Risk and Decision Analysis 2 (2010/2011) 5–32
DOI 10.3233/RDA-2010-0025
IOS Press

Risk in a simple temporal framework for expected utility theory and for SKAT, the Stages of Knowledge Ahead Theory [1]

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Abstract. The paper re-expresses arguments against the normative validity of expected utility theory in Pope (1983, 1991a, 1991b, 1985, 1995, 2000, 2001, 2005, 2006, 2007). These concern the neglect of the evolving stages of knowledge ahead (stages of what the future will bring). Such evolution is fundamental to an experience of risk, yet not consistently incorporated even in axiomatised temporal versions of expected utility. Its neglect entails a disregard of emotional and financial effects on well-being before a particular risk is resolved.

These arguments are complemented with an analysis of the essential uniqueness property in the context of temporal and atemporal expected utility theory and a proof of the absence of a limit property natural in an axiomatised approach to temporal expected utility theory.

Problems of the time structure of risk are investigated in a simple temporal framework restricted to a subclass of temporal lotteries in the sense of Kreps and Porteus (1978). This subclass is narrow but wide enough to discuss basic issues. It will be shown that there are serious objections against the modification of expected utility theory axiomatised by Kreps and Porteus (1978, 1979). By contrast the umbrella theory proffered by Pope that she has now termed SKAT, the Stages of Knowledge Ahead Theory, offers an epistemically consistent framework within which to construct particular models to deal with particular decision situations. A model by Caplin and Leahy (2001) will also be discussed and contrasted with the modelling within SKAT (Pope, Leopold and Leitner, 2006).

Keywords: Temporal utility, expected utility, SKAT, Stages of Knowledge Ahead Theory, normative decision theory, temporal lotteries, resolution times, emotions, risk, substitution axiom, pre-outcome, primary satisfactions, secondary satisfactions

1. Introduction

In their pioneering work on dynamic choice theory, Kreps and Porteus [18,19] explicitly consider the timing of the resolution of risk in their recursive construction of a space of "temporal lotteries". They axiomatize a generalisation of von Neumann–Morgenstern utility to this space.

In the context of preferences on stochastic income streams induced from preferences for consumption streams, the importance of resolution times has received attention (e.g., [8,24,54]). However, with their space of temporal lotteries Kreps and Porteus created a basic decision theoretic structure for an adequate treatment of problems arising from the influence of resolution times on preferences over distributions of time sequences of future events. Undoubtedly this structure was a path-breaking conceptual innovation. However, we shall raise serious objections to the generalisation of von Neumann–Morgenstern utility axiomatized by Kreps and Porteus.

Consider a decision maker who at time point $t = 0$ must decide whether to take a business opportunity or not. At the later time point $t = 10$, this business opportunity will either yield a gain $W > 0$, or a loss $L < 0$. Winning W has the probability α where $0 < \alpha < 1$.

[1] The paper adopts the simple temporal framework of Pope [26] and later including the argument of Pope [37] that epistemic consistency is not achieved in the models of Kreps and Porteus and Caplin and Leahy, given their substitution axioms. The text was written primarily by Reinhard Selten who also contributed the material on the limit property and the modified expectations property, the proofs in the appendices and the examples.

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1569-7371/10/11/$27.50

There are no costs in taking this business opportunity, so the loss L has the complementary probability $1-\alpha$.

Is this everything the decision maker needs to know? The decision maker lacks a further piece of information. When will she know whether she will win W or lose L? Suppose that she will know at a point τ, one of the points $1,\ldots,9$. At τ the risk is resolved, in the sense that, from then on, but not before, the probability of a gain is no longer α but either 0 or 1. At time point τ, her knowledge ahead of whether she will win or lose has changed from being limited to possibilities (probabilities), to being full knowledge ahead, to certainty. Since at time τ the risk is resolved, we term τ the *resolution time*. We assume that the decision maker knows at her point of choice $t=0$ the exact resolution time τ.

If for instance $\tau = 1$, she will learn whether she won or lost a whole nine periods ahead of when she receives her gain or pays out for her loss. An early resolution time (τ small) has planning advantages as regards her response to already known business opportunities at $t=0$ and to new business opportunities that were unknown to her at the point of choice $t=0$ but become known to her after τ. If she learns early that she has made a gain, she is free soon after making the decision to commit herself fully to paying back a loan on the gain in order to finance such new business opportunities.

Optimising the inter-temporal consumption path was the focus of Kreps and Porteus. In having the risk resolved early (τ small), so that she learns soon whether she has made a gain or a loss, the decision maker can plan and implement a beneficially smoother consumption path. If she learns soon that she has made a loss, she can more smoothly reduce her consumption. Conversely she can more smoothly increase her consumption if she learns soon that she has made a gain.

In view of such planning advantages as regards investments and consumption, the business opportunity at time $t=0$ is more attractive the earlier the risk is resolved. However, especially as regards emotional advantages, the opposite preference for a late resolution time is also possible. Consider buying a lottery ticket for 2 dollars that yields a very high gain of say, 1 million dollars, with a small probability 1/1,000,000. In this case the main benefit from the purchase of the ticket may be the hope of gaining one million. Then it may be advantageous to maintain this hope as long as possible, such that a late resolution time increases the attraction of the lottery ticket.

It is of course possible that the risk is gradually resolved at a sequence of resolution times $\tau_1,\tau_2,\ldots,\tau_K$. Her degree of knowledge ahead in this case increases from $\tau_1,\tau_2,\tau_3,\ldots$, and reaches certainty, full knowledge ahead, of whether she won or lost, at time point τ_K. In a modified version of our simple example, the probability α of a gain could change to α_1 at τ_1 and to α_2 at τ_2 etc. until finally $\alpha_K = 0$ or $\alpha_K = 1$ pertains at time point τ_K. Moreover, one could have probabilistic rather than exact knowledge of resolution times. All this is possible within the space of temporal lotteries of Kreps and Porteus.

In this paper we consider only that subset of the space of temporal lotteries that we term simple. This is the subset of lotteries in which all risk is resolved at a single exactly known resolution time and the risk is numerical, in the form of a probability distribution over outcomes. Since in this paper we do not consider decision situations with more complex choices, we can for brevity term simple temporal lotteries 'choices', and the set of all simple temporal lotteries, the 'choice space'. We further assume a finite outcome space and consider decision situations with a finite number of choices. We refer to this set of assumptions and restrictions on what we consider as the 'simple temporal framework'.

Our focus on this narrow subset of the space of temporal lotteries serves the purpose of concentrating on basic conceptual issues without deflecting attention to mathematical detail. We restrict attention to finite sets as much as possible and where this cannot be done, we only look at compact (closed and bounded) subsets of Euclidean space. The presentation is intentionally kept on a very elementary level. We hope that in this way we can also reach readers who are easily discouraged by abstract mathematical formalism.

The objections raised in this paper against the modified expected utility concept of Kreps and Porteus remain valid in their more general framework, e.g., if two properties of a utility function are incompatible in the simple temporal framework, they are also incompatible in the more general framework of Kreps and Porteus. The simple temporal framework is an easily accessible source of examples and counterexamples yielding theoretically important intuitive insights.

In this paper a choice is described as a pair $c = (p_0,\tau)$, namely (i) a probability distribution p_0 over the outcomes, and (ii) a resolution time τ. The distribution p_0 reflects the state of knowledge at the time $t=0$, the point of choice. A decision situation is simply a set of two or any finite larger number of choices.

We shall argue, that the axiomatization of a temporal expected utility by Kreps and Porteus remains unsatisfactory in view of their use of the substitution axiom, on account of the consistency and plausibility issues discussed already in [37] and in [41,42]. As we shall see in Section 7, this axiom lacks an intuitive interpretation, and an example is presented in which plausible preferences violate the substitution axiom. However, it is not the purpose of this paper to propose a new axiomatised temporal utility. On the contrary, we want to convince the reader of our point of view, that axiomatization is far less important than a careful analysis of all relevant satisfactions and dissatisfactions in concrete decision situations, as recommended by SKAT, the Stages of Knowledge Ahead Theory [41, 42]. Nevertheless it is necessary to look at some particular properties of utility functions and their reasonableness without going the full way towards axiomatization.

Sections 2 and 3 present the simple temporal framework and arrive at a complete overview over all the choices within this framework. In Section 3 we look at two statements about expected utility:

(I) the expectation property and
(II) the essential uniqueness property.

Property (I) means that the utility of a distribution over outcomes is the expected value of the utilities of the outcomes with respect to this distribution. Property (II) asserts the uniqueness of the utility function up to the zero point and the unit of measurement. It is shown in Appendix D that for a finite outcome set the essential uniqueness property (II) is a consequence of the expectation property (I).

In the simple temporal framework a utility function defined for outcomes and distributions over outcomes is inadequate. Therefore in Section 6 we look at utility functions defined for choices and also for pairs (x, τ) where x is an outcome and τ is a resolution time. The more general approach of Kreps and Porteus restricted to the simple temporal framework leads to these "modified" utility functions. In turn, insofar as these modified utility functions preserve expected utility maximization, they have the following modified expectation property.

Modified expectation property (I′). The utility $u(p_0, \tau)$ of a choice c is the expected value of the utilities $u(x, \tau)$ where x is distributed according to p_0.

It will be shown in Section 6 that, unlike in the case of ordinary expected utility, the modified expectation property does not imply essential uniqueness.

In the context of axiomatised expected utility theory a natural requirement for a utility function $u(p_0, \tau)$ defined for choices (p_0, τ) in the simple temporal framework is the following limit property.

Limit property (III). The influence on utility of the resolution time τ becomes weaker and weaker as the probability $p_0(x)$ converges to 1 and vanishes in the limit.

Why should it matter much when the risk is resolved, if the decision maker knows at the point of choice that a particular outcome x will be reached with virtual certainty? After all in the limit such a choice is virtually riskless at the point of choice. In our view the limit property (III) is such a natural extension of the usual continuity requirements of axiomatised expected utility that it is hard to imagine how a deviation from this limit property can be intuitively justified by any who accept these other continuity requirements.

Note that limit property (III) concerns a progressively weaker influence on utility of the *resolution time*. It has nothing to do with the so-called certainty effect meaning an "overweighting" of the worst outcome as occurs in prospect theory and other rank dependent generalisations of expected utility theory. This entire class of non-expected utility theories are epistemically atemporal and thus lack reference to the resolution time τ and limit property (III).[1]

In Section 6 it will be shown that a utility function $u(p_0, \tau)$ which satisfies the modified expectation property (I′) cannot satisfy the limit property (III) unless $u(p_0, \tau)$ does not depend on τ. However, dependency on τ is the phenomenon that needs to be explained. Obviously either (I′) or the limit property (III) has to be dropped since it is implausible, even absurd, to propose that a business person's profits and emotional well-being can be independent of τ except in the neighborhood of the limit of the distribution $p_0(x)$ being degenerate. We think that the modified expectation property has to be abolished in favour of limit property (III). The generalised von Neumann–Morgenstern utility of Kreps and Porteus, restricted to the temporal lotteries considered here, has the modified expectation property and therefore cannot depend on the resolution time τ without violating the limit property (III). In our view this is a serious objection against the axiomatic theory of Kreps and Porteus.

An important distinction between primary and secondary satisfactions and dissatisfactions contributing to the evaluation of choices is discussed in Section 8. For some classes of emotional satisfactions, this dis-

[1] For what Savage meant by liking certainty, and its contrast to what Allais and others constructing rank dependent theories today term a certainty effect, see Appendix A.

tinction dates back at least to Canaan [5], Marshall [23] and Ramsey [44]. For the gamut of emotional and financial (material) satisfactions, the distinction is delineated in [26]. The distinction is used to partition all satisfactions, to define the utility of gambling and solve the complementarity paradox whose solution von Neumann and Morgenstern left to future researchers [27, 28,31]. The distinction is given the new terminology of primary and secondary satisfactions in [34]. This paper lists words that had evolved over the preceding century for parts for each sort of satisfaction and describes the confusions associated with each word. The terminology for the overarching theory involving the distinction between primary and secondary satisfactions is changed from that of Pope [26] to being termed SKAT, the Stages of Knowledge Ahead Theory in [42].

Primary satisfactions or dissatisfactions derive from a particular outcome x, regardless of the probability of this outcome x. In our simple example of a business opportunity at $t = 0$, the gain W and the loss L are sources of primary satisfactions connected to the outcomes, winning or losing. The anticipated planning advantages of an early resolution time are secondary satisfactions. Another example of a secondary satisfaction is the hope of winning a big amount after buying a relatively cheap lottery ticket.

Caplin and Leahy [6] have presented an interesting two period model. In this model utility depends only on the 'emotional states' for the two periods. A 'generating function' describes how the emotional state of a period depends on the physical features of an outcome in the current period and in the case of the first period, also on the distribution of emotional states in the second period. A simplified version of this model is presented in Section 9.

