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Single-peakedness and semantic dimensions of preferences

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Abstract

Among the possible solutions to the paradoxes of collective preferences, single-peakedness is significant because it has been associated to a suggestive conceptual interpretation: a single-peaked preference profile entails that, although individuals may disagree on which option is the best, they conceptualize the choice along a shared unique dimension, i.e. they agree on the rationale of the collective decision. In this article, we discuss the relationship between the structural property of single-peakedness and its suggested interpretation as uni-dimensionality of a social choice. In particular, we offer a formalization of the relationship between single-peakedness and its conceptual counterpart, we discuss their logical relations, and we question whether single-peakedness provides a rationale for collective choices.

Keywords: Social choice theory, single-peakedness, deliberation, semantic dimensions, formal dimensions.

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1 Introduction

In social choice theory, Arrow's theorem shows that it is not possible to aggregate individual rational preferences by means of an aggregation procedure that satisfies some reasonable fairness conditions and always returns rational preferences. For instance, the famous Condorcet's paradox shows that there exist profiles of individually rational preferences that return, by majority voting, irrational collective outcomes. Collective irrationality jeopardizes the normative appeal of the aggregative view of democracy endorsed by social choice theory [1, 6, 18] that maintains that individual preferences or values are not subject to normative judgement; the normativity of collective decisions rests solely on the features of the aggregation procedure. Among the well known escape routes to Arrow's impossibility result, Black's solution provides a restriction of the class of all individual preference profiles to *single-peaked profiles* [1, 3]; this restriction is sufficient to guarantee rational outcomes for any possible single-peaked profile. This solution is significant because it has been associated with an apparently convincing interpretation: it amounts to assuming that individuals share a common criterion for voting, a common *dimension* that structures the choice at issue.

This work starts from the observation that there is in fact a huge gap between the conceptual notion of a dimension, according to which decision problems are conceptualized by individuals, and the formal condition of single-peakedness that, as we shall see, merely states a structural property of preference profiles. In this article, following [5], we shall distinguish between a formal notion of a dimension, from now on *formal dimension*, that is the one that appears in the formal definition of single-peakedness and it is merely a linear order of the alternatives, and a semantic notion of a dimension, from now on *semantic dimension*, that is the criterion that agents use to rank the alternatives. Accordingly, we distinguish between *single-peakedness*, the structural property of preference profiles, and *uni-dimensionality*, that is intended as the conceptual counterpart of single-peakedness. For instance, in case of a parliamentary election, voters may conceptualize their choices

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for certain parties in terms of the left-right distinction—thus they appeal to the semantic dimensions of being a left or right party—which corresponds, roughly, to a ranking of the parties along an abstract left-right axis, which provides the corresponding formal dimension.

The motivation for investigating the relationships between single-peakedness and uni-dimensionality are twofold. First, it is important to understand whether uni-dimensionality entails single-peakedness. This entailment has been discussed in particular in [5, 14, 15] in comparison with the deliberative view of democracy [6–8, 11]. The reason is that, as it has been argued in particular in [5], a deliberation that precedes voting is valuable because it is in principle capable of inducing individuals to revise their preferences in such a way to bring about a single-peaked profile. If this is correct, deliberation is indeed a valuable solution to circumvent the paradoxes of collective decisions. The way in which deliberation, via discussion, is supposed to induce preference revision is by making individuals focus on a shared conceptualization of the decision problem at issue, that is by bringing about uni-dimensionality [5]. If deliberation is indeed capable of bringing about a common semantic dimension, then in order to assess whether deliberation can induce single-peaked preferences, it is critical to understand whether uni-dimensionality entails single-peakedness.

Secondly, it is important to assess whether single-peakedness entails uni-dimensionality. The reasons is that any arbitrary or coercive restriction of individual preferences would contradict the principles of an aggregative view of democracy, by arbitrarily excluding possible (rational) individual views. Therefore, the restriction to single-peaked profile has to be at least in principle justified and motivated by appealing to something that is shared among the individuals. Note that it is not satisfactory to claim that single-peakedness is justified by the sole ground that it guarantees rational collective outcomes: many severe restriction of individual preferences can be justified by the same argument. The main independent justification of single-peakedness is in fact its conceptual interpretation, i.e. that it provides individuals with a common criterion that rationalizes the social choice.

The conceptual relations between single-peakedness and uni-dimensionality have been studied in particular in [5, 15] by focusing on the question whether uni-dimensionality entails single-peakedness and whether deliberation is capable of bringing about uni-dimensionality. In this article, I present a formal account of the notion of semantic dimension, I present a formalization of the distinction between formal and semantic dimension, and I discuss the logical relations between single-peakedness and uni-dimensionality. In particular, for the present purposes, it suffices to use a fragment of first-order logic to express and reason about individuals' semantic dimensions. The relationship between semantic and formal dimension is understood here in terms of *denotation*: a semantic dimension is a binary relation that denotes a formal dimension.

Two topics are specifically related to the present work. First, in social choice theory, several possibility results that depend on domain restrictions have been proposed [10]. I focus on single-peakedness because it is the domain restriction that has been better motivated with respect to the principles of the aggregative view of democracy. Secondly, in political philosophy, the relationship between preference aggregation and the justifications of collective choices has been discussed in [14] by highlighting possible mismatches between the winning alternatives and the collective reason for electing it. A conceptual analysis of the relationship between single-peakedness and public justifications of social choices has been proposed in [15]. Moreover, it is worth mentioning that the relations between preferences and reasons supporting them has been approached in particular in [4].

