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# Wittgenstein on Mathematical Advances and Semantical Mutations[1]

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**Abstract**: The objective of this article is to try to elucidate Wittgenstein's extravagant thesis that each and every mathematical advancement involves some "semantical mutation", i.e., some alteration of the very meanings of the terms involved. To do that we will argue in favor of the idea of a "modal incompatibility" between the concepts involved, as they were prior to the advancement, and what they become after the new result was obtained. We will also argue that the adoption of this thesis profoundly alters the traditional way of construing the idea of "progress" in mathematics.

**Keywords**: Wittgenstein, Philosophy of Mathematics, Semantical Mutation, Mathematical Progress.

## 1. Introduction

The topic of this article is going to be Wittgenstein's controversial "*Language mutation thesis*". This is the idea that every new mathematical result alters the very meaning of in terms involved. If there is one claim that possibly deserves

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the title of Wittgenstein's most outrageous, unintuitive proposal is this, his thesis of mathematical progress always involves semantical mutations. As we will see, according to Wittgenstein, there would always be a "semantical gap" between conjecture and the rule it engenders, when is latter demonstrated. The two can never be "leveled up"; a new mathematical advance always represents a *language mutation*.

The goal of this article is to try to make sense of this strange Wittgensteinian ideas. We believe this precept is crucial to a more adequate evaluation of Wittgenstein's philosophy of mathematics. As we will see, Wittgenstein's ideas project an entirely different image of mathematical Progress, one not involving a "glorious discovery of new properties concerning old (eternal) mathematical objects" but rather one involving a "reduction of our previous, past irrationality". But let us not anticipate too much. There is much to be argued, before we can evaluate those claims.

There is no doubt that Wittgenstein did maintain these strange ideas in his so-called middle period:

> ... once a proof has been supplied, it in no way proves what had been conjectured, ... you can't conjecture the proof until you've got it, and not then, either. (WITTGENSTEIN, 2005, p. 418-9)
>
> Why do I say that we don't "discover" a proposition like the fundamental theorem of algebra, but instead "construct" it? – Because in proving it we give it a new sense that it didn't have before. (WITTGENSTEIN, 2005, p. 428)
>
> The "medical proof" didn't incorporate the hypothesis it proved into a new calculus, and so didn't give it a new sense; a mathematical proof incorporates the mathematical proposition into a new calculus and alters its position in mathematics. (WITTGENSTEIN, 2005, p. 426)

There is less agreement whether Wittgenstein did continue to support these ideas in his latter, mature years. Apparently, the majority of the authors (WRIGHT, 1980, Chp 3; SHANKER, 1987, Chp 3; DIAMOND, 1991, Chp 10; Glock, 1996) (MARION & OKADA, 2012; SCHROEDER, 2012; SÄÄTELÄ, 2011) are "continuists": for them Wittgenstein never changed his mind about these odd opinions. But there is a smaller group which suggests the opposite:

> The calculus conception [from the middle period] was unable to account for the change and growth of mathematics, while the language-game conception [from the latter period] emphasizes this. The bizarre views [on language mutation] that have earned many commentators' derision drop out. (GERRARD, 1991, p. 132)[4]

For an exegete trying to make sense of Wittgenstein's ideas, it might look as a gain to classify the semantical mutation thesis as a "temporary speculation" within "transitory period" of the philosopher's development. One would still have to explain the source of these strange ideas – perhaps referring to this period's stringent verificationism – but one would be relieved of any further need of justifying this seemingly outrageous proposal. Unfortunately, the latter textual evidence does not seem to bear this out. In Wittgenstein's *Remarks on the Foundations of Mathematics* one reads:

> When I said that a proof introduces a new concept, I meant something like: the proof puts a new paradigm among the paradigms of the language ...One would like to say: the proof changes the grammar of our language, changes our concepts. (WITTGENSTEIN, 1983, pp. 166, III §31 )

[4] See also Frascolla (2004, pp. 180-1) e Panjvani (2006, pp. 420-1). I thank Severin Schroeder for indicating me some of these earlier sources.

> I should like to say: the proof shows me a new connection, and hence it also gives me a new concept. (WITTGENSTEIN, 1983, pp. 297, V §45)
>
> Mathematics teaches us to operate with concepts in a new way. And hence it can be said to change the way we work with concepts. (WITTGENSTEIN, 1983, pp. 413, VII, § 45)
>
> What is the transition that I make from "It will be like this" to "it *must* be like this"? I form a different concept. One involving something that was not there before. (WITTGENSTEIN, 1983, pp. 237, IV §29)
>
> A new form has been found, constructed. But it is used to give a new concept together with the old one. The concept is altered so that this *had* to be the result. (WITTGENSTEIN, 1983, pp. 248, IV §47)
>
> But in that way every proof, each individual calculation makes new connections! (WITTGENSTEIN, 1983, pp. 181, III §47)
>
> A mathematical proof molds our language. (WITTGENSTEIN, 1983, pp. 196, III §71)

The early dismissal of the language mutation thesis as a "temporary fling" doesn't appear to be really available. The only option for an exegete would be to squarely face the challenge of making sense of the horrible idea. But the task appears daunting. Take the recent discussion on the foundations of mathematics. One finds various proposals for alternative foundations for the entire mathematical knowledge, besides the classical set theoretical one. There are the new categorical foundations, various type-theoretic versions and even intuitionistic reconstructions. But despite such diversity, there is one point about which they all seem to agree: the precept that a mathematician is basically a *proof-producer*. It is precisely this last common ground, perhaps the only common ground shared by all participants of the debate, that Wittgenstein's proposal appears to threaten.

