Title: Objective Logic of Consciousness

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Abstract

We define consciousness as the category of all conscious experiences. This immediately raises the question: What is the essence in which every conscious experience in the category of conscious experiences partakes? We consider various abstract essences of conscious experiences as theories of consciousness. They are: (i) conscious experience is an action of memory on sensation, (ii) conscious experience is experiencing a particular as an exemplar of a general, (iii) conscious experience is an interpretation of sensation, (iv) conscious experience is referring sensation to an object as its cause, and (v) conscious experience is a model of stimulus. Corresponding to each one of these theories we obtain a category of models of conscious experiences: (i) category of actions, (ii) category of idempotents, (iii) category of two sequential maps, (iv) category of brain-generalized figures, and (v) functor categories with intuition as base and conceptual repertoire as exponent, respectively. For each theory of consciousness we also calculate its truth value object and characterize the objective logic intrinsic to the corresponding category of models of consciousness experiences.

Introduction

What is consciousness? Consciousness, according to Koch, "is everything you experience. It is the tune stuck in your head, the sweetness of chocolate mousse, the throbbing pain of a toothache, the fierce love for your child and the bitter knowledge that eventually all feelings will end" (Koch, 2018, p. S9). This raises two foundational questions:

1. What is the nature of conscious experiences?

2. What is the nature of consciousness?

How are we to think of the totality of conscious experiences i.e., consciousness? How are we to think of the constituents of consciousness i.e., conscious experiences? One obvious answer: Conscious experiences are objects of the category of all conscious experiences and consciousness is the category of conscious experiences. In other words, every conscious experience has the essence of the category of conscious experiences, whatever the essence(s) maybe. This characterization is in the spirit of asserting that a chair is an object of the category of chairs.

Let us consider a visual experience: a face. A first-order approximation would represent the experience as a point in a feature-space or as a set of features i.e., Face = {eyes, nose, mouth} (Fodor, 1998). Sensory features are obviously structured, unlike the structureless elements of sets (Lawvere and Rosebrugh, 2003, p. 1). Equally importantly, sensory features of a visual object are related to one another in specific ways resulting in a cohesive object that is conscious experience, which cannot be modelled as a set with its zero internal cohesion (Lawvere and

Schanuel, 2009, p. 146). Elementism, notwithstanding the Gestalt demonstrations (Albright et al., 2000, p. S34), continues to be the default terminology as in analysing "perceptual experience into a collection of simple sensory elements" (Albright, 2013a, p. 19). Along similar lines, mind is defined as a set of brain functions (Bunge, 1981, p. 68; Kandel, 2013, p. 546). The claim that 'mind is a set' is repeatedly asserted in the textbook Principles of Neural Science (Kandel et al., 2013, p. 5, 334, 384), which takes on added significance in light of its pedagogical value in training neuroscientists. Of course, this terminology does not reflect any failure to recognize that, in terms of the above example of face perception, the constituent eyes, nose, and mouth, unlike the structureless elements of a set, are figures of various shapes; these figures constituting a face are related to one another in specific ways (cf. Croner and Albright, 1999). Nevertheless, it does highlight the absence and the significance of having a conceptual repertoire that fits the reality of conscious experiences. Here we put forward mathematical category as a construct suited for the study of consciousness (Lawvere, 1994; Lawvere and Schanuel, 2009, p. 21, 135-148). In line with the commonplace understanding of the notion of category, a mathematical category consists of objects all of which partake in the essence that is characteristic of the category; since every object of the category partakes in the essence, the transformations of objects preserve the essence (e.g. in the category of dogs, a transformation of an young dog into an old dog preserves the "dogness"). We find that defining conscious experience as an object of the category of conscious experiences, instead of as cohesion-less set of structure-less elements, provides the conceptual repertoire—basic shapes, figures, and incidence relations—needed to reason about the essence of conscious experiences and the essence-preserving transformations of conscious experiences.

