration of different sets of situations (Nute, 1980, p. 75). That semantic counterpart is the consequent, which is employed as an additional parameter that allows accounting for a potential context shift on any single occasion of utterance by considering different sets of worlds in the evaluation of conditionals with the same antecedents (for a more formal explanation, see Popieluch, 2019, pp. 32–36). However, as Nute (1980, p. 76) observes, Gabbay's analysis much like the analyses from the contested class will give a single, determinate truth value to the conditional, regardless of the contextual circumstances under which the conditional is evaluated, i.e., the same truth value for all occasions. Nute observes that there may be a relevant difference in the occasions of evaluation, even when both the antecedent and the consequent of the conditional remain the same, however Gabbay's formal semantics fails to offer a semantic mechanism that would allow flexibility in evaluating a conditional in a manner that accounts for distinct contextual considerations more than occasion. So in this sense Gabbay's account fares no better than Lewis'.

In the next section an analysis of counterfactuals is presented that avoids both the linguistic and logical issues described above. The presentation of the aforementioned account intends to be neutral with regard to the matter of subjunctive-indicative distinction and the discussion accompanying the presentation makes no commitments with regard to whether that distinction is fundamental or only apparent, and consequently whether there is a single, unifying analysis for both, or not. Rather, the aim of the article is to offer an analysis that can be applicable whenever context related issues do arise, or have been argued to arise. Because

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[9] Such inferences are valid on Lewis's logic VC and its extensions, which include Stalnaker's logic of conditionals.

subjunctives have been mostly burdened with such issues, much of the article does focus on counterfactuals. Moreover, because the modified account builds on Lewis' account, which is uniquely tailored to counterfactual conditionals (generally uttered in the subjunctive mood), the focus of the proposal has been mostly confined to counterfactuals, but also in the last section an important application of the proposed account to indicative conditionals is discussed. It concerns phenomena of "Gibbardian Stand-Offs", which have been identified by a number of authors to be essentially context related in nature (e.g., Bennett, 2003; Priest, 2016; Santos, 2018).

2. An Alternative

The alternative account, proposed in this article, is developed as a modification of ordering semantics for counterfactuals that proceeds by (i) expanding the formal language by substituting the single conditional connective > with an entire family of indexed connectives $\{>_c: c \in \mathcal{C}\}$, ranging over an index set $\mathcal{C}$ and (ii) subsequently a modified model theory is provided for the evaluation of the logical value of expressions $A >_c B$ interpreted as "In context c: If it were the case that A, then then it would be the case that B". Since the modified account is offered as a modification of ordering semantics for counterfactuals given by Lewis (1974; 1981), I begin by laying out the formal details of the latter. This is required since it is within that formalism that key concepts, such as *ordering frame refinements* are defined, which underlie the main results and the formal foundation for the semantic consequence of the modified account.

The culmination of the modified account is a logic of contextualized counterfactuals, offered in the form of a semantic consequence relation. The idea of *contextual validity*, adds to the standard requirement of *truth preservation* at all possible worlds a second requirement of *contextual information preservation*. A very similar idea—in terms of preserving imported information throughout an inference—is explored in Priest (2016, p. 8). Yet another approach, which proceeds by contextually restricting inferences via a language that contains a certain family of context indexed intensional connectives is outlined in Berto (2017, p. 11).[10]

2.1. Ordering Semantics for Counterfactuals

The resulting logic **CS** that is endorsed in this section is much like Lewis' preferred account save for strict centering being replaced with a weaker centering condition. That is, **CS** is just the logic that Lewis (1973) calls **VW**, which is obtained from his preferred system **VC** (commonly referred to as **C1**) by replacing the *strict centering* condition with the *weak centering* condition, or equivalently, removing the axiom $(A \wedge B) \supset (A > B)$ from the deductive system for **VC**.

[10] For a discussion outlining the similarity between Berto's context indexation suggestion and the approach offered in this paper, see (Popieluch, 2019, pp. 38–40).

2.1.1. Formal language.

Let us start with the basic ingredients for our language, i.e., a set of propositional variables $PV = \{p_n: n \in \mathbb{N}\}$ the elements of which shall be denoted with lowercase Roman letters (p, q, r, …) or subscripted lowercase Roman p's ($p_1, p_2, \ldots, p_k, \ldots$), or lowercase Greek letters (φ, ψ, χ, …); unary connectives: ~ (negation), □ (necessity), ◊ (possibility); and binary connectives: ∧ (conjunction), ∨ (disjunction), ⊃ (material conditional), > (counterfactual conditional). For the metalanguage, upper case letters (A, B, C, …) shall be used as variables ranging over complex formulae and propositional variables.

Definition 1.1. Define the language of interest, denoted $\mathcal{L}$, to be the set: {~, □, ◊, ∧, ∨, ⊃, >}.

Now we define the set of well-formed formulae.[11]

Definition 1.2. Let *For* be the smallest set closed under the following well-formed formula formation rules:

B: All propositional variables are wffs, i.e., $PV \subseteq For$.

R1: If $A \in For$ then $\{{\sim}A, \Box A, \Diamond A\} \subseteq For$.

R2: If $\{A, B\} \subseteq For$ then $\{A \land B, A \lor B, A \supset B, A > B\} \subseteq For$.

Definition 1.3. It will be helpful to define the subset of *For* that contains all and only formulae that contain occurrences of >. Denote that subset with $For_>$.

Definition 1.4. Denote the set $For \backslash For_>$ with For_0, which is just the set of wffs of the basic modal language.

2.1.2. Comparative similarity.

In order to establish the relations in our semantics, we need to introduce their intended meaning and basic properties. The systems of spheres are just a convenient, and intuitive way for representing information about the comparative similarity of worlds (Lewis, 1973, p. 48). We can do the same, directly in terms of comparative similarity of worlds, together with accessibility. To make this explicit let us consider the following definitions.

[11] E.g., the counterfactual "If kangaroos had no tails, they would topple over" would have the form: $p > q$, where p stands for "kangaroos have no tails" and q stands for "kangaroos topple over".

Definition 2.1. A binary relation $R \subseteq S \times S$ on a set S, denoted by $\lesssim$, is a *preorder* iff it is:

(1) transitive: $\forall x,y,z \in S\, ((x \lesssim y \wedge y \lesssim z) \longrightarrow x \lesssim z)$.

(2) reflexive: $\forall x \in S\, (x \lesssim x)$.

If $\lesssim$ satisfies (1), (2), and (3), it is a *total preorder* (also called a *non-strict weak order*).

(3) totality: $\forall x,y \in S\, (x \lesssim y \vee y \lesssim x)$. [12]

Definition 2.2. For any preorder $\lesssim$, denote $(x, y) \notin \lesssim$, i.e., "it is not the case that $x \lesssim y$" with $y < x$, and let us write $x \sim y$ to mean that both $x \lesssim y$ and $y \lesssim x$.

Lemma 2.1. If $\lesssim$ is a *preorder* on S then for no $x \in S$: $x < x$.

P r o o f. This follows directly from reflexivity of $\lesssim$, i.e., $x < x$ means $(x, x) \notin \lesssim$, contradicting reflexivity of $\lesssim$. □

Lemma 2.2. If $\lesssim$ is a *total preorder* on S, then for all $x, y \in S$:

(i) $x < y$ iff $(x, y) \in \lesssim$ and $(y, x) \notin \lesssim$,

(ii) $x \lesssim y$ iff $x < y$ or $x \sim y$.

P r o o f. (i) $(y, x) \notin \lesssim$ follows from definition of $x < y$, and $(x, y) \in \lesssim$ follows from totality of $\lesssim$. (ii) Given totality, either $(x, y) \in \lesssim$ and $(y, x) \notin \lesssim$ or both $(x, y) \in \lesssim$ and $(y, x) \in \lesssim$. The third, totality satisfying option $(x, y) \notin \lesssim$ and $(y, x) \in \lesssim$ is clearly impossible. □

My definition of ordering frames based on comparative similarity closely follows the definition of a *comparative similarity system* in Lewis (1973, p. 48), save for the condition corresponding to what Lewis calls *centering*, i.e.,

(CS3.1) The element i is $<_i$-minimal: $\forall j \in W (j \neq i \longrightarrow i <_i j)$,

which I replace with a weaker condition (CS3) corresponding to *weak centering*.

Definition 2.3. An *ordering frame* based on comparative similarity is a pair $(W, \lesssim)$, where W is a nonempty set and $\lesssim : W \longrightarrow \wp(W) \times \wp(W \times W)$ is a function that assigns to each $i \in W$ a pair $(S_i, \lesssim_i)$, consisting of a set $S_i \subseteq W$, regarded as the set of worlds accessible from i, and a binary relation $\lesssim_i$ on W, regarded as the

[12] Lewis (1973, p. 48) refers to this property as "strongly connected".

ordering of worlds in respect of their comparative similarity to i and satisfying the following conditions, for each $i \in W$:

(CS1) $\lesssim_i$ is a total preorder on S_i.

(CS2) i is self-accessible: $i \in S_i$.

(CS3) i is $\lesssim_i$-minimal: $\forall j \in W(i \lesssim_i j)$.

(CS4) Inaccessible worlds are $\lesssim_i$-maximal: $\forall j,k \in W(k \notin S_i \longrightarrow j \lesssim_i k)$.