## 2. Mathematical Advances

Mathematical advances occur at certain specific moments in time, they are "temporally indexed entities". Alexander Grothendieck and his coworkers settled a major part of Weil's famous conjectures in 1964. Archimedes offered us our first method for approximating Pi in the III$^{rd}$ century BC (providing a function which gave the length edge of circumscribed regular *2n*-gon based on the length of the edge of a circumscribed regular *n*-gon[5]). Shigeru Kondo and Alexander Yee calculated Pi's 5,000,000,000,000th decimal place for the first time in 2010 (American Physical Society, 2010).

What should we say about the mathematical validity of those advancements, though? Should they be "temporalized" as well, in other words, should we say that these results only became valid, say, when they were actually obtained? Or should we suggest instead that they were atemporally valid, i.e., valid through all eternity? In our example above, should we say that Archimedes function for the edge of a *2n*-gon had to wait the III$^{rd}$ BC to become valid, that it wasn't "really valid" before Archimedes? Or should we say that Archimedes function atemporally expressed the relation in length between those two kinds of polygons? Let us suppose Kondo and Yee encountered a "7" as Pi's 5,000,000,000,000?$^{th}$ decimal. Should we again say that the equation "$Pi_{(5,000,000,000,000)} = 7$" became valid only after 2010, that that digit "wasn't really there" before that date, that it just sort of "popped into being" as the result of Kondo and Yee's bizarre undertaking?

[5] $edge_{2n} = \sqrt{2 - \sqrt{4 - (edge_n)^2}}$

Or should we rather insist that “Kondo and Yee’s equation” is “atemporally valid”, i.e., that (under the assumption that Pi’s 5,000,000,000,000?$^{th}$ decimal is 7) that equation was “always valid”, throughout eternity, even before any electronic, or human computer, ever came into being?

As it happens all too often in philosophy, we shall see that neither of the two options – temporalizing or atemporalizing mathematics – is completely free of some rather unsuspected philosophical consequences. But for the moment it might be safe to suggest that the idea of letting the validity of mathematical laws to be depended on the occurrence of some empirical event – say, its realization by some mathematician at some moment in time – certainly seems to be the most unattractive of the two alternatives.

The epistemic act of grasping something, of finally realizing some complex mathematical connection, is completely contingent. Archimedes might not have worked with mathematics at all. He might have even died much before reaching adulthood. But mathematical results, in sharp contradistinction to the events connected to their obtainment, are necessary. The function Archimedes offered us describing the relation between a $2n$-gon and a $n$-gon appears to be universally valid and its validity seems to be completely independent, even of its inceptor. There seems to be no reason to subordinate the necessary and universal validity of a mathematical result like that to the contingency of some empirical occurrence, its realization by some mathematician at some particular moment in time.

There is a further reason to reject any proposal of making mathematical laws depend on empirical reality. In contrast to the natural sciences, even the justification of mathematical results involves no reference to empirical occurrences, say, corroboratory experiments. All the steps of a proof are asserted with "general validity", with no reference whatsoever to specific sets of prior events, statistical data, say. Experimental records can get modified with the advance of time. But if the validity of a mathematical law is completely preserved from such fluctuations, then even that empirical link with reality is severed.

Despite the awkwardness of the proposal of temporalizing mathematics, we all know that some (early) intuitionists were ready to accept that strange option rather wholeheartedly. The very second of the two crucial "acts of intuitionism" according to Brouwer involved precisely such recognition:

> SECOND ACT OF INTUITIONISM which recognizes the possibility of generating new mathematical entities:
>
> *firstly* in the form of infinitely proceeding sequences $p_1, p_2, \ldots$
>
> *secondly* in the form of mathematical species, i.e. properties supposable for mathematical entities previously acquired... (BROUWER, 1975, p. 511)

Not only Brouwer and Heyting, the original intuitionists, but even more contemporary intuitionists appear ready to accept that new results, and even new entities, just "come into being", "begin to be valid" as the result of their realization by some mathematician. The intuitionist Errett Bishop writes:

> A set is not an entity which has an ideal existence: a set exists only when it has been defined.

> It is not true that every countable set is either countably infinite or subfinite. ... This does not rule out the possibility that at some time in the future [a set] A will have become countably infinite or subfinite; it is possible that tomorrow someone will show that A is subfinite. (BISHOP & BRIDGES, 1985, pp. 2, 18)

The intuitionist philosopher Michael Dummett writes even more forcefully:

> It seems that we ought to interpose between the platonist and the constructivist picture [Wittgenstein's] an intermediate picture, say of objects springing into being in response to our probing. We do not make the objects but must accept them as we find them (this corresponds to the proof imposing itself on us); but they were not already there for our statements to be true of false of before we carried out the investigations which brought them into being. (DUMMETT, 1978, p. 185)

## 3. A Realm of Potentially Existing Proofs

Compared to these rather extravagant proposals, the alternative of saying that Mathematical laws are valid forever, all thorough time, seems much more palatable. It sounds reasonable to say that the volume of the Egyptian pyramids is given by the formula "$vol = \frac{side^2 \times height}{3}$", whether or not the Egyptian mathematicians knew that formula.[6] Metrical connections obtained by doubling the sides of polygons didn't have to wait for Archimedes to "become valid". Insisting on "temporalizing" mathematical results sounds gratuitous at best. As we will see below, even some contemporary intuitionists, such as Per Martin-Löf and Dag Prawitz agree on that.

