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Non-normal modalities in variants of linear logic

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This article presents modal versions of resource-conscious logics. We concentrate on extensions of variants of linear logic with one minimal non-normal modality. In earlier work, where we investigated agency in multi-agent systems, we have shown that the results scale up to logics with multiple non-minimal modalities. Here, we start with the language of propositional intuitionistic linear logic without the additive disjunction, to which we add a modality. We provide an interpretation of this language on a class of Kripke resource models extended with a neighbourhood function: modal Kripke resource models. We propose a Hilbert-style axiomatisation and a Gentzen-style sequent calculus. We show that the proof theories are sound and complete with respect to the class of modal Kripke resource models. We show that the sequent calculus admits cut elimination and that proof-search is in PSPACE. We then show how to extend the results when non-commutative connectives are added to the language. Finally, we put the logical framework to use by instantiating it as logics of agency. In particular, we propose a logic to reason about the resource-sensitive use of artefacts and illustrate it with a variety of examples.

Keywords: modal logics; resource-conscious logics; agency; artefacts

1. Introduction

Logics for resources and modalities have each had their share of attention, and have already been studied together. Besides the seminal work on intuitionistic modal logic in Plotkin and Stirling (1986) and Simpson (1994), modalities for substructural implications have been studied in D'Agostino, Gabbay, and Russo (1997), and extensions of intuitionistic linear logic with modalities have been investigated, for example, in Marion and Sadrzadeh (2004) and Kamide (2006). Moreover, modal versions of logics for resources that are related to linear logic have been provided in Pym, O'Hearn and Yang (2004), Pym and Tofts (2006), and Courtault and Galmiche (2013).

Modalities in substructural logics are thus not new although, to our knowledge, modalities in sub-structural logics have always been restricted to *normal* modalities. Modalities that are not normal have been confined to the realm of classical logic.

Normal modalities are the modalities within a logic that is at least as strong as the standard modal logic K. *Non-normal modalities* on the other hand are the modalities that fail to satisfy some of the principles of the standard modal logic K. They cannot be evaluated over a Kripke semantics – but one typical semantics, neighbourhood models, relies on possible worlds. Chellas gave a textbook presentation in Chellas (1980).

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The significance of non-normal modal logics and their semantics in modern developments in logics of agents has been emphasised before (Arló-Costa & Pacuit, 2006). Indeed many logics of agents are non-normal, and neighbourhood semantics allows for defining the modalities that are required to model a number of application domains, including logics of coalitional power (Pauly, 2002), epistemic logics without omniscience (Lismont & Mongin, 1994; Vardi, 1986), and logics of agency (Governatori & Rotolo, 2005).

Let us briefly present some features of reasoning about agency specifically. Modalities of agency aimed at modelling the result of an action have largely been studied in the literature on practical philosophy and multi-agent systems (Belnap, Perloff, & Xu, 2001; Governatori & Rotolo, 2005; Kanger & Kanger, 1966; Pörn, 1977; Troquard, 2014). Logics of agency assume a set of agents, and to each agent i associate a modality E_i. We read $\mathsf{E}_i\varphi$ as 'agent i brings about φ'. With the notable exception of Chellas' (1969) logic, it is generally accepted in logics of agency that no agent ever brings about a tautology. This is because when i brings about φ, it is intended that φ might not have been the case if it were not for i's very agency. Classically, this corresponds to the axiom $\neg\mathsf{E}_i\top$. This principle is enough for the logic of the modality E_i to be non-normal, as it is inconsistent with the necessitation rule. A normal modal logic for agency would also dictate that if i does φ then she also does $\varphi \vee \psi$. This is refuted in general on the same grounds as Ross' paradox. For instance, purposefully bringing about that I am rich should not imply purposefully bringing about that I am rich or unhealthy.

In classical logic, $\mathsf{E}_i\varphi$ allows one to capture that agent i brings about *the state of affairs* φ. Moving from states of affairs to resources, it is then interesting to lift these modalities from classical logic to resource-conscious logic. In doing so, one can capture with $\mathsf{E}_i A$ that agent i brings about *the resource* A.

To make a start with this research program, we will combine intuitionistic fragments of linear logic with non-normal modalities. Linear logic (Girard, 1987) is a resource-conscious logic that allows for modelling the constructive content of deductions in logic.

An *intuitionistic* version of linear logic, such as the one we will work with, has a number of desirable features, one of which is simple and yet greatly appreciated: in intuitionistic sequent calculus, every sequent has a single 'output' formula. This feature favours the modelling of input-output processes. It will prove particularly apt for our application to agency and artefacts as it provides a simple mechanism for the compositionality of individual artefacts' functions into complex input-output processes.

The resource-sensitive nature of linear logic can be viewed as the absence of structural rules in the sequent calculus. Linear logic rejects the global validity of weakening (W), that amounts to a monotonicity of the entailment, and contraction (C), that is responsible for arbitrary duplications of formulas, e.g., $A \to A \wedge A$ is a tautology in classical logic, and so is $A \wedge A \to A$.

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\ (\mathrm{W}) \qquad \frac{\Gamma, B, B \vdash A}{\Gamma, B \vdash A}\ (\mathrm{C}) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, B, A \vdash C}\ (\mathrm{E})$$

Accordingly, the linear implication $\multimap$ encodes resource-sensitive deductions, for example from A and $A \multimap B$ we can infer B, by *modus ponens*, but we are not allowed to conclude B from A, A, and $A \multimap B$. Exchange (E) still holds (although we will later restrict it). Hence, contexts of formulas Γ in sequent calculus are considered multisets. By dropping weakening and contraction, we are led to define two non-equivalent conjunctions with different behaviours: the multiplicative conjunction $\otimes$ (tensor) and the additive conjunction & (with). The intuitive meaning of $\otimes$ is that an action of type $A \otimes B$ can be performed by summing the resources that are relevant for performing A and for performing

B. The unit **1** is the neutral element for $\otimes$ and can represent a null action. A consequence of the lack of weakening is that $A \otimes B$ no longer implies A, namely the resources that are relevant for performing $A \otimes B$ may not be relevant for performing just A. The absence of contraction means that $A \multimap A \otimes A$ is no longer valid. The additive conjunction $A\&B$ expresses an option – the choice to perform A or B. Accordingly $A\&B \multimap A$ and $A\&B \multimap B$ hold in linear logic, as the resources that enable the choice between A and B are also relevant for making A or for making B. The linear implication $A \multimap B$ expresses a form of causality; for example, 'if I strike a match, I light the room'. The action of striking a match is consumed in the sense that it is no longer available after the room is lighted. Linear implication and multiplicative conjunction interact naturally so that $(A \otimes (A \multimap B)) \multimap B$ is valid.

Linear logic operators have been applied to a number of topics in knowledge representation and multi-agent systems such as planning (Kanovich & Vauzeilles, 2001), preference representation and resource allocation (Harland & Winikoff, 2002; Porello & Endriss, 2010a 2010b), social choice (Porello, 2013), action modelling (Borgo, Porello, & Troquard, 2014), and narrative generation (Martens, Bosser, Ferreira, & Cavazza, 2013).

These propositional operators are very useful when talking about resources and agency, as they allow one to capture the following notions:

- Bringing about both A and B together: $\mathsf{E}_i(A \otimes B)$, in such a way that $\mathsf{E}_i(A \otimes B)$ implies $A \otimes B$, but does not imply A alone.
- Bringing about an option between A and B: $\mathsf{E}_i(A\&B)$, in such a way that $\mathsf{E}_i(A\&B)$ implies A and implies B, but does not imply $A \otimes B$.
- Bringing about the transformation of the resource A into the resource B: $\mathsf{E}_i(A \multimap B)$, in such a way that $A \otimes \mathsf{E}_i(A \multimap B)$ implies B.

The only structural rule that holds in linear logic is exchange (E), which is responsible for the commutativity of the multiplicative operators and amounts to forgetting sequentiality of actions, e.g., $A \otimes B \to B \otimes A$. We will see in this paper how to deal with ordered information. We will still admit exchange for the two conjunctions & and $\otimes$ (commutative conjunctions), but we will introduce a non-commutative counterpart $\odot$ to the multiplicative $\otimes$. The formula $A \odot B$ is not equivalent to $B \odot A$. Since $\odot$ is non-commutative, we can have two *order-sensitive* linear implications (noted in Lambek, 1958, $\backslash$ and $/$) .

For agency in multi-agent systems, resources must often become available at key points in a series of transformations. The order of resource production and transformation quickly becomes relevant. It is then interesting to examine the following:

- Bringing about the resource A first, then B: $\mathsf{E}_i(A \odot B)$, in such a way that $\mathsf{E}_i(A \odot B)$ is not equivalent to $\mathsf{E}_i(B \odot A)$.
- Bringing about the order-sensitive transformation of the resource from A into B: $\mathsf{E}_i(A \backslash B)$, in such a way that $A \odot \mathsf{E}_i(A \backslash B)$ implies B, but $\mathsf{E}_i(A \backslash B) \odot A$ does not.

These considerations motivate us to investigate the theoretical underpinnings of modal versions of resource-conscious logics with the listed propositional operators.

We will first concentrate on extensions with *one minimal non-normal* modality. In a second part, where we address modalities of agency, we will exploit our results that will naturally scale up to logics with *multiple non-minimal* (but still non-normal) modalities.

1.1. Outline

In Section 2, we present the Kripke resource models which already exist in the literature. The semantics of all the languages studied in this paper will be adequate extensions of Kripke resource models. We first enrich the Kripke resource models with neighbourhood functions to capture non-normal modalities. We obtain what we simply coin modal Kripke resource models. We define and study a minimal non-normal modal logic, MILL. We introduce a Hilbert system in Section 3 and a sequent calculus in Section 4; both are shown to be sound and complete. Moreover, we can easily show that the sequent calculus admits cut elimination that provides a normal form for proofs. Proof search is proved to be in PSPACE.

We extend our framework in Section 5 to account for partially commutative linear logic that allows for integrating commutative and non-commutative operators.

Then in Section 6 we instantiate the minimal modal logic with a resource-sensitive version of the logics of bringing-it-about called RSBIAT. Again, both a sound and complete Hilbert system and sequent calculus are provided. Proof-search in RSBIAT is PSPACE-easy. We also show how RSBIAT can be extended with the non-commutative language and thus represent sequentiality of actions. In Section 7, we discuss a number of applications of our system to represent and reason about artefacts.

2. MILL and modal Kripke resource models

Let $Atom$ be a non-empty set of atomic propositions. We introduce the most basic language studied in this paper. The language $\mathcal{L}_{\mathsf{MILL}}$ is given by the Backus-Naur Form (BNF)

$$A ::= \mathbf{1} | p | A \otimes A | A \& A | A \multimap A | \Box A,$$

where $p \in Atom$. It is a modal version of what corresponds to the language of propositional intuitionistic linear logic, but without the additive disjunction and the additive units. This propositional part can also be seen as a fragment of BI (O'Hearn & Pym, 1999) without additive disjunction and implication.

Let us first concentrate on the propositional part for which a semantics already exists in the literature. We call the logic ILL and $\mathcal{L}_{\mathsf{ILL}}$ its language. A Kripke-like class of models for ILL is basically due to Urquhart (1972). A *Kripke resource frame* is a structure $\mathcal{M} = (M, e, \circ, \geq)$, where $(M, e, \circ)$ is a commutative monoid with neutral element e, and $\geq$ is a pre-order on M. The frame has to satisfy the condition of *bifunctoriality*: if $m \geq n$, and $m' \geq n'$, then $m \circ m' \geq n \circ n'$. To obtain a *Kripke resource model*, a valuation on atoms $V : Atom \to \mathcal{P}(M)$ is added. It has to satisfy the *heredity* condition: if $m \in V(p)$ and $n \geq m$ then $n \in V(p)$.