Theory of Conscious Experiences

What is the essence of conscious experiences? Continuing with our example of face perception, an experience of a face can be said to consist of figures of various shapes: two eye-shaped figures, one nosed-shaped figure, and one mouth-shaped figure. Of these shapes, we can say that eye, nose, and mouth are the basic shapes, and their incidence relations determine the mutual relations between various basic-shaped figures constituting the face (Lawvere and Schanuel, 2009, pp. 82-83, 250-253, 369-371). When considering conscious experience in general, we may treat sensory features (e.g. colour, shape), modalities (visual, tactile, etc.), and emotion, among others, as basic shapes. For illustration, anger (in conscious experience) can be considered as an emotion-shaped figure (in the experience) just as redness can be thought of as a colour-shaped figure. The mutual relations between basic shapes, say, emotion and colour, determine the mutual relations between figures of the corresponding shapes (anger and redness).

Basic shapes along with their incidence relations constitute the abstract essence or theory of the category of conscious experiences (Lawvere, 2003, p. 215, 217; Lawvere, 2004a, pp. 10-12; Lawvere and Rosebrugh, 2003, pp. 154-155, 235-236; Lawvere and Schanuel, 2009, pp. 149-151, 369-371). First, every experience has the essence [of the category of conscious experiences] given by the basic shapes and their incidence relations. Next, every experience can be represented as a structure formed of basic-shaped figures and their mutual relations induced by the incidences of basic shapes (see Fig. 4 in Posina, Ghista, and Roy, 2017). Since every experience has the essence of experiences, transformations of experiences are required to preserve the essence of experiences, and as such are natural transformations (Lawvere and Schanuel, 2009, p. 378). Geometrically speaking, natural transformations ‘do not tear’ the structure transformed (ibid, p. 210). Philosophically, a natural transformation is: Becoming consistent with Being (e.g. biological growth; Posina, 2016).

What are we to make of the totality of all conscious experiences along with their essence-preserving transformations? Objects along with essence-preserving morphisms of objects form a category. With experiences as objects [with a given structural essence] and essence-preserving transformations of experiences as structure-preserving morphisms of objects, consciousness—the totality of conscious experiences—can be construed as a category of conscious experiences. Note that any experience can remain the same (identity transformation). If I went from sad to happy and from happy to detached, then I went from sad to detached (composition of transformations of experiences). Along these lines, other axioms and laws, which are required to be satisfied in order for us to talk about a category of experiences, can be verified (Lawvere and Schanuel, 2009, p. 21, 149-160). Within this categorical framework, the structure of consciousness is an external reflection of the structural essence of conscious experiences (Lawvere, 1972, p. 10). More immediately, a category embodies a mode of cohesion (Lawvere and Schanuel, 2009, p. 146), which is the most basic attribute of conscious experience. For example, parts of a body (hands, legs, etc.) have a mode of cohesion, which is different from the mode of cohesion of parts of a perceptual object (colour, shape). Note that ‘part’ is both itself and its relationship to the whole (Lawvere, 1994, p. 53). Formally, a part of an object is not merely a subobject, but a monomorphism specifying the inclusion of the subobject into the object (Lawvere and Schanuel, 2009, p. 335).

As illustrations of theory of a category and its basic shapes, we present simple theories (abstract essences) of conscious experiences (in the spirit of Lawvere, 1999). More explicitly, the mathematical method, according to F. William Lawvere, “consists of taking the main structure [of an object] by itself as a first approximation to a theory of the object, i.e. mentally operating as though all further structure of the object simply did not exist” (Lawvere, 1972, pp. 9-10). An example of an abstract theory of conscious experiences is ‘particular as an exemplar of a general’ (cf. categorical perception; Albright, 2013b, pp. 628-630; Grossberg, 1976), whose models form a category of idempotents (Lawvere and Schanuel, 2009, pp. 99-106), with exemplar as its basic shape. The truth value object of the category of idempotents has three global truth values. With ‘interpretation of sensation’ (Albright and Stoner, 1995; Croner and Albright, 1999; Schlack and Albright, 2007) as a theory of conscious experiences, we obtain a category of two sequential processes as the category of conscious experiences. Here, the basic shapes are physical stimuli, neural sensation of stimuli, and conscious interpretation of sensation. With conscious experience as an object of the category of two sequential functions, we find that the objective logic intrinsic to consciousness has four truth values (Posina, Ghista, and Roy, 2017, pp. 172-174). We also consider ‘action of memory on sensation’ (Albright, 2012, Fig. 5, 8; Hopfield, 1982; Lawvere and Schanuel, 2009, p. 218), ‘referring sensation to an object as its cause’ (Albright, 2015, p. 22; Lawvere and Rosebrugh, 2003, pp. 125-126, 148-152), and ‘model of stimulus’ (Chalmers, 2006; Posina, Ghista, and Roy, 2017) as theories of conscious experiences.