Section 10 contains critical remarks on the model of Caplin and Leahy. These remarks concern not only our simplified version but also the original model. In particular the description of the decision process of Caplin and Leahy [6] is contrasted with that of SKAT. It is argued that SKAT offers a more adequate picture. However, we think that Caplin and Leahy [6] is an interesting attempt to portray decision making as a two layer process composed of an automatic emergence of emotional states and the maximization of expected utility depending on the distribution of pairs of emotional states for the two periods.

Caplin and Leahy justify expected utility maximization and timewise additively separable utilities by an axiom system of Fishburn [10]. One of the axioms in this system is the substitution axiom. Our critique of the interpretation of this axiom in Section 7 applies here too. Another axiom in the same system is their 'marginal distribution axiom' that postulates that the preference relationship between two distributions of pairs of emotional states depends only on their marginal distributions with respect to these emotional states. Appendix E shows that the following assertion holds. Assume that the expectation property (I) holds for the utility of distributions over pairs of emotional states and that the marginal distribution axiom holds for the preference relationship represented by this utility function. Then this utility function is timewise additively separable. In addition to this statement it is also pointed out that in view of its interpretation the marginal distribution axiom is by no means obvious. This has been explained already in [10].

The simple temporal framework requires many concepts additional to those of standard expected utility theory and its epistemically atemporal generalisations such as anticipated utility theory, the general theory of random choices [1] and prospect theory [55]. To aid in recalling the notation for the new concepts, Appendix B gives an alphabetised list of the notation giving the name of each concept, and where useful a brief definition or description.

2. Basic notions of the simple temporal framework

The simple temporal framework considers only a narrow class of temporal lotteries namely choices of the form (p_0, τ) where p_0 is a probability distribution over the outcomes at the point of choice time point $t = 0$ and τ is the resolution time point. However, this class is wide enough to discuss the basic issues raised in this paper.

We proceed from the assumption of a finite *outcome set*, X, interpreted as the set of all conceivable outcomes. In order to keep things simple, time is modelled as a sequence of finitely many points. These are $T + 1$ *points* of time, $0, \ldots, T$ combined with T *periods* of time $1, \ldots, T$. Time *period* t begins at the time *point* $t - 1$ and covers all later times *before* the time point t. The time point t does not belong to period t but to the following period $t + 1$. The last time point T does not belong to any period. Consider a probability distribution p over X. The notation $p(x)$ is used for the probability of an outcome $x \in X$.

Outcomes are interpreted as what the decision maker anticipates as possible future events that will affect her utility. Outcomes have a time structure. An outcome

x is a sequence of *segments* $s_1, \dots, s_t, \dots, s_T$ for periods $1, \dots, t, \dots, T$, respectively. These segments s_t are elements of a finite *segment set* S. A segment is interpreted as a description of what, in reaching her choice, the decision maker anticipates might happen in a period. The set X of conceivable outcomes may or may not contain all sequences with T members that can be formed with elements of S. The scope to exclude some sequences allows for irreversibilities like death, after which a person cannot become alive again. Consider an outcome x with the segment sequence $s_1, \dots, s_T$. The subsequence $s_1, \dots, s_\tau$ is called the *part of x before τ*. We further make the simplifying assumption, shared implicitly or explicitly by other temporal choice theories, that from the point of choice, as the future unfolds, the decision maker knows in each period t that segment s_t and derives utility from it.

The time point $t = 0$ is the *point of choice*. At this time point the decision maker has to take a choice available to her. Before she does this she has two items of information about every available choice. One item is an initial probability distribution p_0 over X. The other one is the *resolution time* τ, one of the numbers $0, \dots, T-1$. The resolution time is defined as the first of these time points $0, \dots, T-1$ at which the decision maker knows which outcome is realized.

It is assumed that at the beginning of period t, i.e., at the time point $t-1$, the decision maker sees the segment of period t lying before her. At the time point $T-1$ she sees the last segment and the earlier segments are in the past. Therefore $T-1$ is the latest possible resolution time point.

The probability distribution of the decision maker evolves over time by Bayesian updating. For $t = 0, \dots, T-1$ a probability distribution p_t over X is *degenerate* at time point t if it assigns $p(x) = 1$ to a particular outcome $x \in X$ and $p(y) = 0$ to every outcome $y \in X$ with $y \neq x$. Probability distributions over X without this property are called *non-degenerate*. In this paper's simple temporal framework the decision maker knows that choices which are degenerate at the time of choice $t = 0$ will remain degenerate, and that any non-degenerate probabilities will evolve over time to become degenerate. It is useful to have a distinct symbol for a distribution when it is degenerate as follows. For every $x \in X$, let e_x be the probability distribution with $e_x(x) = 1$ and $e_y(y) = 0$ for every $y \in X$ with $y \neq x$. Every degenerate probability distribution is one of the distributions e_x. A choice c is called *sure* if p_0 is degenerate and *risky* if p_0 is non-degenerate. Here the word 'risky' means that the decision maker perceives a risk connected with the choice. Whether she perceives such a risk depends on her beliefs. It is hardly imaginable that a decision maker with reasonable beliefs perceives a risk connected to a sure choice.

There is a subtle difference between probability 0 and impossibility. Consider, for example, a random variable ξ that is uniformly distributed over the closed unit interval $[0, 1]$. A particular value, say 0.3, has probability 0 under this distribution in spite of the fact that one value must be realised. Even in distributions over finite sets, elements with probability 0 may nevertheless be possible. Thus an outcome z may only occur if the value 0.3 of ξ is realised. However, in the case of distributions over finite sets, a distribution can have the property that an event with probability 0 is impossible. For the sake of simplicity we assume that all probability distributions over X considered in this paper are of this kind. In this way one avoids the cumbersome distinction between absolute certainty and knowledge with probability 1.

For $t = 0, \dots, T-1$, let p_t be the probability distribution at time t. The initial probability distribution p_0, together with the resolution time τ, completely determines which sequences $p_0, \dots, p_{T-1}$ of the decision maker's probability distributions can evolve over time and with which probabilities they can occur. The segment observed at the beginning of a period may lead to an updating. An updating must occur at the resolution time τ. There the decision maker receives the information about the exact outcome realised. If τ is one of the numbers $1, \dots, T-1$, then at this point of time τ the probability distribution $p_{\tau-1}$ changes to e_x with probability $p_0(x)$. The probability distribution p_τ is a random variable just like the outcome x.

The sequences $p_0, \dots, p_{T-1}$ for which p_0 and τ determine positive probabilities are called sequences *generated by the pair* (p_0, τ). Since a pair (p_0, τ) contains all the information necessary for the determination of the consequences of a choice, a choice can be defined as a pair of this kind with some additional properties.

It will be required that p_t changes at most once. If p_0 is non-degenerate and τ is not zero, then the definition of the earliest resolution time (as the earliest point of time at which the exact outcome is known to the decision maker) has the consequence that there must be a change of p_t from $p_{\tau-1}$ to p_τ. This is the only change permitted in a sequence $p_0, \dots, p_{T-1}$ generated by a choice in the simple decision framework. This is expressed by the following definition.

Definition of a choice in the simple temporal framework

A choice c is a pair (p_0, τ), namely an initial probability distribution p_0 over X together with a resolution time τ where τ is one of the numbers $0, \dots, T-1$ such that the following is true for every sequence, $p_0, \dots, p_{T-1}$ generated by (p_0, τ). A change of p_t takes place in the step from $p_{\tau-1}$ to p_τ if p_0 is non-degenerate and $\tau > 0$ holds, but apart from this p_t does not change.

Further definitions

The set of all conceivable choices defined above is the *choice set* C. A decision situation D is a finite subset of C with at least two elements. The choices in D are called *available* in this decision situation. A *utility function* assigns a real number, the utility $u(p_0, \tau)$ to every choice $(p_0, \tau) \in C$.

The decision maker is assumed to have a utility function. If she faces a decision situation D, she will choose an available choice with maximal utility.

3. Consequences of the choice definition

In this section we examine which pairs (p_0, τ) satisfy the choice definition of the preceding section. We first look at the case where p_0 is degenerate, since the decision maker knows the realized outcome already at the point of choice $t = 0$. It follows that every sure choice must have the form $(e_x, 0)$ and that for every risky choice (p_0, τ) the resolution time τ is positive. By the definition of a risky choice p_0 is non-degenerate. We have obtained the following conclusion.

Conclusion (i). Every sure choice has the form $(e_x, 0)$ with $x \in X$. For every risky choice (p_0, τ) the resolution time τ is positive and p_0 is non-degenerate.

Consider a pair of the form $(e_x, 0)$. Since the initial probability distribution $p_0 = e_x$ is already degenerate, it cannot be changed by updating. We must have $p_t = e_x$ for $t = 0, \dots, T-1$. Therefore a pair $(e_x, 0)$ is a choice in the sense of the definition of the preceding section. This together with conclusion (i) leads to the following conclusion.

Conclusion (ii). A pair (p_0, τ) is a sure choice if and only if it has the form $(e_x, 0)$.

Assume that the pair (p_0, τ) is a non-degenerate probability distribution p_0 over X with a positive resolution time. We know by conclusion (i) that a pair denoting a risky choice must have these properties. However, not every pair of this kind is a choice in the sense of the definition of the preceding section.

Following Pope [26] we argue that under our assumptions, in addition to the properties mentioned in conclusion (i), a risky choice $c = (p_0, \tau)$ must satisfy the condition that every outcome $x \in X$ has the same part before τ, a part that we term the *pre-resolution part* of the choice c. Let x and y be two outcomes with $p_0(x) > 0$ and $p_0(y) > 0$ and different parts before τ. Then there must be a smallest positive number h such that the hth segment of x and y is different. Let h be this number. Then the decision maker observes the segment h with $h \leqslant \tau$ at the point $h-1$. If this happens, she can exclude x or y and therefore has to update her probability distribution at the time point $h-1$. The definition of a choice of the preceding section does not permit this. We have obtained the following conclusion.

Conclusion (iii). If a pair (p_0, τ) is a risky choice then all outcomes $x \in X$ with $p_0(x) > 0$ must have the same part before τ, the pre-resolution part of the choice (p_0, τ).

Let (p_0, τ) be a pair with the following three properties: (1) the initial probability distribution p_0 is non-degenerate; (2) the resolution time τ is one of the numbers $1, \dots, T-1$; and (3) all outcomes x with $p_0(x) > 0$ have the same pre-resolution part.

We know already that a risky choice must have the properties (1) and (2) by conclusion (i) and property (3) by conclusion (iii). We shall now show that the three properties of (p_0, τ) have the consequence that (p_0, τ), is a choice c in the sense of the definition of the preceding section.

In view of conclusion (iii), the initial probability distribution is not updated before the time point τ. The decision maker knows already before he takes the choice that the part before τ common to all outcomes x with $p_0(x) > 0$ is guaranteed to occur. Nothing further can be learned from the fact that the decision maker observes these segments as they occur period by period after the point of choice, time point $t = 0$. Therefore we have $p_t = p_0$ for $t = 0, \dots, \tau-1$. However, p_τ is degenerate. Consequently p_t changes in the step from $p_{\tau-1}$ to p_τ from a non-degenerate to a degenerate distribution. A degenerate distribution cannot be updated any more. Therefore we have $p_t = p_\tau$ for $t = \tau, \ldots, T-1$. Consequently (p_0, τ) satisfies the requirement of the choice definition at the end of the preceding section. We can conclude that a pair (p_0, τ) is a risky choice if and only if it has the properties (1), (2) and (3). The following result summarizes what has been shown up to now and gives a complete overview of all choices in the sense of definition at the end of the preceding section.

Results about choices in the simple temporal framework

The set of all "sure" choices is the set of all pairs $(e_x, 0)$. The set of all risky choices is the set of all pairs (p_0, τ) with the following three properties:

(1) the initial probability distribution p_0 is non-degenerate;
(2) the resolution time τ is one of the numbers $1, \ldots, T-1$; and
(3) all outcomes $x \in X$ with $p_0(x) > 0$ have the same part before τ.

4. Remarks on standard expected utility

Standard expected utility theory, from now on abbreviated to EUT, lacks reference to the change in knowledge ahead inherent in the resolution time τ being positive at the point of choice. It is one of the aims of this paper to show that EUT cannot be applied to this paper's simple temporal framework, at least not in a satisfactory way. There are many axiomatizations of EUT. However, here we are not concerned with axioms but rather with conclusions. We will restrict our attention to the case of finitely many possible outcomes.

In EUT, we can use simpler notation. We can omit the subscript 0 to indicate that p is the distribution at the point of choice $t = 0$ and not at a later date, since EUT ignores the issue of when the time point τ occurs when any risk will be resolved. In this section therefore and in Appendix D we use the symbols x, X, p, c, C and D with analogous but *different* meanings from those of Sections 1–3. The set of all conceivable outcomes is denoted by X. This set is non-empty and finite. A choice c is simply a probability distribution p over the outcome set X. The probability of an outcome $x \in X$ under p is denoted by $p(x)$. The symbol C stands for the set of all conceivable choices. A decision problem D is a finite non-empty subset of C. This analogous but simpler set of concepts and associated notation (that are enabled under the restrictive assumptions of EUT and its epistemically atemporal rank dependent generalisations) are used only in Section 4 and Appendix D.