The truth conditions of $\mathcal{L}_{\mathsf{ILL}}$ in the Kripke resource model $\mathcal{M} = (M, e, \circ, \geq, V)$ of the formulas of the propositional part are the following:

$m \models_{\mathcal{M}} p$ iff $m \in V(p)$.
$m \models_{\mathcal{M}} \mathbf{1}$ iff $m \geq e$.
$m \models_{\mathcal{M}} A \otimes B$ iff there exist m_1 and m_2 such that $m \geq m_1 \circ m_2$ and $m_1 \models_{\mathcal{M}} A$ and $m_2 \models_{\mathcal{M}} B$.
$m \models_{\mathcal{M}} A \& B$ iff $m \models_{\mathcal{M}} A$ and $m \models_{\mathcal{M}} B$.
$m \models_{\mathcal{M}} A \multimap B$ iff for all $n \in M$, if $n \models_{\mathcal{M}} A$, then $n \circ m \models_{\mathcal{M}} B$.

Observe that heredity can be shown to extend naturally to every formula, in the sense that:

Proposition 1. *For every formula $A \in \mathcal{L}_{\mathsf{ILL}}$, if $m \models A$ and $m' \geq m$, then $m' \models A$.*

2.1. Notations

An intuitionistic negation can be added to the language. We simply choose a designated atom $\bot \in Atom$ and decide to conventionally interpret $\bot$ as indicating a contradiction. Negation is then defined by means of implication as $\sim A \equiv A \multimap \bot$ (Kanovich, Okada, & Terui, 2006): the occurrence of A yields the contradiction. There will be no specific rule for negation.

Given a multiset of formulas, it will be useful to combine them into a unique formula. We adopt the following notation: $\emptyset^* = \mathbf{1}$, and $\Delta^* = A_1 \otimes \cdots \otimes A_k$ when $\Delta = \{A_1, \ldots, A_k\}$.

Denote $||A||^{\mathcal{M}}$, the extension of A in $\mathcal{M}$, i.e., the set of worlds of $\mathcal{M}$ in which A holds. A formula A is *true* in a model $\mathcal{M}$ if $e \models_{\mathcal{M}} A$.[1] A formula A is *valid* in Kripke resource frames, noted $\models A$, iff it is true in every model.

2.2. Modal Kripke resource models

Now, to give a meaning to the modality, we define a neighbourhood semantics on top of the Kripke resource frame. A neighbourhood function is a mapping $N : M \to \mathcal{P}(\mathcal{P}(M))$ that associates a world m with a set of sets of worlds (see Chellas, 1980). We define:

$m \models \Box A$ iff $||A|| \in N(m)$.

This is not enough, though. It is possible that $m \models \Box A$, yet $m' \not\models \Box A$ for some $m' \geq m$. That is, Proposition 1 does not hold with the simple extension of $\models$ for $\mathcal{L}_{\mathsf{MILL}}$. (One disastrous consequence is that the resulting logic does not satisfy the *modus ponens* or the *cut* rule.) We could define the clause concerning the modality alternatively as $m \models \Box A$ iff there is $n \in M$, such that $m \geq n$ and $||A|| \in N(n)$. However, it will be more agreeable to keep working with the standard definition and instead impose a condition on the models.

We will require our neighbourhood function to satisfy the condition that if some set $X \subseteq M$ is in the neighbourhood of a world, then X is also in the neighbourhood of all 'greater' worlds.[2] Formally, our modal linear logic is evaluated over the following models:

Definition 2. *A* modal Kripke resource model *is a structure $\mathcal{M} = (M, e, \circ, \geq, N, V)$ such that:*

- *$(M, e, \circ, \geq)$ is a Kripke resource frame;*
- *N is a neighbourhood function such that:*

$$\textit{if } X \in N(m) \textit{ and } n \geq m \textit{ then } X \in N(n). \tag{1}$$

It is readily checked thatProposition 1 is true as well for $\mathcal{L}_{\mathsf{MILL}}$ over modal Kripke resource models for modal formulas. We thus have:

Proposition 3. *For every formula $A \in \mathcal{L}_{\mathsf{MILL}}$, if $m \models A$ and $m' \geq m$, then $m' \models A$.*

Proof. We handle the new case $A = \Box B$. Suppose $m \models \Box B$ and $m' \geq m$. By definition, $||B|| \in N(m)$, and thus $||B|| \in N(m')$ follows from Condition (1). Therefore $m' \models \Box B$. $\square$

Table 1. Axiom schemata in H-MILL.

$A \multimap A$
$(A \multimap B) \multimap ((B \multimap C) \multimap (A \multimap C))$
$(A \multimap (B \multimap C)) \multimap (B \multimap (A \multimap C))$
$A \multimap (B \multimap A \otimes B)$
$(A \multimap (B \multimap C)) \multimap (A \otimes B \multimap C)$
$\mathbf{1}$
$\mathbf{1} \multimap (A \multimap A)$
$(A \mathbin{\&} B) \multimap A$
$(A \mathbin{\&} B) \multimap B$
$((A \multimap B) \mathbin{\&} (A \multimap C)) \multimap (A \multimap B \mathbin{\&} C)$

3. Hilbert system for MILL and soundness

This part extends the Hilbert system for ILL from Troelstra (1992) and Avron (1988). We define the Hilbert-style calculus H-MILL for MILL by defining the following notion of deduction.

Definition 4. (Deduction in H-MILL) *A* deduction tree *in* H-MILL $\mathcal{D}$ *is inductively constructed as follows.*

(i) The leaves o f the tree are assumptions $A \vdash_{\mathsf{H}} A$*, for* $A \in \mathcal{L}_{\mathsf{MILL}}$*, or* $\vdash_{\mathsf{H}} B$ *where* B *is an axiom in Table 1 (base cases).*

(ii) We denote by $\Gamma \overset{\mathcal{D}}{\vdash_{\mathsf{H}}} A$ *a deduction tree with conclusion* $\Gamma \vdash_{\mathsf{H}} A$*. If* $\mathcal{D}$ *and* $\mathcal{D}'$ *are deduction trees, then the following are deduction trees (inductive steps).*

$$\frac{\Gamma \overset{\mathcal{D}}{\vdash_{\mathsf{H}}} A \qquad \Gamma' \overset{\mathcal{D}'}{\vdash_{\mathsf{H}}} A \multimap B}{\Gamma, \Gamma' \vdash_{\mathsf{H}} B} \multimap -\text{rule} \qquad \frac{\Gamma \overset{\mathcal{D}}{\vdash_{\mathsf{H}}} A \qquad \Gamma \overset{\mathcal{D}'}{\vdash_{\mathsf{H}}} B}{\Gamma \vdash_{\mathsf{H}} A \mathbin{\&} B} \mathbin{\&} - \text{rule}$$

$$\frac{\overset{\mathcal{D}}{\vdash_{\mathsf{H}}} A \multimap B \qquad \overset{\mathcal{D}'}{\vdash_{\mathsf{H}}} B \multimap A}{\vdash_{\mathsf{H}} \Box A \multimap \Box B} \Box(\text{re})$$

We sometimes refer to the $\multimap$-rule as *modus ponens*. We say that A is deducible from Γ in H-MILL and we write $\Gamma \vdash_{\mathsf{H\text{-}MILL}} A$ iff there exists a deduction tree in H-MILL with conclusion $\Gamma \vdash_{\mathsf{H}} A$. The deduction without assumptions, i.e., $\vdash_{\mathsf{H\text{-}MILL}} A$, is just a special case of the above definition.

In general when defining Hilbert systems for linear logics, we need to be careful in the definition of derivation from assumptions: since Γ is a multiset, we need to handle occurrences of hypothesis when applying for instance *modus ponens*. From Definition 4, every occurrence of assumptions or axioms in a derivation, except for the conclusion, is used exactly once by an application of *modus ponens* (Avron, 1988). With respect to this notion of derivation, the deduction theorem holds.

Theorem 5. (Deduction theorem for H-MILL) *If* $\Gamma, A \vdash_{\mathsf{H\text{-}MILL}} B$ *then* $\Gamma \vdash_{\mathsf{H\text{-}MILL}} A \multimap B$.

Proof. The deduction theorem holds for the propositional fragment ILL (see citealpTroelstra1992). The only case to consider is the rule $\Box$(re). However this trivially holds because the contexts of the sequents in the rule are empty. $\square$

Remark. *We defined the rule $\Box$(re) in that particular way because of the deduction theorem. With the above formulation, the deduction theorem is preserved in* MILL*. Moreover, by our definition of a deduction tree, this version entails the rule $\Box$(re'): from two deductions trees with the conclusions $A \vdash_{\mathsf{H}} B$ and $B \vdash_{\mathsf{H}} A$, we can build a deduction tree with the conclusion $\Box A \vdash_{\mathsf{H}} \Box B$.*

We claim that $\Box$(re) implies $\Box$(re'). Assume the premises of $\Box$(re'): there are two deduction trees with the conclusions $A \vdash_{\mathsf{H}} B$ and $B \vdash_{\mathsf{H}} A$. In virtue of Theorem 5 and the definition of $\vdash_{\mathsf{H\text{-}MILL}}$, we know there are two trees with the conclusions $\vdash_{\mathsf{H}} A \multimap B$ and $\vdash_{H} B \multimap A$. Thus by $\Box$(re), we can build a deduction tree with the conclusion $\vdash_{\mathsf{H}} \Box A \multimap \Box B$. Together with $\Box A \vdash_{\mathsf{H}} \Box A$ (a leaf), the $\multimap$-rule gives us a deduction tree with the conclusion $\Box A \vdash_{\mathsf{H}} \Box B$. With the version $\Box$(re'), the deduction theorem fails, as we do not have any axiom that talks about the modality in this case.

We can prove the soundness of H-MILL w.r.t. our semantics.

Theorem 6. (Soundness of H-MILL) *If $\Gamma \vdash_{\mathsf{H\text{-}MILL}} A$ then, for every model, $\Gamma \models A$ (namely, $e \models (\Gamma)^* \multimap A$).*

Proof. We only give the arguments of the proof of soundness for two representative cases. *Soundness of $\multimap$-rule.* We show now that $\multimap$-rule preserves validity. Namely, we prove, by induction on the length of the derivation tree that if (1) $e \models \Gamma \multimap A$ and (2) $e \models \Gamma' \multimap (A \multimap B)$, then $e \models \Gamma \otimes \Gamma' \multimap B$. The first assumption entails that for all x, if $x \models \Gamma$, then $x \models A$. The second assumption entails that for all y, if $y \models \Gamma'$, then $y \models A \multimap B$. Thus, for all t, if $t \models A$, then $y \circ t \models B$. Let $z \models \Gamma \otimes \Gamma'$, thus there exist z_1 and z_2 such that $z_1 \models \Gamma$, $z_2 \models \Gamma'$ and $z \geq z_1 \circ z_2$. By (1), we have that $z_1 \models A$ and, by (2), $z_2 \models A \multimap B$, thus $z_1 \circ z_2 \models B$, and by Proposition 3 we have that $z \models B$.[3]

Soundness of $\Box$(re). We show that $\Box$(re) preserves validity, namely, if $e \models A \multimap B$ and $e \models B \multimap A$, then $e \models \Box A \multimap \Box B$. Our assumptions imply that, for all x, if $x \models A$, then $x \models B$, and if $x \models B$ then $x \models A$. Thus, $||A|| = ||B||$. We need to show that for all x, if $x \models \Box A$, then $x \models \Box B$. By definition, $x \models \Box A$ iff $||A|| \in N(x)$. Thus, since $||A|| = ||B||$, we have that $||B|| \in N(x)$, that means $x \models \Box B$. $\square$

4. Sequent calculus MILL and completeness

In this section, we introduce the sequent calculus for our logic. A *sequent* is a statement $\Gamma \vdash A$ where Γ is a finite multiset of occurrences of formulas of MILL and A is a formula. The fact that we allow for a single formula in the conclusions of the sequent corresponds to the fact that we are working with the intuitionistic version of the calculus (Girard, 1987).