Given a category of experiences, how do we abstract the theory (essence) of experiences? Theorization begins with measurements of properties of the objects of the given category. Oftentimes, we find that there is small subcategory of properties (and their determinations) within the category of all properties that constitutes the abstract essence shared by all objects of the given category. This abstract essence in which every object of a given category partakes is the theory of the given category (Lawvere, 1994, pp. 44-47; Lawvere and Rosebrugh, 2003, pp. 154-155; Lawvere and Schanuel, 2009, pp. 149-150; see also Fig. 5 in Posina, Ghista, and Roy, 2017). In geometric terminology, we consider a subcategory of basic shapes and their incidence relations, and examine if figures with objects in the subcategory as shapes are adequate to completely characterize every object of the category and tell apart transformations between objects (Lawvere, 1994, p. 49; Lawvere and Schanuel, 2009, pp. 369-371). In the following we focus on the calculation of truth value objects corresponding to various theories of conscious experiences and the subsequent characterization of the objective logic intrinsic to various categories of models of conscious experiences (ibid, pp. 335-357).

Action of Memory on Sensation

Conscious experience of a given physical stimulus can be thought of as an action of memory on the sensation elicited by the stimulus (for a vivid illustration of the action of memory on sensation, see Fig. 5 and 8 in Albright, 2012). A formalization of conscious experience as an action of memory on sensation is provided by Hopfield (1982). Here sensation S is a $1 \times n$ feature vector, with each one of the elements of the vector S representing the activity of each one of the n feature-selective neurons. Memory M, or the $n \times n$ synaptic weight matrix, is a result of associative learning, and can be expressed as a product of the sensation S with its transpose S^T, i.e.

$$M = S^T \times S$$

Conscious experience C(S) corresponding to sensation S is:

$$C(S) = S \times M.$$

For a given memory, conscious experiences corresponding to various sensations have the structure of idempotents (as discussed in detail in appendix A1). Categorical perception, wherein particulars (stimuli) are perceived as exemplars of a general (category; Albright, 2013b, pp. 628-630), also has the structure of idempotents. The abstract essence or theory of the category of idempotents consists of one basic shape: exemplar, along with an idempotent endomap as the structural map. Unlike the classical Boolean logic of sets, we find that the truth value object of the category of idempotents consists of three truth values. Also, two dual forms of negation—*not*, *non*—can be defined (Lawvere, 1986, 1991). We find that double negation can be greater or less than identity depending on the exact nature of negation. Furthermore, the category of idempotents admits logical contradiction (Lawvere, 2003, p. 214-215; or boundary operation defined as the intersection of a part with its negation; Lawvere, 1994, p. 48; Lawvere and Rosebrugh, 2003, p. 201).

Interpretation of Sensation

Conscious experience involves two sequential processes of sensation (of stimulus) followed by interpretation of the sensation. A classic illustration of the two sequential processes involved in conscious experience is R. C. James's image (Miller, 1999). When looking at the image one initially sees black and white blobs of various sizes and shapes, which subsequently, in light of the concept DALMATIAN, is perceptually interpreted as a dog. That conscious experience is mediated by the two processes of sensation followed by interpretation is well-established in various perceptual modalities (e.g. Albright and Stoner, 1995; Croner and Albright, 1999). Thus the abstract theory of consciousness consists of three basic shapes i.e. objects (Physical Stimuli, Neural Codes, and Conscious Experiences) and two incidence relations i.e. maps (*sensation* and *interpretation*) organized as shown below:

Physical Stimuli –*sensation*→ Neural Codes –*interpretation*→ Conscious Experiences

The truth value object of the category of models of conscious experiences of the above theory of consciousness consists of four global truth values (Fig. 6c in Posina, Ghista, and Roy, 2017 depicts the internal diagram of the truth value object; see Appendix A2 in Posina and Roy, 2018 for the calculation of the truth value object; see also Linton, 2005).