The structure of this article is the following. In Section 2, we present the relevant background on single-peaked profiles. In Section 3, we informally exemplify the problems of relating formal and semantic dimensions, as well as single-peakedness and uni-dimensionality. In Section 4, we introduce a model of semantic dimensions of preference viewed as binary relations on alternatives.

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In Section 5, we present results concerning the relationship between single-peaked profiles and uni-dimensionality. Section 6 concludes.[1]

2 Single-peaked profiles

We assume a (finite) set of individuals $\mathcal{N}$ and a (finite) set of alternative A. For $i \in \mathcal{N}$, a (strict) preference ordering P is an *irreflexive*, *transitive* and *complete* relation $P \subseteq A \times A$. A *preference profile* $\mathbf{P}$ is a list of preference orderings $(P_1, \ldots, P_n)$, where n is the number of individuals. Let $\mathcal{L}(A)$ denote the set of all preference ordering on A. A *social welfare function* is a function $F : \mathcal{L}(A)^n \rightarrow \mathcal{L}(A)$, from the set of preference profiles to preference orderings. For example, the majority rule is defined as follows: $F(\mathbf{P}) = P = \{(x,y) \mid |N_{(x,y)}| > n/2\}$, with $N_{(x,y)} = \{i \in N \mid (x,y) \in P_i\}$.

Arrow's theorem shows that every social welfare function that satisfies certain intuitively reasonable fairness constraints must be a dictatorship, namely it always copies the preference ordering of a certain individual. As a corollary, one can infer that interesting non-dictatorial social welfare functions may fail to associate a (rational) collective preference.

A well-known solution to Arrow's impossibility result is to restrict the class of possible preference profiles. In particular, the restriction of the domain to *single-peaked profiles* has been discussed by Duncan Black [1, 3, 10].

Given a linear order $>$, we say that y *is between* x *and* z iff $x > y > z$ or $z > y > x$. Moreover, we define the peak of preference order $\text{PEAK}(P_i)$ as the maximal element wrt P_i.

Definition 1 (Single-peakedness)
A preference profile is *single-peaked* if and only if there exists a linear order $>_\Omega$ (a formal dimension) such that every P_i is compatible with $>_\Omega$ in the following sense: for every alternative y such that $y \neq \text{PEAK}(P_i)$, i prefers any alternative between $\text{PEAK}(P_i)$ and y to y.

A formal dimension may be viewed as a ranking of the alternatives according to some relevant property, e.g. the cost of implementing a certain policy. Single-peakedness means then that such ranking is respected by every individual. In voting theory, $>_\Omega$ is usually interpreted as a *dimension of voting*, one example being the dimension provided by the left-right ordering of candidates for a parliamentary election [2, 12].

In Figure 1, we present an example of single-peaked profile (on the left) and an example of a preference ordering that violates single-peakedness wrt the formal dimension $a >_\Omega b >_\Omega c >_\Omega d >_\Omega e$ (on the right). In Figure 1, the horizontal axis represents the formal dimension and the vertical axis represents individual preferences ranked according to the positions of the alternatives wrt the formal dimension. The preference ordering on the right is not single-peaket wrt $>_\Omega$ because c is between the peak (in this case a) and d wrt $>_\Omega$, but d is preferred to c.

Black's theorem states that, for single-peaked profiles, the majority voting provides an outcome that is consistent wrt the axioms that define preference orderings.

Theorem 1
For every profile that satisfies single-peakedness, the majority rule returns a preference ordering.

In particular, if there is an odd number of $n+1$ voters and we order the agents' peaks according to $>_\Omega$, the *median voter*'s peak, namely the option that has $n/2$ peaks on the right and $n/2$ peaks on the left according to $>_\Omega$ is elected by majority (e.g. in Figure 1, on the right, the median's peak is c).

Moreover, for single-peaked profiles, the median voter's peak is also the Condorcet's winner (i.e. an option that beats every other option in pairwise comparisons).

[1] A short version of this article appeared in [17].

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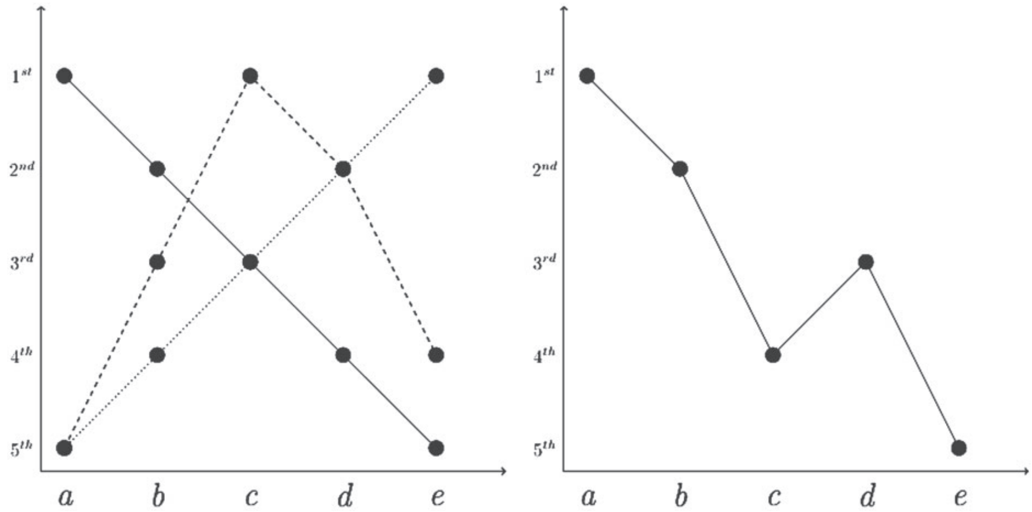

FIG. 1. Single-peaked and non single-peaked preferences wrt $>_\Omega$

We present some simple properties of single-peaked profiles. Given a dimension $>_\Omega$, we define the *opposite dimension* $>'_\Omega$ as the opposite ordering. The next proposition summarizes some simple properties that we are going to use in the next sections.