Since in a sequent $\Gamma \vdash A$ we identify Γ to a multiset of formulas, the exchange rule – the reshuffling of Γ – is implicit.

A sequent $\Gamma \vdash A$ where $\Gamma = A_1, \ldots, A_n$ is *valid* in a modal Kripke resource frame iff the formula $A_1 \otimes \ldots \otimes A_n \multimap A$ is valid, namely $\models \Gamma^* \multimap A$.

We obtain the sequent calculus for our minimal modal logic MILL by extending the language of ILL with modal formulas and by adding a new rule. Saturating the notation, we label the sequent calculus rule like the rule for equivalents in the Hilbert system: $\Box$(re). The calculus is shown in Table 2.

Table 2. Sequent calculus MILL.

$$\dfrac{}{A \vdash A}\,\text{ax} \qquad \dfrac{\Gamma, A \vdash C \qquad \Gamma' \vdash A}{\Gamma, \Gamma' \vdash C}\,\text{cut}$$

$$\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\otimes\text{L} \qquad \dfrac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\,\otimes\text{R}$$

$$\dfrac{\Gamma \vdash A \qquad \Gamma', B \vdash C}{\Gamma', \Gamma, A \multimap B \vdash C}\,\multimap\text{L} \qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,\multimap\text{R}$$

$$\dfrac{\Gamma, A, \Gamma' \vdash C}{\Gamma, A \,\&\, B, \Gamma' \vdash C}\,\&\text{L} \qquad \dfrac{\Gamma, B, \Gamma' \vdash C}{\Gamma, A \,\&\, B, \Gamma' \vdash C}\,\&\text{L}$$

$$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,\&\text{R} \qquad \dfrac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C}\,\mathbf{1}\text{L} \qquad \dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}\text{R}$$

$$\dfrac{A \vdash B \qquad B \vdash A}{\Box A \vdash \Box B}\,\Box(\text{re})$$

To establish a link with the previous section, we first show that provability in sequent calculus is equivalent to provability in the Hilbert system.

Theorem 7. *It holds that* $\Gamma \vdash_{\text{H-MILL}} A$ *iff the sequent* $\Gamma \vdash A$ *is derivable in the sequent calculus for* MILL.

Proof. *(Sketch)* The propositional cases are proved in Avron (1988); we only need to extend it for the case of the modal rules. This is done by induction on the length of the proof. For instance, in one direction, assume that a derivation $\mathcal{D}$ of $\vdash_H \Box A \multimap \Box B$ is obtained by $\mathcal{D}'$ of $\vdash_H A \multimap B$ and $\mathcal{D}''$ of $\vdash_H B \multimap A$. By definition of deduction and of $\vdash_{\text{H-MILL}}$, we have that $A \vdash_{\text{H-MILL}} B$ and $B \vdash_{\text{H-MILL}} A$. By induction hypothesis, we have that $A \vdash B$ and $B \vdash A$ are provable in the sequent calculus. Thus, by $\Box$(re) and $\multimap R$, we have that $\vdash \Box A \multimap \Box B$ is provable in the sequent calculus. □

Crucially, the modal extension does not affect cut elimination. Cut elimination holds for linear logic (Girard, 1987). The proof for MILL largely adapts the proof for linear logic (Troelstra, 1992). Recall that the *rank* of a cut is the complexity of the cut formula. The *cutrank* of a proof is the maximum of the ranks of the cuts in the proof. The *level* of a cut is the length of the subproof ending in the cut (Troelstra, 1992). The proof of cut elimination proceeds by induction on the cutrank of the proof. It is enough to assume that the occurrence of the cut with maximal rank is the last rule of the proof. The inductive step shows how to replace a proof ending in a cut with rank n with a proof with the same conclusion and smaller cutrank. The proof of the inductive step proceeds by induction on the level of the terminal cut, namely on the length of the proof. Note that if one of the premises of a cut is an axiom, then we can simply eliminate the cut. Given a sequent rule R, the occurrence of a formula A in the conclusion of R is principal if A has been introduced by R.

Theorem 8. *Cut elimination holds for* MILL.

Proof. *(Sketch)* As usual, there are two main cases to consider: first, the cut formula is principal in both premises of the terminal cut rule, and second, that is not the case.

First main case. We replace the cut of maximal rank with two cuts of strictly smaller rank. For example, take the case in which $\Box C$ is the cut formula and is principal in both premises (i.e., it has been introduced by $\Box$(re)):

$$\dfrac{\dfrac{B \vdash C \qquad C \vdash B}{\Box B \vdash \Box C}\,\Box(\text{re}) \qquad \dfrac{C \vdash D \qquad D \vdash C}{\Box C \vdash \Box D}\,\Box(\text{re})}{\Box B \vdash \Box D}\,\text{cut}$$

It is reduced by replacing the cut on $\Box C$ by two cuts on C of strictly smaller rank:

$$\dfrac{\text{cut}\,\dfrac{B \vdash C \qquad C \vdash D}{B \vdash D} \qquad \dfrac{D \vdash C \qquad C \vdash B}{D \vdash B}\,\text{cut}}{\Box B \vdash \Box D}\,\Box(\text{re})$$

Second main case. The cut formula is not principal in one of the premises of the cut rule. Suppose R is the rule that does not introduce the cut formula. In this case, we can apply the cut after R. By induction, the proof minus R can be turned into a cut-free proof, since the length of the subproof is smaller and the cutrank is equal or smaller. By applying R again to the cut-free proof, we obtain a proof with the same conclusion of the starting proof and less cuts.

For example,

$$\dfrac{\dfrac{B \vdash C \qquad C \vdash B}{\Box B \vdash \Box C}\,\Box(\text{re}) \qquad \dfrac{\dots}{\Gamma, \Box C \vdash A}\,\text{R}}{\Box B, \Gamma \vdash A}\,\text{cut}$$

can be turned into:

$$\dfrac{\dfrac{\dfrac{B \vdash C \qquad C \vdash B}{\Box B \vdash \Box C}\,\Box(\text{re}) \qquad \dfrac{\dots}{\Gamma', \Box C \vdash A'}}{\Box B, \Gamma' \vdash A'}\,\text{cut} \qquad \Gamma'' \vdash A''}{\Box B, \Gamma \vdash A}\,\text{R}$$

□

By inspecting the rules other than *cut*, it is easy to see that cut elimination entails the subformula property – namely, if $\Gamma \vdash A$ is derivable then there is a derivation containing subformulas of Γ and A only.

The decidability remains to be established. We can show that the proof-search for MILL is no more costly in terms of space than the proof-search for propositional intuitionistic multiplicative additive linear logic (Lincoln, Mitchell, Scedrov, & Shankar, 1992).

Theorem 9. *Proof search complexity for* MILL *is in PSPACE.*

Proof. *(Sketch)* The proof adapts the argument in Lincoln et al. (1992). By cut elimination (Theorem 8), for every provable sequent in MILL there is a cut-free proof with the same conclusion. For every rule in MILL other than *cut*, the premises have a strictly lower complexity w.r.t. the conclusion. Hence, for every provable sequent, there is a proof whose branches have a depth at most linear in the size of the sequent. The size of a branch is at most quadratic in the size of the conclusion. And it contains only subformulas of the conclusion sequent because of the subformula property. This means that one can non-deterministically guess such a proof, and check each branch one by one using only a polynomial space. The proof search is then in NPSPACE = PSPACE. □

We present the proof of completeness of MILL w.r.t. the class of modal Kripke resource frames.

Theorem 10. (Completeness of the sequent calculus). *If* $\models \Gamma^* \multimap A$ *then* $\Gamma \vdash A$.

The proof can be summarised as follows. We build a canonical model $\mathcal{M}^c$ (Definition 11). In particular, the set M^c of states consists in the set of finite multisets of formulas, and the neutral element e^c is the empty multiset. We first need to show that it is indeed a modal Kripke resource model (Lemma 12). Second we need to show a correspondence, the 'Truth Lemma', between $\vdash$ and truth in $\mathcal{M}^c$. Precisely we show that for a formula A and a multiset of formulas $\Gamma \in M^c$, it is the case that Γ satisfies A iff $\Gamma \vdash A$ is provable in the calculus (Lemma 13). Finally, to show completeness, assume that it is not the case that $\vdash \Gamma^* \multimap A$. By the Truth Lemma, in the canonical model $\Gamma^* \multimap A$ is not satisfied at e^c, so $\mathcal{M}^c$ does not satisfy $\Gamma^* \multimap A$ and thus it is not the case that $\models \Gamma^* \multimap A$.

We construct the canonical model $\mathcal{M}^c$, then we prove that $\mathcal{M}^c$ is a modal Kripke resource model, and we prove the Truth Lemma.

In the following, $\sqcup$ is the multiset union. Also, $|A|^c = \{\Gamma | \Gamma \vdash A\}$.

Definition 11. *Let* $\mathcal{M}^c = (M^c, e^c, \circ^c, \geq^c, N^c, V^c)$ *such that:*

- $M^c = \{\Gamma | \Gamma$ *is a finite multiset of formulas*$\}$*;*
- $\Gamma \circ^c \Delta = \Gamma \sqcup \Delta$*;*
- $e^c = \emptyset$*;*
- $\Gamma \geq^c \Delta$ *iff* $\Gamma \vdash \Delta^*$*;*
- $\Gamma \in V^c(p)$ *iff* $\Gamma \vdash p$*;*
- $N^c(\Gamma) = \{|A|^c | \Gamma \vdash \Box A\}$.

Lemma 12. $\mathcal{M}^c$ *is a modal Kripke resource model.*

Proof. 1. $(M^c, e^c, \circ^c, \geq^c)$ is the 'right type' of ordered monoid: (i) $(M^c, e^c, \circ^c)$ is a commutative monoid with the neutral element e^c; (ii) $\geq^c$ is a pre-order on M^c; and (iii) if $\Gamma \geq^c \Delta$ and $\Gamma' \geq^c \Delta'$ then $\Gamma \circ^c \Gamma' \geq^c \Delta \circ^c \Delta'$.

For (i), commutativity (and associativity) follow from the definition of $\circ^c$ as the multiset union, and the neutrality of e^c follows from it being the empty multiset – the neutral element of the multiset union.

For (ii), $\geq^c$ is reflexive because $\{A_1, \ldots, A_n\} \vdash \{A_1, \ldots, A_n\}^*$ can be proved from the axioms (ax) $A_k \vdash A_k$, $1 \leq k \leq n$, and by applying $\otimes$R $n-1$ times. The key rule to establish that $\geq^c$ is transitive is *cut*.

For (iii), assume $\Gamma \geq^c \Delta$ and $\Gamma' \geq^c \Delta'$, that is, $\Gamma \vdash \Delta^*$ and $\Gamma' \vdash \Delta'^*$. By $\otimes$R we have $\Gamma, \Gamma' \vdash \Delta^* \otimes \Delta'^*$. By applying the definitions we end up with $\Gamma \sqcup \Gamma' \vdash (\Delta \sqcup \Delta')^*$ and the expected result follows.

2. V^c is a valuation function and satisfies heredity: if $\Gamma \in V(p)$ and $\Delta \geq^c \Gamma$ then $\Delta \in V(p)$. To see this, suppose $\Gamma \vdash p$ and $\Delta \vdash \Gamma^*$. By applying $\otimes$L enough times, we have $\Gamma^* \vdash p$. By *cut*, we obtain $\Delta \vdash p$.

3. N^c is well-defined. Suppose that $|A|^c = |B|^c$. We need to show that $|A|^c \in N^c(\Gamma)$ iff $|B|^c \in N^c(\Gamma)$.