The decision maker is assumed to have a complete *preference order* $\geqslant$ over C. The expression $p \geqslant q$ means that the decision maker prefers p to q or is indifferent between p and q. Strict preference of p to q is expressed by $p > q$. A utility function u is a function which assigns a real number $u(p)$ to every $p \in C$. We say that a utility function u *represents* the preference order $\geqslant$ if the following is true: $u(p) \geqslant u(q)$ holds if and only if we have $p \geqslant q$. A utility function v is a *positive linear transformation* of a utility function u, if there are real numbers a and b with $a > 0$ and

$$v(p) = au(p) + b \quad \forall p \in C.$$

For every $x \in X$ let e_x be the probability distribution of X that assigns 1 to x and 0 to every other outcome in X. The distribution e_x is the *sure choice* of x. Strictly speaking e_x is an element of C, but x itself does not belong to C.

However, as explained in [26,35], to avoid introducing additional inconsistencies, in EUT the utilities of outcomes must be interpreted in one of two ways, that of Ramsey [44], or that of Friedman and Savage [11]. See Appendix A for more details. The upshot is that it is not necessary to distinguish between e_x and x in EUT. However, this identification is problematic since it hides severe restrictions in the interpretation of EUT and in its axiomatizations.

Beginning with Marschak [22] the axioms in most EUT axiomatizations are requirements imposed on the preference order $\geqslant$. Axioms were in terms of utilities in von Neumann and Morgenstern [56]. However, a translation to the language of preferences is feasible in view of the lack of necessity to distinguish between e_x and x in EUT. In the following it will be assumed that the axioms express postulated properties of the preference order. Most axiomatizations of this kind lead to a *representation theorem* of the following form.

If the axioms are satisfied then the following two assertions (I) and (II) hold.

(I) *Expectation property.* The preference order $\geqslant$ can be represented by a utility function u of the following form

$$u(p) = \sum_{x \in X} p(x)u(x).$$

(II) *Essential uniqueness property.* If two utility functions u and v represent the same preference order $\geqslant$ then one is a positive linear transformation of the other.

A positive linear transformation of a utility function shifts the zero point and the unit of measurement like the transition from Fahrenheit to Celsius. Obviously the difference between the two temperature scales is not essential. In this sense a utility function u with the expectation property (I) is *essentially unique* if it also satisfies (II).

It can be seen without difficulty that a positive linear transformation v of a utility function u with the expected value property also has this property. As has been said before, this paper is restricted to the case of finitely many possible outcomes. For this case it is relatively easy to show that (I) implies (II). A proof will be given in Appendix D. Nevertheless it is important to keep in mind that a representation theorem has the two parts (I) and (II). As we shall see in Section 5, in the context of the simple temporal framework a utility function may not be essentially unique even if it has a modified expectation property, analogous to property (I).

5. Expected utility in the simple temporal framework

From now on we revert to the simple temporal framework and thus to the denotation of x, X, p, c, C and D as defined in Section 2 and summarised in Table A2.1 of Appendix B. As we have seen in Section 2 from conclusions (i) and (ii), in the simple temporal framework, choosing (p_0, τ) is making a sure choice if and only if $\tau = 0$. Otherwise it is making a risky choice. In this framework a utility function u can have the expectation property provided that $u(p_0, \tau)$ does not depend on τ such that instead of $u(p_0, \tau)$ we can write:

$$u(p_0, \tau) = u(p_0) \quad \forall (p_0, \tau).$$

The expectation property also implies the conditions under which an outcome x can be identified with the distribution e_x.

Suppose that the decision maker does not care about the resolution time τ and evaluates outcomes as if certain so that we can omit the subscript 0 to indicate that distributions p, q and r are at the point of choice $t = 0$ and not at a later date. However, even if this is the case, there is still an important difference between EUT and the simple temporal framework. This difference concerns the definition of a choice.

In EUT a choice is simply a probability distribution over X. Let p, q and r be three probability distributions over X with

$$r(x) = \alpha p(x) + (1 - \alpha) q(x) \quad \forall x \in X.$$

Here α is a real number with $0 \leqslant \alpha \leqslant 1$. If p, q and r can be related in this way, and simplifying our notation to the EUT convention – wherein all probabilities are recorded solely with their values at the time of choice – the subscript 0 can be left implicit

$$r = \alpha p + (1 - \alpha) q$$

and we say that r is a "mixture" of p and q. If we have $0 < \alpha < 1$ then r is called a *proper mixture* of p and q. In EUT every proper mixture of two choices is again a choice or in other words, the set C of all choices is convex.

By contrast, the choice set in this paper's simple temporal framework does not have a similar convexity property. Suppose that (p_0, τ) and (q_0, τ) are risky choices with different pre-resolution parts, i.e., with different outcome segments prior to τ, the time point when the risk concerning the last outcome segment is resolved. Let r_0 be a proper mixture of p_0 and q_0. Then the conceivable choice (r_0, τ) satisfies conclusions (i) and (ii) of the simple temporal framework identified in Section 2, but not conclusion (iii). Consequently (r_0, τ) fails to be a choice. This is the reason why a utility function $u(p_0, \tau)$ defined on the set of all conceivable choices C in the simple temporal framework may have the expectation property, but nevertheless not be essentially unique. This will now be illustrated by an extremely simple example.

Example 1. The time horizon is $T = 3$. The set of segments is $S = \{a, d\}$. The symbol a stands for "alive" and d means "dead". Somebody who dies in period t cannot become alive later. Death will happen for sure in period 3, if it does not happen earlier. Accordingly there are only three possible outcomes:

$$x_1 = (a, a, d),$$
$$x_2 = (a, d, d),$$
$$x_3 = (d, d, d).$$

Here the outcomes are described by vectors with the jth segment as the jth component ($j = 1, 2, 3$). For

$i = 1, 2, 3$, let e_i be the probability distribution which assigns 1 to x_i and zero to each of the two other outcomes.

In our example a risky choice must have the form $(p_0, 1)$. At $\tau = 2$ no uncertainty is left. Conclusion (ii) in Section 2 restricts the probability distribution p_0 in $(p_0, 1)$ to a proper mixture of e_1 and e_2. Otherwise the outcomes with positive probability under p_0 would lack a common pre-resolution part. Table 1 provides a complete overview over all possible choices.

Given that "a" is alive and "d" is dead, the following suggests itself.

(P1) *Preference order assumption in Example 1.* Let $(p_0, 1)$ and $(q_0, 1)$ be risky choices with $p_0(x_1) > q_0(x_1)$. Then following is true:

$$(e_1, 0) > (p_0, 1) > (q_0, 1) > (e_2, 0) > (e_3, 0).$$

Assumption (P1) completely describes the preference order $\geqslant$. All preference relations between two different choices are strict. From Table 1 it can be seen that the resolution time in a choice $(p_0, \tau) \in C$ is uniquely determined by p_0. Therefore a utility function defined on C has the property that the utility of a choice $(p_0, \tau) \in C$ can be described as a function of p_0 alone, as $u(p_0)$. Starting from three fixed but arbitrary real numbers u_1, u_2 and u_3 with

$$u_1 > u_2 > u_3$$

we now construct such a utility function $u(p_0)$ which represents $\geqslant$ and has the expectation property (I). Define

$$u(e_i) = u_i \quad \text{for } i = 1, 2, 3$$

and

$$u(p_0) = p_0(x_1)u_1 + p_0(x_2)u_2$$

for

$$p_0 = \alpha e_1 + (1 - \alpha)e_2 \quad \text{with } 0 < \alpha < 1.$$

It can be seen without difficulty that $u(p_0)$ represents the preference order $\geqslant$ defined by (P1). Moreover, we obviously have

Table 1

The choice set C for Example 1

Sure choices	Risky choices
$c_1 = (e_1, 0)$	$c = (p_0, 1)$
$c_2 = (e_2, 0)$	with
$c_3 = (e_3, 0)$	$p_0(x_i) > 0$ for $i = 1, 2$
	and
	$p_0(x_3) = 0$

$$u(p_0) = \sum_{i=1}^{3} p_0(x_i)u(x_i) \quad \forall (p_0, \tau) \in C.$$

This shows that $u(p_0)$ has the expectation property (I).

We now show that $u(p_0)$ is not essentially unique. Let v be a positive linear transformation of u

$$v(p_0) = au(p_0) + b.$$

Define

$$v_i = v(e_i).$$

We have

$$v_i = au_i + b \quad \text{for } i = 1, 2, 3$$

and therefore

$$\frac{v_1 - v_2}{v_2 - v_3} = \frac{u_1 - u_2}{u_2 - u_3}.$$

The ratio of the two utility differences $u_1 - u_2$ and $u_2 - u_3$ is not changed by a positive linear transformation. However, in the same way as u has been constructed we can construct a utility function w starting from real numbers w_1, w_2 and w_3 with

$$w_1 > w_2 > w_3.$$

Such that we have

$$\frac{w_1 - w_2}{w_2 - w_3} \neq \frac{u_1 - u_2}{u_2 - u_3}.$$

This utility function w cannot be a positive linear transformation of u. Therefore u does not satisfy the essential uniqueness property (II).

The lack of the essential uniqueness in utility functions with the expectation property is due to the fact that only relatively few of the probability distributions p over the outcome set X are parts of choices $(p_0, \tau) \in C$.

This is the reason why Proposition 1 in Appendix D does not hold for the simple temporal framework. In the case of Example 1, the proof of Appendix D's Proposition 1 would require a choice (p_0, τ) such that p is a proper mixture of e_1 and e_3 for which the decision maker is indifferent to e_2. There is no such choice in Example 1.

6. Resolution time dependence

It is a highly unrealistic assumption that preferences do not depend on resolution times. Many people would like to know as soon as possible whether they have passed an exam. However, early resolution of risk is not always preferred. Many people would not like to know the exact day of their death years in advance. Suppose that somebody belongs to a family in which there have been cases of a deadly hereditary disease. Whether she will get the disease ten years from now depends on whether she has a certain genetic disposition. This can be found out by a gene test. Would she want to take the test? How much would she willing to pay for it or would she refuse to take it even if money is paid to her for taking it? Presumably most people would not be indifferent between an early or late resolution of the risk in this situation.

If resolution times matter, utility cannot depend on the outcome alone. In the simple temporal framework one may want to pursue an approach which looks at a pair (x, τ) with $x \in X$ and $\tau = 0, \dots, T-1$ as the basic unit to which a utility $z(x, \tau)$ is attached. One can then postulate a utility function u defined on the set C of all choices in the simple temporal framework such that u has the following property.

(I′) *Modified expectation property.* For every choice $(p_0, \tau) \in C$ we have

$$u(p_0, \tau) = \sum_{x \in X} p(x) z(x, \tau),$$

where z is a function defined on the set of all pairs (x, τ) with $x \in X$ and $\tau = 0, \dots, T-1$.

It must be emphasized that for $\tau = 1, \dots, T-1$ the pair (e_x, τ) is not a choice but a pseudo choice. If x is reached with probability 1 then the resolution time τ must be zero. Therefore $z(x, \tau)$ should not be interpreted as the utility of a choice but as a limit of choice utilities. Consider an arbitrary choice (p_0, τ) and an outcome x with $p_0(x) > 0$. Define

$$q_\varepsilon = (1 - \varepsilon) e_x + \varepsilon p_0 \quad \text{for } 0 < \varepsilon < 1.$$

Obviously (q_ε, τ) is a choice. This implies that we have

$$z(x, \tau) = \lim_{\varepsilon \to 0} u(q_\varepsilon, \tau) \quad \text{if (I}'\text{) holds.}$$

It is a basic property of expected utility that it depends linearly on the probabilities of the outcomes wherever it is defined. If this is the case then the limit of $u(q_\varepsilon, \tau)$ for $\varepsilon \to 0$ exists and is independent of p_0. The representation required by (I′) is then possible. Obviously the modified expectation property is the only natural adaptation of the expected utility property (I) to the simple temporal framework.

The modified expectation property permits a dependence of $u(p_0, \tau)$ on the resolution time τ and on the common pre-resolution part of the outcomes x with $p_0(x) > 0$ in the choice (p_0, τ). Nevertheless we shall now argue that the modified expectation property is not really satisfactory.

Consider again the choices of the form $q_\varepsilon = (1 - \varepsilon) e_x + \varepsilon p_0$ with $0 < \varepsilon < 1$ where (p_0, τ) is an arbitrary choice. As ε approaches zero, the risk involved in q_ε becomes smaller and smaller and vanishes in the limit $\varepsilon \to 0$. There is little difference between the sure choice $(e_x, 0)$ and (q_ε, τ) with a very small ε, say $\varepsilon = 10^{-100}$. It is reasonable to assume that the difference does not matter in the limit and that therefore the limit of $u(q_\varepsilon, \tau)$ for $\varepsilon \to 0$ is equal to $u(e_x, 0)$. This leads to the following limit property for a utility function $u(p_0, \tau)$ defined on the set C of all conceivable choices (p_0, τ) in the simple temporal framework.