(CS5) Accessible worlds are more similar to i than inaccessible worlds:

$$\forall j,k \in W((j \in S_i \wedge k \notin S_i) \longrightarrow j <_i k)$$

On the intended interpretation, elements of W are possible worlds, S_i is regarded as the set of worlds accessible from i, and $\lesssim_i$ is regarded as the ordering of worlds in respect of their comparative similarity to i, with the following intended meaning:

$j \lesssim_i k$: j is at least as similar to i as k is,

$j <_i k$: j is more similar to i than k is,

$j \sim_i k$: j and k are equally similar to i.[13]

Definition 2.4. Denote the *class of ordering frames from Definition 2.3* by **CS**.

Note that since centering implies weak centering, the class of ordering frames satisfying (CS3.1) instead of (CS3) is a proper subclass of **CS**.[14]

Definition 2.5. Given some $F \in$ **CS**, let W^F denote the domain of F and let $\lesssim^F$ denote F's ordering assignment on F's domain, i.e., $W^F \longrightarrow \wp(W^F) \times \wp(W^F \times W^F)$ as defined in 2.3. Also, let S_i^F and $\lesssim_i^F$ denote the elements of the image $\left(S_i^F, \lesssim_i^F\right)$ of $i \in W^F$ under $\lesssim^F$.

Definition 2.6. A *model based on comparative similarity* is the triple $(W, \lesssim, V)$ such that $(W, \lesssim)$ is an ordering frame and for each $i \in W$, V_i: $PV \longrightarrow \{0, 1\}$ is a function from PV to $\{0, 1\}$. Informally we think of $\{i \in W\colon V_i(p) = 1\}$ as the set of worlds in the model where is p true, and $\{i \in W\colon V_i(p) = 0\}$ as the set of worlds in the model where is p false.

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[13] Lewis' (1981, p. 220) definition of $\sim_i$ in terms of a strict comparative similarity relation $<_i$ is logically equivalent to the one he gave earlier, in (Lewis, 1973, p. 48)—the one I choose to use in this article. In terms of $<_i$ the comparative similarity equivalence $\sim_i$ is defined as follows: $j \sim_i k$: neither $j <_i k$ nor $j <_i k$.

[14] Since, if $j <_i^F k$, then $j \lesssim_i^F k$ for any $i, j, k \in W$, by totality and definition of $<_i^F$.

Truth in a model is defined in terms the satisfiability relation $\Vdash \subseteq W \times For$. We read $i \Vdash A$ as "A is true at i". Given a model $(W, \lesssim, V)$ and any $i \in W$, define $\Vdash$ as follows:

(1)	$i \Vdash p$	iff	$V_i(p) = 1$
(2)	$i \Vdash \sim A$	iff	not $i \Vdash A$
(3)	$i \Vdash A \wedge B$	iff	$i \Vdash A$ and $i \Vdash B$
(4)	$i \Vdash A \vee B$	iff	$i \Vdash A$ or $i \Vdash B$
(5)	$i \Vdash A \supset B$	iff	$i \Vdash \sim A$ or $i \Vdash B$
(6)	$i \Vdash \Box A$	iff	$\forall j \in W: j \Vdash A$
(7)	$i \Vdash \Diamond A$	iff	$\exists j \in W: j \Vdash A$
(8)	$i \Vdash A > B$	iff	(i) $\sim\exists k \in S_i: k \Vdash A$, or
			(ii) $\exists k \in S_i: k \Vdash A$ and $\forall j \in S_i\ (j \lesssim_i k \longrightarrow j \Vdash A \supset B)$

For convenience, let us introduce the following notation: $i \Vdash \Sigma$ iff $i \Vdash A$ for all $A \in \Sigma$.

When we want to explicitly refer to truth at a world in some model $\mathfrak{A}$, we shall employ the following notation: $\mathfrak{A}, i \Vdash A$ and $\mathfrak{A}, i \Vdash \Sigma$. Also, write $\mathfrak{A} \Vdash A$ when $\mathfrak{A}, i \Vdash A$ for all $i \in W^{\mathfrak{A}}$.

Definition 2.7. It will also be convenient to define $[A]^{\mathfrak{A}} := \{i \in W: \mathfrak{A}, i \Vdash A\}$ for any model $\mathfrak{A}$ with domain W. The superscript will be omitted in cases when its absence will not lead to ambiguity.

Definition 2.8. Let $\vDash_{CS} \subseteq \wp(For) \times For$, and define $\Sigma \vDash_{CS} A$ iff for all models $(W, \lesssim, V)$, and all $i \in W$, if $i \Vdash B$ for all $B \in \Sigma$, then $i \Vdash A$. We say an inference from Σ to A is valid iff $\Sigma \vDash_{CS} A$. That is, valid inference is defined as truth preservation at all worlds in all **CS**-models. A formula $A \in For$ is said to be valid iff $\emptyset \vDash_{CS} A$. Call this logic **CS**.

Note that since the truth conditions for $\Box$ and $\Diamond$ formulae are defined in terms of unrestricted quantification over possible worlds, i.e., only >-formulae truth conditions contain accessibility restrictions, the above validity conditions give the modal logic **S5** for the basic modal language.

Just as we have relativized formula validity to a model $\mathfrak{A} \Vdash A$ it will be of use to define valid inference relativized to a model.

Definition 2.9. Let $\vDash_{\mathfrak{A}} \subseteq \wp(For) \times For$, and given a **CS** model $\mathfrak{A} = (W, \lesssim, V)$, write:

(i) $\vDash_{\mathfrak{A}} A$ iff $\mathfrak{A} \Vdash A$

(ii) $\Sigma \vDash_{\mathfrak{A}} A$ iff for all $i \in W$, if $\mathfrak{A}, i \Vdash B$ for all $B \in \Sigma$, then $\mathfrak{A}, i \Vdash A$.

This allows us to give a more succinct definition of semantic consequence:

$$\Sigma \vDash_{\mathbf{CS}} A \text{ iff for all } \mathbf{CS} \text{ models } \mathfrak{A}\text{: } \Sigma \vDash_{\mathfrak{A}} A$$

Note that it is immediate from the above definitions that $\vDash_{\mathbf{CS}} \subseteq \vDash_{\mathfrak{A}}$, for any **CS** model $\mathfrak{A}$.

2.1.3. Ordering frame refinements and dilutions.

Let us now turn to defining ordering frame refinements and dilutions, which are the key protagonists in the account of ordering semantics presented here.[15]

Definition 3.1. Let $\mathcal{R} \subseteq \mathbf{CS} \times \mathbf{CS}$ and call an ordering frame G a *refinement* of ordering frame F iff $(F, G) \in \mathcal{R}$. And define $(F, G) \in \mathcal{R}$ iff:

(i) $W^G = W^F$,

and for all $i \in W^F$:

(ii) $\lesssim_i^G \subseteq \lesssim_i^F$

(iii) $S_i^G = S_i^F$

Definition 3.1.1. A *proper refinement* of F is a refinement G, such that $G \neq F$.

Definition 3.1.2. Let $\mathcal{R}[F] \coloneqq \{G \in \mathbf{CS}\colon (F, G) \in \mathcal{R}\}$ denote the *image* of F under $\mathcal{R}$, i.e., the set of all refinements of F.

Definition 3.2. Let $\mathcal{D} \subseteq \mathbf{CS} \times \mathbf{CS}$ and call an ordering frame G a *dilution* of ordering frame F iff $(F, G) \in \mathcal{D}$. And define $(F, G) \in \mathcal{D}$ iff:

(i) $W^G = W^F$,

and for all $i \in W^F$:

(ii) $\lesssim_i^F \subseteq \lesssim_i^G$

(iii) $S_i^G = S_i^F$

---

[15] The essential idea of refinements is based on (Lewis, 1981, pp. 226–227). However, Lewis (1981) defines refinements on strict preorder relations: if $j <_i^F k$, then $j <_i^G k$ (where G is a refinement of F). Given the way I have defined refinements (using total preorders) Lewis' definition is a derived property of refinements, i.e., Lemma 4.1.

Definition 3.2.1. A *proper dilution* of F is a dilution G of F, such that $G \neq F$.

Note: the orderings of refinements and dilutions are *total*, by definition of ordering frames.

Definition 3.2.2. Let $\mathcal{D}[F] \coloneqq \{G \in \mathbf{CS}\colon (F, G) \in \mathcal{D}\}$ denote the *image* of F under $\mathcal{D}$, i.e., the set of all dilutions of F.

2.1.4. Elementary properties of refinements and dilutions.

Now we prove some elementary yet crucial properties of refinements and dilutions. Frame refinements *preserve the strict ordering* of original ordering frames in the following sense:

Lemma 4.1. If G is a refinement of F, then if $j <_i^F k$ for any i, j, k according to some comparative similarity assignment $(S_i^F, \lesssim_i^F)$, then $j <_i^G k$ according to $(S_i^G, \lesssim_i^G)$.

P r o o f. It suffices to note that, since $\lesssim_i^F$ is total and $\lesssim_i^G \subseteq \lesssim_i^F$ for each i, then if $(j, k) \in \lesssim_i^F$ and $(k, j) \notin \lesssim_i^F$, i.e., $j <_i^F k$, then it follows that both $(j, k) \in \lesssim_i^G$ and $(k, j) \notin \lesssim_i^G$, i.e., $j <_i^G k$. Denying $(k, j) \notin \lesssim_i^G$ contradicts the subset property, and denying $(j, k) \in \lesssim_i^G$ contradicts totality. □

We have a dual result to Lemma 4.1 for frame dilutions. That is, frame dilutions *preserve the non-strict ordering* of original ordering frames in the following sense:

Lemma 4.2. If G is a dilution of F then if $j \lesssim_i^F k$ for any i, j, k according to some comparative similarity assignment $(S_i^F, \lesssim_i^F)$, then $j \lesssim_i^F k$ according to $(S_i^G, \lesssim_i^G)$.