Facts 1

The following statements hold:

(1) Every individual preference ordering P_i is compatible with a dimension.
(2) If **P** is compatible with a dimension $>_\Omega$, then it is compatible with the opposite dimension $>'_\Omega$.
(3) Given a single-peaked profile, if the top of P_i is equal to the top of $>_\Omega$ ($>'_\Omega$), then P_i is equal to $>_\Omega$ ($>'_\Omega$).
(4) If **P** is single-peaked wrt $>_\Omega$ and P_i differs from $>_\Omega$ and $>'_\Omega$, then $>_\Omega$, $>'_\Omega$ and **P** are not single-peaked with respect to P_i.

Point 1 is immediate, as we can take as dimension $>_\Omega$ the same individual preference ordering P_i. Point 4 can be easily shown by considering that, if we take the ordering P_i as a formal dimension, then either $>_\Omega$ or $>'_\Omega$ violate single-peakedness wrt P_i.

3 Formal and semantic dimensions

In this section, we discuss the relationship between formal dimensions and conceptualizations of choices. We start by presenting a simple social choice problem where three agents (1, 2 and 3), for instance three members of a parliamentary committee, decide by majority whether to promote one among the three alternative policies a, b and c. Consider the preference profile in Table 1.

The profile is single-peaked, as it is compatible for example with the formal dimension $c > a > b$. Therefore, the majority rule returns a rational preference ordering that is $a > b > c$ and the winning alternative is a. In this situation, everything works fine from the perspective of an aggregative view

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TABLE 1. Agents' single-peaked preference profile

$c >_1 a >_1 b$
$a >_2 b >_2 c$
$b >_3 a >_3 c$
Maj. $a > b > c$

TABLE 2. Agents' semantic dimensions 1

	(a,b)	(b,c)	(a,c)
1	aPrb	cPrb	cPra
2	aCob	bCoc	aCoc
3	bFaa	bFac	aFac

of collective decisions. However, if we ask now whether the preference profile is uni-dimensional—i.e. whether the agents share a common reason for choosing—we see that single-peakedness does not tell us much about semantic dimensions.

Consider the following scenario where individuals provide reasons for their preferences. Suppose that the first agent ranks c above a because she/he believes that c improves productivity more than a, a more than b and b more than c; hence she/he appeals to the semantic dimension of *productivity*, which is the rationale for her/his choices. We view semantic dimensions as binary predicates, we label the semantic dimension of productivity by Pr, and we write xPry to formalize the sentence 'x is more productive than y'. The second agent appeals to the the semantic dimension of the *cost* (Co) of implementing a given policy. The third agent ranks the alternatives according to the extent to which they promote *fairness* (Fa). Thus, for each pair of alternatives, the semantic dimensions that the agents are appealing to are Pr, Co and Fa and they provide reasons for agents' preferences of an option over another. By listing all the semantic dimensions used by the agents, we obtain a profile of reasons, cf. Table 2.

A preliminary informal definition of uni-dimensionality of a profile of reasons is the following: a profile of reasons is uni-dimensional if every agent appeals to the same semantic dimension. Does a single-peaked preference profile entail an uni-dimensional profile of reasons? In general, the answer is negative (see also [5]). Single-peakedness merely states that there exists a formal dimension that is compatible with every individual preference.[2] Thus, as the previous example shows, in a single-peaked profiles, agents may appeal to heterogeneous semantic dimensions.

In the scenario of Table 2, agents agree on a collectively acceptable choice—i.e. on the alternative a—however, they do not agree on a collective reason for choosing a, as they do not share a common semantic dimension. In the example, the profile is single-peaked, but it is *not* uni-dimensional, as it does not manifest a single semantic dimension.

Although single-peakedness does not entail uni-dimensionality, we ask whether it is possible to reduce the possibly heterogeneous semantic dimensions used by individuals to a number of collectively sharable semantic dimensions.

We will show that it is not possible, in general, to associate a *single* semantic dimension (i.e. a single binary predicate), such as productivity, to any single-peaked profile. However, we will see

[2] Moreover, there may be several dimensions wrt which a preference profile is single-peaked [9]

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TABLE 3. Agents' semantic dimensions 2

	(a,b)	(b,c)	(a,c)
1	$a\mathsf{Pr}b$	$c\mathsf{Pr}b$	$c\mathsf{Pr}a$
2	$a\mathsf{Pr}b$	$b\mathsf{Fa}c$	$a\mathsf{Fa}c$
3	$b\mathsf{Fa}a$	$b\mathsf{Fa}c$	$a\mathsf{Fa}c$

that it is possible to associate to any single-peaked profile *two* semantic dimensions that are related to each other.

Note that, in the previous example, agent 2 may in principle provide reasons for her/his preferences by appealing to the semantic dimensions of productivity and fairness. For instance, agent 2 was preferring a to b on the ground of the cost, but a may also be viewed as better than b on the ground of productivity. Moreover, agent 2 was preferring a to c on the ground of the cost, but a is better than c also on the ground of fairness. Thus, in principle, agent 2's preference ordering can be rationalized also by $a\mathsf{Pr}b$, $a\mathsf{Fa}c$, and $b\mathsf{Fa}c$. Suppose that agent 2 is convinced to revise his conceptualization and to rationalize his preferences by using the semantic dimensions of fairness and productivity. We obtain the profile of semantic dimensions in Table 3.