(III) *Limit property.* Let (p_0, τ) be a risky choice in C and let x be an outcome with $p_0(x) > 0$. Then for

$$q_\varepsilon = (1 - \varepsilon) e_x + \varepsilon p_0 \quad \text{with } 0 < \varepsilon < 1$$

the following equation

$$\lim_{\varepsilon \to 0} u(q_\varepsilon, \tau) = u(e_x, 0)$$

holds. This is true for every choice (p_0, τ) and every outcome x with $p_0(x) > 0$.

Consider a utility function u which satisfies (I′) and (III). Then it follows by limit property (III) that we have

$$u(x, \tau) = z(e_x, 0) \quad \text{for } \tau = 0, \dots, T-1.$$

Consequently the modified expectation property yields

$$u(p_0, \tau) = \sum_{x \in X} p_0(x) z(x, 0).$$

This means that such a utility function does not depend on the resolution time but only on the probability distribution p_0. If we write $u(p_0)$ instead of $u(p_0, \tau)$ and $u(x)$ instead of $z(x, 0)$ we obtain the expectation property (I).

The modified expectation property is an attempt to save as much as possible from the expectation property. The limit property (III), however, has a strong intuitive appeal. As has been argued at the beginning of this section, resolution times matter in serious decision problems. If one wants to model resolution time dependent utility one cannot have both (I′) and (III). In our view (I′) should be dropped in favour of (III).

We do not want to suggest a functional form of a utility function defined on the set C of all conceivable choices (p_0, τ). Instead of this we propose identifying the causes of satisfactions and dissatisfactions in every concrete application context and making them explicit in the modelling of the utility function $u(p, \tau)$. Preferences for and against an early resolution of risk may be motivated by curiosity, hope, or fear or even the fear of losing the hope. Emotional factors of this kind are important for the wellbeing of the decision maker [3]. Moreover, in suitable dosages, such emotions enhance decision making. Decision makers experiencing and anticipating too little fear and regret in contemplating risky choices, for instance, undertake foolhardy choices [4] and [7]. Those taking choices yielding too little in the way of thrills and hope for the brain's needed stimulation often compensate with other choices that involve socially and personally destructive behaviour such as juvenile delinquency and excessive social gambling [34,40,48,49]. Since they affect utility in the current choice, and the appropriateness of future choice, emotions should enter the utility evaluation of each choice. In this respect SKAT, the Stages of Knowledge Ahead Theory [42] can be useful as a modelling guide.

The following result states the most important conclusion of this section.

Result

Let $u(p_0, \tau)$ be a utility function defined for all $(p_0, \tau) \in C$ and assume that u has the modified expectation property (I′) and the limit property (III). Then $u(p_0, \tau)$ neither depends on τ, nor on the pre-resolution part of the outcomes with $p_0(x) > 0$, and $u(p_0, \tau)$ is a function of p_0 alone.

7. Critique of the substitution axiom

In their pioneering *Econometrica* paper, Kreps and Porteus [18] developed a decision theoretic approach in which resolution times of probabilities have an important role. Other authors like Klibanoff and Ozdenoren [17] have built on their work. We also appreciate Kreps and Porteus, even if we present a different point of view.

As has been shown, the modified expectation property together with the limit property exclude preferences depending on the resolution time. Nevertheless in their *Econometrica* paper Kreps and Porteus [18] axiomatize utility functions that may depend on the resolution time. The restrictions of these utility functions to the simple temporal framework satisfy the modified expectation property. As we have seen these restrictions cannot satisfy the limit property unless they do not depend on the resolution time and also not on the common pre-resolution part of all outcomes with positive probability in the probability distribution of the choice.

If the limit property holds for a utility function $u(p_0, \tau)$ defined on the set C of all choices in the simple temporal framework and $u(p_0, \tau)$ depends on the resolution time τ, then the modified expectation property must be violated. We may express this by saying that the modified expectation property, resolution time dependence and the limit property are incompatible.

In the theory of Kreps and Porteus, the modified expectation property is a consequence of Axioms 4.1–4.3 (p. 195). Axiom 4.1 requires that the preference order over temporal lotteries be complete and transitive. Axiom 4.2 postulates continuity of the preference order. These two axioms are standard. We do not want to criticize them in this paper. The crucial axiom is the substitution Axiom 4.3. In the language of our simple temporal framework, this axiom can be expressed as follows:

Substitution axiom. Let (p, τ) and (p', τ) be two choices with the property that all outcomes x with $p(x) > 0$ or $p'(x) > 0$ have the same pre-resolution part. Assume

$$(p, \tau) > (p', \tau).$$

Moreover, for some β with $0 < \beta < 1$. Let p'' be probability distribution

$$p'' = \beta p + (1 - \beta)p'.$$

Then,

$$(p'', \tau) > (p', \tau)$$

holds.

Interpretation of the substitution axiom

Whether an axiom is plausible or not depends on the interpretation of the formal terms appearing in it. In the substitution axiom p'' is defined as $\beta p + (1 - \beta)p'$. An interpretation of β and $1 - \beta$ as probabilities in a two-stage lottery suggests itself. In a first stage two events, say Y and Z come about with probabilities, β and $(1 - \beta)$ and then in the second stage (p, τ) follows in the case of Y and (p', τ) in the case of Z. However, this interpretation does not seem to be adequate in the framework of a theory in which every probability is connected to a resolution time. What is the resolution time of β? Suppose that τ_1 is the resolution time of β. Since the first stage of the two-stage lottery precedes the second one, we must have $\tau_1 < \tau$. The substitution axiom, however, mentions only one resolution time, namely τ. For a two-stage lottery, the resolution time τ_1 of β and $(1-\beta)$ must precede the resolution time τ of (p, τ) and (p', τ). We must have $\tau_1 < \tau$. As observed in [37], this precludes β and $1 - \beta$ being interpreted as probabilities: β and $1 - \beta$ are simply coefficients in a convex linear combination of p and p'. See Appendix C.

Let $X'(s_1, \ldots, s_{\tau-1})$ be the set of all outcomes with the same part $s_1, \ldots, s_{\tau-1}$ before τ. Moreover, let $P'(s_1, \ldots, s_{\tau-1})$ be the set of all probability distributions over $X'(s_1, \ldots, s_{\tau-1})$. The set $P'(s_1, \ldots, s_{\tau-1})$ is convex, in the sense that every convex linear combination of probability distributions in this space is also in this space. Under the conditions specified for p and p' by the substitution axiom, (p'', τ) is therefore always a choice. However, the mere fact that every convex linear combination of p and p' is in $P'(s_1, \ldots, s_{\tau-1})$ does not provide a justification of the mathematical operation of forming a convex linear combination of p and p' implying the preference relation of the substitution axiom.

Consider a preference order over a convex set of consumption bundles. In this case convex linear combinations are also in the set. An axiom analogous to the substitution axiom combined with transitivity and continuity would yield the conclusion that indifference curves must be linear, contrary to the usual assumptions of consumption theory that a rational decision maker can have non-linear indifference curves.

An intuitive justification of the substitution axiom would require a substantive interpretation of the weights β and $1 - \beta$ in the definition of p''. Moreover, as noted above, this interpretation cannot be based on the idea that β and $1 - \beta$ are probabilities of having the choices of respectively (p, τ) and (p', τ), since no resolution time is attached to them.

Unlike the substitution axiom, the limit property has a strong intuitive appeal. Since the substitution axiom, together with completeness, transitivity and continuity, leads to the modified expectation property, our objections against the modified expectation theory also apply to the substitution axiom. There is a conflict between the substitution axiom and the limit property if one wants to model resolution time dependent utility. In our view one cannot uphold the substitution axiom.

It is however also important to understand what is not intuitively compelling in the substitution axiom given that the interpretation of its mixture β and $1 - \beta$ is restricted to being a convex linear combination of p and p'. For this purpose we present an example. Note that in this example we avoid impossible sequences in a set of segments as in Example 1 (from being dead in period 1 to being alive in period 2) because the distributions lacked a common pre-resolution part. The substitution axiom is confined to distributions with a common pre-resolution part. Nevertheless as Example 2 below shows, one cannot uphold the substitution axiom as a pre-requisite for reasonable choice.

Example 2. Suppose that after an examination a patient is told that he has a very rare nerve disease. There is a chance of 90% for either blindness or a complete loss of balance after 50 months. Both developments of the disease have the same probability, 45%. With a probability of 10% nothing bad happens. It will be known after 46 months, but not earlier, which of the three possibilities will be realized.

There is a treatment available that eliminates the danger of a loss of balance but also increases the probability of blindness to 90%. The treatment is quick and not disagreeable. The cost is carried by the health insurance. It will be known after 46 months whether the patient will become blind after 50 months. The health status in the first 50 months is the same regardless of whether the patient takes the treatment or not. The patient has to decide for or against the treatment. He has no other choices.

The decision problem permits three outcomes x_1, x_2 and x_3:

x_1 – blindness;
x_2 – complete loss of balance;
x_3 – no adverse effects.

All 3 outcomes have the same resolution time and the same part (the same sequence of segments) before the resolution time. The names listed above refer to what happens in the last outcome segment. We shall look at the probability distribution p over these three outcomes as a vector $p = (p_1, p_2, p_3)$ with the probabilities p_i of the x_i as parts. Define

$$p = (0.9, 0, 0.1),$$
$$p' = (0, 0.9, 0.1) \quad \text{and}$$
$$p'' = \frac{1}{2}p + \frac{1}{2}p' = (0.45, 0.45, 0.1).$$

The length of one period is assumed to be one month. The resolution time of both choices of the patient is $\tau = 47$, the point of time at which period 46 has just ended. Let c_1 be the choice of taking the treatment and c_2 be the choice of not taking it. Moreover, let c_0 be the unavailable choice $(p, 47)$:

$$c_0 = (p, 47), \qquad c_1 = (p', 47) \quad \text{and}$$
$$c_2 = (p'', 47).$$

The decision situation is portrayed in Table 2.

Suppose that the patient would prefer $(p, 47)$ to $(p'$, 47) as assumed by the substitution axiom. This means that he would prefer a 90% probability of a complete loss of balance to a 90% probability of blindness if in both cases with 10% probability there are no bad effects. Although $(p, 47)$ is not available as a choice, p'' mixes p and p' with equal probabilities and therefore he should prefer c_2 to c_1 according to the substitution axiom.

Table 2
The decision situation of Example 2

	Outcomes		
	Loss of sense of balance x_1	Blindness x_2	No adverse effects x_3
$c_0 = (p, 47)$ not available	0.9	0.0	0.1
$c_1 = (p', 47)$ available	0.0	0.9	0.1
$c_2 = (p'', 47)$ available	0.45	0.45	0.1

Notes: Outcomes named by what happens in the last outcome segment – all have the same sequence of segments prior to the resolution time $\tau = 47$. The entries in the field are the probabilities of outcomes (column) following a choice (row).

However, is it really plausible to conclude that c_2 is preferred to c_1 as required by the substitution axiom, regardless of how strong is the preference of $(p'$, 47) over $(p, 47)$? Suppose for a moment that the decision maker were instead indifferent between $(p'$, 47) and $(p, 47)$. Then the patient may take the treatment since he values the gain in knowledge ahead. If he takes the treatment he does not have to worry which of the two dreadful events will happen with high probability. This gives him an opportunity to adjust emotionally to his likely fate. Moreover, it will be easier for him to prepare for the future after 50 months. He can learn to read and write Braille in this time and does not have to think about how to cope with the complete loss of balance.

We now return to the assumption that the patient would prefer the unavailable choice c_0 over c_1. Suppose that the intensity of this preference is low and the dislike of not knowing which of the two bad events will occur is so strong that it outweighs the influence of the preference of c_0 over c_1. The patient will then choose to take the treatment since he hates the uncertainty of c_2 so much that he is willing to choose blindness with probability 0.9 in spite of a slight preference for a complete loss of balance. We can conclude that violations of the substitution axiom can be quite reasonable. They cannot be excluded as irrational.

Note that the patient's dislike of not knowing which bad outcome will occur has nothing to do with ambiguity aversion. Ambiguity aversion means that the patient dislikes not knowing the distribution. In the example of Table 2 the patient knows the distribution of each choice exactly. What the patient dislikes is that the exactly known distribution c_2 in which there are two different bad possible outcomes with substantial probabilities.

Our example is sufficient as a direct refutation of the idea that a rational decision maker must obey the substitution axiom. This direct refutation complements an earlier finding in this section. Our earlier finding is that the substitution axiom of Kreps and Porteus [18] and the limit property, together with some uncontroversial assumptions on the preference order, jointly preclude preferences from depending on resolution times.