Brain-generalized Figures

Everyday experience of effectively interacting with objects of conscious experience indicates that conscious experience of objects is recovery of the objects based on the sensation elicited by the objects (i.e. constructing an object isomorphic to the cause of sensation; Albright, 2015, p. 22; Lawvere and Rosebrugh, 2003, pp. 125-126, 148-152). In reconstructing objects in conscious experience, we encounter the possibility of not only the commonplace isomorphism between objects and conscious experience of objects, but also illusory conscious experiences with no correspondent underlying objects. We now present a mathematical framework rich enough to capture both veridical perception and illusions (Lawvere, 2004b). Given a stimulus A, we define sensation p as a brain V-valued property of the stimulus A, i.e.

$$p\colon A \to V$$

Next, we consider brain V-valued properties of sensations, i.e.

$$q\colon V^A \to V$$

which may be considered as a special case of interpretation of sensation, i.e. a neural measure of sensation. Perceived element is defined as a brain V-generalized point of A satisfying naturality conditions (discussed in detail in appendix A2). Within this mathematical framework, we find that defining brain V as neuronal states of ‘firing’ and ‘not firing’, i.e. as a two-element property type, can give rise to illusions. However, defining brain V in terms of all possible changes in neuronal firing rate, i.e. as a three-element set V = {decreased firing rate, constant firing rate, increased firing rate}, ensures illusion-free perception, or isomorphism between elements of a stimulus set and perceived elements. As we have been emphasizing, physical stimuli, neural sensations, and conscious experiences are much more structured than structureless sets; as such we are working on refining the mathematical framework to accommodate conscious experience defined as brain-generalized figure of stimuli (for basic shapes and their incidences of the categories corresponding to physical stimuli and neural sensation).

Model of Physical Stimulus

Of the various abstract essences (theories) of conscious experiences, the most basic characterization of conscious experience is: conscious experience of a stimulus is a model of the stimulus (Chalmers, 2006). This immediately suggests functorial semantics, which provides a mathematical account of constructing models of particulars, as an abstract theory of conscious experience (Lawvere, 1994, 2004a). Given a category of physical stimulus, a model or conscious experience of the stimulus is calculated by abstracting the essence (mental concepts) of its brain-valued properties i.e. sensation. Thus abstracted mental concepts are then interpreted into a background category of intuition to obtain models of the physical stimuli or conscious experiences. The objective logic of conscious experiences construed as functor categories, with intuition as base and mental concepts as exponent, is not classical (Posina, Ghista, and Roy, 2017). Furthermore, subjectivity (understood as viewpoint; cf. Sen, 1993) is captured by the framework of functorial semantics; more specifically, the mathematical construct of monad determines how a category of particulars is [subjectively] generalized into the adjoint pair of functors: mental concepts (theories) and conscious percepts (defined as functorial interpretation of concepts into a background of intuition or models; Eilenberg and Moore, 1965; Lawvere, 1994, 2004a). As such, functorial semantics is the objective logic (as defined in Lawvere, 1994, p. 43) of consciousness.

Conclusions

We defined conscious experience as an object of the category of conscious experiences, which aligns with the intuitions engendered by our everyday experiences with objects (cf. a table is an object of a category of tables). It is fascinating to note that the most advanced scientific understanding of object (as an object of a category of objects; Lawvere, 2015) is in accord with our ordinary experience. The category of conscious experiences provides the conceptual repertoire—basic shapes, figures, and incidences—needed to develop an adequately explicit theory of conscious experience. Given that the objective logic intrinsic to conscious experiences is not classical for a variety of abstract essences of consciousness that we considered, it would be interesting to compare the objective logic of consciousness with quantum logic that was found to better account for cognition compared to the classical logic (Roy, 2016).

Acknowledgments

We dedicate our paper to the memory of Professor B. V. Sreekantan. VRP is truly grateful to Professors: Andrée C. Ehresmann and F. William Lawvere for invaluable help in learning category theory. Thanks also to Professor Narasimhan Marehalli and Dr. Ruadhan O'Flanagan for helpful discussions. One of the authors (Sisir Roy) greatly acknowledges Homi Bhabha Trust for financial assistance for this work.