[6] They did know the formula, cf. Neugebauer (1969, p. 78).

But if we accept that mathematical laws are "atemporally valid", we are immediately confronted with an extra challenge: what should one say about future mathematical advances? If we've agreed that the validity of present mathematical advances retroact throughout the past (and through the future as well), should we say that future, yet unrealized results are also "eternally valid", that they also "retroact their validity" over our time (just like the validity of presently obtained proofs retroact their validity over the past)? We will refer to this as the Problem of the Symmetry between future/present and present/past.

As we've already pointed out, some contemporary intuitionists are ready to reject earlier intuitionistic ideas of "temporalizing" mathematical laws and talk about a tenseless validity of mathematical results (just like Platonists are known to have always urged).[7] But once we accept atemporal validity, the question of symmetry becomes urgent. Should we also accept that future yet unrealized mathematical advances are already valid in the present, despite our unawareness of them? And if we accept the symmetry, should we also follow the platonic lead and postulate some form of "abstract existence" of proofs, alongside with our concrete possession of the texts "in our hands"?

Perhaps surprisingly so, new intuitionists are also ready to make such extra concession to platonism. They are ready to postulate an abstractly existing sort of "potential truth", mathematical facts which, though their proofs may end up

[7] In a volume presenting Martin-Löf's Type Theory, Sommaruga even talks about a form of "intentional platonism" (SOMMARUGA, 2000, pp. 17-8). Cf. also Richman (2018).

being obtained in the future, have not yet been realized, much as if those proofs "just stood there, waiting to be discovered". Martin-Löf writes:

> It has often been pointed out that it is very counter-intuitive to say that a proposition becomes true when it is proved, and it has often been held against the intuitionists that they construe the notion of truth in that way.... of course, even an intuitionist cannot fail to understand this objection, and ought to answer it by saying that there is, not only the notion of actual truth, but also the notion of potential truth, and that, even before the proposition was proved, it could be proved, which is to say that, although not yet actually true, it was potentially true. Thus the notion of potential truth is not tensed in the way the notion of actual truth is. (MARTIN-LÖF, 1990, p. 142)
>
> ... clearly there are true propositions whose truth has not been experienced, that is, propositions which can be shown to be true in the future, although they have not been proved to be true now. (MARTIN-LÖF, 1996, p. 14)

The strategy is clear, similar to that of the Platonists. To avoid temporalizing mathematical validity, one posits some sort of "abstract reality" instead, parallel to our "empirical reality". True, in the case of the new intuitionists, this "abstract reality" is still qualified as "potential" in the strong sense that every (future) mathematical truth can, either be known to be true, or known to be false (MARTIN-LÖF, 1995, p. 195). The new intuitionists maintain the rejection of any idea of "absolutely undecidable propositions" preached by Classical Computability Theory. In fact, right from the beginning the intuitionists have been known to reject all kinds of "transcendental mathematical entities". Not only do they reject the classical idea of "transcendental mathematical facts" (i.e., "necessarily unprovable mathematical truths"), but much earlier than that, they also rejected Cantorian proposals of "transcendental mathematical properties and relations", (necessarily undefinable concepts) and "transcendental mathematical objects" (necessarily unnamable individuals). Thus, despite the recourse to the idea of "abstract existence" by the new intuitionists, their version of this notion is still firmly connected to (theoretically) obtainable mathematical proofs and calculi.

## 4. Analytical Truth and the idea of a "Prefiguration of Mathematical Future"

As we've seen in our last section, in an effort to avoid temporalizing mathematical validity contemporary intuitionists are ready to postulate some sort of "potential mathematical facts" which are said to exist in an abstract realm much before they are ever realized by some mathematician at some particular moment in time. Despite Wittgenstein's strong reservations towards these proposals, one has to agree that introducing the notion of "potency" into mathematics is an exceedingly natural idea, not only regarding the problem of time in mathematics, but, as we shall see, to elucidate the notion of proof and of calculi.

One initial point to be stressed: concerning the modality we've been discussing, it is extremely natural to accept the step $P \rightarrow \Diamond P$, i.e., the principle which says that, if something does occur (even for the first time ever), the possibility of such occurrence had somehow to be given much before its realization. One needs only to further accept what we've called present/past, future/present symmetry to also admit that even upcoming advances, if they are to be realized at some point in our future, would already have to exists as a possibility even now, in our present time. Martin-Löf writes:

> ... If something has been, is being or will be done, then it can be done, .... In the case of proving a proposition, this means that, if a proposition has been, is being or will be proved, then certainly it can be proved, that is, it is potentially true.... The principle just spelled out is again a principle which had a succinct scholastic formulation: it is the principle, *Ab esse ad posse valet consequentia (illatio)*. (MARTIN-LÖF, 1990, p. 143)

The modal point is quite persuasive, but there is a further, much more potent, epistemic component involved in favor of the idea of "potency". This is the idea of an "argumentative force", a "persuasive power" which would be behind the very cogency of our mathematical proofs and calculi, the capacity which (correct) logical arguments seem to have on us to simply "force us" to accept their conclusions.