P r o o f. It suffices to observe that, since $\lesssim_i^F \subseteq \lesssim_i^G$ for each i, if $(j, k) \in \lesssim_i^F$ then $(j, k) \in \lesssim_i^G$. □

Corollary 4.2.1. If $j \sim_i^F k$ for any i, j, k according to some comparative similarity assignment $(S_i^F, \lesssim_i^F)$ on a frame F, then $j \sim_i^G k$ according to any dilution G of F.

P r o o f. Immediate from Lemma 4.2 and definition of $\sim_i$. □

The dual relationship between frame refinements and frame dilutions, although implicit in the definition, deserves highlighting.

Lemma 4.3. For any ordering frames $F, G \in \mathbf{CS}$, $(F, G) \in \mathcal{R}$ iff $(F, G) \in \mathcal{D}$.

P r o o f. It is immediate from definitions of refinements and dilutions. □

Lemma 4.4. For any ordering frames $F = (W^F, \lesssim^F)$, $G = (W^G, \lesssim^G)$, and any V:

If $W^F = W^G$ and $A \in For_0$, then $(F, V), i \Vdash A$ iff $(G, V), i \Vdash A$.

P r o o f. It suffices to observe that the truth of formulae in For_0 is independent of $\lesssim$. □

2.1.5. Semantic properties of refinements and dilutions.

The following result is central to the key applications of the modified account. It is difficult to overstate its importance. It is the main result of ordering semantics for counterfactuals presented in this article. Refinements are *truth-preserving* in the following sense:

Proposition 5.1. If a counterfactual $A > B$ (such that $A, B \in For_0$) is *true* at a world according to some ordering frame F, then it is true at that world according to all refinements of F. That is, for all $F = (W^F, \lesssim^F) \in \mathbf{CS}$, and for all $A, B \in For_0$, $i \in W^F$, and V:

$$(F, V), i \Vdash A > B \text{ iff } (\forall G \in \mathcal{R}[F])((G, V), i \Vdash A > B)$$

P r o o f. ($\Longleftarrow$) Is immediate, since $F \in \mathcal{R}[F]$. ($\Longrightarrow$) Consider some $F \in \mathbf{CS}$, $A \in For_0$, $i \in W^F$, V such that $(F, V), i \Vdash A > B$. Hence, for all $A, B \in For_0$, $i \in W^F$, V either $\sim\exists k \in S_i^F$: $(F, V), k \Vdash A$ or $\exists k \in S_i^F$: $(F, V), k \Vdash A$ and $\forall j \in S_i^F$ ($j \lesssim_i^F k \longrightarrow (F, V), j \Vdash A \supset B$). Let us start with the vacuous case and assume for arbitrary $A \in For_0$, $i \in W^F$, and V that $\sim\exists k \in S_i^F$: $(F, V), k \Vdash A$. From this, Lemma 4.4, and the fact that $S_i^G = S_i^F$ we can infer that $\sim\exists k \in S_i^G$: $(G, V), k \Vdash A$. Next, let us assume that $\exists k \in S_i^F$: $(F, V), k \Vdash A$ and $\forall j \in S_i^F$ ($j \lesssim_i^F k \longrightarrow (F, V), j \Vdash A \supset B$). To distinguish it from other assumptions call this assumption *the main hypothesis*. It follows that $\exists k \in S_i^G$ and $(G, V), k \Vdash A$ for all $G \in \mathcal{R}[F]$, by Lemma 4.4 and the fact that $S_i^G = S_i^F$. Now, to show that $\forall j \in S_i^G$ ($j \lesssim_i^G k \longrightarrow (G, V), j \Vdash A \supset B$) we will proceed by assuming $j \lesssim_i^G k$ for arbitrary $j \in S_i^G$, $G \in \mathcal{R}[F]$, and show $(G, V), j \Vdash A \supset B$. So, let us assume $j \lesssim_i^G k$ for arbitrary $j \in S_i^G$, $G \in \mathcal{R}[F]$, and note that since G is a refinement of F, then F is a dilution of G, by Lemma 4.3. Also, it should be noted that dilutions are $\lesssim$-preserving in the sense of Lemma 4.2. Hence, we conclude $j \lesssim_i^F k$, by Lemma 4.2 and Lemma 4.3. From this and *the main hypothesis*, we infer $(F, V), j \Vdash A \supset B$, which in conjunction with the fact that $W^F = W^G$ gives $(G, V), j \Vdash A \supset B$, by Lemma 4.4. Therefore, we finally conclude that $\forall j \in S_i^G\ (j \lesssim_i^G k \longrightarrow (G, V), j \Vdash A \supset B)$, by conditional proof. This completes the proof. [16] □

We have a dual result for dilutions, which are *falsity-preserving* in the following sense:

Corollary 5.2. For all frames $F, G \in$ **CS** and for all $A, B \in For_0$, and V:

$$(G, F) \in \mathcal{D} \longrightarrow \forall i \in W^G)((G, V), i \nVdash A > B \longrightarrow (F, V), i \nVdash A > B)$$

Proof. We have the following from Proposition 5.1, for all $F, G \in$ **CS**, $A, B \in For_0$, and V:

1. $(F, G) \in \mathcal{R} \longrightarrow (\forall i \in W^F)((F,V), i \Vdash A > B \longrightarrow (G, V), i \Vdash A > B)$

Contraposing the consequent yields:

2. $(F, G) \in \mathcal{R} \longrightarrow (\forall i \in W^F)((G,V), i \nVdash A > B \longrightarrow (F, V), i \nVdash A > B)$

Finally, we obtain 3 by substituting an equivalent term in the antecedent of 2, by Lemma 4.3,

3. $(G, F) \in \mathcal{D} \longrightarrow (\forall i \in W^G)((G,V), i \nVdash A > B \longrightarrow (F, V), i \nVdash A > B)$

and note that whenever the antecedents of 2 and 3 are true, then $W^F = W^G$ is true, and the consequents of 2 and 3 are identical. If the antecedents of 2 and 3 are false, then both 2 and 3 are vacuously true, so the quantifier change is justified. □

2.1.6. Interpretation: contextual information.

Ordering frames, which constitute the basis of **CS** model theory are—much like systems of spheres—a means of carrying information about the comparative similarity of worlds, relative to any other world where a counterfactual's truth is being evaluated. On Lewis' (1981, §2) conception of comparative similarity, ordering frames, being largely determined by contextual considerations are to be viewed as *carriers of contextual information*.[17]

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[16] Lewis (1981, pp. 226–227) has proven a very similar result. His result is more general than Proposition 5.1 in one sense, and less general in another. Whereas Proposition 5.1 holds only for a class of frames based on *total preorderings*, Lewis has proven a similar result for ordering frames based on *partial orderings* (where only refinements are required to be based on total preorderings). On the other hand, whereas Lewis has proven this only for (strongly) *centered* ordering frames, Proposition 5.1 holds for *weakly centered* orderings frames, i.e., satisfying (CS3), so *a fortiori* it holds for ordering frames satisfying the (stronger) *centering* restriction (CS3.1). Also, the employment of frame dilutions and Lemmas 4.2 and 4.3. makes the proof of Proposition 5.1 substantially simper than Lewis' proof.

[17] Following (Lewis, 1973, §2.3; 1981, §2) in that regard.

> The ordering that gives the factual background depends on the facts about the world, known or unknown; how it depends on them is determined—or underdetermined—by our linguistic practice and by context. We may separate the contribution of practice and context from the contribution of the world, evaluating counterfactuals as true or false at a world, and according to a frame determined somehow by practice and context. (Lewis, 1981, p. 218)

Refinements, whilst containing more contextual information (when we refine, we add contextual information by making additional distinctions), preserve the contextual information of the original ordering frame. Another way of looking at this is to view those distinctions (absent from the original ordering frame) as becoming relevant on the context represented by the refinement. Dilutions do the opposite—they remove previously existing distinctions, so when we dilute we are removing contextual information (irrelevant information), i.e., distinctions that have been relevant on the context represented by the original frame are no longer relevant on the context represented by its dilution.

Usually we tend to think of submodels as providing less information than their extensions. But in this case, there is a sense in which the opposite seems to be happening. When we refine, we are taking submodels, and we can keep going until we get to a linear ordering: that direction feels like we are adding information. On the other hand, if we take supermodels (dilute), the limit is the case where everything is related to everything else, which feels like we are losing information. This tends to go against the usual intuitions.[18]

2.2. The Modified Account

2.2.1. Introduction.

The following sections constitute the model theory of the proposed analysis of contextualized counterfactuals, consisting of context representation, a formal language and its semantics. Setting up the basics of the semantics for the contextualized language, I designate (by way of proposal) the role of context representation to **CS** ordering frames (which constitute the basis of the **CS** account of counterfactuals) and argue that they are adequate for that purpose. The formal language for contextualized counterfactuals, introduces context-indexed connectives $>_c$ for each context c. That is, expressions like $A >_c B$ in the formal language intend to model contextualized counterfactuals of the form "In context c: If it were the case that A, then then it would be the case that B", where A and B express propositions. The corresponding semantics (**CS**+ model theory) of a language contextualized in that manner allows making distinctions in the truth value of counterfactuals with the same antecedents (and even the same antecedents and consequents), by appeal to contextual considerations explicitly indicated by their respective context indices.