From $|A|^c = |B|^c$, we have $\Gamma \vdash A \Rightarrow \Gamma \vdash B$. In particular, we have $A \vdash A \Rightarrow A \vdash B$. Hence, $A \vdash B$ is provable (by rule (ax)). We show symmetrically that $B \vdash A$ is provable.

From $A \vdash B$ and $B \vdash A$, we have by rule $\Box$(re) that $\Box A \vdash \Box B$ is provable, and also that $\Box B \vdash \Box A$ is provable.

Now suppose that $\Gamma \vdash \Box A$. Since $\Box A \vdash \Box B$ is provable, we obtain by *cut* that $\Gamma \vdash \Box B$ is provable. Symmetrically, suppose that $\Gamma \vdash \Box B$. Since $\Box B \vdash \Box A$ is provable, we obtain by *cut* that $\Gamma \vdash \Box A$ is provable.

Hence, we have that $\Gamma \vdash \Box A$ iff $\Gamma \vdash \Box B$. By definition of N^c, this means that $|A|^c \in N^c(\Gamma)$ iff $|B|^c \in N^c(\Gamma)$.

4. If $X \in N^c(\Gamma)$ and $\Delta \geq^c \Gamma$ then $X \in N^c(\Delta)$. To see that this is the case, the hypotheses are equivalent to $\Gamma \vdash \Box A$ for some A such that $|A|^c = X$, and $\Delta \vdash \Gamma^*$. By repeatedly applying $\otimes$L to obtain $\Gamma^* \vdash \Box A$ and by using *cut*, we infer that $\Delta \vdash \Box A$, which is equivalent to the statement that $X \in N^c(\Delta)$. □

Let us then denote by $\models_c$ the truth relation in $\mathcal{M}^c$.

Lemma 13. $\Gamma \models_c A$ *iff* $\Gamma \vdash A$.

Proof. The proof is obtained by structural induction on the form of A. For the base case, we have for every atom p that $\Gamma \models_c p$ iff $\Gamma \in V^c(p)$ iff $\Gamma \vdash p$. For induction, we suppose that the lemma holds for a formula B (Induction Hypothesis). The cases of the propositional connectives are found in Kamide (2003). We prove here the case $A = \Box B$.

We have the following sequence of equivalences of $\Gamma \models_c \Box B$:

iff $||B||^{\mathcal{M}^c} \in N^c(\Gamma)$, by definition of $\models_c$;
iff $\{\Delta | \Delta \models_c B\} \in N^c(\Gamma)$, by definition of $||.||^{\mathcal{M}^c}$;
iff $\{\Delta | \Delta \vdash B\} \in N^c(\Gamma)$, by the Induction Hypothesis;
iff $|B|^c \in N^c(\Gamma)$, by definition of $|.|^c$;
iff $\Gamma \vdash \Box B$, by definition of N^c. □

We could now prove that the sequent calculus is sound, and we could adapt our proof of Theorem 10 to prove that the Hilbert system H-MILL is complete – but we are already there. We have the following:

- if $\Gamma \vdash_{\mathsf{H\text{-}MILL}} A$ then $e \models \Gamma^* \multimap A$ (Theorem 6);
- if $e \models \Gamma^* \multimap A$ then $\Gamma \vdash A$ (Theorem 10);
- if $\Gamma \vdash A$ then $\Gamma \vdash_{\mathsf{H\text{-}MILL}} A$ (Theorem 7).

Therefore, the completeness of the Hilbert system and the soundness of the sequent calculus both follow.

Corollary 14. *We have:*

- *if* $e \models \Gamma^* \multimap A$ *then* $\Gamma \vdash_{\mathsf{H\text{-}MILL}} A$*;*
- *if* $\Gamma \vdash A$ *then* $e \models \Gamma^* \multimap A$*.*

5. Adding non-commutativity

Systems that integrate a commutative linear logic with a non-commutative one have been studied in De Groote (1996), Abrusci & Ruet (1999), and Retoré (1997). Note that the purely non-commutative version of intuitionistic linear logic is basically the calculus developed by Lambek (1958) with two order-sensitive implications. The basic propositional logic which we use is that of De Groote (1996), known as partially commutative linear logic (PCL). The main novelty is that the structural rule of exchange no longer holds in general. The context of a sequent is now only partially commutative, and is now built by means of two constructors. Thus, we essentially use *context* as a shorthand for *partially commutative context*. Every formula (to be defined shortly after) is a context. Then, for every context Γ and Δ, we can build their *parallel* (Γ, Δ) composition (primarily commutative context) and their *serial* $(\Gamma; \Delta)$ composition (primarily non-commutative context; see Béchet, de Groote, & Retoré, 1997). A context can thus be seen as a finite tree with non-leaf nodes labelled with ';' or ',' and with leaves labelled with formulas. Two branches emanating from a ',' commute with each other, while the branches emanating from a ';' node do not. We write () for the empty

Table 3. PCMILL: extending the sequent calculus MILL.

Structural rules

$$\frac{\Gamma[\Delta_1,(\Delta_2,\Delta_3)]\vdash A}{\Gamma[(\Delta_1,\Delta_2),\Delta_3]\vdash A}\ ,\mathrm{a1} \qquad \frac{\Gamma[(\Delta_1,\Delta_2),\Delta_3]\vdash A}{\Gamma[\Delta_1,(\Delta_2,\Delta_3)]\vdash A}\ ,\mathrm{a2}$$

$$\frac{\Gamma[\Delta_1;(\Delta_2;\Delta_3)]\vdash A}{\Gamma[(\Delta_1;\Delta_2);\Delta_3]\vdash A}\ ;\mathrm{a1} \qquad \frac{\Gamma[(\Delta_1;\Delta_2);\Delta_3]\vdash A}{\Gamma[\Delta_1;(\Delta_2;\Delta_3)]\vdash A}\ ;\mathrm{a2}$$

$$\frac{\Gamma[\Delta_1,\Delta_2]\vdash A}{\Gamma[\Delta_2,\Delta_1]\vdash A}\ ,\mathrm{com} \qquad \frac{\Gamma[\Delta_1;\Delta_2]\vdash A}{\Gamma[\Delta_1,\Delta_2]\vdash A}\ \mathrm{ent}$$

Non-commutative connectives

$$\frac{\Gamma[A;B]\vdash C}{\Gamma[A\odot B]\vdash C}\ \odot\mathrm{L} \qquad \frac{\Gamma\vdash A \qquad \Gamma'\vdash B}{\Gamma;\Gamma'\vdash A\odot B}\ \odot\mathrm{R}$$

$$\frac{\Gamma\vdash A \qquad \Delta[B]\vdash C}{\Delta[\Gamma;A\setminus B]\vdash C}\ \setminus\mathrm{L} \qquad \frac{A;\Gamma\vdash B}{\Gamma\vdash A\setminus B}\ \setminus\mathrm{R}$$

$$\frac{\Gamma\vdash A \qquad \Delta[B]\vdash C}{\Delta[B/A;\Gamma]\vdash C}\ /\mathrm{L} \qquad \frac{\Gamma;A\vdash B}{\Gamma\vdash A/B}\ /\mathrm{R}$$

context, and assume that it acts as the identity element of the parallel and serial composition, that is: $((),\Gamma) = (\Gamma,()) = (();\Gamma) = (\Gamma;()) = \Gamma$. We denote by $\Gamma[-]$ a context 'with a hole', and $\Gamma[\Delta]$ denotes this very context with the 'hole filled' with the context Δ.

The language of PCMILL extends the language of MILL by adding the following operators: the non-commutative tensor noted $\odot$ and the two order-sensitive implications noted $\setminus$ and $/$:

$$A ::= \mathbf{1}|p|A\otimes A|A\&A|A\multimap A|A\odot A|A\setminus A|A/A|\Box A,$$

where $p \in Atom$.

In order to blend together commutative and non-commutative sequences, we have to decide upon an interpretation of commutativity, namely if we view a commutative concurrent process such as $A\otimes B$ as entailing that either direction is allowed (Béchet et al. 1997; De Groote, 1996). That is, parallel composition is weaker than serial composition, so it shall hold that $A\otimes B \multimap A\odot B$. This means that if two resources can be combined with no particular order, then they can be combined sequentially. This choice is reflected by the structural rule of *entropy* (ent) below. The first two lines of Table 3 state the associativity of serial and parallel compositions, while the third line states the commutativity of parallel composition and the entropy principle. The meaning of the two implications can be expressed in terms of *pre-conditions* and *post-conditions*. $A\setminus B$ requires that A occurs *before* (i.e., on the left of) the implication: $A;\ A\setminus B\vdash B$. By contrast, B/A requires that A occurs *after* the implication, accordingly: $B/A;\ A\vdash B$.

5.1. Semantics and completeness

In order to define a class of modal Kripke resource models for PCMILL, we extend the models we have considered in Section 2. We add to a modal Kripke resource model $(M, e, \circ, \geq, N, V)$ an associative, non-commutative operation $\bullet$ such that e is neutral also for $\bullet$. Thus, a Kripke resource model is now specified by $\mathcal{M} = (M, e, \circ, \bullet, \geq, N, V)$. Bifunctoriality is assumed also for $\bullet$: if $m \geq n$, and $m' \geq n'$, then $m \bullet m' \geq n \bullet n'$. Moreover, the entropy principle is captured in the models by means of the following constraint: for all $x, y, x \circ y \geq x \bullet y$. If $\mathcal{M}$ satisfies all these conditions, we call it a *partially commutative modal Kripke resource model*.

The new truth conditions are as follows:

$m \models_{\mathcal{M}} A \odot B$ iff there exist m_1 and m_2 such that $m \geq m_1 \bullet m_2$ and $m_1 \models_{\mathcal{M}} A$ and $m_2 \models_{\mathcal{M}} B$.
$m \models_{\mathcal{M}} A \setminus B$ iff for all $n \in M$, if $n \models_{\mathcal{M}} A$, then $n \bullet m \models_{\mathcal{M}} B$.
$m \models_{\mathcal{M}} B/A$ iff for all $n \in M$, if $n \models_{\mathcal{M}} A$, then $m \bullet n \models_{\mathcal{M}} B$.

Note that, if $m \models A \otimes B$, then by $m_1 \circ m_2 \geq m_1 \bullet m_2$ and heredity, we have that $m \models A \odot B$.

We shall prove soundness and completeness of PCMILL. We start by discussing the partially commutative version of MILL. The soundness of PCMILL w.r.t. the semantics above is just an extension of the induction for the soundness of MILL with the new rules for the non-commutative connectives. For completeness, we need to extend the construction of the canonical model in order to account for the non-commutative structure.

As before, a context can be associated to a unique formula by means of a recursive operation, here '.+'. We adopt the following definitions:

$$()^+ = \mathbf{1};$$
$$(A)^+ = A;$$
$$(\Gamma, \Delta)^+ = (\Gamma^+ \otimes \Delta^+);$$
$$(\Gamma; \Delta)^+ = (\Gamma^+ \odot \Delta^+).$$

Let $\mathcal{M}^c_\bullet = (M^c, e^c, \circ^c, \bullet^c, \geq^c, N^c, V^c)$ such that: $M^c = \{\Gamma | \Gamma$ is a partially commutative context$\}$; $e^c = ()$; $\Gamma \circ^c \Delta = (\Gamma, \Delta)$; $\Gamma \bullet^c \Delta = (\Gamma; \Delta)$; $\Gamma \geq^c \Delta$ iff $\Gamma \vdash \Delta^+$; $\Gamma \in V^c(p)$ iff $\Gamma \vdash p$; $N^c(\Gamma) = \{|A|^c | \Gamma \vdash \Box A\}$.