> Why do certain inferences have the epistemic power to confer evidence on the conclusion when applied to premises for which there is evidence already? We take for granted that some inferences have such a power, and there is no reason to doubt that they have. But what is it that gives them this power? This should be explained. (PRAWITZ, 2015, p. 73)

We are prone to accept the idea that a proof is like a transition, some sort of a "path" (say, from ignorance to knowledge). And like any path, it would have to be already "open to traffic" before anyone could ever ride along them. In other words, the opposite idea seems strange, the idea of a path which "keeps popping into being" as the result of moving through it. This suggestion would seem to destroy all necessity behind mathematical laws. It would be as if mathematical results would just sort of "fall out of the sky", as if merely by chance, without any previous predetermination. As Dummett protest many years ago: mathematical proofs do not just "*drives us along willy-nilly until we arrive at the theorem*" (DUMMETT, 1978, p. 170).

The tendency to postulate some kind of "inferential force", some sort of "prefiguration of mathematical truths" seems overwhelming.

> ... it would seem to follow that there is already something in virtue of which the sentence is true, and then it seems that the corresponding fact is also there before we establish it or unknowingly come into the possession of effective means to do so. (PRAWITZ, 1998, p. 286)

And the usual elucidation of such "inferential force" is, of course, analytic. It is said that the cogency of our proofs, the supposed "epistemic force" behind our arguments and our calculi, is derived from the very meanings of the terms involved, as they were laid down (explicitly or implicitly) before any conjecture could ever be settled, or operation, executed.

> What is it that makes an argument valid and thus compels us, by necessity of thought, to hold the conclusion true, given the truth of the premises? It is difficult to think of any answer that does not bring in the meaning of the sentences in question. In the end it must be because of the meaning of the expressions involved that we get committed to holding one sentence true, given the truth of some other sentences. (PRAWITZ, 2005, p. 678)

To sum up what we've obtained so far: the natural attituded seems to be insisting that mathematical advances do not occur just "by chance", that they are somehow "prefigured" much before they ever do take place. This "prefiguration" would be incarnated in a sort of "inferential force" which would "run ahead of us" and open up such proof-paths much before these routes were ever traversed by any flesh and blood mathematician. Further on, such "force" would derive all its power of persuasion analytically, from the very meanings of the words employed. Thus, mathematical terms would be loaded with a "mathematical content", a sort of "meaning reservoir", the core content of the notions involved, from which all further mathematical demonstrations would extract their power of persuasion, like "liquid flowing from a jar".

Anyone familiar with Wittgenstein's mature writings recognizes all these ideas and images. But it is quite striking that in his writings these ideas and images, instead of being praised as "natural", are condemned without reservations. Before we finally move into Wittgenstein's proposals, though, let us quickly review something which may come as a surprise to a less attentive observer. Full-blooded classicists tend to be equally critical of any such imagery such as "potency" and "inferential force" in mathematics.

## 5. The Classicist's Rejection of the idea of "Potency" in Mathematics

As we've insisted in our previous section, the constructivist's introduction of the idea of "potency" into mathematics is an exceedingly natural proposal. We do normally think of an operation, a function, as "abstractly generating" its results. Thus, in the case of Pi's decimals, for example, we tend to imagine a sort of "dormant potency", a "calculating force" precisely as if this force were to run ahead of us and determine the correct decimal places, much before they are ever reached. In Dummett's words from the quote above, "*we do not make the objects*" – Pi's decimals – we "*must accept them as we find them*". And, in Prawitz's even stronger image, "*the corresponding fact is there before we establish it or unknowingly come into the possession of effective means to do so*". Much the same would take place in mathematical arguments. Correct inferences would have, again in Prawitz's words, a power to "*compel us, by necessity of thought, to hold the conclusion true*".

But, as we've anticipated in the end of our last section though, a fully coherent classicist would tend to reject all these modal ideas and images. The reason is simple: in classical mathematics, as founded on Zermelo-Fraenkel's Set Theory, there is no place for modality. In an amusing, as yet unpublished paper, Wilfrid Hodges writes:

> ... one sufficient condition for the correctness of a mathematical argument is that it should be formalizable as a proof in Zermelo-Fraenkel set theory. Zermelo-Fraenkel set theory has just two primitive notions, 'set' and 'is a member of'. Neither of these notions is modal. (HODGES, 2007, pp. 1-2)

Still, mathematicians – the ordinary "working mathematicians" – insist on introducing modal notions into mathematical writing, much to Hodges' surprise. The logician goes as far as evoking what he calls "a paradox" in ordinary mathematical writing:

> Facts A and B below seem at first sight to be inconsistent with each other. So, we have a paradox.
>
> FACT A: Mathematics contains no modal notions.