[18] I owe this observation to Toby Meadows.

2.2.2. The modified language.

Each modified language is just like $\mathcal{L}$ given in Definition 1.1 that generates *For*, but instead of the single connective >, each contains a family of indexed connectives.

Definition 7.1. Let $\mathcal{L}^{\mathcal{C}} := \{\sim, \Box, \Diamond, \wedge, \vee, \supset\} \cup \{>_c : c \in \mathcal{C}\}$, where $\mathcal{C}$ is a set, regarded as a set of contexts.

The set of well-formed formulae $For^{\mathcal{C}}$ will reflect the intended analysis, so context-indices will not vary across nested $>_c$-formulae. I propose that the context-index of the main conditional connective $>_c$ of a nested conditional, e.g., $A >_c (B >_c C)$ should settle the matter of what information is imported into counterfactual worlds when evaluating its subformulae. I do this in Definition 7.3 by stipulating that nested indexed-conditionals inherit the context-index of the outermost indexed conditional.[19] The thought is that the information imported in evaluating the inner conditional is *contextually the same*, i.e., restricted by what information is imported in evaluating the outer conditional. But the information *is not the same simpliciter*, since the inner conditional need not have the same antecedent as the outer conditional, and its truth may not be evaluated at the same world as the outer conditional—both highly relevant factors that contribute to determining what information should be imported.

To define the set $For^{\mathcal{C}}$ of well-formed formulae of interest, it will be easier to first define a larger set, and subsequently apply the required (intended) restrictions.

Definition 7.2. Let $for^{\mathcal{C}}$ be the smallest set closed under the following well-formed formula formation rules:

B: All propositional variables are wffs, i.e., $PV \subseteq for^{\mathcal{C}}$.

R1: If $A \in for^{\mathcal{C}}$, then $\{\sim A, \Box A, \Diamond A\} \subseteq for^{\mathcal{C}}$.

R2: If $A, B \in for^{\mathcal{C}}$, then $\{A \wedge B, A \vee B, A \supset B\} \subseteq for^{\mathcal{C}}$.

R3: If $A, B \in for^{\mathcal{C}}$ and $c \in \mathcal{C}$, then $A >_c B \in for^{\mathcal{C}}$.

As mentioned earlier, indexed conditionals nested within other indexed conditionals inherit the indices of the outermost indexed conditional. It just does not make sense in this picture to speak of embedded conditionals whose indices vary. Below is the restriction on $for^{\mathcal{C}}$ that reflects this motivation.

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[19] The proposed approach may be interpreted as going some way of addressing a question posed by Priest (2018, Section 3.1, Endnote 14), regarding what information from the world where the counterfactual is evaluated should be imported into counterfactual worlds, when evaluating nested conditionals (counterfactuals).

Definition 7.3. Let $For^{\mathcal{C}}$ be the subset of $for^{\mathcal{C}}$ with the following restriction: for any single, nested formula $A >_c B$ where A or B contain instances of an indexed connective $>_x$ for some $x \in \mathcal{C}$, then $x = c$.

Example. Formulae such as $p >_a (q >_b r)$ or $(q >_b r) >_a p$, where $a \neq b$, are not elements of $For^{\mathcal{C}}$. However, the following are: $p >_a (q >_a r)$, $(q >_b r) >_b p$, $(p >_a q) \vee (r >_b s)$.

The following couple of definitions establish useful restrictions on $For^{\mathcal{C}}$, to which the key results will apply—namely unnested formulae. The following definition characterizes the part of $For^{\mathcal{C}}$ whose elements contain no nested indexed conditionals. If indexed conditionals exist, their antecedents and consequents are basic modal logic formulae.

Definition 7.4. Let $For^{\mathcal{C}}_{>0}$ be the subset of $For^{\mathcal{C}}$ such that for any formula of the form $A >_c B$, the following restriction applies: $A, B \in For_0$.

Example: $\sim(p >_a (q \supset r)) \wedge (((p \wedge \sim q) >_b r) \vee (q >_c r)) \in For^{\mathcal{C}}_{>0}$ for any a, b, $c \in \mathcal{C}$. But $p >_c (p >_c p) \notin For^{\mathcal{C}}_{>0}$ for no $c \in \mathcal{C}$.

The following definition characterizes the part of $For^{\mathcal{C}}$ whose elements have an indexed conditional connective as the main connective.

Definition 7.5. Define $For^{\mathcal{C}}(>) \coloneqq \{A >_c B\colon A, B \in For^{\mathcal{C}}, c \in \mathcal{C}\}$. That is, $For^{\mathcal{C}}(>)$ is just the set of $For^{\mathcal{C}}$ formulae whose main connective is $>_c$, for some $c \in \mathcal{C}$.

2.2.3. Modified model theory.

The semantics for the contextualized language draws heavily on **CS** model theory (intended to serve as the foundation for **CS**+ model theory) by developing a formalism that reduces the truth conditions for $A >_c B$ on a **CS**+ model to those for $A > B$ on a corresponding **CS** model whose underlying ordering frame is taken to represent context c. That is, contextual considerations underlying a context-indexed expression are cashed out in terms of contextual information carried by ordering frames. Some tentative suggestions to that effect can be found in Nolan's (1997, n. 28).

The formula $A >_c B$ is intended to be read as an explicitly contextualized version of $A > B$. That is, the model theory in this section gives an analysis of $A >_c B$, which is to be read as: "In context c: If it were the case that A, then it would be the case that B".

The following definition will play a key role in the defining the truth conditions for indexed counterfactuals, i.e., for the truth conditions of formulae like $A >_c B$.

Definition 7.8. Let $\underline{\quad}$: $For^{\mathcal{C}} \longrightarrow For$ be the function that transforms all formulae with indexed connectives $>_c$ for any $c \in \mathcal{C}$ into unindexed ones $>$, in all subformulae of a formula. That is, it "strips" any $For^{\mathcal{C}}$ formula of its indices, leaving its index-less *For* counterpart.

B: $\underline{p} = p$ for all $c \in PV$.

R1: $\underline{*\,A} = *\,\underline{A}$ for each $* \in \{\sim, \Box, \Diamond\}$ and $A \in For^{\mathcal{C}}$.

R2: $\underline{A \circ B} = \underline{A} \circ \underline{B}$ for each $\circ \in \{\wedge, \vee, \supset\}$ and $A, B \in For^{\mathcal{C}}$.

R3: $\underline{A >_c B} = \underline{A} > \underline{B}$ for each $c \in \mathcal{C}$ and $A, B \in For^{\mathcal{C}}$.

E x a m p l e. $\underline{{\sim}p >_c (q \vee r)} = {\sim}p > (q \vee r)$.

It will be useful to extend the above definition of the "index-elimination function" to sets of formulae. No ambiguity should arise whether the argument is a formula or a set of formulae.

Definition 7.9. For any $\Sigma \subseteq For^{\mathcal{C}}$, let $\underline{\Sigma} := \{\underline{A} \in For\colon A \in \Sigma\}$.

Definition 8.1. A **CS**+ frame of the modified language is a triple $(W, \mathcal{C}, r)$, where $W \neq \emptyset$ and $\mathcal{C} \neq \emptyset$ are sets, and $r\colon W \times \mathcal{C} \longrightarrow \wp(W) \times \wp(W \times W)$ is a function such that $r(i, c)$ satisfies conditions CS1–CS5 for each $c \in \mathcal{C}$.[20]

On the intended interpretation W is regarded as a set of possible worlds, $\mathcal{C}$ is regarded as a set of contexts, and r is regarded a comparative similarity assignment to world i in context c, i.e., $r(i, c) = (S_{i,c}, \lesssim_{i,c})$ such that $S_{i,c}$ and $\lesssim_{i,c}$ satisfy conditions CS1–CS5. Informally, the set of all similarity assignments restricted to some index $c \in \mathcal{C}$, i.e., $\{r(i, c)\colon i \in W\} = r[W \times c]$ is regarded as *representing* (reflecting) a context indexed by c.[21]

Putting it another way, in line with Lewis, we could say that $r[W \times c]$ is the ordering frame determined somehow by practice and context c. It should be observed that collecting such assignments for all $i \in W$ and some fixed $c \in \mathcal{C}$ will give $\{r(i, c)\colon i \in W\} = \lesssim^F [W]$, where $F \in$ **CS**, i.e., a comparative similarity assignment that is identical to one given by some **CS** ordering frame. Hence, the image of r over all elements of $W \times \mathcal{C}$ would be identical to a collection of com-

[20] It should be noted that from a purely formal perspective, **CS**+ frames are essentially Lewisean in spirit. Not only does the domain of r (the extended counterpart to Lewisean \$ or $\lesssim$) satisfy the condition of the general formalism envisaged by Lewis (1973, p. 119) of being a set, but the image of r contains total preorders, which are just the Lewisean (1973; 1981) ordering semantics counterparts to systems of spheres.