In this case, productivity and fairness constitute in fact opposite dimensions, in the sense that $x\mathsf{Pr}y$ holds iff $y\mathsf{Fa}x$ holds. Thus, by means of productivity and fairness, every individual can rationalize any preference for one option over the other: if a may be viewed as better then b on the ground of productivity, b may be viewed to be better than a on the ground of fairness. In fact agent 2 is referring to both semantic dimensions, he is somehow balancing between the two opposite reasons: agent 2's reason for preferring a over b is that a is better than b according to productivity, whereas a is preferred to c because of fairness. In this case, there is indeed a single formal dimension $c >_\Omega a >_\Omega b$ that provides the denotation of both the semantic dimensions of productivity and fairness. The two semantic dimensions are thus interrelated as opposite dimensions. Therefore, there is in fact a common conceptualization for electing a as the social choice, the reasons for a are expressed in terms of productivity *and* fairness, namely, a is a good compromise between productivity and fairness.[3]

We will see that this example indeed provides a general way to associate semantic dimensions to single-peaked profiles. Note that we are only discussing the possibility for the individuals to agree on a common conceptualization. We are not discussing nor modelling the motivations or the incentives that individuals may have for revising the conceptualization of their preferences. I leave that for future work. In the remainder of this article, I shall develop a formal modelling to make the previous points precise.

4 A model of semantic dimensions

We introduce a number of relations on alternatives that are intended as means to rationalize, or explain, a preference ordering according to a certain relevant criterion. This relations provide the denotation of the semantic dimensions that we are going to introduce as binary predicate in a first-order language.[4]

[3] If we were to vote by majority on the profile of judgments in Table 3, we obtain that the collectively accepted reasons are: $a\mathsf{Pr}b$, $b\mathsf{Fa}c$, and $a\mathsf{Fa}c$. Actually, this may be viewed as a judgment aggregation problem [13].

[4] An analogous treatment of preference relations has been presented in [16].

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Given a preference ordering P, a *reason* for P is a *transitive* and *irreflexive* relation D such that $D \subseteq P$. A set of reasons $R = \{D_1, ..., D_m\}$ *rationalizes* a preference ordering P if and only if D_j are reasons for P and, for every $(x,y) \in P$, there exist a single D_j in J such that xD_jy. Thus, a set of reasons is a partition of P.

For example, the preference ordering $aPbPc$ can be rationalized by two relations as in $R = \{\{(a,b)\}, \{(b,c), (a,c)\}\}$. The meaning of this definition is that sub-realtions of P provide the denotation of semantic dimensions; so, saying that P is rationalized by two relations means that the preference of a over b is motivated by a certain semantic dimension, e.g. productivity, whereas the preference of b over c and of a over c is motivated by another semantic dimension, e.g. fairness.

The assumptions on sets of reasons have the following meaning. We assume that each pair of alternatives $(x,y) \in P$ is rationalized by some D_j in R. As we assume that each reason is included in P, an agent cannot have both reasons aDb and bDa. The fact that D_i are not necessarily complete means that some pair (a,b) can be rationalized by D while some other pair (c,d) by a distinct D'. For example, an agent can prefer a policy a over b because 'a promotes GDP's growth better that b' and c over d because "c is more liberal than d". The transitivity of the Ds means that if an agent rationalizes aPb on the ground of D (e.g. 'a promotes the conditions of the worst off more than b') and bPc on the same ground of D ('b promotes the conditions of the worst off more than c') , then she/he rationalizes aDc on the same ground.

We say that the set of justifications R *uniformly rationalizes* P iff there exists a single relation D that rationalizes P. In that case, D is also complete and coincides with P. For example, $R = \{\{(a,b),(b,c),(a,c)\}\}$.[5]

We assume that each agent i has a set of reasons R_i of her/his preference orderings P_i. A profile of reasons is a list $\mathbf{R} = (R_1, \ldots, R_n)$. The next definitions provide a slight generalization of single-peakedness to several partial preference relations. First, we define when a set of reasons R is compatible with a (formal) dimension.

Definition 2
A set of reasons R is compatible with a dimension $>_\Omega$ iff, whenever $x >_\Omega y >_\Omega z$ or $x >'_\Omega y >'_\Omega z$, if, for $D_j \in R$, xD_jy, then for no $D_k \in R$, zD_ky.

The definition says that no relation in R violates single-peakedness wrt the given dimension. In case of a single complete relation in R, the definition provides the usual notion of single-peakedness.

The notion of single-peakedness restricted to a single agent is not demanding, as every set of reasons is compatible with at least one dimension (e.g. the union of the D_j in R).

Definition 3
A profile of reasons is single-peaked if and only if there exists a dimension $>_\Omega$ such that for every set of reasons R_i in $\mathbf{R}$, R_i is compatible with $>_\Omega$.

We can easily see that if $\mathbf{P}$ is single-peaked wrt $>_\Omega$, then for every profile of reasons $\mathbf{R}$ such that each R_i rationalises P_i, $\mathbf{R}$ is single-peaked wrt $>_\Omega$. Moreover, the properties we listed in Fact 1 hold also for sets of reasons.

Consider the following example (Table 4). We have a single-peaked preference profile on the left and the rationalization of preferences by the corresponding sets of relations on the right.

As we will see in the next section, the relations in R_i provide the extensional interpretation of the semantic dimensions intended as binary predicates in a given language. Modelling semantic

[5] Our assumptions entails that, for instance, an agent cannot prefer at the same time a to b on the ground of productivity and b to a on the ground of fairness. This is motivated by the fact that we are modelling rationalization of preference, thus we assume that an agent can always choose whether a is better than b or vice versa.