We will show that $\mathcal{M}^c_\bullet$ is actually a partially commutative modal Kripke resource model. It suffices to adapt and extend the proof of Lemma 12. The important parts are as follows:

(1) e^c is neutral for $\bullet^c$;
(2) associativity of $\bullet^c$;
(3) for all $\Gamma, \Delta \in M^c$: $\Gamma \circ^c \Delta \geq^c \Gamma \bullet^c \Delta$;
(4) if $\Gamma \geq^c \Delta$ and $\Gamma' \geq^c \Delta'$ then $\Gamma \bullet^c \Gamma' \geq^c \Delta \bullet^c \Delta'$;
(5) if $X \in N^c(\Gamma)$ and $\Gamma \geq^c \Delta$ then $X \in N^c(\Delta)$.

We sketch the arguments here. Items 1 and 2 follow from the definition of partially commutative contexts. We look at the case of entropy (item 3) in more detail. By repeated use of (ax), $\otimes$R, and $\odot$R, we can show $(\Gamma; \Delta) \vdash (\Gamma; \Delta)^+$. By (ent), we obtain $(\Gamma, \Delta) \vdash (\Gamma; \Delta)^+$, and apply the definition of $\geq^c$ to have $(\Gamma, \Delta) \geq^c (\Gamma; \Delta)^+$. By definition of $\circ^c$ we have $(\Gamma \circ^c \Delta) \geq^c (\Gamma; \Delta)^+$. Let us define In as the latter inequality. Now, by (ax), we have $(\Gamma; \Delta)^+ \vdash (\Gamma; \Delta)^+$, which by definition of $\geq^c$ is equivalent to $(\Gamma; \Delta)^+ \geq^c (\Gamma; \Delta)$.

By definition of $\bullet^c$, we have $(\Gamma; \Delta)^+ \geq^c (\Gamma \bullet^c \Delta)$. Together with In, we have $(\Gamma \circ^c \Delta) \geq^c (\Gamma \bullet^c \Delta)$.

For item 4, assume $\Gamma \geq^c \Delta$ and $\Gamma' \geq^c \Delta'$, that is by definition of $\geq^c$, $\Gamma \vdash \Delta^+$ and $\Gamma' \vdash \Delta'^+$. By $\odot$R, we have $(\Gamma; \Gamma') \vdash \Delta^+ \odot \Delta'^+$. By the definition of $.^+$, it is the case that $(\Gamma; \Gamma') \vdash (\Delta; \Delta')^+$. Again by definition of $\geq^c$, we have $(\Gamma; \Gamma') \geq^c (\Delta; \Delta')$. Finally by definition of $\bullet^c$ we conclude that $(\Gamma \bullet^c \Gamma') \geq^c (\Delta \bullet^c \Delta')$.

Item 5 is almost identical to the same case in the proof of Lemma 12, but we explicitly adapt it here. The hypotheses are equivalent to $\Gamma \vdash \Box A$ for some A such that $|A|^c = X$ and $\Delta \vdash \Gamma^+$. By repeatedly applying $\otimes$L and $\odot$L to obtain $\Gamma^+ \vdash \Box A$ and by using *cut*, we infer that $\Delta \vdash \Box A$, which is equivalent to $X \in N^c(\Delta)$.

The truth lemma can be checked by routine induction, and thus we can conclude.

Theorem 15. PCMILL *is sound and complete w.r.t. the class of partially commutative modal Kripke resource models.*

6. Resource-sensitive 'bringing-it-about'

We present the (non-normal modal) logic of agency of *bringing-it-about* (Elgesem, 1997; Governatori & Rotolo, 2005), and propose two versions of it in linear logic, coined resource-sensitive 'bringing-it-about' (RSBIAT) and resource-sensitive 'bringing-it-about' with sequences of actions (SRSBIAT). RSBIAT and SRSBIAT are respectively extensions of MILL and PCMILL. In Section 7, we will illustrate the logic by representing a few actions of agents, functions of artefacts, and their interactions.

We apply a specialisation of the minimal modality studied in the previous sections to a modality agency. In fact, for each agent a in a set $\mathcal{A}$, we define a modality E_a, and $\mathsf{E}_a A$ specifies that agent $a \in \mathcal{A}$ brings about A. As previously, to interpret them in a modal Kripke resource frame, we take one neighbourhood function N_a for each agent a that obeys Condition (1) in Definition 2. We have $m \models \mathsf{E}_a A$ iff $||A|| \in N_a(m)$.

6.1. *Bringing-it-about in classical logic*

The four following principles typically constitute the core of logics of agency (Pörn, 1977; Elgesem, 1997; Belnap, Perloff, & Xu, 2001):

(1) If something is brought about, then this something holds.
(2) It is not possible to bring about a tautology.
(3) If an agent brings about two things concomitantly then the agent also brings about the conjunction of these two things.
(4) If two statements are equivalent, then bringing about one is equivalent to bringing about the other.

Briefly, we explain how these principles are captured in classical logic. Item 1 is a principle of success which corresponds to the axiom T: $\mathsf{E}_i A \to A$. Item 2 has been open to some debate, although Chellas (1969; 1992) is essentially the only antagonist, and it corresponds to the axiom $\neg \mathsf{E}_i \top$ (notaut). Item 3 corresponds to the axiom $\mathsf{E}_i A \wedge \mathsf{E}_i B \to \mathsf{E}_i (A \wedge B)$; that is, if i is bringing about A while also bringing about B, then we can deduce that i is bringing about $A \wedge B$. The other way round does not need to be true. Item 4 refers to the concept of bringing about the quality of being a modality, effectively obeying the rule of equivalents: if $\vdash A \leftrightarrow B$ then $\vdash \mathsf{E}_i A \leftrightarrow \mathsf{E}_i B$.

Table 4. Axiom schemata in H-RSBIAT.

all axioms of H-MILL
$\mathsf{E}_a A \multimap A$
$\mathsf{E}_a A \otimes \mathsf{E}_a B \multimap \mathsf{E}_a(A \otimes B)$
$\mathsf{E}_a A \,\&\, \mathsf{E}_a B \multimap \mathsf{E}_a(A \,\&\, B)$

6.2. Resource-sensitive 'bringing-it-about'

We now detail the logic of RSBIAT. We capture the four principles, adapted to the resource-sensitive framework, by means of rules in the sequent calculus (see Table 5).

The principle of item 1 is captured by E_a(refl), which entails the linear version of T: $\mathsf{E}_a A \multimap A$. In our interpretation, it means that if an agent brings about *A*, then *A* affects the environment.

Because of the difference between the unities in linear logic and classical logic, the principle of item 2 requires some attention. In classical logic all tautologies are provably equivalent to the unity $\top$. Say *A* is theorem ($\vdash A$), we have $\vdash A \leftrightarrow \top$. Hence, from the rule of equivalents, and the axiom $\vdash \neg \mathsf{E}_a \top$, which indicates that no agent brings about the tautological constant, one can deduce $\vdash \neg \mathsf{E}_a A$ whenever the formula *A* is a theorem. In linear logic, the unity **1** is *not* provably equivalent to all theorems. Thus, the axiom of 'bringing-it-about' must be changed into an inference rule ($\sim$nec) in RSBIAT: if $\vdash A$, then $\mathsf{E}_a A \vdash \bot$. It is effectively a sort of 'anti-necessitation rule'. So, if a formula is a theorem, if an agent brings it about, then the contradiction is entailed. This amounts to negating $\mathsf{E}_a A$, according to intuitionistic negation, for every tautological formula *A*.

The principle of 'bringing-it-about' for combining actions (item 3 in the list) is open to two interpretations here: a multiplicative one and an additive one. The additive combination means that if there is a choice for agent *a* between bringing about *A* and bringing about *B*, then agent *a* can bring about a choice between *A* and *B*. $\mathsf{E}_a \otimes$ means that if an agent *a* brings about action *A* and brings about action *B* then *a* brings about both actions $A \otimes B$. Moreover, in order to bring about $A \otimes B$, the sum of the resources for *A* and the resources for *B* is required.

Finally, the logics of the minimal modality already satisfy the rule of equivalents for E_a: from $A \vdash B$ and $B \vdash A$ we infer $\mathsf{E}_a A \vdash \mathsf{E}_a B$. This is inherited by RSBIAT, and it is all that is needed to capture the principle of item 4.

We enrich H-MILL, the Hilbert sytem for MILL, to obtain H-RSBIAT which contains the axioms of Table 4.

The definition of deduction in H-RSBIAT extends the definition of deduction in H-MILL (Definition 4) with the following possible rule to consider for the inductive steps:

$$\frac{\begin{array}{c}\mathcal{D}\\ \vdash_{\mathsf{H}} A\end{array}}{\vdash_{\mathsf{H}} \mathsf{E}_a A \multimap \bot}\ (\sim \text{nec}).$$

On the side of the semantics, we propose the following conditions on modal Kripke resource frames $(M, e, \circ, \geq, \{N_a\}, V)$. The rule ($\sim$nec) requires:

$$\text{if } (X \in N_a(w)) \text{ and } (e \in X) \text{ then } (w \in V(\bot)). \tag{2}$$

The rule (E_a(refl)) requires:

$$\text{if } \ X \in N_a(w) \text{ then } w \in X. \tag{3}$$

Table 5. RSBIAT (extends MILL).

$$\frac{\vdash A}{\mathsf{E}_a A \vdash \bot}\sim\text{nec} \qquad \frac{\Gamma \vdash \mathsf{E}_a A \qquad \Gamma \vdash \mathsf{E}_a B}{\Gamma \vdash \mathsf{E}_a(A \,\&\, B)}\,\mathsf{E}_a\&$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \mathsf{E}_a A \vdash B}\,\mathsf{E}_a(\text{refl}) \qquad \frac{\Gamma \vdash \mathsf{E}_a A \qquad \Delta \vdash \mathsf{E}_a B}{\Gamma, \Delta \vdash \mathsf{E}_a(A \otimes B)}\,\mathsf{E}_a\otimes$$

The condition corresponding to the multiplicative version of action combination ($\mathsf{E}_a\otimes$) is the following, where $X \circ Y = \{x \circ y | x \in X \text{ and } y \in Y\}$, and $X^{\uparrow} = \{y | y \geq x \text{ and } x \in X\}$:

$$\text{if } X \in N_a(x) \text{ and } Y \in N_a(y) \text{ , then } (X \circ Y)^{\uparrow} \in N_a(x \circ y). \tag{4}$$

The condition on the frames corresponding to the additive version is the following:

$$\text{if } X \in N_a(x) \text{ and } Y \in N_a(x), \text{ then } X \cap Y \in N_a(x). \tag{5}$$

Next, we introduce a sequent calculus for RSBIAT. The rules of RSBIAT for $\mathsf{E}_a\&$ show that in order to bring about the choice between $A\&B$ is enough to use the resources for one of the two. On the contrary, in order to bring about $A \otimes B$, the sum of the resources for A and the resources for B is required.

We can prove that H-RSBIAT and the sequent calculus for RSBIAT are equivalent.

Proposition 16. *It holds that* $\Gamma \vdash_{\text{H-RSBIAT}} A$ *iff* $\Gamma \vdash A$ *is derivable in* RSBIAT.

Proof. *(Sketch)* The proof is again an induction on the length of derivations. For example, in one direction, axiom $\mathsf{E}_a A \otimes \mathsf{E}_a B \multimap \mathsf{E}_a(A \otimes B)$ is derivable in the sequent calculus by simply applying $\otimes L$ and $\multimap R$ to $\mathsf{E}_a A, \mathsf{E}_a B \vdash \mathsf{E}_a(A \otimes B)$ which has been obtained from axioms by means of $\mathsf{E}_a\otimes$. □

We can now prove the soundness and completeness of RSBIAT.

Theorem 17. RSBIAT *is sound and complete w.r.t. the class of modal Kripke frames that satisfy Conditions (2), (3), (4), and (5).*

Proof. *(Sketch)* We just show two correspondences.