[21] This is standard notation for the image of a set under some function, i.e., $r[W \times c] = \{r(i, c)\colon i \in W\}$, and similarly $\lesssim^F [W] = \{\lesssim^F_i\colon i \in W\}$.

parative similarity assignments given by a subset of **CS** frames. That is $r[W \times \mathcal{C}] \subseteq \{\lesssim^F [W]: F \in \mathbf{CS}\}$. Informally and succinctly, we could say that each **CS**+ frame acts like a set of **CS** frames.

Some of the above informal observations can be made precise in the following lemma, which will have some important applications in developing a precise account of a contextualized consequence relation.

Lemma 8.1. For any $\mathfrak{F} = (W, \mathcal{C}, r) \in$ **CS**+ and any $c \in \mathcal{C}$ there exists a unique ordering frame $F = (W, \lesssim^F) \in$ **CS** such that $r(i, c) = (S_i^F, \lesssim_i^F)$ for each $i \in W$, or equivalently the following holds: $\{r(i, c): i \in W\} = r[W \times c] = \lesssim^F [W]$. Also, for any $F = (W, \lesssim^F) \in$ **CS** there exists a unique family $\mathbb{F}$ of **CS**+ frames whose assignments for some fixed $c \in \mathcal{C}$ and all worlds are exactly the comparative similarity assignments of F. Formally, this family would be $\mathbb{F} = \{(W, \mathcal{C}, r) \in \mathbf{CS}+: \exists c \in \mathcal{C}(\{r(i, c): i \in W\} = \lesssim^F [W])\}$ for some $F \in$ **CS**.

Proof. Observe that each $r(i, c)$ satisfies CS1–CS5 by definition, making $F = (W, \lesssim^F) \in$ **CS**, as required. Moreover, F is unique since $W^F = W^{\mathfrak{F}}$. To see that there exists a unique family of **CS**+ frames for any $F \in$ **CS**, it suffices to see that $W^F = W^{\mathfrak{F}}$ and that $\lesssim^F [W]$ is just a collection of comparative similarity assignments and if some $\mathfrak{F} = (W, \mathcal{C}, r) \in$ **CS**+ contains a context index $c \in \mathcal{C}$ such that $\{r(i, c): i \in W\} = \lesssim^F [W]$, then $\mathfrak{F} \in \mathbb{F}$, else $\mathfrak{F} \notin \mathbb{F}$. □

Given the above result we can establish some useful notation, which will be crucial in giving a succinct expression of contextualized validity as well as an important theorem.

Definition 8.2. Given *Lemma 8.1*, and given a $\mathfrak{F} = (W, \mathcal{C}, r) \in$ **CS**+, and any $c \in \mathcal{C}$, denote with $F_{\mathfrak{F}}(c)$ the unique $F = (W, \lesssim^F) \in$ **CS** such that $r[W \times c] = \lesssim^F [W]$.

The motivation for the following definition stems from expressing contextualized validity as succinctly and clearly as possible. Here we introduce notation that bridges semantic notions such as ordering frames and ordering frame refinements with the corresponding syntactic notions of context indices, explicitly present in the formal language. This notation will be key in the formulation of Theorem 8.6 and Definition 9.1 (contextualized consequence relation).

Definition 8.3. For any $\mathfrak{F} = (W, \mathcal{C}, r) \in$ **CS**+ and $a, b \in \mathcal{C}$ let $b \leq a$ iff $(F_{\mathfrak{F}}(a), F_{\mathfrak{F}}(b)) \in \mathcal{R}$. That is, $b \leq a$ iff $F_{\mathfrak{F}}(b)$ is a refinement of $F_{\mathfrak{F}}(a)$.

Definition 8.4. A **CS**+ model of the modified language is the quadruple:

$$(W, \mathcal{C}, r, V)$$

where $(W, \mathcal{C}, r)$ is a **CS**+ frame and V is as in Definition 2.6.

Definition 8.4.1. Truth in **CS**+ models is defined via a satisfiability relation $\Vdash^{\mathcal{C}} \subseteq W \times For^{\mathcal{C}}$. We read $i \Vdash^{\mathcal{C}} A$ as "A is true at i". Given a **CS**+ model $(W, \mathcal{C}, r, V)$ and any $i \in W$, define $\Vdash^{\mathcal{C}}$ as follows:

(1) $i \Vdash^{\mathcal{C}} p$ iff $V_i(p) = 1$

(2) $i \Vdash^{\mathcal{C}} {\sim}A$ iff not $i \Vdash^{\mathcal{C}} A$

(3) $i \Vdash^{\mathcal{C}} A \wedge B$ iff $i \Vdash^{\mathcal{C}} A$ and $i \Vdash^{\mathcal{C}} B$

(4) $i \Vdash^{\mathcal{C}} A \vee B$ iff $i \Vdash^{\mathcal{C}} A$ or $i \Vdash^{\mathcal{C}} B$

(5) $i \Vdash^{\mathcal{C}} A \supset B$ iff $i \Vdash^{\mathcal{C}} {\sim}A$ or $i \Vdash^{\mathcal{C}} B$

(6) $i \Vdash^{\mathcal{C}} \Box A$ iff $\forall j \in W{:}\ j \Vdash^{\mathcal{C}} A$

(7) $i \Vdash^{\mathcal{C}} \Diamond A$ iff $\exists j \in W{:}\ j \Vdash^{\mathcal{C}} A$

(8) $i \Vdash^{\mathcal{C}} A >_c B$ iff ${\sim}\exists k \in S_{i,c}{:}\ k \Vdash \underline{A}$, or
$\exists k \in S_{i,c}{:}\ k \Vdash \underline{A}$ and $\forall j \in S_{i,c}(j \lesssim_{i,c} k \longrightarrow j \Vdash \underline{A \supset B})$

Note that if this definition is restricted to $For^{\mathcal{C}}_{>0}$, as indeed many of our results are, then resorting to the index-elimination function is unnecessary, and we can just give the following, simpler expression:

(8') $i \Vdash^{\mathcal{C}} A >_c B$ iff ${\sim}\exists k \in S_{i,c}{:}\ k \Vdash A$, or
$\exists k \in S_{i,c}{:}\ k \Vdash A$ and $\forall j \in S_{i,c}(j \lesssim_{i,c} k \longrightarrow j \Vdash A \supset B)$

We have an analogous result to Proposition 5.1 for **CS**+ models, if we observe, as Lemma 8.1 shows, that **CS** models are embedded within **CS**+ models. Let us consider a relationship much like refinements but defined between collections of comparative similarity assignments for some fixed $c \in \mathcal{C}$, and all $i \in W$, i.e., $r[W \times c]$, which Lemma 8.1 shows to be identical to **CS** ordering frames. That is, let us extend the notion of refinements to **CS**+ frames as follows.

Definition 3.1.1. For any **CS**+ frame $(W, \mathcal{C}, r)$ call $r[W \times b]$ a *refinement* of $r[W \times a]$ iff for all $i \in W$:

(i) $\lesssim_{i,b} \subseteq \lesssim_{i,a}$

(ii) $S_{i,a} = S_{i,b}$

Note that condition (i) of domain identity in Definition 3.1 of refinements on **CS** frames on this definition is automatically satisfied, since it is defined on a single **CS**+ frame. Let us abbreviate $r[W \times a]$ with r_a for any **CS**+ frame. We could borrow the notation $(r_a, r_b) \in \mathcal{R}$ to say that r_b is a *refinement* of r_a. Now we get the **CS**+ counterpart of Proposition 5.1 for free.

Corollary 5.1.1. If a counterfactual $A >_a B$ (such that $A, B \in For_0$) is true at some world and r_b is a refinement of r_b, then $A >_b B$ true at that world. That is, for all $\mathfrak{A} = (W, \mathcal{C}, r, V)$, $A, B \in For_0$, $i \in W$, $a, b \in \mathcal{C}$, and V:

$$\mathfrak{A}, i \Vdash^{\mathcal{C}} A >_a B \quad \text{iff} \quad (\forall r_b \in \mathcal{R}[r_a])(\ \mathfrak{A}, i \Vdash^{\mathcal{C}} A >_b B)$$

Proof. Each r_c for any $c \in \mathcal{C}$ has the properties of a **CS** ordering frame, by Lemma 8.1, so the result follows by Proposition 5.1. □

Let us introduce further notation that will make subsequent, key expressions more succinct.

Definition 8.5. As in the case of **CS** models, let us introduce the following notation for convenience: $\mathfrak{A}$, $i \Vdash^{\mathcal{C}} \Sigma$ iff $\mathfrak{A}$, $i \Vdash^{\mathcal{C}} A$ for all $A \in \Sigma$. Also denote with $\mathfrak{A} \Vdash^{\mathcal{C}} A$ when $\mathfrak{A}$, $i \Vdash^{\mathcal{C}} A$ for all $i \in W^{\mathfrak{A}}$.

Just as we have relativized *formula validity* to a model $\mathfrak{A} \Vdash^{\mathcal{C}} A$ in the definition above, it will be of use to define *valid inference* relativized to a model.

Definition 8.6. Let $\vDash^{\mathcal{C}}_{\mathfrak{A}} \subseteq \wp(For^{\mathcal{C}}) \times For^{\mathcal{C}}$, and given a **CS**+ model $\mathfrak{A} = (W, \mathcal{C}, r, V)$, write

$\vDash^{\mathcal{C}}_{\mathfrak{A}} A$	iff	$\mathfrak{A} \Vdash^{\mathcal{C}} A$,
$\Sigma \vDash^{\mathcal{C}}_{\mathfrak{A}} A$	iff	for all $i \in W$: if $\mathfrak{A} \Vdash^{\mathcal{C}} \Sigma$, then $\mathfrak{A} \Vdash^{\mathcal{C}} A$.