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TABLE 4. A profile of reasons

	Preferences:	Sets of reasons:
1	aP_1bP_1c	$\{\{(a,b)\},\{(b,c)\},\{(a,c)\}\}$
2	bP_2cP_2a	$\{\{(b,c)\}\{(c,a),(b,a)\}\}$
3	cP_3bP_3a	$\{\{(c,b),(b,a),(c,a)\}\}$

dimensions as mere relations on the set of alternatives is not sufficient: for instance, an agent may think that b is better than c because b increase entrepreneurs' freedom more than c, another agent might think that b is better than c because it is better in reducing taxation. The point is that extensionally equal relations may still express distinct semantic dimensions. For this reasons, we have to introduce a language to model semantic dimensions.

4.1 *A language of semantic dimensions*

We introduce a fragment of first-order logic in order to model semantic dimensions. The alphabet contains a set of constants $\mathcal{A}$—that are intended to correspond to the alternatives A of given a decision problem—and a set of *predicates* $\mathcal{D}=\{\bar{D}_1, \ldots, \bar{D}_m\}$, where $\bar{D}_j$ is a binary relational symbol.[6] A *semantic dimension* is a predicate in $\mathcal{D}$.[7]

The domain in which we interpret the individual constants is given by the set of the alternatives A at issue. Moreover, we assume that the interpretation $\mathcal{I}$ of the individual constants is fixed and determined by a bijective function $\mathcal{I}_0(a)$. By slightly abusing the notation, we denote the alternatives in A and the individual constants in $\mathcal{A}$ by the same letters. As usual, the interpretation of the relational symbols is given by $\mathcal{I}(D_j) \subseteq A \times A$.

The following first-order sentences express rationality constraints on $\bar{D}_j$:

- *irreflexivity*: $\forall x \neg x\bar{D}_j x$,
- *transitivity*: $\forall xyz(x\bar{D}_j y \wedge y\bar{D}_j z \rightarrow x\bar{D}_j z)$.

The *language* $\mathcal{L}_{\mathcal{D}}$ is the set of all sentences $a\bar{D}_j b$ with $a,b \in \mathcal{A}$ and $\bar{D}_j \in \mathcal{D}$.[8] We simply label the sets of sentences expressing reasons by judgment sets.

DEFINITION 4
A *judgment set* $\bar{J}$ is a subset of $\mathcal{L}_{\mathcal{D}}$ such that:

(1) Each predicate in $\bar{J}$ is consistent wrt irreflexivity and transitivity (i.e. it denotes a transitive and irreflexive relation);[9]
(2) $\bar{J}$ is *complete*: for every pair of constants a, b, either there is a sentence $a\bar{D}_j b \in \bar{J}$ or there is a sentence $b\bar{D}_j a \in \bar{J}$.
(3 $\bar{J}$ is *univocal*: if $a\bar{D}_j b$ and $a\bar{D}_h b$ are in $\bar{J}$, then $i=h$.

[6] Intuitively, such predicates represent watchwords like 'more productive than', 'more liberal than'..., that usually express semantic dimensions in political contexts.

[7] By means of this definition, we stress the linguistic nature of semantic dimension. Accordingly, two co-extensive binary predicates may still express distinct semantic dimensions.

[8] The language does not contain any logical operator. We leave a proper treatment of reasoning about semantic dimensions for future work.

[9] Namely, we check whether $\bar{J} \cup \{\forall x \neg x\bar{D}_j x, \forall xyz(x\bar{D}_j y \wedge y\bar{D}_j z \rightarrow x\bar{D}_j z)\}$ is consistent.

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Note that, given our assumptions, a set containing $a\bar{D}_j b$ and $b\bar{D}_j a$ is not a judgment set, as it is not consistent wrt transitivity and irreflexivity. Moreover, a set containing $a\bar{D}_j b$ and $a\bar{D}_h b$ is excluded by the third condition. The second condition impose that every ranking of alternatives is rationalized by some sentence. This conditions entail that the model of a judgment set is a set of reasons that rationalizes a certain preference order.

The model of $\bar{J}$, $\mathcal{M}(\bar{J})$, is the set of alternatives A endowed with relations that are transitive and irreflexive. Moreover, the completeness and univocity of $\bar{J}$ entail that the models of the relational symbols in $\bar{J}$ provide a partition of a preference relation P.

On the other hand, each set of reasons R, in the sense of Section 4, can be represented by means of a set of judgments as follows. We associate with each relation D_i in R a set of sentences $x\bar{D}_i y \in \mathcal{L}_{\mathcal{D}}$ such that $(x,y) \in D_i$ iff $x\bar{D}_i y \in \bar{J}_i$. Moreover different relations in R are associated with different binary predicates in $\bar{J}$. The translation is not unique, as it depends on the choice of the relational symbols in $\mathcal{D}$. However the models of each set of judgments $\bar{J}_i$ that comes from the same set of reasons R coincide with the starting set of reasons R.

A profile of judgment sets $\bar{\mathbf{J}}$ is a vector $(\bar{J}_1,\ldots,\bar{J}_n)$. Moreover, the vector of models $(\mathcal{M}(\bar{J}_1),\ldots,\mathcal{M}(\bar{J}_n))$ satisfies the following conditions:

- (i) the interpretation of the constants in each $\mathcal{M}(\bar{J}_i)$ is defined by $\mathcal{I}_0$,
- (ii) for every relational symbol D that occurs both in $\bar{J}_i$ and $\bar{J}_j$, $\mathcal{I}$ associates the same relation on A.