> FACT B. Mathematical writing is full of modal notions. …
> To be more objective I took the first hundred pages of a well-known textbook, Birkhoff and Mac Lane's *A Survey of Modern Algebra*, and listed all the instances of modality. I included for example 'allow', 'can', 'cannot', 'could, 'essential', 'have to', 'impossible', 'inevitably', 'may', 'might', 'must', 'necessarily', 'need not', 'need only', 'possibility', 'possible', 'will'. I found 340 examples, or 3.4 examples per page. (HODGES, 2007, pp. 1-2)

Consistently with his fully abstract notion of "pure mathematical existence", Hodges also rejects all dynamical imagery of a function "generating", "producing" its results. He writes:

> With any function $f\,(x,y)$ we think of someone taking the arguments $(a,b)$ and turning them into the value $f\,(a,b)$. … There is a cost in metaphors of this kind. Generally, what they say isn't literally true. We can't cause a mathematical structure A to be embedded in another structure *B*; in general, the most we can do is to describe an embedding. But very often we can't even do that, even when an embedding exists; it would take more than a lifetime to write out the description, or to compute what it is. Some embeddings can't be defined at all with the notions available to us. So, if these metaphors were taken literally, they would imply we have magical powers.
> In spite of the attempts of set theorists to persuade us to think of functions as sets of ordered pairs, we persist in thinking of them as things that a person can do (i.e., has the power of doing). (HODGES, 2007, pp. 6, 7)

A function does not "generate its results". For a classicist like Hodges, a function should be construed as "nothing but an abstractly existing set of ordered tuples", the last member of these tuples to be called "the result of the function". There is no "generation", no "production", just pure abstract existence. Even the modal idea of the "necessity" behind our mathematical arguments and proofs is deemed "distracting" by Hodges, and should stricken off our mathematical jargon:

> The basic objection is the point at the beginning of this paper: the necessity of the conditional is irrelevant to the argument. Mathematical authors wouldn't want to distract the reader by even mentioning this necessity. (HODGES, 2007, pp. 9-10)

As a certified classicist, Hodges would rather reject any modal ingredient in the notion of "logical consequence" and would deal with it within strict Zermelo-Frankel's jargon as "preservation of all (set theoretical) models". Hodges concludes by warning us against use of any such modal and dynamical imagery:

> I can see only one reason [for including such imagery. [It] adds a certain human coloring, by suggesting that part of the mathematics is carried out by a human being. This adds nothing to the mathematical content, but somehow it helps the readability. Mathematical writers know they have to be careful about adding human content. Anything that distracts from the argument will offend some readers. (HODGES, 2007, p. 4)

## 6. Entering Wittgenstein's ideas

Let us finally move into Wittgenstein's proposals. There is one initial option which should be stressed right from the beginning, so as to avoid any possible equivocation regarding it. Just like Platonists and the latter intuitionists, the philosopher sharply rejects any idea of temporalizing mathematical validity, of accepting strange linguistic formulations such as mathematical laws "becoming valid from some moment in time onwards", say. For the philosopher mathematical laws – his "rules" – are crucially characterized as being "atemporally valid". In fact, he considers any attempt to temporalize mathematical laws (by clamming that some law could possibly be "non-valid" at some point in time, but "acquire validity" at some latter moment) as utter absurdity. The textual evidence is impressive:

> Matters of fact always involve time; mathematical facts or propositions do not. (WITTGENSTEIN, 1979, p. 184)
>
> In mathematical propositions, the "is" is not temporal. It is absurd to say, "6 × 6 is 36 at 3 o'clock". (WITTGENSTEIN, 1976, pp. 41, Lect IV)
>
> "The 100 apples in this box consist of 50 and 50" – here the non-temporal character of "consists" is important. For it doesn't mean that now, or just for a time, they consist of 50 and 50. (WITTGENSTEIN, 1983, pp. 74, I, §101)
>
> In mathematics we have propositions which contain the same symbols as, for example, "Write down the integral of ...", etc. with the difference that when we have a mathematical proposition time doesn't enter into it and in the other it does. (WITTGENSTEIN, 1979, pp. 34, Lect III)
>
> When we say: "This proposition follows from that one" here again "to follow" is being used *non-temporally*. (And this shows that the proposition does not express the result of an experiment). (WITTGENSTEIN, 1983, pp. 75, I, §103)

The intuition is clearly modal. One should never allow any mixture of the pristine necessity of mathematical laws and the mundane contingency involved in empirical occurrences. Just like the classicists, Wittgenstein sharply differentiates the contingent epistemic act, the event of its mathematical realization by some mathematician at some moment in time, from the necessary, atemporal validity of the (new) mathematical law thus obtained. But what about what we've referred above as the "Problem of Symmetry", the idea of projecting atemporal validity also towards future, yet unrealized mathematical laws? Does Wittgenstein also subscribe to a "prefiguration of future mathematics already operative in our present and even our past"? Should we say that at least part of our future mathematical achievements is already "potentially there" in our present conceptual structures as they are incarnated in our deductive systems, i.e., as logical consequences "just waiting to be adumbrated" by some young able mathematician? In short, should we accept that there is some sort of "abstract potency" linking our present epistemic state to our future achievements?

Wittgenstein's sharp answer to these questions is, as we know, a resounding "No!". His counter-proposal involves two components. To begin with, he sharply rejects any trace whatsoever of a "logical prefiguration of our mathematical future already operative in our present and in our past". Just like the classicist logician Wilfrid Hodges above, Wittgenstein insists that however we may construe the idea of "conceptual cogency", of a "psychological force" leading us towards future mathematical results, that should be sharply differentiated from the "mathematical content" of the laws finally obtained. All of this is, of course, in strict agreement with the Fregean precept of "always separating the psychological from the logical". Instead of the image of some "continuous path" connecting the epistemic stage preceding a mathematical advancement and its final, later realization, Wittgenstein proposes the very opposite idea: mathematical achievements always involve "ruptures". As we'll see below, for him mathematical advances – any mathematical advance, even the most mundane calculation such as Kondo and Yee's

– should be construed as involving a discontinuity, some sort of cleavage!