Appendices

A1. Category of Idempotents

Let's consider, within the framework of Hopfield model (1982), a neural network consisting of two neurons coding for two features. For a given stimulus, a neuron can respond with a decrease, constant, or increase in its firing rate. So, we have nine 1 × 2 vectors as possible sensations S:

S_1 = [-1 -1]

S_2 = [-1 0]

S_3 = [-1 +1]

S_4 = [0 -1]

S_5 = [0 0]

S_6 = [0 +1]

S_7 = [+1 -1]

S_8 = [+1 0]

S_9 = [+1 +1]

We define memory as a synaptic weight matrix M with entries given by associative learning:

$$m_{ij} = s_i \times s_j$$

where (subscripts) i, j index the two neurons. For example, memory M_1 of sensation S_1 is:

$$M_1 = S_1^T \times S_1$$

(T denotes transpose), which is a 2 × 2 weight matrix. Substituting the values of sensation S_1, we find:

$$M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Next, we define perception P as an action of memory M on sensation S:

$$P = S \times M$$

With memory $M = M_1$, we find the percepts resulting from the action of memory M_1 on the nine sensations:

$P_1 = S_1 \times M_1 = 2[-1\ -1]$

$P_2 = S_2 \times M_1 = [-1\ -1]$

$P_3 = S_3 \times M_1 = [0\ 0]$

$P_4 = S_4 \times M_1 = [-1\ -1]$

$P_5 = S_5 \times M_1 = [0\ 0]$

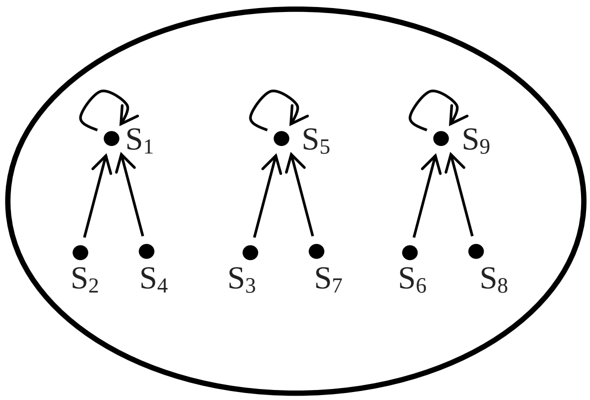
$P_6 = S_6 \times M_1 = [+1\ +1]$

$P_7 = S_7 \times M_1 = [0\ 0]$

$P_8 = S_8 \times M_1 = [+1\ +1]$

$P_9 = S_9 \times M_1 = 2[+1\ +1]$

Note that P_1, P_5, and P_9 are fixed-points (with, say, S_1 as initial state and P_1 (= S_1) as final state in the language of dynamical systems), while sensations S_1, S_2, and S_4 perceived as P_1; sensations S_3, S_5, and S_7 perceived as P_5; sensations S_6, S_8, and S_9 perceived as P_9. The dynamics of action of memory on sensation has the structure of idempotents as displayed below:



The above neural network can be formalized with action *P* of memory M on sensation S defined as a map:

$$P: S \times M \rightarrow S$$

For a neural network of N neurons, memories M are N × N matrices, which when thought of endomaps N → N, and with matrix multiplication as composition (of endomaps; MacLane, 1998, p. 11), form a monoid (Lawvere and Rosebrugh, 2003, p. 167). The N × N matrix with all diagonal elements as 1 serves as monoid identity (ibid, p. 77), while the N × N matrix with all entries as 0 is a constant C, since $C \times M = C$ for all memories M. Since this is the only constant of the monoid of memories, the category of actions (Lawvere and Schanuel, 2009, p. 360):

$$P\colon S \times M \rightarrow S$$

forms a topos of idempotents (ibid, p. 367), i.e. a category with truth value object. The truth value object of the category of idempotents has three truth values:

$$V = \{\mathit{false}, u, \mathit{true}\}$$

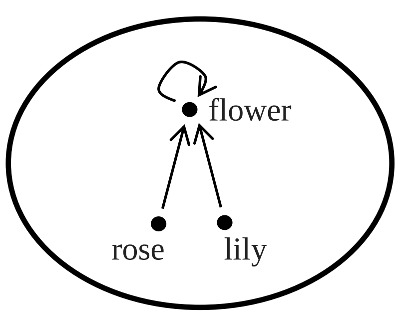
equipped with an idempotent endomap:

$$v\colon V \rightarrow V$$

defined as

$v(\mathit{false}) = \mathit{false}$, $v(u) = \mathit{true}$, and $v(\mathit{true}) = \mathit{true}$.