8. Primary and secondary satisfactions and their role in SKAT

A basic distinction between primary and secondary satisfactions is made by SKAT, the Stages of Knowledge Ahead Theory [26,34,35,37,41,42]. The term knowledge ahead refers to what the decision maker knows about the future. A degenerate probability distribution means full knowledge ahead – certainty. A non-degenerate probability distribution means limited knowledge ahead – risk, uncertainty. Each stage in the theory ends with a change in knowledge ahead. The term *satisfactions* is used in a wide sense, also including dissatisfactions. (One may think of dissatisfactions as negative satisfactions.)

Beginning with Pascal in the seventeenth century, Pope [34] surveys the numerous terminologies used for the partition of satisfactions anticipated to be experienced *during chronological time* into primary and secondary, the misleading connotations, the confusions and associated misclassifications to which each terminology was subject, and decided that a new terminology would be helpful.

Primary satisfactions derive from specific outcomes of a choice and depend neither on probabilities nor their resolution time. They are independent of knowledge ahead at any stage. Examples of primary satisfactions are pleasures and displeasures from consumption and state of health. Of course, what is considered to be a source of primary satisfaction depends on the concrete decision situation and the degree of detail in which it is modelled.

Secondary satisfactions depend on the probabilities of specific outcomes and/or on resolution time, in other words on what is known about the future. They include certainty effects experienced in chronological time from full knowledge ahead, and uncertainty effects experienced in chronological time from limited knowledge ahead. The concept of secondary satisfactions (in an earlier terminology "utility of chance") can be defined with respect to the satisfaction derived from the distribution, as in [27], or more simply with respect to the satisfaction from each outcome as that satisfaction is affected by the distribution as in [31]. When the term risk includes the border case of risk reduced to zero (and thus certainty effects experienced in chronological time), both definitions partition the set of satisfactions identically between primary and secondary satisfactions.

For the new terminology, the term secondary was not selected in order to connote that these satisfactions are of lesser importance. Rather it was selected in order to connote that they derive from primary satisfactions – from concern at what will or may be primary satisfactions in the future, and from what were and might have been primary satisfactions in the past. Secondary satisfactions may arise from emotional responses to the uncertain evolution of the choice situation over time, like fear of or hope for some future events. We call such secondary satisfactions *emotional*. SKAT also considers non-emotional secondary satisfactions such as inter-temporal planning inefficiencies and inability to commit unconditionally to repay a loan or complete a task when the future is uncertain. Such secondary satisfactions are called *material*.

SKAT describes how knowledge ahead, or in other words what is known about the future, changes over time. In the course of dealing with a problem there is an evolution in what is unknown at each stage and what progressively becomes known about the future. It is necessary to distinguish at least three stages of *knowledge ahead*.

Stage 0: *The pre-decision stage* – before a choice is made and thus when the decision maker has limited knowledge ahead of which choice he will make.

Stage 1: *The pre-outcome stage* – before the risk is resolved of which of the various possible outcomes in the choice will occur and thus when the decision maker has limited knowledge ahead about what will be the outcome of the choice.

Stage 2: *The post-outcome stage* when the risk concerning which outcome of that choice will occur has been resolved even if the outcome has not yet occurred, and thus when the decision maker will have full knowledge ahead (certainty) of the outcome of the (previously risky) choice.

At the beginning of stage 0 the decision maker is ignorant of what choices are available and ignorant of what he will choose. By the end of stage 0 the decision maker knows what choices are available and what the utility of each choice is. Then at $t = 0$ the decision is made and his knowledge ahead enters stage 1 of knowing what he has chosen. In stage 1 the decision is irrevocably fixed but risk is still unresolved since he has limited knowledge ahead of which outcome will eventuate. All risk concerning which particular outcome will occur from his choice is resolved at the resolution time τ so that there is full knowledge ahead concerning the outcome. That is, stage 2 begins at this point.

Pope [33, pp. 292–293] subdivides the pre-decision stage into two processes (i) generating options and (ii) generating decision rules for evaluating options and evaluating options. Pope, Leopold and Leitner [42, Fig. 5.2, p. 28] subdivide the pre-decision stage into two, namely into a pre-choice set stage and the remainder a pre-decision stage during which the available choices are evaluated. The phrase "pre-choice set" refers to the process of ascertaining by search and negotiation the set of available choices.

The subdivision of stage 0 is based on a picture of the decision process involving consecutive activities: first the construction of the set of available choices; and then the construction of rules for evaluating choices and the evaluation of these choices by those rules. However, SKAT does not require that this process is strictly sequential. Rather the process will normally involve a degree of toing and froing since, apart from anything else, the matter of evaluating options typically results in needing to specify the options more precisely in some respects than was initially realised.

Further SKAT does not exclude path-dependent satisficing and aspiration–adaptation in the pre-decision stage 1. Satisficing [53] is a search for alternatives in which, as soon as it is found, a choice is evaluated and, if on evaluation it reaches the satisficing threshold, chosen without expending time and resources to ascertain if there are available other better choices. Under aspiration adaptation theory, before encountering a particular decision situation, the chooser has identified a set of conceivable options for altering his current situation and made an order of urgency as regards improvements, and as regards a retreat variable. Thus each time a firm completes its search procedure of discovering what is feasible, it has no additional evaluation to do. If it discovered that moving up is feasible, it already knows that if it has to choose between different upward directions, and already knows which upward directions are higher on its urgency scale. Again, if retreating is all that is feasible, our firm already knows its retreat variable. It has no need to do an evaluation in order to discover its desired advance or retreat steps [45,50–52].

In the case of a sure choice, all risk is resolved at $t = 0$. Therefore in this case stage 1 is missing. Stage 0 is immediately followed by stage 2. In the following it will be assumed that the choice under consideration is risky. This has the consequence $\tau > 0$. There is a pre-outcome stage of at least one period. In stage 1 the decision maker knows the choice she has taken but she does not yet know which outcome will be realized. This becomes known at the resolution time τ, when the post-outcome stage 2 begins.

We now turn our attention to the evaluation of choices in SKAT. The evaluation procedure determines a *value* $u(p_0, \tau)$ for every available choice $c = (p_0, \tau)$. In this paper we restrict our attention to the case where $u(p_0, \tau)$ is a real number, the *utility* of $c = (p_0, \tau)$. Multi-dimensional values could also be considered, but they are beyond the scope of this paper.

The value $u(p_0, \tau)$ of a choice (p_0, τ) is the final result of a stepwise procedure in which successively broader aggregates of satisfactions are formed. Table 3 provides a list of the notation of the parts of $u(p_0, \tau)$ entering this aggregation. Table 4 orders these parts by stage and level of aggregation to provide an overview of how these parts combine to form $u(p_0, \tau)$.

With respect to a resolution time τ, with $0 < \tau < T$, every outcome $x \in X$ can be split into two parts, $x_{1\tau}$ and $x_{2\tau}$. The first part is $x_{1\tau}$, the pre-resolution part of x that occurs before τ in the pre-outcome stage 1. The second part is $x_{2\tau}$, the post-resolution part of x that occurs during the post-outcome stage 2 that begins at τ. The pre-resolution part $x_{1\tau}$ consists of the first τ segments $s_1, \ldots, s_{\tau-1}$ of x and the post-resolution part consists of the remaining segments $s_\tau, \ldots, s_T$ of x.

Every x with $p_0(x) > 0$ has the same pre-outcome stage 1 part $x_{1\tau}$. (See part 3 of the result at the end of Section 3.) Primary satisfactions received in stage 1 are a function $v_1(x_{1\tau})$ of this common pre-resolution part. Even if this is not formally expressed, $v_1(x_{1\tau})$ should be thought of as an aggregate of disparate primary satisfactions anticipated to be experienced in the segments of $x_{1\tau}$. Primary satisfactions anticipated to be experienced in the post-outcome stage 2 for a particular outcome x are a function $v_2(x_{2\tau})$ of the post-resolution part $x_{2\tau}$ of x, and are sub-aggregates, aggregated over each mutually exclusive x with $p_0(x) > 0$ to form $\overline{v}_2(p_0, \tau)$. How this is done is left unspecified.

The primary satisfactions $v_1(x_{1\tau})$ of the pre-outcome stage 1 and $\overline{v}_2(p_0, \tau)$ of the post-outcome stage 2 are then combined to form total primary satisfactions $v(p_0, \tau)$. Again, how this is done is left unspecified.

Secondary satisfactions can only be anticipated and formulated after primary satisfactions have been formed. In order to fear a bad outcome or hope for a

Table 3
Notation for parts of the utility of a choice in SKAT

$x_{1\tau}$	Pre-resolution part of an outcome x in the pre-outcome stage 1, common to all x with $p_0(x) > 0$.
$x_{2\tau}$	Post-resolution part of an outcome x in the post-outcome stage 2, different for each different x.
$v_1(x_{1\tau})$	Primary satisfactions in pre-outcome stage 1, common to all x with $p_0(x) > 0$.
$v_2(x_{2\tau})$	Primary satisfactions of a particular outcome x in post-outcome stage 2.
$\overline{v}_2(p_0, \tau)$	Aggregate primary satisfactions in post-outcome stage 2.
$v(p_0, \tau)$	Total primary satisfactions.
$w_1(x_{1\tau}, p_0)$	Secondary satisfactions in pre-outcome stage 1, common to all x with $p_0(x) > 0$.
$w_2(x_{2\tau}, p_0)$	Secondary satisfactions of a particular outcome x in post-outcome stage 2.
$\overline{w}_2(p_0, \tau)$	Aggregate secondary satisfactions in post-outcome stage 2.
$w(p_0, \tau)$	Total secondary satisfactions.
$u(p_0, \tau)$	Utility of the choice (p_0, τ).

Table 4
Parts of the utility of a choice

	Pre-outcome stage 1	Post-outcome stage 2		Total
		Particular outcome x	Aggregate over $x \in X$	
Primary	$v_1(x_{1\tau})$	$v_2(x_{2\tau})$	$\overline{v}_2(p_0, \tau)$	$v(p_0, \tau)$
Secondary	$w_1(x_{1\tau}, p_0)$	$w_2(x_{2\tau}, p_0)$	$\overline{w}_2(p_0, \tau)$	$w(p_0, \tau)$
				$u(p_0, \tau)$

good one, it has to be known which outcomes are good or bad and this is judged in terms of their primary satisfactions.

Unlike the primary satisfactions $v_1(x_{1\tau})$ and $v_2(x_{2\tau})$, the secondary satisfactions $w_1(x_{1\tau}, p_0)$ and $w_2(x_{2\tau}, p_0)$ do not depend solely on $x_{1\tau}$ and $x_{2\tau}$ respectively, but also on p_0. In the pre-outcome stage 1 a bad outcome may be feared less the smaller is its probability in stage 1, and in the post-outcome stage 2, elation at a good outcome may be more intense the smaller its probability was in stage 1.

Actually $x_{1\tau}$ is redundant in the notation for secondary satisfactions $w_1(x_{1\tau}, p_0)$ in pre-outcome stage 1 since $x_{1\tau}$ is uniquely determined by p_0. However, for the sake of an intuitive notation we retain $x_{1\tau}$ as an argument of w_1.

In the same way as the primary satisfactions $v_1(x_{1\tau})$ are an aggregate, the secondary satisfactions $w_1(x_{1\tau}, p_0)$ is an aggregate of the disparate secondary satisfactions of the segments of $x_{1\tau}$ in the pre-outcome stage 1. An analogous statement holds for secondary satisfactions $w_2(x_{2\tau}, p_0)$ in stage 2.

The secondary satisfactions $w_2(x_{2\tau}, p_0)$ are aggregated over $x \in X$ to $\overline{w}_2(p_0, \tau)$. Then the secondary satisfactions $w_1(x_{1\tau})$ of stage 1 and $\overline{w}_2(p_0, \tau)$ of stage 2 are combined to form total secondary satisfactions $w(p_0, \tau)$.

Finally the total utility of the choice $u(p_0, \tau)$ is formed from $v(p_0, \tau)$ and $w(p_0, \tau)$. Again the precise way in which these aggregations and combinations are done is intentionally left unspecified in order to leave room for adjustments to specific contexts.

Remarks on aggregation of mutually exclusive outcomes in the post-outcome stage 2

In the post-outcome stage 2, in the case of a risky choice, satisfactions differ for each mutually exclusive x with $p_0(x) > 0$. The formation of the aggregates of primary satisfactions $\overline{v}_2(p_0, \tau)$ and secondary satisfactions $\overline{w}_2(p_0, \tau)$ thus involve mutually exclusive outcomes x. This means that, compared to the stage 1 aggregates $v_1(x_{1\tau})$ and $v_2(x_{1\tau})$ an additional tier of aggregation is required, that over each mutually exclusive x with $p_0(x) > 0$. This merits additional attention in view of the widespread confusion in the literature of aggregation procedures involving probability weights with the inclusion of secondary satisfactions in the mapping from outcomes into utilities.