Condition (2) and rule ($\sim$nec). ($\sim$nec) is sound. Assume that for every model, $e \models A$. We need to show that $e \models \mathsf{E}_a A \multimap \bot$. That is, for every x, if $x \models \mathsf{E}_a A$, then x models $\bot$. If $x \models \mathsf{E}_a A$, then by definition, $||A|| \in N_a(x)$. Since A is a theorem, $e \in ||A||$, thus by Condition (2), $x \in V(\bot)$, so $x \models \bot$. For completeness, it suffices to adapt our canonical model construction. Build the canonical model for RSBIAT as in Definition 11 (we now have more valid sequents). Now suppose (1) $X \in N_a^c(\Gamma)$, and (2) $e^c \in X$. By definition of N_a^c and of $|.|^c$, there is A s.t. $|A|^c = X$, (1) $\Gamma \vdash \mathsf{E}_a A$ and (2) $\vdash A$. From (2) and ($\sim$nec), $\mathsf{E}_a A \vdash \bot$. From (1), and the previous, we obtain $\Gamma \vdash \bot$ using *cut*. By definition of V^c, $\Gamma \in V^c(\bot)$. *Condition (4) and rule ($\mathsf{E}_a\otimes$).* ($\mathsf{E}_a\otimes$) is sound. Assume $e \models \Gamma^* \multimap \mathsf{E}_a A$ and $e \models \Delta^* \multimap \mathsf{E}_a B$. Then, for all x that make Γ true, $||A|| \in N_a(x)$, and for all y that make Δ true, $||B|| \in N_a(y)$. By Condition (4), $||A|| \circ ||B|| \in N_a(x \circ y)$, so for any $x \circ y$ that make $(\Gamma, \Delta)^*$ true, $x \circ y \models \mathsf{E}_a(A \otimes B)$. For completeness, suppose $X \in N_a^c(\Gamma)$ and $Y \in N_a^c(\Delta)$. By definition of N_a^c and of $|.|^c$, there is A and B, with $|A|^c = X$ and $|B|^c = Y$ s.t. $\Gamma \vdash \mathsf{E}_a A$ and $\Delta \vdash \mathsf{E}_a B$. By ($\mathsf{E}_a\otimes$), we obtain $\Gamma, \Delta \vdash \mathsf{E}_a(A \otimes B)$ and thus $|A \otimes B|^c \in N_a^c(\Gamma \sqcup \Delta)$ by definition of $|.|^c$. The definition of $\circ^c$ gives us $|A \otimes B|^c \in N_a^c(\Gamma \circ^c \Delta)$. By the Truth Lemma, we have that $||A \otimes B||^{\mathcal{M}^c} \in N_a^c(\Gamma \circ^c \Delta)$. Thus $(X \circ^c Y)^{\uparrow} \in N_a^c(\Gamma \circ^c \Delta)$. □

Therefore, our extensions of Hilbert systems and sequent calculus are also sound and complete w.r.t. the modal Kripke resource frames restricted to the relevant conditions given in this section.

Corollary 18. H-RSBIAT *is sound and complete w.r.t. the class of modal Kripke frames that satisfy Conditions (2), (3), and (4).*

Moreover, RSBIAT enjoys cut elimination.

Theorem 19. *Cut elimination holds for* RSBIAT.

Proof. *(Sketch)* We extend the proof of Theorem (8) by presenting a number of new cases for the cut formula being principal in both premises of the cut rule. The other cases can be treated similarly. The cut formula has been introduced by E_a(re) and ($\sim$nec).

$$\dfrac{\dfrac{A \vdash B \qquad B \vdash A}{\mathsf{E}_a A \vdash \mathsf{E}_a B}\,\mathsf{E}_a(\text{re}) \qquad \dfrac{\vdash B}{\mathsf{E}_a B \vdash \bot}\sim\text{nec}}{\mathsf{E}_a A \vdash \bot}\,cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash B \qquad B \vdash A}{\vdash A}\,\text{cut}}{\mathsf{E}_a A \vdash \bot}\sim\text{nec}$$

The cut formula has been introduced by $\mathsf{E}_a\otimes$ and E_a(refl).

$$\dfrac{\dfrac{\dfrac{\Gamma' \vdash A \qquad \ldots}{\Gamma \vdash \mathsf{E}_a A} \qquad \dfrac{\Delta' \vdash B \qquad \ldots}{\Delta \vdash \mathsf{E}_a B}}{\Gamma, \Delta \vdash \mathsf{E}_a(A \otimes B)} \qquad \dfrac{\dfrac{\Sigma', A \vdash C' \qquad \Sigma'', B \vdash C''}{\Sigma, A \otimes B \vdash C}}{\Sigma, \mathsf{E}_a(A \otimes B) \vdash C}}{\Gamma, \Delta, \Sigma \vdash C}\,\text{cut}$$

It can be reduced by pushing the cut upwards.

$$\dfrac{\dfrac{\dfrac{\Gamma' \vdash A \qquad \Sigma', A \vdash C'}{\Gamma', \Sigma' \vdash C'}\,\text{cut} \qquad \dfrac{\Delta' \vdash B \qquad \Sigma'', B \vdash C''}{\Delta', \Sigma'' \vdash C''}\,\text{cut}}{\ldots}\,\ldots}{\Gamma, \Delta, \Sigma \vdash C}\,\ldots$$

□

Once again, it is easy to see that cut elimination entails the subformula property for RSBIAT. Using the same arguments as for Theorem 9, it is clear that we can decide polynomial space whether a sequent is valid in RSBIAT.

Theorem 20. *The proof search complexity for* RSBIAT *is in PSPACE.*

6.3. RSBIAT *with sequences of actions*

So far, we have discussed how to control weakening and contraction in order to provide a resource-sensitive account of agency. There is one important structural rule that we did not discuss, namely the exchange rule. In this section, we extend RSBIAT by introducing the *non-commutative* multiplicative conjunction $\odot$, and its two associated order-sensitive implications.

The significance of this move goes beyond the technical aspect, which is rather straightforward at this point. Indeed, a recurring point of contention against the logics of ‘bringing-it-about’ is the absence of a basic notion of time. The non-commutative composition of formulas will provide us with an immediate and natural way of at least talking about sequences of actions, if not about time proper. Reading $A \odot B$ as ‘first A then B’, we will also read $(\mathsf{E}_a A) \odot (\mathsf{E}_b B)$ as ‘first a brings about A then b brings about B’.

We obtain SRSBIAT by adding the non-commutative versions of $\mathsf{E}_a\otimes$ and rephrasing the commutative rules by means of the context notation. The rule E_a(refl) can now oper-

Table 6. Resource 'bringing-it-about' with sequences of actions (extends PCL and H-RSBIAT).

$$\frac{\Gamma[A] \vdash B}{\Gamma[\mathsf{E}_a A] \vdash B}\ \mathsf{E}_a(\text{refl}) \qquad \frac{\Gamma \vdash \mathsf{E}_a A \qquad \Delta \vdash \mathsf{E}_a B}{\Gamma; \Delta \vdash \mathsf{E}_a(A \odot B)}\ \mathsf{E}_a\odot$$

ate both in commutative and non-commutative contexts. The new rules are presented in Table 6.

Note that the presentation of $\sim$nec, $\mathsf{E}_a\otimes$, and $\mathsf{E}_a\&$ is not affected by the generalisation to partially commutative. E_a(refl) states that we can introduce the modality also in non-commutative contexts. $\mathsf{E}_a\odot$ adapts the principle of composition of actions (cf. item 3 Section 6.1) to sequences; it states that we can compose two ordered actions into one sequence of actions.

In order to offer a semantics to our extension of RSBIAT with sequences of actions, we need to add a condition on the neighbourhood functions that deals with the non-commutative operator. Specifically, we need the following condition:

$$\text{if } X \in N_a(x) \text{ and } Y \in N_a(y)\text{ , then } (X \bullet Y)^{\uparrow} \in N_a(x \bullet y). \tag{6}$$

We obtain naturally the semantic determination.

Theorem 21. SRSBIAT *is sound and complete w.r.t. the class of partially commutative modal Kripke resource models that satisfy Conditions (2), (3), (4), (5), and (6).*

7. Application: manipulation of artefacts

7.1. Artefacts

Our application lies in the reasoning about artefact function and tool use. By endorsing what we may call an *agential* stance, we view artefacts as special kind of agents. They are characterised by the fact that they are designed by some other agent in order to achieve a purpose in a particular environment. An important aspect of the modelling of artefacts is their interaction with the environment and with the agents that *use* the artefact to achieve a specific goal (Borgo & Vieu, 2009; Garbacz, 2004; Houkes & Vermaas, 2010; Kroes, 2012). Briefly, we can view an artefact as an object that in the presence of a number of preconditions $c_1, \ldots, c_n$ produces the outcome o. In this work, we want to represent the function of artefacts by means of logical formulas and to view the correct behaviour of an artefact by means of a form of reasoning. When reasoning about artefacts and their outcomes, we need to be careful in making all the conditions of use of the artefact explicit, otherwise we end up facing the following unintuitive cases. Imagine we represent the behaviour of a screwdriver as a formula of classical logic that states that if there is a screw S, then we can tighten it T. We simply describe the behaviour of the artefact as a material implication $S \to T$. In classical logic, we can infer that by means of a single screwdriver we can tighten two screws: $S, S, S \to T \vdash T \wedge T$. Worse, we do not even need to have two screws to begin with: $S, S \to T \vdash T \wedge T$. Thus, without specifying all the relevant constraints on the environment (e.g., that a screwdriver can only handle one screw at the time) we end up with unintuitive results. Another possible drawback of classical logic is that it is commutative, i.e., the order of formulas does not matter. For example, if we describe the process of hammering a nail by means of the implication if I place a nail N and I provide the right force F, then I can drive a nail in (D), that is $N \wedge F \to D$, that would also entail that one can put a force before placing the nail.

Moreover, we need to specify the relationship between the artefact and the agents: for example, there are artefacts that can only be used by one agent at a time. Since a crucial point in modelling artefacts is their interaction with the environment, either we carefully list all the relevant conditions or we need to change the logical framework that we use to represent the artefact's behaviour. In this paper, we propose to pursue this second strategy. Our motivation is that, instead of specifying for each artefact the precondition of its application (e.g., that there is only one screw that a screwdriver is supposed to operate on), the logical language that encodes the behaviour of the artefact already takes care of preventing unintuitive outcomes. Thus, the formulas of linear logic shall represent actions of agents and functions of artefacts, and the non-normal modality shall specify which agent or artefact brings about which process.

7.2. Functions

The concept of a *function* of an artefact aims to capture the description of the behaviour of an artefact in an environment with respect to its goals: artefacts are not living things but have a purpose, attributed by a designer or a user (Borgo & Vieu, 2009; Kroes, 2012). We model a function of an artefact by means of a formula A in RSBIAT or $\mathsf{SRSBIAT}$. If A is a function of an artefact t, then one can represent t's behaviour as $\mathsf{E}_t A$ (t brings about A) in a conceptually consistent manner, namely an artefact brings about its function A.