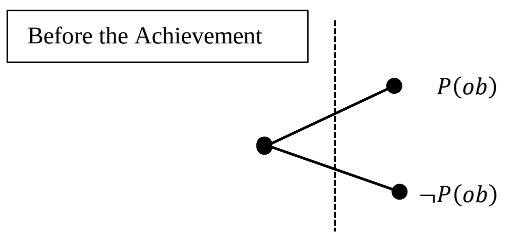
We arrive here at the second crucial component of his proposal. This supposed cleavage, this rupture, should be construed, not as alteration in our state of knowledge regarding some form of "abstract mathematical reality", but as mutation in our very concepts, our shared language, in our views regarding the limits of what could and could not possibly be conceived. For Wittgenstein, mathematical advances are not the discovery of new mathematical facts concerning some platonic realm, they are modifications in our ideas regarding what should count as "rational" and what should be dismissed as "entirely irrational", "utter absurdity".

As we've already anticipated in the beginning of our paper, Wittgenstein suggests that we should not view mathematical progress as a "glorious discovery of new mathematical properties of atemporal mathematical objects". As we've remarked above, he would rather insist on a more sober idea of viewing them as the "final shedding of absurd, mathematically incompatible suppositions we've once entertained". Thus, as we'll cover in more details below, instead of a forward inferential force, prefiguring our future, Wittgenstein proposes a backward, retroactive impact of our new present achievements over our epistemic past.

## 7. A Modal Incompatibility

Before we move on with our presentation of Wittgenstein's ideas concerning mathematical progress, it is important to acknowledge an important historical source for them: Brouwer. Despite their sharp differences regarding "temporalization of mathematical validity", Wittgenstein's proposals are clearly tributary to the intuitionist's general views on mathematical advancements as involving a sort of "branching of alternatives". As we know, these ideas on mathematical progress were latter further articulated by Beth and even more recently by Kripke into what is now know in the literature as the "Brouwer-Beth-Kripke Schema".[8]

The proposed model offers us a quite natural image of mathematical progress. According to it, we should construe our mathematical advancements as a kind of "branching structure". Simplifying the diagram all we can by considering the case of just a single mathematical conjecture, we are encouraged to construe the general form of a mathematical advancement, as viewed *before* the mathematical achievement, as involving a fork:

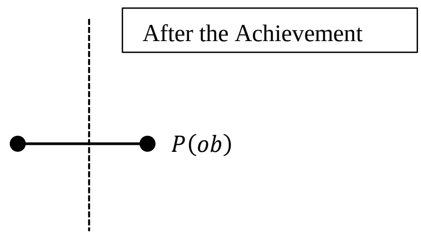



As we said above, the image is very natural. We have a property *P* concerning some mathematical object *ob*, say. And are also bound to have some sort of epistemic alternative: either the object *ob* does have the property *P*, or it doesn't, thus the fork. To be sure, that epistemic alternative had to be present, otherwise there could be no conjecture, and thus no mathematical progress. Thus, the fork

[8] Cf. Van Atten (Forthcoming).

represents a semantical condition which allow us to even introduce the idea of a "mathematical conjecture waiting to be settled".[9]

What happens to this diagram when we position ourselves *after* the mathematical advancement taken place, though? Should we still viewed it as a branching structure? Here the answer is of course, "No". Suppose we've stablished that the object *ob* does have the property *P*. Then, since all mathematical laws are strictly necessary ($\Box P$), we can safely assert that "by actually enjoying the property *P*, that object *ob* could not possibly lack such property *P*" ($\neg\Diamond\neg P$). Thus, instead of figure involving a fork, we should now rather represent our situation (from a vantage point after the mathematical advancement) as a direct line:




This striking "collapsing of the previous branching structure" (as it was represented before the advancement) into a new single mathematical necessary route (after the achievement) was vividly rendered in the following illustration taken from a recent article by (MARTIN-LÖF, 2008, p. 243):[10]

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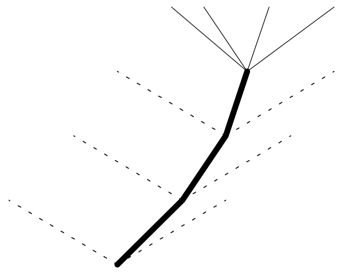
[9] This way of construing conjecture-settling is of course not exactly the one delineated by the famous BHK interpretation and which is supposed to have been incarnated in Martin-Löf's Intuitionistic Type Theory. As we'll argue bellow, the primary notion there is not of ordinary mathematical objects (numbers, functions, etc.), but of proof-objects which can in turn make-true propositional-domains ("propositions", in Martin-Löf's terminology, i.e., conjectures).

[10] I thank Luiz Carlos Pereira for always insisting with me on the importance of this point.



Let us return now to Wittgenstein's ideas. As we've seen above, in a clear opposition to the Swedish intuitionists and to the rather natural ordinary views on mathematical progress, Wittgenstein sharply rejects any kind of prefiguration of future mathematical advances as already "potentially operative" in our present. In fact, we've emphasized that the philosopher rejects any idea of "continuity", some sort of "path" linking the stages before and after the advancement. The proposed alternative is expressed in his insistence that we should construe mathematical advancements as "ruptures" instead, i.e., as not involving any sort of "non-actualized potency", some sort of "inferential force" linking the mathematical past to its future advances. We are now ready to begin accessing the modal intuitions behind such strange proposals.