This second tier of aggregation can take the form of an atemporal weighting procedure that gives a fractional weight at $t = 0$, that, depending on x and p_0, is allocated to each of these mutually exclusive outcomes. Such weights can take many different forms. There can be an ultra conservative of putting all the weight on the worst possible outcome and ignoring all the other possible outcomes. There can be an ultra optimistic weighting of putting all the weight on the best possible outcome. There can be some intermediate form of weighting in between the extremes such as the various sorts of probability weighting of rank dependent theories that give more weight to bad outcomes as in most rank dependent theories [1,43,55], or to both good and bad outcomes [20,21].

Decision scientists generally fail to make the distinction between:

(a) the atemporal derivative usage of probabilities for aggregating alternatives in rank dependent theories, and
(b) the denotation of probabilities as the decision maker's degree of knowledge ahead at particular times.

Pope [26,31] demonstrate how to make this distinction, and how, once the plausibility and reasonableness of including secondary satisfactions is admitted, there are neither descriptive nor normative grounds for performing the atemporal aggregation with probability weights. Rather the appropriate weights should be determined on a case by case basis concerning the range of outcomes and the decision maker's preferences.

The distinction between (a) the atemporal derivative usage of probabilities as weights and (b) probabilities as measures of the decision maker's degree of knowledge ahead means that the matter of atemporal weighting of mutually exclusive alternatives is distinct from the issue of consistently including secondary satisfactions within SKAT, as explained in [26,31] and [42, pp. 55–57]. This is because atemporal weighting by probabilities may or may not occur forming the aggregate of primary satisfactions $\overline{v}_2(p_0, \tau)$. Thus we may have

$$\overline{v}_2(p_0, \tau) = \sum_{x \in X} p(x) v_2(x_{2\tau}).$$

Likewise these are examples of a special model within SKAT in which probability weights $p_0(x)$ are used in stage 2 also for the aggregation of the mutually exclusive secondary satisfactions $w_2(x, \tau)$ for each different x into $w_3(p_0, \tau)$,

$$\overline{w}_2(p_0, \tau) = \sum_{x \in X} p(x) w_2(x_{2\tau}).$$

But SKAT does not recommend aggregation by probability weights for most decision situations. The above examples are given as in earlier papers to assist in ending the widespread current practice of speaking about the utility of a choice outside EUT as "non-linear in probabilities" without distinguishing between the two classes of theories. One class of theories, namely standard rank dependent theories, generalises EUT by allowing generalisations of EUT's atemporal probability weights but excludes secondary satisfactions. The other class of theories, of which SKAT is an epistemically consistent example, includes secondary satisfactions. Within this other class, SKAT is eclectic, an umbrella theory that permits all secondary satisfactions and all atemporal aggregation rules for the mutually exclusive outcomes in stage 2. By contrast, many other theories, such as Caplin and Leahy [6], Kreps and Porteus [18] and Klibanoff and Ozdenoren [17] are less eclectic both as regards the secondary satisfactions included (excluding important sets of secondary satisfactions due to the axiomatic restrictions of their substitution axiom as itemised in [42]), and non-eclectic as regards the atemporal aggregation weights (imposing atemporal aggregation by simple probability weights). For understanding and researching what influences choice and giving policy advice, it is crucial that these two classes of theories need to be kept distinct. Note that in both stage 1 and in stage 2, aggregates may be formed by automatic intuitive judgment processes inaccessible to conscious inspection.

Many details are left unspecified in the description of the evaluation of a choice. Details may depend on the context and may vary from decision maker to decision maker. A careful analysis of every concrete decision situation is much more important than a premature specification of a precise mathematical form.

9. A model by Caplin and Leahy

In this section we want to describe a version of a model proposed by Caplin and Leahy [6]. The additional notation introduced to analyse the Caplin and Leahy model is in an alphabetical list of symbols, Table A2.2 of Appendix B. Their approach is similar to that of SKAT, even if they apply axiomatized expected utility theory. They take the point of view that all satisfactions and dissatisfactions are emotional and that therefore the utility of a choice should only depend emotional states. The emotional state is assumed to depend on the current physical outcome segment and on anticipations of the future. Caplin and Leahy develop a two-period model based on these ideas.

They assume that the outcome set is infinite. We shall present a simplified version of the model of Caplin and Leahy in which the outcome set is finite. This will permit us to connect the model with the simple temporal framework and use of the definitions and notations of Section 2. However, some additional definitions and notations will have to be introduced.

Since the model of Caplin and Leahy has two periods we have $T = 2$. For every outcome $x \in X$ let $s_1(x)$ be the first segment and $s_2(x)$ be the second segment of x. Let m_1 and m_2 be emotional states of periods 1 and 2, respectively. The emotional state depends on the currently experienced segment and the anticipated emotional states in the future and their probabilities. In period 2 there are no anticipated future emotional states. Therefore the emotional state m_2 depends only on the segment $s_2(x)$ of the realized outcome x. This enables us to identify m_2 with $s_2(x)$. Without any loss of generality:

$$m_2 = s_2(x).$$

One may think of an outcome x as specified by a first physical segment $s_1(x)$ and a second emotional segment $m_2 = s_2(x)$.

Suppose the decision maker takes a choice $c = (p_0, \tau)$. As we have seen in Section 2, every outcome $x \in X$ with $p_0(x) > 0$ has the same first segment. If c is a sure choice, this is due to the fact that there is only one outcome x with $p_0(x) > 0$. For a risky choice this follows from Conclusion (iii) in Section 3. The first segment common to all outcomes $p_0(x) > 0$ is uniquely determined. We call this segment the *first segment realized* by p_0 and denote it $s_1(p_0)$. Of course, the function $s_1(p_0)$ is only defined for probability distributions which can appear in choices $c = (p_0, \tau)$.

The emotional state m_1 for the first period depends on the first segment $s_1(p_0)$ and on the probability distribution over $m_2 = s_2(x)$ which is uniquely determined by p_0:

$$m_1 = \Phi(s_1(p_0), p_0).$$

Consequently m_1 can be described as a function of p_0 alone:

$$m_1 = \psi(p_0).$$

We call ψ the *generating function* for m_1. For a given choice $c = (p_0, \tau)$ this function directly connects the emotional state m_1 in the first period to the probability distribution p_0.

We shall refer to a pair $m = (m_1, m_2)$ of two emotional states for periods 1 and 2 as an *emotional future.* Caplin and Leahy think of emotional states as vectors in a Euclidean space. The components may be intensities of emotions or similar variables which at least in principle permit physiological measurement. In their model the emotional states m_1 and m_2 for periods 1 and 2 are elements of two convex subsets M_1 and M_2 of a Euclidean space. In our version of the model, we also assume that M_1 and M_2 are compact (closed and bounded). The set of all $m = (m_1, m_2)$ is denoted by M. Obviously all emotional futures are elements of M. Caplin and Leahy assume that the decision maker has a complete and transitive preference order over all probability distributions M.

Consider a choice $c = (p_0, \tau)$. What distribution g over M results from taking the choice? We shall look at this question. For any distribution p_0 over the outcome set, let $R_2(p_0)$ be the set of all second segments s_2 such that

$$x = (s_1(p_0), s_2)$$

is an outcome in X with $p_0(x) > 0$. We call $R_2(p_0)$ the *second segment reservoir* or for brevity the reservoir of p_0. It can be seen that the probability $g(m)$ of $m = (m_1, m_2)$ is as follows:

$$g(m) = \begin{cases} p_0(s_1(p_0), s_2) \\ \quad \text{for } m_1 = \psi(p_0) \text{ and } m_2 \in R_2(p_0), \\ 0 \quad \text{else.} \end{cases}$$

This distribution $g(m)$ is called the distribution over M *generated* by the choice $c = (p_0, \tau)$. The set of all g generated by a choice is denoted G.

Let f be a probability distribution specifying positive probabilities $f(m)$ for all m in a finite subset $K(f)$ of M and specifying a probability of zero for all subsets of M not intersecting $K(f)$. The set $K(f)$ is called the *carrier* of f. Probability distributions of this kind are called *distributions with a finite carrier.* This definition excludes continuous or partially continuous distributions that assign probability densities rather than probabilities in some regions of M. The set of all distributions over M with a finite carrier is denoted F. Obviously G is a subset of F. For our version of the model it is therefore sufficient to look at the preference relations $\geqslant$ defined for pairs of distributions in F.

Let u be a utility function defined on M and extended to F by (I)

$$u(f) = \sum_{m \in K(f)} f(m)u(m). \tag{I}$$

It is clear that this equation simply restates the expectation property introduced in Section 4. We say that a utility function u defined on M is *timewise additively separable* if for every $m = (m_1, m_2)$ in M we have

$$u(m) = u_1(m_1) + u_2(m_2), \tag{IV}$$

where u_1 and u_2 are continuous partial utility functions defined on M_1 and M_2 and respectively.

Caplin and Leahy proceed from a system of axioms imposed on the preference relationship $\geqslant$ over general distributions over M. The axioms yield the conclusion that a preference relationship which can be represented by a utility function with the expectations property (I) can also be represented by a utility function u satisfying (I) and (IV).

In this paper we do not want to describe the axiom system of Caplin and Leahy for their utility function in detail. However, it is important to say something about the crucial role of two of these axioms.

The first one of these two axioms is the substitution axiom [6, Assumption I(ii), p. 61], essentially the same one as discussed in Section 6. The substitution axiom is crucial for the expectation property (I). The second of these two axioms assumes indifference between two probability distributions over M with the same marginal distributions on M_1 and M_2. A precise statement of this *marginal distribution axiom* can be found in Appendix E. There a proposition is proved that for our version of the model of Caplin and Leahy the marginal substitution axiom together with (I) implies the timewise additive separability property (IV).

The approach of Caplin and Leahy looks at decision making as a two layer process. The first layer describes how a distribution of physical outcomes connected to a choice leads to a distribution over emotional futures. The second layer concerns the selection of a choice by the maximization of expected utility. In this way they achieve a clear separation of psychological assumptions about the pre-rational genesis of emotional states on the one hand and the postulated rationality of the preference order over distributions of emotional futures on the other hand.

The two layer structure results in a high degree of flexibility. In their paper Caplin and Leahy apply their model to the influence of anxiety on asset pricing. Moreover, they discuss many interesting experimental results in the psychological literature, and interpret them in the light of their model. Undoubtedly their paper is an important contribution to decision theory. Nevertheless some critical points need to be raised. This will be done in the next section.

10. Remarks on the model by Caplin and Leahy

In the following we shall present three remarks on the model by Caplin and Leahy. In these remarks we shall refer to our versions of the model of Caplin and Leahy as presented in the previous section and to the axioms stated in Appendix E. It will be clear to every reader familiar with the paper by Caplin and Leahy that the objections also apply to their more general form of the model.

10.1. The region on which preferences are defined

In the model of Caplin and Leahy the decision maker is assumed to have preferences over arbitrary probability distributions over M. In our version of their theory we consider only distributions f in the set F of distributions with finite carriers.

Consider the set G of distributions that result from the choice set C. As we have seen the distributions in G have very special carriers. All emotional futures $m = (m_1, m_2)$ in such a carrier must have the same first part m_1. For any reasonable measure over M the probability that a randomly selected carrier with n elements ($n \geqslant 2$) is a carrier of a distribution in G is zero. In this sense almost all distributions are not in G.

It is sufficient for the decision maker to have preferences defined on G. Why should she be forced to form preferences over all pairs of distributions f and g in the vastly larger set F? Distributions in G but not in F arise from pseudo choices rather than choices. Any interpretation in terms of a possible choice is bound to be contradictory. Preference judgments involving pseudo choices are not natural. It is not reasonable to require that a rational decision maker should make preference judgments concerning impossible distributions over emotional futures.

10.2. Use of the substitution axiom

Caplin and Leahy [6] make use of a substitution axiom (Assumption 1(ii), p. 61). As has been explained in Section 7 there are serious objections against the substitution axiom. This axiom lacks a reasonable interpretation. The weights β and $1 - \beta$ of p and p' in the substitution axiom (see Section 7) cannot be interpreted as probabilities with a temporal extension.

Example 2, presented in Section 7, was a direct argument against a substitution axiom in the theory of Kreps and Porteus [18,19]. This argument loses its force in the context of the theory of Caplin and Leahy, since they do not apply their axioms directly to temporal lotteries, but to distributions over emotional futures. Nevertheless the more basic objection concerning the lack of a reasonable interpretation remains valid.

10.3. Formation of preferences

The model of Caplin and Leahy conveys the impression that the generating function is looked upon as the description of an automatic process not involving evaluative judgment or deliberation. This process transforms a choice to a probability distribution over emotional futures. Preference judgments concerning such distributions are then used in order to select one of the available choices. The preference relationship is assumed to satisfy a system of rational axioms. This suggests that the formation of preferences involves rational cogitation.

The emergence of an emotional future is modelled behaviourally but preferences over distributions of emotional futures are dealt with in the usual rationalistic fashion of axiomatized expected utility theory. This extreme difference between the two layers of the model of Caplin and Leahy does not seem to be plausible.