With linear logic, we are equipped with a formalism to represent and reason about processes and resources. In classical and intuitionistic logic, if one has A and A implies B, then one has B, but A still holds. This is fine for mathematical reasoning but often fails to be acceptable in the real world where implication is *causal*. Girard (1989, p. 72) remarks that a 'causal implication cannot be iterated since the conditions are modified after its use; this process of modification of the premises (conditions) is known in physics as *reaction*'. That is, linear logic allows for modelling how the function of an artefact can actually be realised in a certain environment. At an abstract level, an artefact can be seen as an agent t. It takes resource-sensitive actions by reacting to the environment. For any artefact t with function A, $\mathsf{E}_t A$, we say that t accomplishes a certain goal O in the environment Γ if and only if the sequent $\Gamma[\mathsf{E}_t A] \vdash O$ is provable. The context Γ describes a number of preconditions that specify the environment resources as well as the actions of the agents that are interacting with the artefact. The proof of $\Gamma[\mathsf{E}_t A] \vdash O$ exhibits the execution of the function of t in the environment Γ: when t is an artefact, and $\Gamma[\mathsf{E}_t A] \vdash O$ is provable, then the occurrence of A in the proof of $\Gamma[\mathsf{E}_t A] \vdash O$ is the concrete instantiation of the function of t in Γ. Since we are using intuitionistic versions of linear logic, every proof of a sequent is a process that behaves like a function in the mathematical sense.[4] Hence, the rules of sequent calculus provide instructions to compose basic functions of artefacts to obtain complex functions and to model how the composition interacts with the environment. We will see examples of complex functions in the next paragraphs. For that reason, our view of artefacts simply generalises to a number of artefacts that interact in an environment as follows:

$$\Gamma[\mathsf{E}_1 A_1, \ldots, \mathsf{E}_m A_m] \vdash O.$$

Again, if the above sequent is provable, then the combination of artefacts $\mathsf{E}_1 A_1$, …, $\mathsf{E}_m A_m$ can achieve the goal O in Γ by executing their functions $A_1, \ldots, A_m$ in Γ.

Defining the function of an artefact as a formula demands some care because in this way functions do not have a unique formulation. The functions $(A \otimes B) \multimap C$, and $A \multimap (B \multimap C)$ are provably equivalent. However, the rule E_a(re) ensures that bringing about a function is provably equivalent to bringing about any of itsequivalent forms.

By means of sequent calculus provability, we can view the problem of using artefacts in an environment to achieve a goal as a decision problem that is related to the AI problem of planning (Kanovich & Vauzeilles, 2001). Note that the complexity of deciding whether a goal is achievable depends only on the fragment of the logic that we use to model the formulas in the sequent. In the next paragraphs, we shall instantiate the descriptive features of our calculus by means of a number of toy examples.

7.3. Simple examples of functions

To take a very simple example, we can represent the function of a screwdriver s as an implication that states that if there is a screw (formula S) and some agent brings about the right rotational force (F), then the screw gets tightened (T). The formula corresponding to the function of the screwdriver is $S \otimes F \multimap T$. The formula that captures the screwdriver as an agent of the system is $\mathsf{E}_s(S \otimes F \multimap T)$.

Suppose the environment provides S and an agent i is providing the right force $\mathsf{E}_i F$. We can show by means of the following proof in RSBIAT that the goal T can be achieved:

$$\dfrac{\dfrac{\dfrac{S \vdash S \qquad \dfrac{F \vdash F}{\mathsf{E}_i F \vdash F}\,\mathsf{E}_i(\text{refl})}{S, \mathsf{E}_i F \vdash S \otimes F}\,\otimes R \qquad T \vdash T}{S, \mathsf{E}_i F, S \otimes F \multimap T \vdash T}\,\multimap \mathrm{L}}{S, \mathsf{E}_i F, \mathsf{E}_s(S \otimes F \multimap T) \vdash T}\,\mathsf{E}_s(\text{refl})$$

Our calculus is resource sensitive, thus, as expected, we cannot infer for example that two agents can use the same screwdriver at the same time to tighten two screws, as this would in fact require two screwdrivers and thus an extra $\mathsf{E}_s(S \otimes F \multimap T)$ at the left of the sequent:

$$S, S, \mathsf{E}_i F, \mathsf{E}_j F, \mathsf{E}_s(S \otimes F \multimap T) \nvdash T \otimes T.$$

Often, to be effective for some goal B, an artefact's function transforming a resource A into a resource B should not be realised before the resource A is available.

In the case of our example, the description of a screwdriver should exclude that the screw can be tightened *before* a loose screw and a rotational force, in this order, are provided. Thus, we may reconsider our screwdriver as a 'non-commutative' screwdriver $s\bullet$ and write its function as $S \odot F \setminus T$. The screwdriver is now defined as $\mathsf{E}_{s\bullet}(S \odot F \setminus T)$:

$$\dfrac{\dfrac{\dfrac{S \vdash S \qquad \dfrac{F \vdash F}{\mathsf{E}_i F \vdash F}\,\mathsf{E}_i(\text{refl})}{S; \mathsf{E}_i F \vdash S \odot F}\,\odot R \qquad T \vdash T}{(S; \mathsf{E}_i F); S \odot F \setminus T \vdash T}\,\setminus\mathrm{L}}{S; \mathsf{E}_i F; \mathsf{E}_{s\bullet}(S \odot F \setminus T) \vdash T}\,\mathsf{E}_{s\bullet}(\text{refl})$$

The meaning of entropy (ent) is the following. By means of (ent), we can infer a fully commutative context:

$$S, \mathsf{E}_i F, \mathsf{E}_{s\bullet}(S \odot F \setminus T) \vdash T.$$

This means that it is the description of the function of the artefact which takes care of specifying how the resources have to be ordered. Of course, our screwdriver formula correctly excludes, for instance, that the force is applied after using the screwdriver: $S; \mathsf{E}_{s\bullet}(S \odot F \setminus T); \mathsf{E}_i F \nvdash T$. Moreover, $\mathsf{E}_i F; \mathsf{E}_{s\bullet}(S \odot F \setminus T); S \nvdash T$, and $\mathsf{E}_i F; S; \mathsf{E}_{s\bullet}(S \odot F \setminus T) \nvdash T$.

The interaction of commutative and non-commutative operators is exemplified as follows. Suppose there are two screws S, S, two screwdrivers s and s' and two agents a, b. The goal of tightening two screws can be achieved by using the screwdrivers in whatever order, as the following proof shows:

$$\dfrac{\dfrac{\dfrac{\dfrac{S \vdash S \qquad \dfrac{F \vdash F}{\mathsf{E}_a F \vdash F}\,\mathsf{E}_a(\text{refl})}{S; \mathsf{E}_a F \vdash S \odot F}\,\odot R \qquad T \vdash T}{(S; \mathsf{E}_a F); S \odot F \setminus T \vdash T}\,\setminus\text{L}}{S; \mathsf{E}_a F; \mathsf{E}_s(S \odot F \setminus T) \vdash T}\,\mathsf{E}_s(\text{refl}) \qquad \dfrac{\dfrac{\dfrac{S \vdash S \qquad \dfrac{F \vdash F}{\mathsf{E}_b F \vdash F}\,\mathsf{E}_b(\text{refl})}{S; \mathsf{E}_b F \vdash S \odot F}\,\odot R \qquad T \vdash T}{(S; \mathsf{E}_b F); S \odot F \setminus T \vdash T}\,\setminus\text{L}}{S; \mathsf{E}_b F; \mathsf{E}_{s'}(S \odot F \setminus T) \vdash T}\,\mathsf{E}_{s'}(\text{refl})}{[S; \mathsf{E}_a F; \mathsf{E}_s(S \odot F \setminus T)], [S; \mathsf{E}_b F; \mathsf{E}_{s'}(S \odot F \setminus T)] \vdash T \otimes T}\,\otimes\text{R}$$

7.4. Function composition

By extending the previous example, we can demonstrate how the output of some artefact's function can naturally be fed into another function so as to construct a new complex artefact.

An *electric screwdriver* has two components. Firstly, the *power-pistol* creates some rotational force F when the button is pushed (P): $P \setminus F$. Secondly, what is typically called the screwdriver *bit* is for all intents and purposes effectively a screwdriver as specified before: it tightens a loose screw when a rotational force is applied. We define the electric screwdriver by means of $\mathsf{E}_e((P \setminus F) \odot (S \odot F \setminus T))$

Now suppose the environment provides a loose screw S and an agent i is pushing the button of the power pistol: $\mathsf{E}_i P$. We can show again that the goal T of having a screw tightened can be achieved, this time by using the electric screwdriver:

$$\dfrac{\dfrac{\dfrac{\dfrac{S \vdash S \qquad \dfrac{\dfrac{P \vdash P}{\mathsf{E}_i P \vdash P}\,\mathsf{E}_i(\text{refl}) \qquad F \vdash F}{\mathsf{E}_i P; P \setminus F \vdash F}\,\setminus\text{L}}{S; \mathsf{E}_i P; P \setminus F \vdash S \odot F}\,\odot R \qquad T \vdash T}{S; \mathsf{E}_i P; P \setminus F, S \odot F \setminus T \vdash T}\,\setminus\text{L}}{S; \mathsf{E}_i P; (P \setminus F) \odot (S \odot F \setminus T) \vdash T}\,\odot\text{L}}{S; \mathsf{E}_i P; \mathsf{E}_e((P \setminus F) \odot (S \odot F \setminus T)) \vdash T}\,\mathsf{E}_e(\text{refl})$$

7.5. Complex interactions between agents and artefacts

The function of an artefact may require specification as to how agents use it. A number of aspects of the interaction between agents' actions and artefacts can be captured by means of the rules $\mathsf{E}_i\otimes$, $\mathsf{E}_i\odot$, and $\mathsf{E}_i\&$. Recall that $\mathcal{A}$ is a set of agents. We write $\&_{x\in\mathcal{A}} \mathsf{E}_x A$ as a shorthand for $\mathsf{E}_{i_1} A \& \ldots \& \mathsf{E}_{i_m} A$. The latter formula means that any agent can perform A, so for example $\mathsf{E}_{i_1} A \& \ldots \& \mathsf{E}_{i_m} A \vdash \mathsf{E}_{i_j} A$.

An artefact that is defined by $\mathsf{E}_t(\&_{x\in\mathcal{A}}(\mathsf{E}_x(A \otimes B) \multimap O))$ requires the *same* agent x to perform both actions A and B in order to get O. As an example, let us consider a single-person rowing boat which requires a single agent to operate on both oars (R_1) and (R_2), in whichever order, so as to produce movement (M). This is can be modelled by means of our $\mathsf{E}_i\otimes$ rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{E}_i R_1 \vdash \mathsf{E}_i R_1 \qquad \mathsf{E}_i R_2 \vdash \mathsf{E}_i R_2}{\mathsf{E}_i R_1, \mathsf{E}_i R_2 \vdash \mathsf{E}_i(R_1 \otimes R_2)}\,\mathsf{E}_i\otimes \qquad M \vdash M}{\mathsf{E}_i R_1, \mathsf{E}_i R_2, \mathsf{E}_i((R_1 \otimes R_2) \multimap M) \vdash M}\,\multimap\text{L}}{\mathsf{E}_i R_1, \mathsf{E}_i R_2, \&_{x\in\mathcal{A}} \mathsf{E}_i((R_1 \otimes R_2) \multimap M) \vdash M}\,\&L\text{enough times}}{\mathsf{E}_i R_1, \mathsf{E}_i R_2, \mathsf{E}_t(\&_{x\in\mathcal{A}} \mathsf{E}_i((R_1 \otimes R_2) \multimap M)) \vdash M}\,\mathsf{E}_t(\text{refl})$$

On the other hand, by specifying the function by $\mathsf{E}_t(\&_{x,y\in\mathcal{A},x\neq y}(\mathsf{E}_x A \otimes \mathsf{E}_y B) \multimap O)$, we are forcing the agents who operate tool t to be different (e.g., a crosscut saw). If an artefact's function does not determine whether the actions must be performed by the same agent, we can write $\mathsf{E}_t(\&_{x,y\in\mathcal{A}}(\mathsf{E}_x A \otimes \mathsf{E}_y B) \multimap O)$.