We can now define single-peakedness for sets of judgments by saying that the models of the judgment sets provide sets of reasons that are compatible wrt the formal dimension (in the sense of Definition 2).

Definition 5
A set of judgments $\bar{J}$ is compatible with a formal dimension $>_\Omega$ iff the model of $\bar{J}$ is.

A profile of judgments $\bar{\mathbf{J}}$ is single-peaked iff there exists a formal dimension $>_\Omega$ such that each $\bar{J}_i$ is compatible with $>_\Omega$.

It is useful to provide a definition of compatibility of a judgment set wrt a formal dimension in terms of the properties of the sentences that belong to each judgment set. We define the following notion of *coherence* wrt a formal dimension. The following definition simply rephrases the notion of compatibility of a set of reasons for sentences of $\mathcal{L}_D$.

Definition 6
A set of sentences $S \subset \mathcal{L}_{\mathcal{D}}$ is *coherent* wrt $>_\Omega$ iff S is a judgment set and it satisfies the following condition:

Whenever $x >_\Omega y >_\Omega z$ or $x >'_\Omega y >'_\Omega z$, if $xD_j y \in S$, then there is no $zD_k y$ in S.

Thus, by using Definition 2 and 4, we can state the following facts that link the syntactic properties of the sets of judgments with the compatibility of their models wrt a formal dimensions.

Facts 2

The following statements hold:

- A judgment set $\bar{J}$ is compatible wrt $>_\Omega$ iff it is coherent wrt $>_\Omega$.
- A profile of judgments $\bar{\mathbf{J}}$ is compatible wrt $>_\Omega$ iff every $\bar{J}_i$ is coherent wrt $>_\Omega$.

Coherence wrt a formal dimensions has the following meaning. Suppose that an agent is using $a\bar{D}b$ to rationalize her preference of a over b and such reason is meant to refer to the formal dimension

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Table 5. Non-shared semantic dimensions

	Preferences:	Justifications:
1	aP_1bP_1c	$\bar{J}_1=\{a\bar{D}_1b, b\bar{D}_2c, a\bar{D}_3c\}$
2	bP_2cP_2a	$\bar{J}_2=\{b\bar{D}_4c, c\bar{D}_5a, a\bar{D}_5c)\}$
3	cP_3bP_3a	$\bar{J}_3=\{c\bar{D}_6b, b\bar{D}_6a, c\bar{D}_6a\}$

Table 6. Shared semantic dimensions (uni-dimensionality)

	Preferences:	Justifications:
1	aP_1bP_1c	$\{a\bar{D}b, b\bar{D}c, a\bar{D}c\}$
2	bP_2cP_2a	$\{b\bar{D}c, c\bar{D}'a, a\bar{D}c)\}$
3	cP_3bP_3a	$\{c\bar{D}'b, b\bar{D}'a, c\bar{D}'a\}$

$a >_\Omega b >_\Omega c$. Then, the agent cannot rationalize her preference of c over b by appealing to a semantic dimension D': $c\bar{D}'b$. If someone views that a is better than b on the ground of productivity, she claims that productivity is the sole relevant criterion for choosing a collective option, and she knows that b is also more productive than c, then she would fail to be coherent with her criterion, if she were to say that c is better than b on a different ground.

The example of the previous section (Table 5) can be now represented as follows (Table 6). Suppose that agents rationalize their preference by means of sentences defined in three different language $\mathcal{L}_{\mathcal{D}_1}$, $\mathcal{L}_{\mathcal{D}_1}$ and $\mathcal{L}_{\mathcal{D}_3}$.

Each set of formulas is coherent wrt a single formal dimension; however, in principle, each agent may refer to the formal dimension by using different semantic dimensions.

In the situation depicted above, the preference profile is single-peaked but agents do not agree on a single relevant semantic dimension, as the reasons they are providing may be in principle conceptually unrelated.

5 Single-peakedness and uni-dimensionality

The following result shows that, at least in principle, it is possible that the agents agree on a shared semantic dimension. Namely, we show that there exists a language that can faithfully express agents' reasons whose alphabet contains two predicates (two semantic dimensions) D and D'.

Definition 7
We say that a judgment set $\bar{J}_1$ is *consistent* with $\bar{J}_2$ if $\bar{J}_2$ is obtained from $\bar{J}_1$ as follows: for every sentence $x\bar{D}_1y$ in $\bar{J}_1$, there exists a D_2 such that $x\bar{D}_2y \in \bar{J}_2$.

Moreover, a profile $\bar{\mathbf{J}}$ is consistent with $\bar{\mathbf{J}}$' if each $\bar{J}_i$ is consistent with $\bar{J}_i'$

We can prove the following result.

Theorem 2
For every profile of judgments $\bar{\mathbf{J}}$ (which is coherent wrt $>_\Omega$) there is a profile of judgments $\bar{\mathbf{J}}'$ (which is coherent wrt $>_\Omega$) such that each $\bar{J}_i$ is consistent with $\bar{J}_i'$ and every $\bar{J}_i'$ in $\bar{\mathbf{J}}'$ is defined on the language $\mathcal{L}_{\{\bar{D},\bar{D}'\}}$. Moreover, $\bar{D}$, $\bar{D}'$ denote $>_\Omega$ and $>'_\Omega$ (respectively).