In contrast to the model of Caplin and Leahy, SKAT describes the utility of a choice as the result of a stepwise procedure of forming aggregates of primary and secondary satisfactions (see Section 8). Instead of the neat separation into an automatic step and a rational step in the decision process as modelled by Caplin and Leahy, the evaluation procedure of SKAT permits a combination of emotional response, intuitive judgment and rational deliberation at every step, without being too precise about the role of each of these mental activities. Satisfactions and dissatisfactions certainly have an emotional aspect as indicated by the everyday meaning of the words but they are also influenced by intuitive judgment and rational deliberation.

Axiomatized expected utility theory remains silent about the way in which preferences are formed. The decision maker is assumed to have a preference book in his head in which preference intensities can be looked up without much effort, like numbers in a telephone book. This picture also seems to underlie the upper layer of the model by Caplin and Leahy. SKAT rejects the axiomatic approach and instead of this tries to gain insight into how preferences are formed.

11. Discussion

This paper has discussed conceptual issues concerning the generalization of axiomatized expected utility to temporal lotteries in the sense of Kreps and Porteus [18]. For this purpose it was not necessary to go beyond a narrow subclass of temporal lotteries in which all risk is resolved at a resolution time τ and not before. A temporal lottery of this kind can be described by a pair (p_0, τ) where p_0 is a probability distribution over the outcomes at the point of choice and τ is the resolution time. However, such a pair must satisfy an additional condition to be a choice in a decision situation. The resolution time τ is zero if and only if p_0 assigns probability 1 to one outcome and zero to all other outcomes. Therefore the set of all possible choices is not convex.

Axiomatizations of expected utility usually derive a representation theorem asserting two properties, the expectation property and the essential uniqueness property (see Section 4). As we have seen in Section 5, the essential uniqueness does not hold for the simple temporal framework. The reason for this lack of convexity is not removed by the more general framework of Kreps and Porteus.

The essential uniqueness property is important for the interpretation of utility as a cardinal quality. However, as long as the expectation property holds, it is still possible to describe decision making as maximizing expected utility, even if the utility function is not unique up to the origin and the unit of measurement. Therefore one can take the point of view that the essential uniqueness property is not really an indispensable feature of a generalized expected utility of temporal lotteries.

However, the essential uniqueness property may matter for welfare judgments involving interpersonal comparisons of expected utilities. Harsanyi [12,15] has argued that interpersonal preference judgments can be formed by empathy. A person with perfectly good eyesight must be able to form preference judgments comparing his or her present situation with that of a blind person. This is necessary for judging decision alternatives involving the risk of loosing one's eyesight. Such judgments are made by empathy, i.e., "by putting oneself into the other's shoes". Empathy can also be used to form counterfactual preference judgments, e.g., about what it would be worth to be 10 cm taller. This presumably impossible increase in size is as much in the power of imagination as a possible loss of eyesight. At least in principle a "universal utility function" applicable to every conceivable situation of a human being can be constructed.

In Harsanyi's utilitarianism welfare judgments are based on the average of the universal utilities of all living human beings. No difficulties arise in the theory if the universal utility function has the expectation property but not the essential uniqueness property. However, if welfare depends not only on the average universal utility but also on a measure of dispersion like the variance of the distribution of universal utilities, then the essential uniqueness property becomes important.

As has been argued in Section 3, the modified expectation property (I′) is the only natural generalization of the expectation property (I). In the same section a further requirement, the limit property (III) has been introduced. The limit property is nothing more than a natural extension of continuity. It has been shown that a utility function for the simple temporal framework cannot depend on τ if it has the modified expectation property (I′) and the limit property (III). However, the whole purpose of a utility theory for temporal lotteries is to capture the influence of resolution times. Therefore, one of the two properties has to be dropped. In our view the limit property has a greater intuitive appeal. The modified expectation property has to be given up.

In our view the limit property is a decisive argument against the utility theory for temporal lotteries axiomatized by Kreps and Porteus [18]. Their theory permits resolution time dependence, but not in a reasonable way. The restriction of their theory to the simple temporal framework satisfies the modified expectation property and therefore does not have the limit property unless utility does not depend on resolution time, but only on the initial probability over outcomes.

The axiomatization of Kreps and Porteus [18] makes use of a substitution axiom. In Section 7 we argued that this substitution axiom has no reasonable interpretation. The axiom makes an assertion about a mixture of two choices without offering a substantive interpretation of the weights in the mixture. In the theory of temporal lotteries every probability is connected to a resolution time. Therefore the weights cannot be interpreted as probabilities.

Section 7 also presents an example indicating why it would be wrong to impose the substitution axiom even if it had a substantive interpretation.

Sections 6 and 7 have shown that there are serious objections to the axiomatic theory of Kreps and Porteus [18]. The same criticism also applies to later papers, e.g., Klibanoff and Ozdenoren [17]. In our view the axiomatic approach to choice among temporal lotteries remains unsatisfactory. Axiomatic theories assume that the preference relationship satisfies some plausible properties and then derive a representation theorem on this basic. Maybe a fundamentally different approach to modelling rational choice of a temporal lottery in more promising. Instead of taking plausible properties of the preference relationship as a point of departure one can ask oneself how preferences are constructed from more basic elements. SKAT, the Stages of Knowledge Ahead Theory [41,42] provides a picture of the decision process proceeding from primary to secondary satisfactions and finally to evaluations of choices in the simple temporal framework. SKAT is described in Section 8.

SKAT does not aim at a full description of the decision process but leaves much room for modelling detail to be developed as appropriate in the application to concrete cases. This avoids assuming that for all decision situations the same model details and way of reaching a solution will be identical. However, in SKAT the distinction between primary and secondary satisfactions is fundamental in guiding the modelling of the description and analysis of particular concrete decision problems.

The two-period model by Caplin and Leahy [6] portrays the decision process as composed of two levels. The first layer determines emotional futures for every choice in the decision situation by a pre-rational psychological mechanism. On the second level the final choice is determined rationally by expected utility maximization. In a sense all satisfactions are secondary in the model. In our view the neat separation of the two layers does not sufficiently take into account the complexity of the interaction of automatic psychological reactions and rational deliberation. In this respect SKAT seems to offer a better picture of the decision process. Our critical objections against the model of Caplin and Leahy have been made and need not be repeated here. However, undoubtedly that model is an important contribution to the literature.

Appendix A: Alternative concepts of a certainty effect

In his responses to two choices posed by Allais in 1952, Savage violated expected utility theory. In his historical retrospective on why he violated a theory to which he had recently converted, Savage [46] explained that he was attracted to a guaranteed good outcome, that he liked to be certain of this. He reported that he still felt an intuitive attraction to the choice of a guaranteed good outcome, even though he had subsequently decided that this was a mistake. The liking of Savage and many others for a guaranteed good outcome is something excluded by the axioms of expected utility theory, including Savage's own 1954 dual axiomatization of utilities and probabilities under expected utility theory.

Allais' first asked Savage to choose between being guaranteed a good outcome, and almost guaranteed a better one. Savage's choice was the guaranteed good outcome. Allais next asked Savage to choose between two risky outcomes, one a little more risky but with an even better possible outcome. Savage's choice was the riskier one.

Allais' method of convincing Savage that his pair of answers to the two choices violated expected utility theory was the following. Given his choice of the riskier option only when no good sure choice was available revealed a disproportionate increment in utility from being sure of the good outcome – disproportionate compared to what expected utility theory implies.

This disproportionate increment in utility revealed by Allais' pair of questions might be termed a certainty effect. It can arise in two main ways or by a combination of the two main ways [31]. One way is outside chronological time and does not require an epistemically temporal framework in order to discern its operation. The second way requires an epistemically temporal framework.

This first way is through the atemporal aggregation of the mutually exclusive outcomes. Additional weight is placed on the bad outcome relative to its probability. When all the weight is placed on reducing the risk of the worst possible outcome, one has minimax theory. Savage however did not conform to this theory in his answers to Allais. If Savage could not eliminate entirely the risk of the worst outcome, his choice was the riskier option with its possibility of an even better outcome.

When merely more weight is placed on it than its probability implies, one has subjective expected utility theory of Edwards [9]. Edward's "overweighting" of the worst outcome assumption recurs in the original prospect theory of Kahneman and Tversky [16] and in rank dependent models such as Allais [1] and Tversky and Kahneman [55]. This atemporal "overweighting" of the worst outcome has come to be termed a certainty effect since it puts more weight on avoiding the worst outcome than under expected utility theory. As emphasised by Allais [1] the "overweighting" can be reasonable, not a consequence of the person being irrational, but rather a consequence of the person choosing a different more conservative atemporal weighting system than assumed under expected utility theory.

While such atemporal "overweighting" the worst outcome can be reasonable – especially when the worst outcome entails death or bankruptcy, such "overweighting" does not conform with Savage's [46] account of what he meant by liking a guarantee. Nor does it accord with his account how he employed his sure thing principle to reconcile this liking with changing his choices to conform to expected utility theory. Savage changed his preferences by inventing what he termed the "extra logical" "loose" "clarifying" sure-thing principle to convert an almost sure alternative into a sure one. As analysed in [29], the sure thing principle involves truncating distributions until the distribution with a higher probability of a good outcome is sure. This truncation is done by the decision maker in his evaluation process. The real choice (e.g., that offered by Allais) still involves risk. In short, applying the sure thing principle involves creating a delusion of certainty.

Terming atemporal overweighting a certainty effect therefore has nothing to do with what appealed to Savage in the guaranteed good outcome. Savage only persuaded himself to alter his choices when he used his sure thing principle to generate a delusion of complete certainty (and to declare that this let him better understand probability differences). What then did Savage intuit as the "disproportionate" advantages of the guaranteed good outcome, disproportionate relate to expected utility theory? Savage reports no specifics.

The specifics may well have been those that can only be discerned in theories employing epistemically temporal frameworks that Savage [46, Chapter 2] decided to ignore since like his sure thing principle, they involve knowledge and possibility, concepts he wished to avoid. The specifics may well have been the emotional and/or material benefits of certainty anticipated to be experienced during chronological time under the sure choice. Examples are feeling secure (an emotional satisfaction) and being able to commit 100% on a job or to repay a loan or to more efficiently intertemporally plan (material satisfactions). Yet other examples are in [35,36,39–42]. These benefits are absent when the good outcome is less than certain.

Such benefits from certainty drive a wedge between the "as if certain" interpretation of expected utility in [11], and the "independent of probabilities" interpretation in [35,44]. These benefits of a guaranteed good outcome are excluded under the Ramsey interpretation of expected utility but are included under the Friedman and Savage "as if certain" interpretation of expected utility theory. Clearly Savage did not so apply his interpretation – or he would have disagreed with Allais that he had violated expected utility theory in his pair of choices.

There are two obvious possibilities of why Savage did not think of this interpretation. One is that Savage had not immersed himself in the specifics of his "as if certain" interpretation. This accords with the fact that he accepted Allais' algebraic proof. Allais' proof in fact holds only for the Ramsey interpretation of expected utility theory.

Another possibility is that Savage realised that his "as if certain" interpretation involves treating risky outcomes as if certain and thus illusorily attributing to risky outcomes benefits of certainty (examples are in [35]). But Savage chose not to mention these awkward issues of impossible events being assumed by adherents to expected utility theory. In this regard, on a related issue of impossible events being imagined, Aumann asked Savage how he dealt with the matter that his axiomatization of expected utility theory involves a states split in which a person could simultaneously be hanged on the gallows and in good health and reputation. Savage's response was:

> To some – perhaps to you – it will seem grotesque if I say that I should not mind being hung so long as it be done without damage to my health or reputation, but I think it desirable to adopt such language so that the danger of being hung can be contemplated in the framework of F. of S. [47, p. 307].

The abbreviation F. of S. refers to Savage's famous book *Foundations of Statistics*.

Appendix B: Alphabetical listing by symbol

Notation introduced Sections 1–3 for the simple temporal framework is in Table A2.1. Where it seemed desirable, inclusive of a brief definition of the term, or description of its features is included. For fuller descriptions and definitions, see the text, Sections 1–3. The additional concepts introduced in Section 8, are alphabetically listed in Tables 3 and 4 of that section.

In Section 4 and Appendix D in analysing epistemically atemporal expected utility theory we used the symbols $\alpha, \alpha_1, \alpha_2, x, y, X, p, c, C$ and D with analogous but *different* meanings from those above of the simple temporal framework. This is because that atemporal framework ignores when risk is resolved in defining a choice and an outcome. These analogous concepts for the atemporal framework of expected utility theory are defined in the second paragraph of Section 4.

In this appendix, Table A2.2 has the additional notation introduced in Section 9 and used also in Section 10 and Appendix E to analyse the model of Caplin and Leahy and one additional symbol introduced in Section 7 to analyse the substitution axiom (essentially the same in the axiomatizations of Kreps and Porteus and in Caplin and Leahy). In the case of these additional symbols, the matter that they are only introduced and used in Sections 7 or in 9, 10 or in Appendix E is noted. This avoids readers searching the initial sections when seeking a definition or fuller description of the concepts distinctive to Caplin and Leahy's model.