In the non-commutative case, $\mathsf{E}_t(\&_{x\in\mathcal{A}} \mathsf{E}_x(A \odot B) \multimap O)$ forces the same agent to perform first A and then B, whereas $\mathsf{E}_t(\&_{x,y\in\mathcal{A},x\neq y}(\mathsf{E}_x A \odot \mathsf{E}_y B) \multimap O)$ forces the agents to be different. For example, the function of a hanging ladder is described as follows: first, one agent holds the ladder (Ho), then another agent climbs up the ladder (Cl) and reaches a certain position R: $\mathsf{E}_t(\mathsf{E}_a Ho \odot \mathsf{E}_b Cl \setminus \mathsf{E}_b R)$:

$$\dfrac{\dfrac{\dfrac{\mathsf{E}_a Ho \vdash \mathsf{E}_a Ho \qquad \mathsf{E}_b Cl \vdash \mathsf{E}_b Cl}{\mathsf{E}_a Ho; \mathsf{E}_b Cl \vdash \mathsf{E}_a Ho \odot \mathsf{E}_b Cl}\odot\mathrm{R} \qquad \mathsf{E}_b R \vdash \mathsf{E}_b R}{\mathsf{E}_a Ho; \mathsf{E}_b Cl; \mathsf{E}_a Ho \odot \mathsf{E}_b Cl \setminus \mathsf{E}_b R \vdash \mathsf{E}_b R}\setminus\mathrm{L}}{\mathsf{E}_a Ho; \mathsf{E}_b Cl; \mathsf{E}_t(\mathsf{E}_a Ho \odot \mathsf{E}_b Cl \setminus \mathsf{E}_b R) \vdash \mathsf{E}_b R}\mathsf{E}_t(\mathrm{refl})$$

By means of $\mathsf{E}_i\&$, we can describe a function which requires an agent's choice. For example, a monkey wrench can tighten two sizes of nuts/bolts (N_1, N_2) provided that an agent chooses the right measure (M_1, M_2): $\mathsf{E}_t((\mathsf{E}_i(M_1\&M_2) \multimap \mathsf{E}_i N_1)\&(\mathsf{E}_i(M_1\&N_2) \multimap \mathsf{E}_i N_2))$. The following proof shows that if a single agent can choose the right measure for the nut (e.g., M_1), then the same agent can tighten the right type of nut (e.g., $\mathsf{E}_i N_1$):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{E}_i M_1 \vdash \mathsf{E}_i M_1}{\mathsf{E}_i M_1 \,\&\, \mathsf{E}_i M_2 \vdash \mathsf{E}_i M_1}\&L \qquad \dfrac{\mathsf{E}_i M_2 \vdash \mathsf{E}_i M_2}{\mathsf{E}_i M_1 \,\&\, \mathsf{E}_i M_2 \vdash \mathsf{E}_i M_2}}{\mathsf{E}_i M_1\&\mathsf{E}_i M_2 \vdash \mathsf{E}_i(M_1\&M_2)}\begin{matrix}\&\mathrm{R}\\ \mathsf{E}_i\&\end{matrix} \qquad \mathsf{E}_i N_1 \vdash \mathsf{E}_i N_1}{\mathsf{E}_i M_1\&\mathsf{E}_i M_2, \mathsf{E}_i(M_1\&M_2) \multimap \mathsf{E}_i N_1 \vdash \mathsf{E}_i N_1}\multimap\mathrm{L}}{\mathsf{E}_i M_1\&\mathsf{E}_i M_2, (\mathsf{E}_i(M_1\&M_2) \multimap \mathsf{E}_i N_1)\&(\mathsf{E}_i(M_1\&M_2) \multimap \mathsf{E}_i N_2) \vdash \mathsf{E}_i N_1}\&\mathrm{L}}{\mathsf{E}_i M_1\&\mathsf{E}_i M_2, \mathsf{E}_t((\mathsf{E}_i(M_1\&M_2) \multimap \mathsf{E}_i N_1)\&(\mathsf{E}_i(M_1\&M_2) \multimap \mathsf{E}_i N_2)) \vdash \mathsf{E}_i N_1}\mathsf{E}_t(\mathrm{refl})$$

In a similar way, we can represent functions of artefacts that require any number of actions and agents to achieve a goal (of course, if we want to express that any subsets of $\mathcal{A}$ can operate the tool, then we need an exponentially-long formula).

7.6. Function warranty and reuse of artefacts

RSBIAT is resource sensitive, as the non-provable sequent in our screwdriver example illustrated (see section 7.3):

$$S, S, \mathsf{E}_i F, \mathsf{E}_j F, \mathsf{E}_s(S \otimes F \multimap T) \nvdash T \otimes T.$$

The screwdriver *cannot be reused*, despite the fact that an additional screw is available and an appropriate force is brought about. This is perfectly fine as long as our interpretation of resource consumption is *concurrent*: all resources are consumed at once. And indeed, one cannot tighten two screws at once with only one screwdriver.

Abandoning a concurrent interpretation of resource consumption, we may specialise the modality E_a when a is an artefactual agent in such a way that the function of an artefact can be used at will. After all, using a screwdriver once does not destroy the screwdriver. Its function is still present after a single use. It seems that we are after a property of *contraction* for our operator E_s:

$$\dfrac{\Gamma, \mathsf{E}_s A, \mathsf{E}_s A \vdash B}{\Gamma, \mathsf{E}_s A \vdash B}\, c(\mathsf{E}_s)$$

Now, if we adopt the rule c(E_s), we can easily see that the following is indeed provable:

$$S, S, \mathsf{E}_i F, \mathsf{E}_j F, \mathsf{E}_s(S \otimes F \multimap T) \vdash T \otimes T.$$

There are several issues with this solution to 'reuse' as a duplication of assumptions, some of which are technical and some of which are conceptual. The main technical issue is that we lose a lot of control on the proof search, as contraction is the main source of non-termination (of bottom-up proof search). Another technical (or theoretical) issue is that trying to give a natural condition on our frames that would be canonical for contraction is out of question. The conceptual issue is the same as the one posed by Girard (1987) in creating linear logic: duplication of assumptions should not be automatic. Similarly, *ad lib* reuse of an artefact does not reflect a commonsensical experience. In general, although they are not used up after the first use, tools will nonetheless eventually become so worn out that they will not realise their original function. The point is that, in order to keep track of the relevant resources, the reuse of artefacts should not be arbitrary and should be allowed in a controlled manner instead.

We can capitalise on the 'additive' feature of linear logic language: employing the 'with' operator &, we can specify a sort of warranty of artefact functions. Denote $A^n = A \odot \cdots \odot A$, for n times. We present the treatment by focusing on our example of screwdriver. A sequentially reusable screwdriver is defined as follows:

$$(S \odot F \setminus T)^{\leq n} = (S \odot F \setminus T) \& (S^2 \odot F^2 \setminus T^2) \& \ldots \& (S^n \odot F^n \setminus T^n).$$

For example, with three screws, and three agents (a, b, and c) providing the appropriate force, then using a decently robust screwdriver, one can obtain three tightened screws:

$$S;\, S;\, S;\, \mathsf{E}_a F;\, \mathsf{E}_b F;\, \mathsf{E}_c F;\, \mathsf{E}_{s\bullet}((S \odot F \setminus T)^{\leq 10{,}000}) \vdash T \odot T \odot T,$$

where $s\bullet$ is our 'non-commutative' screwdriver, now with a 10,000-use warranty:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{S \vdash S \qquad S \vdash S}{S; S \vdash S \odot S} \qquad S \vdash S}{S; S; S \vdash S \odot S \odot S} \qquad \cfrac{\cfrac{\mathsf{E}_a F \vdash F \qquad \mathsf{E}_b F \vdash F}{\mathsf{E}_a F; \mathsf{E}_b F \vdash \mathsf{E}_a F \odot \mathsf{E}_b F} \qquad \mathsf{E}_c F \vdash F}{\mathsf{E}_a F; \mathsf{E}_b F; \mathsf{E}_c F \vdash F \odot F \odot F}}{S; S; S; \mathsf{E}_a F; \mathsf{E}_b F; \mathsf{E}_c F \vdash S \odot S \odot S \odot F \odot F \odot F} \qquad T^3 \vdash T^3}{S; S; S; \mathsf{E}_a F; \mathsf{E}_b F; \mathsf{E}_c F; S^3 \odot F^3 \setminus T^3 \vdash T^3}\,\&L}{S; S; S; \mathsf{E}_a F; \mathsf{E}_b F; \mathsf{E}_c F; (S \odot F \setminus T)^{\leq 10000} \vdash T^3}\,\mathsf{E}_s(\mathrm{re})}{S; S; S; \mathsf{E}_a F; \mathsf{E}_b F; \mathsf{E}_c F; \mathsf{E}_s((S \odot F \setminus T)^{\leq 10000}) \vdash T^3}$$

Note that, the goal $T \otimes T \otimes T$ is not provable, and this reflects our view on reusability as a sequential operation.

By pushing this analogy of warranty of artefact functions further, we can model a 'refurbishing' function that augments the warranty of the function A of a tool t. For instance, consider the refurbishing function which at the cost of consuming a resource R, transforms a worn out (but not too worn out!) screwdriver t into a screwdriver t with a function with extended warranty. It can be written as:

$$R \otimes \mathsf{E}_s((S \odot F \setminus T)^{\leq 50}) \multimap \mathsf{E}_s((S \odot F \setminus T)^{\leq 7000}),$$

and is for example a function that a bench grinder would have.

8. Conclusion

The semantics of all the languages studied in this paper are adequate extensions of Urquhart's 1972 Kripke resource models for intuitionistic substructural logic. We first enriched the Kripke resource models with a neighbourhood function to give a meaning to a minimal (non-normal) modality and obtain what we simply coin modal Kripke resource models. We thus defined and studied a minimal non-normal modal logic. The non-normal minimal modality $\Box$ is defined in the usual way where $\Box A$ holds at a point of evaluation iff the extension of A is in the neighbourhood of the point of evaluation. With Condition (1) on the models we enforced a sort of heredity (or monotonicity) on the neighbourhood function, without which the logic would not even validate *modus ponens*.

We introduced a Hilbert system and a sequent calculus. We showed that both are sound and complete with respect to the class of modal Kripke resource models. Moreover, we showed that the sequent calculus admits cut elimination. We used this fact to establish that proof search can be done in PSPACE. The soundness and completeness of the sequent calculus, and the cut elimination, show that the rule $\Box$(re)

$$\frac{A \vdash B \qquad B \vdash A}{\Box A \vdash \Box B},$$

which is atypical in a sequent calculus, yields the expected results and presents no logical issue. Moreover, the new semantics and calculi are general enough to extend the framework, as we did, to account for partially commutative linear logic.

The significance of non-normal modal logics and their semantics in modern developments in logics of agents has been emphasised before in the literature. In classical logic, logics of agency in particular have been widely studied and used in practical philosophy and multi-agent systems. Moving from classical logic to resource-sensitive logics allows us to lift the study of agents bringing about states of affairs to the study of agents bringing about resources, or of artefacts bringing about resource transformations.

We thus instantiated the minimal modal logics (commutative and partially commutative) with a resource-sensitive version of the logics of 'bringing-it-about'. Again, a sound and complete Hilbert system and sequent calculus are provided. Proof-search in the commutative version is shown to be PSPACE-easy. We finally presented a number of applications of the resulting resource-conscious logics of agency to reason about the resource-sensitive manipulations of technical artefacts.

The perspectives for future research are extensions of our treatment to further non-normal modalities for logics of Belief-Desire-Intention (BDI) agents. For instance, introducing resource-sensitive operators of beliefs will enable us to model agents' beliefs that depend on the amount of available information. Moreover, we are particularly interested in a resource-sensitive view of the strategic power of agents and coalitions, where the social interactions are mediated by the available resources.

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Notes

1. When no confusion can arise we will write $||A||$ instead of $||A||^{\mathcal{M}}$, and $m \models A$ instead of $m \models_{\mathcal{M}} A$.
2. An analogous yet less transparent condition was used in D'Agostino et al. (1997) for a normal modality.
3. Remember that Proposition 3 holds for the modal language $\mathcal{L}_{\mathsf{MILL}}$ because of Condition (1) on modal Kripke resource models.
4. This suggestion can be made mathematically precise. It is possible to associate terms in the (linear) λ-calculus to proofs in intuitionistic linear logic (Benton, Bierman, De Paiva, & Hyland 1993).

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