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PROOF. Suppose that $\bar{\mathbf{J}}'$ is defined on $\mathcal{L}_{\{\bar{D},\bar{D}'\}}$ and that D and D' denote $>_\Omega$ and $>'_\Omega$. As we observed in Section 4, every relation is included in $>_\Omega$ or $>'_\Omega$. Thus, we have that, for every $x\bar{D}_j y \in \bar{J}_i$, $(x,y) \in \mathcal{I}(\bar{D}) = >_\Omega$ or $(x,y) \in \mathcal{I}(\bar{D}') = >'_\Omega$. Thus, every $x\bar{D}_j y \in \bar{J}_i$ can be expressed by xDy or $xD'y$.

We show that if $\bar{\mathbf{J}}$ is single-peaked wrt $>_\Omega$, then also $\bar{\mathbf{J}}'$ is single-peaked wrt $>_\Omega$. If $\bar{\mathbf{J}}$ is single-peaked, then every $\bar{J}_i$ is compatible with $>_\Omega$, namely, no $\bar{J}_i$ contains both sentences xD_jy and yD_kz, when $x >_\Omega y >_\Omega z$ or $x >'_\Omega y >'_\Omega z$. Since, $\bar{\mathbf{J}}'$ is consistent with $\bar{\mathbf{J}}$, no $\bar{J}'_i$ contains $x\bar{D}y$ and $y\bar{D}'z$ or $x\bar{D}'y$ and $y\bar{D}z$. Thus, if $\bar{\mathbf{J}}$ is coherent wrt $>_\Omega$, then $\bar{\mathbf{J}}$' is. ∎

The theorem shows that there is a way to express a profile of agents' reasons by means of a language consisting of two predicates. Note that a language consisting of a single predicate (a single semantic dimension), e.g. a language such that $\mathcal{D} = \{D\}$, would not be enough to express the justifications relative to any possible individual preference P_i. For example, the judgment sets $\{xD_1y, yD_1z, xD_1z\}$ and $\{zD_2y, yD_2x, zD_2x\}$ cannot be consistently translated into sets of judgments that are defined on $\mathcal{L}_{\{D\}}$: once we have decided to associate xD_1y to xDy we cannot consistently associate any sentence to yD_2x.

Thus, if agents are interested just in preserving the coherence of their judgment sets wrt a formal dimension, Theorem 2 states that it is possible for agents to agree on a shared language that contains two semantic dimensions without revising any of their preferences.

Moreover, by using Theorem 2, we can directly associate a preference profile with a judgment profile that *rationalizes* it in the following sense.

DEFINITION 8
A preference profile $\mathbf{P}$ is *rationalized* by a profile of judgments $\bar{\mathbf{J}}$ iff each $\bar{J}_i$ rationalizes P_i, i.e. the set of relations in $\mathcal{M}(\bar{J}_i)$ rationalizes P_i (cf. Section 4).

We can prove that for every preference profile $\mathbf{P}$ that is single-peaked wrt $>_\Omega$, there exists a judgment profile $\bar{\mathbf{J}}$ such that $\bar{\mathbf{J}}$ is coherent wrt to $>_\Omega$, $\bar{\mathbf{J}}$ is defined with $\mathcal{D} = \{\bar{D}, \bar{D}'\}$, and $\bar{\mathbf{J}}$ rationalizes $\mathbf{P}$.

THEOREM 3
For every (single-peaked) preference profile $\mathbf{P}$, there exists a judgment profile $\bar{\mathbf{J}}$ that rationalizes $\mathbf{P}$ and that is defined on $\mathcal{L}_{\{\bar{D},\bar{D}'\}}$.

Given a preference profile and a possible profile of reasons (cf Definition 3), we can associate judgments sets whose models coincide with the profile of reasons. We can retrieve a suitable profile of judgment, by means of the same strategy as in Theorem 2. Of course there might be several ways of associating preference profiles to semantic dimensions $\bar{D}$ and $\bar{D}'$. In particular, any formal dimension with respect to which $\mathbf{P}$ is compatible provides a way to reconstruct the sets of reasons by means of judgments defined using $\bar{D}$ and $\bar{D}'$.

Note that *any* profile (i.e. also non single-peaked) can be rationalized by means of a language that uses two predicates; the important point is that, when the profile is single-peaked, our definition of compatibility wrt a dimension entails that the judgment sets are coherent wrt a formal dimension. In a non single-peaked profile, there exists at least one agent whose judgment set contains a sentence $a\bar{D}b$, where D is meant to refer to a formal dimension $a >_\Omega b >_\Omega c$, and a sentence $c\bar{D}'b$ that violates coherence wrt $>_\Omega$.[10]

[10]The interpretation of single-peakedness as a form of discursive agreement on reasons is interesting from the point of view of deliberative democracy. According to deliberativists, an agent may change his mind or his preferences only by the 'force of the better argument'. Thus, a blame of incoherence wrt a possibly shared semantic dimension may trigger the revision of an agent's judgments or preferences.

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We can now define the concept of *uni-dimensionality* of a profile of judgment sets. Single-peakedness is a property of preference profiles that states the compatibility with a formal dimension, while uni-dimensionality is a property of judgment profiles that refer to a shared semantic dimension.

Definition 9
A profile of judgment is *uni-dimensional* iff either every $\bar{J}_i$ in **J** is defined over the same language $\mathcal{L}_{\{\bar{D}\}}$ or every $\bar{J}_i$ in **J** is defined over the same language $\mathcal{L}_{\{\bar{D},\bar{D}'\}}$ where each $\bar{J}_i$ is consistent with the following constraint:

- $\forall x \forall y \, x\bar{D}y \leftrightarrow y\bar{D}'z$.