Mathematical progress, any mathematical advancement, even an ordinary calculation, such as Kondo and Yee's, is bound to involve two temporal stages, one before, and one after the new mathematical law has been obtained. The difficulty is that these two moments are in some sense "modally incompatible" with each other. As we've seen, we cannot even introduce the idea of a "conjecture waiting to be settled" unless we have some sort of alternative (in our case above, the object *ob* either enjoying, or lacking, the property *P*). But that had to be our situation before the mathematical advance. Because after the new law was determined, all but one of these various alternatives are suddenly gone, vanished forever. This is precisely what Martin-Löf's illustration so vividly aims to represent above.

Exactly because we are dealing here with mathematical laws (and not merely with empirical findings), the effect of the new law is that, not only other previous alternatives turn out to be false propositions, (but still viable), rejected mathematical alternatives are actually necessarily false, they involve impossible combinations between "objects" and "properties". If we go back to our example above, once we've determined that our object *ob* does have the mathematical property *P*, we can also safely assert that it could not possibly lack it (as the previous "alternative" had suggested). But then, what exactly were we up to when, before the new law had been obtained, we were conjecturing on the (absurd) alternative $\neg P(ob)$? Exactly how can one "conjecture something which could not possibly be the case"? How can we even perform a "conceptual operation" which involves entertaining an utterly impossible, absurd state of affairs"?

The important point to be emphasized here is what we've called the "mismatch" between the epistemic stage prior to the advent of a new mathematical law, and that of its final adoption. Prior to its settlement, a mathematical conjecture involves alternatives (say, "$\Diamond P \vee \Diamond \neg P$"). But after the adoption of the new law, all these alternatives are gone in the strongest possible sense that they are proven to be, strictly speaking, absurd, pseudo-alternatives. They just could not possibly *be* anymore the case (for example, "$\Box P \wedge \neg\Diamond\neg P$").

One could try to dismiss the whole point by saying that this is nothing but the old distinction between merely epistemic possibilities and alethic possibilities. Quite correct. But Wittgenstein's point is that we only get to segregate the two kinds of modalities after the advanced has taken place. With each new mathematical result, the frontier between the alethic and the merely epistemic modalities gets retraced. And then, goes on the philosopher, faced with this "modal incompatibility", how can we conceptually connect the two moments, the one preceding the advance, and the one after the advance has taken place? The philosopher's surprising reply is that we simply cannot do that. Thus, his idea that mathematical progress would always involve conceptual ruptures, conceptual mutations. We cannot retrocede to our previous "epistemic state" any more than we can suddenly decide to be, once again, illiterates.

## 8. Rejection of the idea of a "Mathematical Substance"

While arguing in favor of what we've called a "mismatch" between the moments preceding and following an adoption of a new mathematical law we've contrasted the situation of a mathematical advancement with that of a simple empirical experimentation. The philosopher was ready to go as far as claiming that:

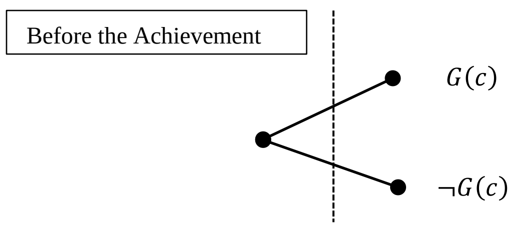
> Nothing is more disastrous to philosophical understanding than the notion of proof and experience as two different – yet still comparable – methods of verification. (WITTGENSTEIN, 2005, p. 419)

> Mathematicians, when they begin to philosophize, always make the mistake of overlooking the difference in function between mathematical propositions and non-mathematical propositions.
>
> These discussions have had one point: to show the essential difference between the uses of mathematical propositions and the uses of non-mathematical propositions which seem exactly analogous to them. (WITTGENSTEIN, 1976, p. 111)

In order to make things a bit more tangible in this initial discussion of the topic, let us illustrate our argument with a favorite example of a "mathematical conjecture" of Wittgenstein involving the operation of "dividing 1÷3".

> Suppose a person divides 1 by 3 to see whether 4 turns up in the development. I tell him "You will never get 4; it is hopeless", and draw his attention to the fact that the dividend and remainder are the same. This may never have struck him. (WITTGENSTEIN, 1979, p. 182)

And as an example of an empirical enquire, let us pick a very ordinary interrogation concerning a single empirical object, say, a question about the location of my car (whether it is in my garage, or not). Once again, we are faced with the familiar forking image of our epistemic alternatives.




Let us move this time directly onto the consideration of the moment *following* the settling of the two conjectures, both the empirical and the mathematical one. The important point we want to bring out here is the question of the advance's impact on the identity of the concerned object. Even in the actual presence of my car, right in front of us, peacefully parked in my garage, one could still insist that that very car, which is in fact here in front of me, could have been elsewhere. "Spatial location" is a contingent property of an automobile and it can thus be altered at will without any impact on the identity of the object involved. Wherever it could possibly be, it would not fail to be that very vehicle which I've once purchased.