Categorical perception, wherein a particular (rose) is perceived as an exemplar of a general (flower), also has the structure of idempotents (see also Lawvere and Schanuel, 2009, p. 106):



Given the splitting of idempotent endomaps into section-retract pairs, the category of idempotents can also be characterized in terms of the opposite pair of section-retract maps, wherein the basic shapes are particulars and generals, while sorting and exemplifying are the incidence relations between the two basic shapes (Kathpalia, Posina, and Nagaraj, 2017).

We now calculate the truth value object of the category of idempotents. Truth value object of the category of idempotents is an object of the category of idempotents (just as in the case of sets, where a two-element set **2** = {false, true} is the truth value object of the category of sets). An object of the category of idempotents (modelled in the category of sets) is a set A equipped with an idempotent endomap

$$a\colon \mathrm{A} \rightarrow \mathrm{A}$$

satisfying

$$a \circ a = a.$$

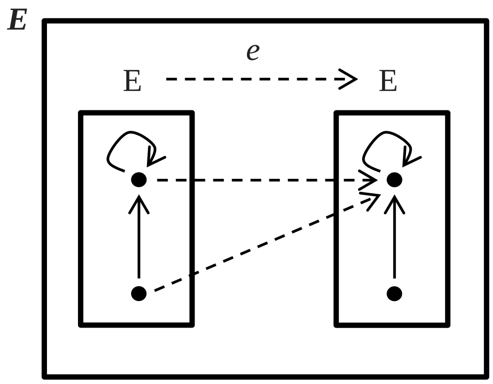
where '∘' denotes composition of maps. A map f from an idempotent a to an idempotent b (where b: B → B satisfying $b \circ b = b$) is a function

$$f\colon \mathrm{A} \rightarrow \mathrm{B}$$

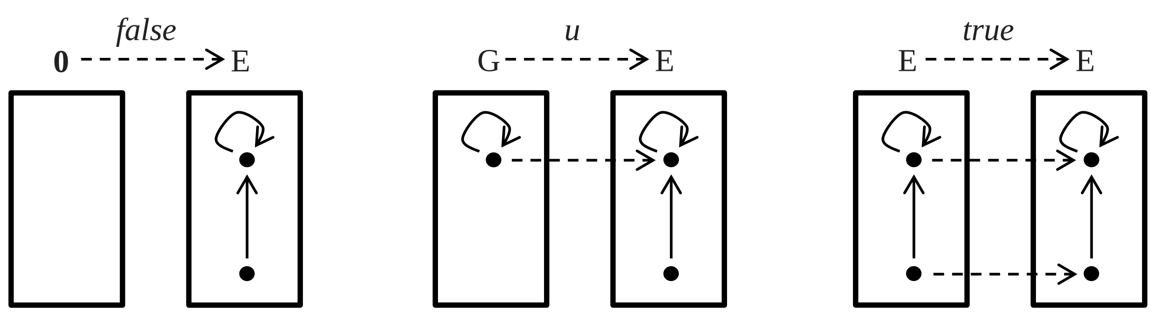
satisfying

$$f \circ a = b \circ f.$$

The truth value object of a category is defined as an object representing every part (monomorphism) of any object of the category. The truth value object can be calculated in terms of the inverse images of parts of basic shapes along incidence relations (structural maps). Basic shapes along with incidence relations constitute the abstract essence or the theory of a given category. What are the basic shapes and their incidences constituting the theory of category of idempotents? The theory of idempotents consists of a generic idempotent E, along with an [non-identity] idempotent endomap *e* on E (as shown below):

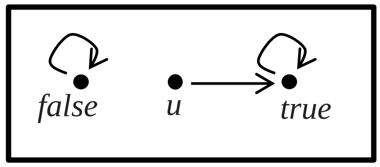


The basic shape E along with the incidence relation *e* together constitute the theory subcategory ***E*** of the category of idempotents. The basic shape E has three parts *false*, *u*, and *true* (shown below), which, by the definition of truth value object, correspond to three E-shaped figures (truth values) in the truth value object of the category of idempotents.