Although in an uni-dimensional profile two semantic dimensions may be present, they in fact constitute one the negation of the other. For instance, in case productivity and fairness are the two semantic dimensions, one can say that x promote productivity more than y if and only if it is not the case that x promotes fairness more than y. Once a social choice problem is conceptualized in terms of productivity-fairness, every pair of option can be ranked according to one, and only one, of the two semantic dimensions.

Note that uni-dimensionality does not entail single-peakedness. This is consistent with the view in [5, 15] and it is motivated by the idea that agents may fail to associate a suitable ranking of the option (a formal dimension) to the semantic dimension. Accordingly, Definition (9) does not require that the profile **J** is discursively coherent or compatible with a formal dimension, i.e. our definition of uni-dimensionality does not presuppose single-peakedness.

By Theorem 3, the example in Table 5 can now be rephrased by means of an uni-dimensional judgment profile, cf. Table 6.

In this case, each agent's justification set is using shared semantic dimensions and each set is coherent wrt a formal dimension. Moreover, the formal dimension may be retrieved from the judgment sets of each individuals by taking the models of the binary predicates $\bar{D}$ and $\bar{D}'$ that appear in $\bar{J}_i$.

$$>_\omega \quad = \quad \begin{array}{c} \bigcup\{(x,y) \mid x\bar{D}y \in \bar{J}_i\} \\ \cup \\ \bigcup\{(y,x) \mid x\bar{D}'y \in \bar{J}_i\} \end{array}$$

That is, the formal dimension is given by the union of the interpretation of the D-semantic dimension involved the individual judgment sets together with the opposite pairs included in the interpretation of the D'-semantic dimension.

To conclude, if the agents agree to express their reasons by appealing to two shared opposite semantic dimensions, it is possible to associate a single-peaked profile of preferences with an uni-dimensional profile of judgments.

5.1 The median voter's reasons

The example in Table 6 shows an interesting fact. The judgment set of agent 2, who is the *median* voter wrt the preference profile at issue (cf. Section 2), contains *both* sentences built by $\bar{D}$ and $\bar{D}'$, whereas the other agents have sets of judgments that employ either $\bar{D}$ or $\bar{D}'$. Thus, in case the preference profile contains sufficiently heterogeneous preference, we can prove the following results.

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THEOREM 4

Given a preference profile $\mathbf{P}$ that is single-peaked wrt $>_\Omega$, there is no profile of judgments $\bar{\mathbf{J}}$ defined over $\mathcal{L}_{\{\bar{D},\bar{D}'\}}$ that rationalizes $\mathbf{P}$ where the median voter preference P_i are rationalized by a set of judgments such that xP_iy iff $x\bar{D}y \in \bar{J}_i$ or xP_iy iff $x\bar{D}'y \in \bar{J}_i$.

PROOF. Suppose P_i is rationalized by a set $\bar{J}_i$ such that, for every x, y in A, xP_iy iff $xEy \in \bar{J}_i$, where E is either $\bar{D}$ or $\bar{D}'$. Then, by Theorem 2, either $\bar{D}$ or $\bar{D}'$ shall denote the dimension $>_\Omega$. By Fact 1.4, $>_\Omega$, $>'_\Omega$ and $\mathbf{P}$ are not single-peaked wrt P_i. Therefore, there exists a P_j that violates the compatibility of $\bar{J}_j$ with $>_\Omega$. Thus $\bar{\mathbf{J}}$ cannot rationalize $\mathbf{P}$. ∎

Theorem 4 means that the median voter cannot uniformly rationalize her preferences by appealing to the underlying dimension. The interpersonal agreement on a shared semantic dimension forces the median voter to balance between the two opposite judgments expressed by $\bar{D}$ and $\bar{D}'$. In this sense, the median voter, whose most preferred option is the median of the peaks of the others' preferences, has to rationalize her/his preference by using both semantic dimensions. Theorem 4 shows that, in general, it is not possible that agents agree on any other formal dimension with respect to which the median voter has a single semantic dimension that uniformly rationalizes his/her preferences.

6 Conclusions

We have discussed the formal condition of single-peakedness and confronted it with its suggestive interpretation in terms of a shared rationale of collective choices. We have distinguished between a formal dimension and a semantic dimension, where semantic dimensions are construed as reasons that rationalize preferences. In order to model semantic dimensions and their relations with formal dimension, we have introduced a simple framework based on first-order logic where semantic dimensions are modelled by means of binary predicates that denote possible formal dimensions. The concept of uni-dimensionality has been defined then as a property of profiles of judgments that endorse shared semantic dimensions. The aim of this article was to investigate the relationship between single-peakedness and its conceptual counterpart, uni-dimensionality. By relying on the previous analysis, we conclude that a single-peaked profile is non-necessarily associated with uni-dimensionality, but it can in principle be associated to an uni-dimensional profile of judgments, where agents agree on two interrelated semantic dimensions. This means that single-peakedness is not sufficient to provide a rationale for collective decisions; the agreement on the semantic dimensions is crucial as well.

On the other hand, uni-dimensionality does not entail single-peakedness, since agents may still disagree on how to rank the options according to the semantic dimensions, cf. [5]. Thus, a shared semantic dimension in an uni-dimensional profile may fail to provide a shared formal dimensions with respect to which agents' preferences are single-peaked. Therefore, uni-dimensionality is not sufficient for providing a rationale for collective choices as well. The conclusion is that we need both conditions in order to provide a rationale for collective choices.

Future work shall concentrate on the relationship between semantic dimensions and agents' preferences. In particular, the idea is to view preferences as *justified* by appealing to semantic dimension, that is, the point is to model sentences such as 'an agent prefers a to b on the ground that a is better than b according to the semantic dimension $\bar{D}$'.

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