Now we proceed to define formula validity and semantic consequence of the contextualized language on the proposed **CS**+ model theory.

Definition 8.7 (**CS**+ validity)**.** Define the relation $\vDash^{\mathcal{C}}_{\mathbf{CS+}} \subseteq \wp(For^{\mathcal{C}}) \times For^{\mathcal{C}}$, as follows: $\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS+}} A$ iff for all **CS**+ models $\mathfrak{A}$ and $i \in W$: if $\mathfrak{A}$, $i \Vdash^{\mathcal{C}} \Sigma$, then $\mathfrak{A}$, $i \Vdash^{\mathcal{C}} A$.

We say an inference from Σ to A is **CS**+ valid iff $\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS+}} A$. That is, valid inference is defined as truth preservation at all worlds in all **CS**+ models. A formula $A \in For^{\mathcal{C}}$ is said to be **CS**+ valid iff $\emptyset \vDash^{\mathcal{C}}_{\mathbf{CS+}} A$. Notation from Definition 8.6 allows us to express the **CS**+ semantic consequence more succinctly: $\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS+}} A$ iff for all **CS+** models $\mathfrak{A}$: $\Sigma \vDash^{\mathcal{C}}_{\mathrm{A}} A$.

Note that it is immediate from the above definitions that $\vDash^{\mathcal{C}}_{\mathbf{CS+}} \subseteq \vDash^{\mathcal{C}}_{\mathfrak{A}}$ for any **CS**+ model $\mathfrak{A}$.

It should be also noted that since the truth conditions for □ and ◊ formulae are defined in terms of unrestricted quantification over possible worlds, i.e., only $>_c$-formulae truth conditions depend on $\mathcal{C}$ and r, the above validity conditions give the modal logic **S5** for the basic modal language.

The part of the basic modal language is indistinguishable between the two classes of models in the following sense.

Lemma 8.2. If for any **CS** model $\mathfrak{A}$, **CS**+ model $\mathfrak{B}$ such that $W^{\mathfrak{A}} = W^{\mathfrak{B}}$ and $V^{\mathfrak{A}} = V^{\mathfrak{B}}$, then for any $A \in For_0$, and $i \in W$: $\mathfrak{A}, i \Vdash A$ iff $\mathfrak{B}, i \Vdash^{\mathcal{C}} A$.

Proof. It suffices to note that elements of For_0 depend only on W and V. □

That is, the classes of **CS** models and **CS**+ models validate exactly the same formulae of For_0.

Theorem 8.3. If $\Sigma \cup \{A\} \subseteq For_0$: then $\Sigma \vDash_{\mathbf{CS}} A$ iff $\Sigma \vDash_{\mathbf{CS+}} A$.

Proof. Immediate from Lemma 8.2 and the definitions of $\vDash_{\mathbf{CS}}$ and of $\vDash_{\mathbf{CS+}}$. □

2.2.4. Main results of the modified account.

Aside from the formulation of the contextualized consequence relation given in the next section, Theorem 8.5 and Theorem 8.6, formulated and proved in this section are the main results of the modified account. Lemma 8.2 and Theorem 8.3 sanction Theorem 8.5, which captures our intuition regarding the contextualized language—if we restrict our discourse to a single context on any occasion, then we should expect **CS**+ analysis (indexed account) reduce to the **CS** analysis (unindexed account).

The second of the two main theorems, Theorem 8.6—sanctioned by the application of Proposition 5.1 in a key step of its proof—states that part of the logic given by **CS** semantic consequence relation can be preserved on **CS**+ models if the conclusion context preserves the contextual information of the contexts over which the premises range.[22]

Definition 8.8. Call frame $H \in$ **CS** a *mutual refinement* of frames F and G iff $(F, H) \in \mathcal{R}$ and $(G, H) \in \mathcal{R}$. Note that H is a mutual refinement of F and G iff $H \in \mathcal{R}[F] \cap \mathcal{R}[G]$.[23]

[22] It is worthwhile following the proof of Theorem 8.6 to see how other results are employed, but in particular how Proposition 5.1 is applied in securing contextual information preservation—from the premises to the conclusion—as that should may offer insight to understanding the formulation of contextualized validity.

[23] For a reminder of the meaning of $\mathcal{R}$ and $\mathcal{R}[F]$, see Definitions 3.1 and 3.1.2, respectively.

There is another important fact, that goes beyond For_0 that we will need in proving Theorems 8.5 and 8.6. Informally speaking it states that **CS**+ models behave much like collections of **CS** models. That is, whenever we restrict our discourse to a single context, modelled by **CS**+ models restricted to some single context index, there is a **CS** model that gives us the same analysis. This result and subsequently Theorem 8.5 can be viewed as a formal vindication of the objection expressed in Section 1, that analyses on the contested class are already restricted in that manner. The following lemma establishes the above informal observation.

Lemma 8.4. Given $(\mathfrak{F}, V) \in$ **CS**+ and $(F_{\mathfrak{F}}(c), V)$, for any $c \in \mathcal{C}$ and any $A >_c B \in For^{\mathcal{C}}$:

$$(\mathfrak{F}, V), i \Vdash^{\mathcal{C}} A >_c B \text{ iff } (F_{\mathfrak{F}}(c), V), i \Vdash \underline{A >_c B}$$

P r o o f. The result follows directly from the definition of $F_{\mathfrak{F}}(c)$ and **CS**+ truth conditions for indexed formulae. That is, $(\mathfrak{F}, V)$, $i \Vdash^{\mathcal{C}} A >_c B$ is given in terms of $\underline{A >_c B}$ being true according to the comparative similarity assignment $r(i, c)$, but the comparative similarity assignment at i on $F_{\mathfrak{F}}(c)$ is just $r(i, c)$ by Definition 8.2. □

Definition 8.9. Denote $For^{\mathcal{C}}_{>0} \cap For^{\mathcal{C}}(>)$ with $For^{\mathcal{C}}_{>0}(>)$.[24]

Definition 8.10. Let Ind: $\wp(For^{\mathcal{C}}) \longrightarrow \wp(\mathcal{C})$ be the function that outputs the set of all indices appearing in a set of formulae; e.g., $Ind(\{p >_c q\}) = \{c\}$, $Ind(\{p >_a q, p >_b q\}) = \{a, b\}$.

All valid **CS** inference patterns are preserved on the modified account, whenever the premises and conclusions range over at most a single context index. This makes sense intuitively, and the semantics manages to align with our intuition in this regard. This can almost be stated without proof, as a corollary of Lemma 8.4, but I provide one anyway, only if to highlight some important relationships between **CS** and **CS**+ models.

Theorem 8.5. For all $\Sigma \cup \{A\} \subseteq For^{\mathcal{C}}$: If $\underline{\Sigma} \vDash_{\mathbf{CS}} \underline{A}$ and $|Ind(\Sigma \cup \{A\})| \leq 1$, then $\Sigma \vDash_{\mathbf{CS+}} A$.

In other words, if the unindexed inference is **CS** valid, and if the premises and conclusion range over at most one context-index, then the inference is **CS**+ valid.

P r o o f. An informal argument should suffice, if we observe that restricting the inference to at most a single context index, effectively restricts the analysis to

[24] See Definition 7.4 and Definition 7.5.

a basic modal language with a single indexed conditional connective $>_c$. That is, with Lemma 8.4 and Theorem 8.3 it can be shown that such a restriction makes **CS**+ models behave exactly like **CS** models. □

E x a m p l e. For all $A, B \in For^{\mathcal{C}}$, and all $c \in \mathcal{C}$:

$\vDash_{\mathbf{CS}+} A >_c A$
$A, A >_c B \vDash_{\mathbf{CS}+} B$
$\Box(A \supset B) \vDash_{\mathbf{CS}+} A >_c B$
$\vDash_{\mathbf{CS}+} \sim((A >_c B) \wedge (A >_c \sim B))$

An important generalization of Theorem 8.5 can be given by employing truth preserving properties of ordering frame refinements, given by Proposition 5.1. This generalization is a step toward establishing a notion of contextualized validity, to which we shall turn our attention to in the next section.[25]

Theorem 8.6. For all $\Sigma \cup \{A\} \subseteq For^{\mathcal{C}}_{>0}(>) \cup For_0$:

If (1) $\underline{\Sigma} \vDash_{\mathbf{CS}} \underline{A}$ and
(2) if $Ind(\{A\}) = \{a\}$, then $a \leq b$ for all $b \in Ind(\Sigma)$ for each **CS**+ frame,
then $\Sigma \vDash_{\mathbf{CS}+} A$.

In other words, if the unindexed inference is **CS** valid and the conclusion index corresponds to an ordering frame that is a refinement of all ordering frames that correspond to the indices over which the premises range, then the inference is also **CS**+ valid. We interpret condition (2) as saying that the context on which the conclusion is evaluated is not independent of the contexts on which the premises are evaluated. That is, the conclusion context is supposed to *preserve the contextual information* carried by contexts on which the premises are evaluated. It is hoped that the following, informal proof will be insightful.