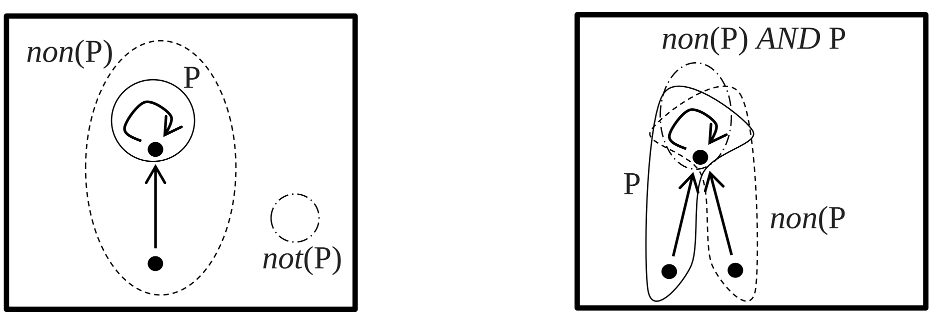


The idempotent endomap *v*: V → V, where V = {*false*, *u*, *true*}, is given by the inverse images of the three parts along the incidence relation *e*: E → E. The inverse image of the part *false* along *e* is *false*; the inverse image of the part *u* along *e* is *true*; and the inverse image of the part *true* along *e* is also *true*. Putting it all together we obtain the truth value object *v* (depicted below) of the category of idempotents.

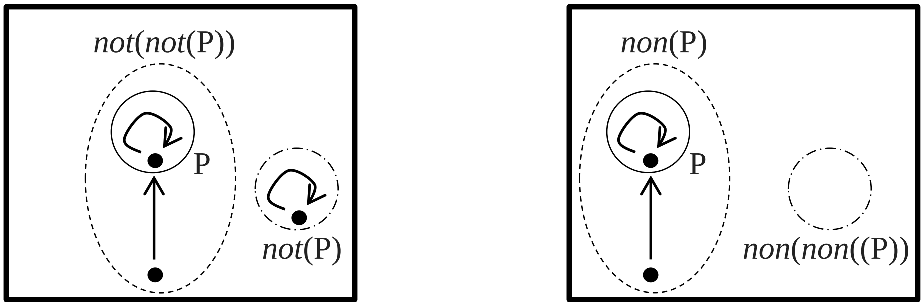
v: V → V



Based on the truth value object *v*, logical operations negation, *AND*, and *OR* can be defined. Given a part P of an object A (of the category of idempotents), the familiar negation *not*(P) is defined as the largest part of A whose intersection with P is empty. Dually, another negation *non*(P) is defined as the smallest part of A whose union with P is the whole object A. In the category of sets, both negation operations—*not*, *non*—are identical. However, in the category of idempotents, these two negation operations can give different results (as shown below). The negation operation *non* allows logical contradiction: P *AND non*(P), which is boundary in geometric terminology (Lawvere, 1986, 1991; Lawvere and Rosebrugh, 2003, p. 201).



Furthermore, double negation can be bigger or smaller than identity depending on the nature of negation as shown below:



A2. Illusion-free Perception

Consider a set A and a set V of values of properties of A. A function

$$p: \mathrm{A} \to \mathrm{V}$$

is a V-valued property of the set A. The set of all V-valued properties of A (or more broadly, the set of all functions from the domain set A to the codomain set V) is the map set V^A. The set V^A of properties, in turn, has properties, which are functionals

$$q: \mathrm{V}^{\mathrm{A}} \to \mathrm{V}$$

where the functional q is a V-valued property of the set V^A of all V-valued properties of A. Our objective is to reconstruct the set A from the set $V^{(V^A)}$ of properties of its V-valued properties (Lawvere and Rosebrugh, 2003, pp. 125-126, 148-152). Towards this end, we define generalized point. First recollect that a point of a set A is a function

$$a: \mathbf{1} \to \mathrm{A}$$

where **1** = {*} is the terminal set of the category of sets, i.e. a single-element set; hence points of a set correspond to its elements. Since elements of set completely characterize a set, we define generalized point so as to establish a 1-1 correspondence between points and generalized points (perceived elements).

A V-generalized point of the set A is a functional

$$q\colon \mathrm{V}^{\mathrm{A}} \rightarrow \mathrm{V}$$

such that for each V-valued property of A

$$p\colon \mathrm{A} \rightarrow \mathrm{V}$$

and for every endomap of the property type V

$$e\colon \mathrm{V} \rightarrow \mathrm{V}$$

the following equation is satisfied:

$$q\,(e \circ p) = e\,(q\,(p))$$

where '∘' denotes composition of functions and parentheses denote evaluation of functions.