That is not, of course, what goes on in the case of a mathematical advance. The new mathematical properties are immediately incorporated into the identity of the object in question. Returning to our example above, after having realized the (necessary) presence of the twin cycles (quotient = 3, remainder = 1, ...), it simply does not make sense anymore for us to even talk about an implementation of the division 1÷3 and yet insist on conceiving a different cycle, or in fact any different result but the one we've determined. To be sure, we could still imagine a faulty division, in which, say, that digit "4" could somehow have managed to sneak in. But regarding such implementations, one could always retort that a faulty procedure is not, strictly speaking, the intended operation. It could at best *resemble quite closely* that division, it could not *be* that operation, for if it were that very division, well, then it could not fail to exhibit the twin cycles. As Wittgenstein writes:

> To find a 2 in the division of 1 by 7 you might say is easy: here it is. ... 'Finding', however, should mean finding by correct calculation.
>
> A mathematical process is not such that it could be what it is

> and the result be a different one. (WITTGENSTEIN, 1979, pp. 183, 186)

Evoking an old terminology reminiscent of Aristotle, we could say that an automobile is a sort of "substance", an entity which manages to maintain its self-identity even while undergoing change of some of its properties through time. Not so with "mathematical objects". In the case of mathematical entities, all their properties could not fail to count also as necessary attributes of those entities. Evoking Aristotle's terminology we could say that Wittgenstein's strange suggestion is that the semantical components of our mathematical laws, mathematical "objects", could not possibly be substances, for they do not maintain their identities through mathematical innovations. His reasoning behind such strange proposals is quite direct. In sharp contrast to the empirical discoveries, the acquisition of new mathematical prosperities is always immediately reconceptualized as necessary traits of the objects in question. Mathematical "objects" could not be those very entities they are and yet somehow fail to exhibit the entire range of their (necessary) properties, even the one which were only recently determined, previously utterly unforeseen new attributes. Wittgenstein writes:

> It impossible for us to discover rules of a new type that hold for a form with which we are familiar. If they are rules which are new to us, then it isn't the old form. For, only the group of rules *defines* the sense of our signs, and any alteration (e.g., supplementation) of the rules means an alteration of the sense. Just as we can't alter the marks of a concept without altering the concept itself. (Frege) (WITTGENSTEIN, 1975, pp. 182, §154)

## 9. A New Construal of the Idea of "Mathematical Progress"

We've been insisting all along on two antithetical views on mathematical progress. According to the more traditional view, mathematical advances should be construed as involving a "continuity", a "path" which would link the moments preceding the new law, and those of its final realization. We are thus invited to view mathematical progress as the final realization of some sort of "inferential potency" which would already be operative before the advancement took place and which would have been thus "merely actualized" by its "discoverer" (as a sort of "America" waiting for its own "Columbus").

In direct opposition to this idea, Wittgenstein insists on rejecting any sort of link between the period preceding the advancement and that of its final realization. As we've seen above, the philosopher urges us to view mathematical advancements as "ruptures", "discontinuities". In our last section we've began finding out just whence this insistence comes from. The proposal is based on modal claims. In an acute contrast to that of empirical discoveries, both the rejection of old epistemic alternatives and the adoption of new mathematical laws are all necessities. Consequently, these new properties (and the rejection of previous "hypothetical alternatives") cannot but be incorporated into the very demarcation of what could count as being the "objects" involved.

A crucial result is then the rejection of the notion of a "mathematical substance", the idea of an "abstract pivot", an entity which, by being capable of maintaining its self-identity throughout the process, would thus be able to provide the very link connecting the moments preceding and following mathematical advances. The rejection of the idea of a "mathematical object", of an "abstract substance", is thus a further, ultimate rejection of any "mathematical link" connecting occasions, before and after the advance. Not even the very objects involved remain the same. Thus, the idea we've already mentioned right in the beginning of our argument, urging us to view mathematical progress, not as a "glorious discovery of new mathematical properties concerning old, atemporal mathematical objects", but as the "final shedding of absurd, mathematically incompatible suppositions we've once somehow manage to entertain".

As we've remarked above, the impact of these ideas to Wittgenstein's philosophy of mathematics is immense and utterly contrary to our ordinary views of mathematics. As an early example of these latter, extravagant proposals we could mention the idea that, strictly speaking one could never settle a mathematical conjecture just because by settling it one would end up changing the very meanings of the terms involved in the original interrogation:

> Wouldn't this imply that we can't learn anything new about an object in mathematics, since, if we do, it is a new object? (WITTGENSTEIN, 1975, p. 183 §155)
>
> Now how about this — ought I to say that the same sense can only have *one* proof? Or that when a proof is found the sense alters?
>
> Of course, some people would oppose this and say: "Then the proof of a proposition cannot ever be found, for, if it has been found, it is no longer the proof of *this* proposition". (WITTGENSTEIN, 1983, pp. 366, VII, §10)

These strange views are of course completely foreign to usual conceptions regarding what "proofs" are, on how to construe the notion of "inference". But these will have to be dealt with in another place.

**Resumo**: O objetivo deste artigo é tentarmos elucidar a extravagante tese de Wittgenstein de que todo e qualquer avanço matemático envolve alguma "mutação semântica", ou seja, alguma alteração nos próprios significados dos termos envolvidos. Para isso, argumentaremos a favor da ideia de uma "incompatibilidade modal" entre os conceitos envolvidos, como eram antes do avanço, e o que se tornam após a obtenção do novo resultado. Também argumentaremos que a adoção dessa tese altera profundamente nossa maneira tradicional de construir a ideia de "progresso" em matemática.

**Palavras-chave**: Wittgenstein, Filosofia da Matemática, Mutação Semântica, Progresso Matemático.

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