P r o o f. Let $(\mathfrak{F}, V)$ be a **CS**+ model where all $B \in \Sigma$ are true at some world i. Now, each $\underline{B} \in \underline{\Sigma}$ originally indexed by $b \in Ind(\Sigma)$ is also true at i according to $(F_{\mathfrak{F}}(b), V)$ by Lemma 8.4. Next, given that $(F_{\mathfrak{F}}(a), V)$ is a *mutual refinement* of

[25] Note that the restriction to $For^{\mathcal{C}}_{>0}(>)$ stems from the fact that ordering frame refinements are only truth preserving, and that is the part of $For^{\mathcal{C}}$ to which Proposition 5.1 applies. Just to be clear, if $A >_c B \in For^{\mathcal{C}}_{>0}(>)$, then $A, B \in For_0$. That is, $A \in For^{\mathcal{C}}_{>0}(>)$ iff $|Ind(\{A\})| = 1$. In other words, this result apples to a language restricted to the basic propositional modal language with indexed conditionals appearing only as the main connectives to formulae that do not contain any other indexed conditionals as proper subformulae. I have stated this in the overview of the current section.

each ($F_{\mathfrak{F}}(b)$, V) by condition (2), Proposition 5.1 grants that each $\underline{B} \in \underline{\Sigma}$ is also true at i according to ($F_{\mathfrak{F}}(a)$, V). Therefore, $\underline{A}$ is also true at i according to ($F_{\mathfrak{F}}(a)$, V), since (1) is assumed. Finally, we see that A is also true at i according to ($\mathfrak{F}$, V), by Lemma 8.4. Hence, the inference is **CS**+ valid, as required. □

2.2.5. Contextualized validity.

We close the discussion by giving the definition of contextualized validity, and show that it fares well with inference patterns that motivated this account. **CS**+ is very weak since on the current definition of **CS**+ validity via **CS**+ semantic consequence relation, there are no conditions placed on the relationship between context-indices appearing in the premises and the conclusion. But this is inadequate if we wish to fashion a logic that is sensitive to explicit contextual content. That is, we have developed an analysis of the contextualized language but have only included *truth preserving* conditions for validity in that definition—naturally, we also want a notion of *contextual information preserving* conditions on the new, contextualized notion of valid inference. That is, currently, by Definition 8.4 we have the following condition for **CS**+ valid inference:

$$\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS+}} A \text{ iff } \Sigma \vDash^{\mathcal{C}}_{\mathfrak{A}} A \text{ for all } \mathbf{CS}\text{+ models } \mathfrak{A}.$$

Clearly, these validity conditions are no different from those for **CS** and as such inadequate for the notion of a consequence relation that takes into account relationships that may exist between the context indices of formulae in the premises and conclusion. Consequently, such conditions make **CS**+ unacceptably weak, because for every **CS** valid inference there will be a counterexample by choice of indices for the premises and conclusion such that the premises are true, and the conclusion is false.

Theorem 8.6 captures some of the contextual information preserving features that hint at how contextual constraints could be fashioned. The theorem tells us that if we restrict the language in a way that Proposition 5.1 can be implemented, then **CS** validity and valid inference is preserved if additional conditions on the relationship between the premise indices and conclusion index are satisfied, i.e., conditions that correspond to what we mean by contextual information preservation. This opens a possibility for defining a notion of valid inference that those conditions underlie. That is, we could fashion a notion of contextualized inference by adding condition (2) of Theorem 8.6 to the current definition of **CS**+ valid consequence. The key definition that requires modification is of $\Sigma \vDash^{\mathcal{C}}_{\mathfrak{A}} A$, defined (below) since $\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS+}} A$ is defined in terms of it.

Definitions 9.1 and 9.2 establish a proper logic of contextualized counterfactuals. That is, a logic where valid inference is not defined merely in terms of truth preservation but also in terms of *contextual information preservation*.

Definition 9.1. For a **CS**+ model $\mathfrak{A}$ let $\vDash^{\mathcal{C}}_{\mathfrak{A}} \subseteq \wp(For^{\mathcal{C}}_{>0}(>) \cup For_0) \times For^{\mathcal{C}}_{>0}(>) \cup For_0$, be defined as: $\Sigma \vDash^{\mathcal{C}}_{\mathfrak{A}} A$ iff

(1) $\forall i \in \mathfrak{A}[i \Vdash^{\mathcal{C}} \Sigma \longrightarrow i \Vdash^{\mathcal{C}} A]$ and

(2) $\forall F \in \mathbf{CS}+ [Ind(\Sigma) \neq \emptyset \longrightarrow \exists a \in Ind(\{A\})\ \forall b \in Ind(\Sigma)[a \leq b]]$

Condition (1) demands *truth preservation* at all worlds in a model whereas condition (2) requires *contextual information preservation* at all worlds in a model, making it the uniquely characteristic feature of the proposed, contextualized account. The requirement in condition (2) that for each model there must exist a mutual refinement of all the premise contexts intends to capture the idea of the necessary condition of there being a context that preserves some of the contextual information that is present in the premises. If there is no such context, then it makes little sense to speak of the conclusion being true anywhere else.[26]

Definition 9.2. Now we can finally define contextualized validity as follows:

$$\Sigma \vDash^{\mathcal{C}}_{\mathbf{CS}+} A \text{ iff } \Sigma \vDash^{\mathcal{C}}_{\mathfrak{A}} A \text{ for all } \mathbf{CS}+ \text{ models } \mathfrak{A}.$$

Part of the motivation for the contextualized account was to invalidate inferring $A > (B \wedge C)$ from $A > B$ and $A > C$, due to some obvious counterexamples. We know that it is **CS** valid and on other conditional logics, because for the kind of instances that we view as worrisome, the premises can never be true, and as such go through vacuously.

3. Advantages of the Modified Account

3.1. Accounting for Contextual Differences

The proposed analysis overcomes all the shortcomings of the analyses that have been identified in Section 1. We can (i) have models that allow the evaluation both elements in each pair (1)–(2) and (3)–(4) as true at a single world. Moreover, (ii) in each case, such evaluation does not commit the analysis to the truth of the conditional with the same antecedent and the conjunction of the consequents of the conditionals in each pair. That is, *Adjunction of Consequents* is not valid if context indices that range over the premises are allowed to vary, i.e.,

[26] It may be worthwhile sharing the observation that the requirement of the existence of a context index that is said to preserve some information mutual to all premise indices (present in condition (2) of Definition 9.1) resembles in its form the syntactic, propositional variable sharing condition for valid relevant conditionals in the definition of relevant logic validity: "A propositional logic is relevant iff whenever $A \longrightarrow B$ (where '$\longrightarrow$' denotes logical implication) is logically valid, A and B have a propositional variable in common" (Priest, 2008, Section 9.7.8).

we can never infer $A >_c (B \wedge {\sim}B)$, and for similar reasons we need not commit to (5) from the truth of (1) and (2). Moreover, Theorem 8.5 guarantees that the analysis does not invalidate ${\sim}((A >_c B) \wedge (A >_c {\sim}B))$, i.e., the principle of *Conditional Non-Contradiction* for any context variable c (that principle is **CS**+ valid).

Example. To illustrate both (i) and (ii), let us consider (3) and (4) once more:

3. If I was Julius Caesar, I (Caesar) would not be alive in the 21^{st} century.
4. If I was Julius Caesar, he (Caesar) would be alive in the 21^{st} century

On the modified account we can evaluate both (3) and (4) as true, albeit relative to their contexts. This is done by reformulating them in the contextualized language with the enthymematic/nominal contextual content revealed, as follows:

3.1. In context *a*: If I was Julius Caesar, I (Caesar) would not be alive in the 21^{st} century.
4.1. In context *b*: If I was Julius Caesar, he (Caesar) would be alive in the 21^{st} century.

That is, (3) and (4) are effectively analysed as (3.1) and (4.1). It can be easily checked that it is possible to have both evaluated as true at a single world on **CS**+ models, since the set of relevant antecedent worlds (where I am Caesar) in context *a* is not the same as the set of relevant antecedent worlds (where I am Caesar) in context *b*. So, the joint truth of (3.1) and (4.1) does not force nor require the existence of relevant antecedent worlds where I (Caesar) am both alive and not alive in the 21^{st} century.

To illustrate (ii), an informal argument will suffice, followed by a formal counterexample to *Adjunction of Consequents*. Let us formalize (3.1) and (4.1) by recourse to $For^{\mathcal{C}}$ and denote (3.1) with $A >_a B$ and (4.1) with $A >_b C$. Now it will be shown that $A >_c (B \wedge C)$ need not follow from $A >_a B$ and $A >_b C$, which is certainly desired. Although on the contextualized analysis there are now contexts *a* and *b* such that both premises $A >_a B$ and $A >_b C$ can be evaluated as true at some possible world *i*, there is no context *c* such that $c \leq a$ and $c \leq b$. In other words, it is not possible to integrate the contextual information of contexts *a* and *b*, carried by the corresponding comparative similarity ordering assignments $r(i, a)$ and $r(i, b)$ in a manner that corresponds to some possible context *c* whose information would be carried by the comparative similarity ordering assignment $r(i, c)$. The following counterexample to *Adjunction of Consequents*, where the consequent is an explicit contradiction, formally spells out the above informal argument.

Proposition 9.1. $p >_a q, p >_b {\sim}q \nvDash^{\mathcal{C}}_{\mathbf{CS+}} p >_c (q \wedge {\sim}q)$.