Appendix C: Discussion of the substitution axiom's mixing coefficient in Kreps and Porteus

> The feature that most clearly distinguishes our treatment from previous work is its focus on the temporal aspect of uncertainty. Our approach to dynamic choice problems and temporal lotteries explicitly models uncertainty as "attached" to a certain time. Although reduction of compound uncertainty at a single time is implicit, reduction of uncertainty at several different times is not allowed (Kreps and Porteus [18, opening three sentences of Section 7]).

Comment

Since the mixing coefficient in the substitution axiom has no attached uncertainty resolution time, it is not a probability since as Kreps and Porteus themselves in their model, a probability has uncertainty attached to a certain time. However, the third sentence of the above citation from Kreps and Porteus claims

Table A2.1
Notation for the simple temporal framework (Sections 1–3)*

c	$c = (p_0, \tau)$ is a choice specified in terms of a probability distribution p_0 over the outcomes, and a resolution time τ.
C	The set of all conceivable choices.
D	A decision situation, a finite subset of C with at least two elements.
e_x	The degenerate probability distribution with $e_x(x) = 1$ and $e_y(y) = 0\ \forall y \in X$ with $y \neq x$.
p_0	The probability distribution at the point of choice $t = 0$.
S	A finite *segment set of the conceivable* segments that can be elements of an outcome x.
s_t with $t = 1, \ldots, T$	The tth segment of an outcome $x = (s_1, \ldots, s_T)$.
τ	A resolution time when the when the distribution p_0 of x becomes degenerate.
t with $t = 0, \ldots, T$	The time point t. The point of choice is $t = 0$. For $t = 0, \ldots, T - 1$, the period t begins at $t - 1$.
	The last time point T does not belong to any period.
$u(p_0, \tau)$	The expected value of the utilities $u(x, \tau)$ of a choice c where x is distributed according
	to p_0 becomes degenerate at resolution time τ.
$u(x, \tau)$	The utility of a pair (x, τ) of an outcome x and a resolution time τ.
(x, τ)	A pair where x is an outcome and τ, is a resolution time.
X	The finite set of all outcomes x.

Note: *Additional definitions of concepts for the simple temporal framework are in Tables 3 and 4 of Section 8.

Table A2.2
Notation for the analysis of Caplin and Leahy's model (Sections 9, 10)

f	A distribution over M with a finite carrier.
F	The set of all distributions over M with a finite carrier.
g	A distribution over M generated by a choice $c = (p_0, \tau)$.
G	The set of all distributions generated by choices $c = (p_0, \tau)$.
$K(f)$	The carrier of f, the set of all $m \in M$ with $f(m) > 0$, if $K(f)$ is finite.
m	$m = (m_1, m_2)$, an emotional future, a pair of two emotional states for periods 1 and 2.
m_i	An emotional state for period i with $i = 1, 2$.
M_i	The set of all possible emotional states for period m with $i = 1, 2$.
M	The set of all $m = (m_1, m_2)$ with $m_1 \in M_i$ for $i = 1, 2$.
$\Phi(s_1(p_0), p_0) = \psi(p_0)$	The generating function for the emotional state m_1 for the first period.
$R_2(p_0)$	The second segment reservoir, the set of all second segments s_2 such that $x = (s_1(p_0), s_2) \in X$.
$s_1(p_0)$	The common first segment of all x with $p_0(x) > 0$ for a choice $c = (p_0, \tau)$.
$u(m)$	The utility of an emotional future, with $u(m) = u_1(m_1) + u_2(m_2)$.
$u_i(m_i)$	The partial utility of an emotional state $m_i \in M_i$, with $i = 1, 2$.
$u(f)$	The expected utility of a distribution f over M with a finite carrier.

that reduction of compound uncertainty at a single time is implicit. This claim creates the impression that the mixing coefficient is somehow implicitly a compound gamble when the remainder of their sentence owns that it cannot be a compound gamble since that involves reduction of uncertainty at several different times, something not allowed in their model.

Appendix D: The essential uniqueness of utility in atemporal expected utility theory

This appendix uses the notation of standard expected utility theory of Section 4.

Proposition 1. *Let X be a finite non-empty outcome set and let C be the set of all probability distributions over X. Moreover, let u be a utility function over C representing $\geqslant$ with property* (I). *Then u also has property* (II).

Proof. Without loss of generality we can assume

$$X = \{x_1, \ldots, x_n\}$$

and

$$u(x_1) \leqslant u(x_2) \leqslant \cdots \leqslant u(x_n).$$

Consider first the trivial case of a constant utility $u(x_i) = u_0$ for $i = 1, \ldots, n$. It is clear that (II) holds in this case. From now on assume $u(x_n) > u(x_1)$. For $i = 1, \ldots, n$ let α_i be the number with

$$u(x_i) = (1 - \alpha_i)u(x_1) + \alpha_i u(x_n).$$

In view of $u(x_1) \leqslant u(x_i) \leqslant u(x_n)$ we must have

$$0 \leqslant \alpha_i \leqslant 1.$$

Therefore for the probability distribution p_i with $p_i(x_1) = 1 - \alpha_i$ and $p_i(x_n) = \alpha_i$, the decision maker is indifferent between this (non-degenerate) probability distribution and x_i for sure. This yields

$$u(p_i) = u(x_i).$$

Let v be another utility function over C with the expectation property (I), which represents $\geqslant$. In view of $x_i \sim p_i$ we have:

$$v(x_i) = (1 - \alpha_i)v(x_1) + \alpha_i(x_n).$$

We obtain the following equation for α_i

$$\alpha_i = \frac{u(x_i) - u(x_1)}{u(x_n) - u(x_1)} = \frac{v(x_i) - v(x_1)}{v(x_n) - v(x_1)}.$$

Define

$$a = \frac{v(x_n) - v(x_1)}{u(x_n) - u(x_1)}$$

and

$$b = v(x_1) - au(x_1).$$

Since u and v represent the same preference order $\geqslant$, we must have $v(x_n) > v(x_1)$ in view of $u(x_n) > u(x_1)$. Therefore $a > 0$ holds. Multiplication of the equation for α_i by

$$v(x_n) - v(x_1) = a(u(x_n) - u(x_1))$$

yields

$$v(x_i) = au(x_i) + b.$$

Therefore we have

$$v(p) = \sum_{x \in X} p(x)v(x) = \sum_{x \in X} p(x)[au(x) + b]$$

and consequently

$$v(p) = au(p) + b \quad \forall p \in C.$$

It follows that v is a positive linear transformation of u as required by the essential uniqueness property (II). This completes the proof of the proposition. □

Remark. It is of crucial importance for the proof of Proposition 1 that the probability distribution p_i always is an element of C. The proof does not work unless C contains all probability distributions that assign positive probabilities to x_1 and x_n and to no other outcomes. However, the proof does not require that C contain other probability distributions.

Appendix E: The marginal distribution axiom in Caplin and Leahy

It is the purpose of this appendix to throw light on the crucial role of the marginal distribution axiom for the additive separability property (V) of the utility function u in the model of Caplin and Leahy (see Section 9). As explained in Section 9, in our version of this model we assume that the sets M_1 and M_2 of possible emotional states for periods 1 and 2, respectively are compact convex subsets of Euclidean space and M is the set of all emotional futures $m = (m_1, m_2)$ with $m_1 \in M_1$ and $m_2 \in M_2$. Moreover, the set of all probability distributions over M with finite carrier is denoted by F.

Consider a probability distribution $g \in F$. Let K be the carrier of g. For every first part m_1 of an emotional future $m = (m_1, m_2) \in K$, let $L_2(m_1)$ be the set of all m_2 with $(m_1, m_2) \in K$. Similarly for every second part m_2 of an $m \in \tau$, let $L_1(m_2)$ be the set of all m_1 with $(m_1, m_2) \in K$. Then the *marginal distributions* g_1 and g_2 of g with respect to M_1 and M_2 are defined as follows:

$$g_1(m_1) = \sum_{m_2 \in L_2(m_1)} g\big[(m_1, m_2)\big],$$

$$g_2(m_2) = \sum_{m_1 \in L_1(m_2)} g\big[(m_1, m_2)\big].$$

Let g and h be two distributions in F with the property that $g_1 = h_1$ and let $g_2 = h_2$ hold for the marginal distributions g_1, g_2 of g and h_1, h_2 of h. If this is the case we say that g and h agree with respect to their marginal distributions.

Caplin and Leahy state the marginal distribution axiom as a property of the preference relation $\geqslant$ represented by u [6, Assumption 1(iv), p. 61]. For our purpose an equivalent version in terms of u is more convenient.

Marginal distribution axiom. Let g and h be two distributions in F such that g and h agree with respect to their marginal distributions, then $u(g) = u(h)$ holds.

It is the aim of this appendix to prove a proposition about a continuous utility function u defined on M and extended to F by the expectation property (I) in Section 4. This Proposition 2 connects the timewise additive separability property of Section 9 to the marginal distribution axiom.

Proposition 2. *Let M_1 and M_2 be compact subsets of Euclidean space and let M be the set of all $m = (m_1, m_2)$ with $m_1 \in M_1$ and $m_2 \in M_2$. Let u be a continuous utility function defined on M and extended to F by the expectation property (I) in Section 4. Moreover, let $\geqslant$ be the preference relation over M represented by u. Assume that u satisfies the marginal distribution axiom. Then continuous functions u_1 and u_2 defined on M_1 and M_2, respectively, exist, such that*

$$u(m) = u_1(m_1) + u_2(m_2)$$

holds for every $m = (m_1, m_2) \in M_1$.

Proof. Since M is compact and u is continuous, the function u has a minimum u_0 over M:

$$u_0 = \min_{m \in M} u(m).$$

Let $m^0 = (m_1^0, m_2^0)$ be a fixed minimizer of u in M. (There may be more than one minimizer.) For $m = (m_1, m_2)$ define

$$u_1(m_1) = u((m_1, m_2^0)) + u_0$$

and

$$u_2(m_2) = u((m_1^0, m_2)).$$

We shall show that $u(m) = u_1(m_1) + u_2(m_2)$ holds for the *partial utility functions* u_1 and u_2 defined in this way. Let $m^* = m(m_1^*, m_2^*)$ be a fixed but arbitrary element of M. Let g and j be the following distributions:

$$g(m) = \begin{cases} \dfrac{1}{2} & \text{for } m = (m_1^*, m_2^0), \\ \dfrac{1}{2} & \text{for } m = (m_1^0, m_2^*), \\ 0 & \text{else,} \end{cases}$$

$$h(m) = \begin{cases} \dfrac{1}{2} & \text{for } m = m^0, \\ \dfrac{1}{2} & \text{for } m = m^*, \\ 0 & \text{else.} \end{cases}$$

It can be seen immediately that g and h have the same marginal distributions $g_1 = h_1$ and $g_2 = h_2$. The distribution $g_1 = h_2$ assigns probability $\frac{1}{2}$to m_1^0 and m_1^* and zero to all other elements of M_1. Similarly $g_2 = h_2$ assigns probability $\frac{1}{2}$to m_2^0 and m_2^* and zero to all other elements of M_2. It follows by the marginal distribution axiom that we have $u(g) = u(h)$. This yields

$$\frac{1}{2}u((m_1^*, m_2^0)) + \frac{1}{2}u((m_1^0, m_2^*)) \\ = \frac{1}{2}u_0 + \frac{1}{2}u(m^*).$$

Division by $\frac{1}{2}$ and rearrangement of terms leads to

$$u(m^*) = u_1(m_1^*) + u_2(m_2^*).$$

Since m^* is an arbitrary element of M this equation holds for every $m^* \in M$. This completes the proof of the proposition. $\square$

Comment

Caplin and Leahy make use of an axiomatization of timewise additive expected utility taken from Fishburn [10]. In his book Fishburn explains that the marginal distribution axiom is not really compelling. He points out that a decision maker may not be indifferent between the distributions g and h described in the proof of Proposition 2. Suppose that m_1^* and m_2^* are better than m_1^0 and m_2^0, respectively, in the sense that (m_1^*, m_2^0) and (m_1^0, m_2^*) as well as (m_1^*, m_2^*) are strictly preferred to (m_1^0, m_2^0). Then in g the decision maker is sure to get the better emotional state in at least one of the two states where as in h she may end up with the worse emotional state in both periods.

It is of course possible that the preferences of the decision maker are adequately represented by the expectation of a timewise additively separable utility function, but whether this the case or not has to be judged on the basis of the concrete context. The marginal distribution axiom fails to be a property that must always be satisfied by the preferences of a rational person.

Acknowledgements

We thank for very helpful comments an anonymous referee of this journal, and for auxiliary assistance Amira Allam, Andreas Orland, Kanishka Hiththetiya and Patrick Partington of Bonn University.

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