Proof. It suffices to provide a countermodel. Let $\mathfrak{A} = (W, \mathcal{C}, r, V)$ be a **CS**+ model as follows:

$$W = \{i, j, k\}, \mathcal{C} = \{a, b, c\}$$

Below, in the characterization of $\lesssim_{i,a}$ and $\lesssim_{i,b}$ the ellipses indicate the reflexive cases.

$$r(i, a) = (\{i, j, k\}, \{(i, j), (i, k), (j, k), \ldots\})$$
$$r(i, b) = (\{i, j, k\}, \{(i, k), (i, j), (k, j), \ldots\})$$
$$V_i(p) = 0$$
$$V_j(p) = 1 \qquad V_j(q) = 1$$
$$V_k(p) = 1 \qquad V_k(q) = 0$$

It is easy to check that both $i \Vdash^{\mathcal{C}} p >_a q$ and $i \Vdash^{\mathcal{C}} p >_b {\sim}q$ and that there is no ordering assignment $r(i, c)$ corresponding to index c that would be a mutual refinement of both $r(i, a)$ and $r(i, b)$. In particular there is no $\lesssim_{i,c}$ such that both $\lesssim_{i,c} \subseteq \lesssim_{i,a}$ and $\lesssim_{i,c} \subseteq \lesssim_{i,b}$. The only mutual information that $\lesssim_{i,a}$ and $\lesssim_{i,b}$ share is $\{(i, i), (j, j), (k, k)\}$, which fails to be a similarity assignment, since it is not total. Hence, the existence requirement of condition (2) of Definition 9.2 of **CS**+ contextualized validity is not satisfied. Hence, the above is a counterexample to *Adjunction of Consequents*, as required. □

What is paradigmatic about those counterexamples is that they highlight precisely what is really at play in contextualized validity when we explore limit cases, i.e., where the premises are true in radically different contexts (up to inconsistency). That is, we can have possible premises true for any contexts, but the inference is valid only if the conclusion can always be true in a contextually meaningful way—one that is not independent of the contextual information by virtue of which the premises are true. If there is no mutual refinement of comparative similarity assignments that represent context-indices over which the premises range, then there is no contextually meaningful way of speaking of the conclusion following from those premises. Therefore, the inference is contextually invalid. It should be noted that the inference fails in limit cases as exemplified in Proposition 9.1, but may very well go through on some **CS**+ models if the divergence of contexts over which the premises range is not completely incompatible, as the case may be with Quine's example of Caesar using both nuclear weapons and catapults. It could be argued that such contextual incompatibility of premises—all true but on contexts that do not have a mutual refinement—should be treated in the manner that inconsistent sets of premises are treated, i.e., the conclusion should follow vacuously. Perhaps this needs some more thought, but instances such as Proposition 9.1—which appear to be legitimate counterexamples to *Adjunction of Consequents*—seem to speak against such an approach (we want the inference to fail). The inference is invalid, and the contextualized account presented in this article gives the corresponding correct analysis.

3.2. A Note on Indicatives

The contextualized account lends itself to broader applications. It can be shown to serve as an explanatory device whilst offering a satisfactory analysis of a class of common phenomena, related to indicative conditionals, known as "Gibbardian Stand-Offs". The modified account fares better than the accounts in the contested class for the same reasons it did in the case of counterfactuals (subjunctive) conditionals. Indicative conditionals give rise to so called "stand-offs" when there are equally good reasons for two speakers to assert (on a single occasion) two conditionals that are in stark disagreement (a stand-off) with each other in the following way: the conditionals have identical antecedents and contradictory consequents. Moreover, no third party would have a reason to choose between the two conditionals (deeming one as wrong and the other one as right), because each of the assertions seems to be equally justified (Santos, 2008, Section 1). The phenomenon has been shown to be widespread by a number of authors and so there are numerous examples (Gibbard, 1981; Santos, 2008).[27] For the purposes of the present discussion I will analyse one particular example, given by Bennett.

> Top Gate holds back water in a lake behind a dam; a channel running down from it splits into two distributaries, one (blocked by East Gate) running eastwards and the other (blocked by West Gate) running westwards. The gates are connected as follows: if east lever is down, opening Top Gate will open East Gate so that the water runs eastwards; and if west lever is down, opening Top Gate will open West Gate so that the water will run westwards. On the rare occasions when both levers are down, Top Gate cannot be opened because the machinery cannot move three gates at once. Just after the lever-pulling specialist has stopped work, Wesla knows that west lever is down, and thinks "If Top Gate is open, all the water will run westward"; Esther knows that east lever is down, and thinks "If Top Gate is open, all the water will run eastward". (Bennett, 2003, p. 85)

Clearly both Wesla and Esther speak the truth, yet appear to disagree with each other.[28] Moreover, we can imagine there being a third party, who does not know the settings of the levers, but hears what Esther and Wesla say, and has

[27] Bennett (2003, p. 87) argues that the vast majority of acceptable conditionals with a false antecedent are based upon stand-off situations.

[28] Let us make an informal observation. The proposed analysis indexes conditionals by contexts, but it should be noted that on the assumption that in any given situation an individual (speaker) need not justify their assertion by recourse to the same context as other individuals (in general those will differ), the proposed analysis could just as well be indexed by individuals. Consequently, epistemic considerations could be employed in explaining the differences in the justifications for asserting one conditional instead of its *stand-off counterpart*.

good reasons to believe them. This leads the third party to correctly conclude that the antecedent must be false, i.e., that the Top Gate is in fact closed (see also Priest, 2018, p. 5). It will now be shown how the proposed analysis accommodates such situations. What we essentially want is an analysis that evaluates both (6) and (7) as true at the same world, but according to different contexts. This is precisely what **CS**+ has been tailored to do and what has been argued in Section 1 to be an unattainable feat on the analyses in the contested class. An explanation that involves contextual considerations would be desired, and as it will become clear, the proposed account does this naturally.

6. If Top Gate opens, all the water will run westwards.
7. If Top Gate opens, all the water will run eastwards.

Both Wesla and Esther speak the truth, albeit relative to their contexts, so (6) and (7) can be formulated, with the enthymematic/nominal contextual content revealed, as follows:

6.1. In context *w*: If Top Gate opens, all the water will run westwards.
7.1. In context *e*: If Top Gate opens, all the water will run eastwards.

We can explain how the third party infers that Top Gate is in fact (actually) closed, upon hearing (6) and (7) analysed as (6.1) and (7.1) respectively.[29] What follows is an informal argument, which will be subsequently followed by providing a **CS**+ model that reflects it closely. The third party knows that both conditionals are true, relative to their context. So there is contextual information carried by similarity assignments $r(@, w)$ and $r(@, e)$ corresponding to contexts w and e, respectively ($@$ denotes the actual world), such that in any (closest) world according to $r(@, w)$ where Top Gate is open, all the water flows West, and in any (closest) world according to $r(@, e)$ in which Top Gate is open, all the water will flow East. The third party gathers from what Wesla and Esther say that both lower gates must be open, and from that alone it follows that Top Gate must be closed. If Top Gate were actually open, all the water would flow West and all the water would flow East, which is impossible.[30]

To explicitly demonstrate how the formal model theory fares in giving an account of this scenario, we construct a **CS**+ model. Let us formalize (6.1) and (7.1) by recourse to the set of formulae $For^{\mathcal{C}}$ of the extended language: denote (6.1) with $T >_w W$, (7.1) with $T >_e E$. Let $\mathfrak{A} = (W, \mathcal{C}, r, V)$ be as follows: $W = \{@, j, k\}$,

[29] The reasoning couched on the proposed semantics parallels one given by Priest (2018, pp. 4–5). Whereas Priest appeals directly to information importation in the explanation of contextual disparities, I appeal to similarity assignments that are interpreted as carriers of contextual information.

[30] This explanation mirrors one given by Priest (2018, p. 5) given in terms of information importation.

$\mathcal{C} = \{w, e\}$. Below, in the characterization of $\lesssim_{@,w}$ and $\lesssim_{@,e}$ the ellipses indicate the reflexive cases.

$r(@, w) = (\{@, j, k\}, \{(@, j), (@, k), (j, k), \ldots\})$	Wesla's context.
$r(@, e) = (\{@, j, k\}, \{(@, k), (@, j), (k, j), \ldots\})$	Esther's context.

$V_@(W) = V_@(E) = 1$, $V_@(T) = 0$	Top Gate is closed, since the other two are open.
$V_j(T) = V_j(W) = 1$	Top Gate and West Gate are open.
$V_k(T) = V_k(E) = 1$	Top gate and East Gate are open.

It is easy to check that both $T >_w W$ and $T >_e E$ are true at the actual world, as required. Not only is the analysis adequate, but it also avoids the pitfall of committing to $T >_c (W \wedge E)$ at the actual world for any context c, which is certainly desirable. The argument for this runs along the same lines as the counterexample to *Adjunction of Consequents* given in the previous section.

4. Conclusion

It should be clear that the proposed account in this article merely extends the traditional accounts, e.g., the Stalnaker-Lewis analyses of conditionals. That is, it merely provides an extension of the traditional accounts in a manner that amends their context related difficulties. However, there is no fundamental tension between what is proposed here and the traditional accounts, other than accounting for the contextual differences that these accounts fail to accommodate in their respective formal semantics.[31]

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[31] As a vindication of this we could take what Stalnaker (2017) himself stated in a seminar in Milan. He affirmed the suggestion of an audience member that his solution to the difficulties raised by Gibbardian Stand-Offs would be by allowing the world selection function to differ from speaker to speaker. This is essentially what I propose, but in a more general fashion.

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