Let's first consider the left-hand side

$$q\,(e \circ p)$$

The V-valued property of A

$$p\colon \mathrm{A} \rightarrow \mathrm{V}$$

is an element of the set V^A of all V-valued properties of A. Furthermore, since the codomain V of p is same as the domain V of

$$e: V \rightarrow V$$

we can compose them to obtain a composite

$$e \circ p = A \rightarrow V \rightarrow V = A \rightarrow V$$

which is also an element of the set V^A of all V-valued properties of A; and hence when the functional

$$q: V^A \rightarrow V$$

is evaluated at ‘$e \circ p$’ gives an element ‘v’ of the set V of values

$$q\,(e \circ p) = v$$

Let us now consider the right-hand side

$$e\,(q\,(p))$$

Once again

$$p: A \rightarrow V$$

is a V-valued property of A, i.e. an element of the set V^A of all V-valued properties of A. Hence evaluating the functional

$$q: V^A \rightarrow V$$

at p gives a value

$$q\ (p)$$

which is an element of the set V of values. Hence the endomap

$$e\colon \mathrm{V} \rightarrow \mathrm{V}$$

when evaluated at the element ‘$q\ (p)$’ of domain set V gives an element of V, i.e.

$$e\ (q\ (p)) = \mathrm{v}'$$

Summing up, the functional

$$q\colon \mathrm{V}^{\mathrm{A}} \rightarrow \mathrm{V}$$

is a V-generalized point of A (perceived element of A) if for every

$$p\colon \mathrm{A} \rightarrow \mathrm{V}$$

and for every

$$e\colon \mathrm{V} \rightarrow \mathrm{V}$$

we find that

$$q\ (e \circ p) = e\ (q\ (p))$$

or in terms of our above example

$$\mathrm{v} = \mathrm{v}'.$$

Returning to our main objective, i.e., establishing an isomorphism between points and perceived elements (generalized points) involves finding a property type V such that for every set A, the V-generalized points of A, i.e. the functionals

$$q: V^A \rightarrow V$$

satisfying

$$q\ (e \circ p) = e\ (q\ (p))$$

for every

$$p: A \rightarrow V$$

and for every

$$e: V \rightarrow V$$

are in 1-1 correspondence with the points

$$a: \mathbf{1} \rightarrow A$$

of the set A.

We give an example of perceived elements (generalized points) corresponding to points of a set. Let A = {a1, a2} be the object of our investigation and V = {0, 1} be the property type. There are two points in A

$$a1: \mathbf{1} \rightarrow A$$

and

$$a2: \mathbf{1} \rightarrow A$$

Now let's calculate the number of generalized points. First, there are four functions from A to V, i.e., four V-valued properties of A, p: A → V ($V^A = 2^2 = 4$)

$$p1: A \rightarrow V;\ p1(a1) = 0,\ p1(a2) = 0$$

$$p2: A \rightarrow V;\ p2(a1) = 1,\ p2(a2) = 0$$

$$p3: A \rightarrow V;\ p3(a1) = 0,\ p3(a2) = 1$$

$$p4: A \rightarrow V;\ p4(a1) = 1,\ p4(a2) = 1$$

Thus,

$$V^A = \{p1, p2, p3, p4\}$$

Next, there are 16 functionals, q: $V^A \rightarrow V$ ($V^{(V^A)} = 2^{(2^2)} = 16$)

$$q1: V^A \rightarrow V;\ q1(p1) = 0,\ q1(p2) = 0,\ q1(p3) = 0,\ q1(p4) = 0$$

$$q2: V^A \rightarrow V;\ q2(p1) = 1,\ q2(p2) = 0,\ q2(p3) = 0,\ q2(p4) = 0$$

$$q3: V^A \rightarrow V;\ q3(p1) = 0,\ q3(p2) = 1,\ q3(p3) = 0,\ q3(p4) = 0$$

$$q4: V^A \rightarrow V;\ q4(p1) = 1,\ q4(p2) = 1,\ q4(p3) = 0,\ q4(p4) = 0$$

$$q5: V^A \rightarrow V;\ q5(p1) = 0,\ q5(p2) = 0,\ q5(p3) = 1,\ q5(p4) = 0$$

$$q6: V^A \rightarrow V;\ q6(p1) = 1,\ q6(p2) = 0,\ q6(p3) = 1,\ q6(p4) = 0$$

$$q7: V^A \rightarrow V;\ q7(p1) = 0,\ q7(p2) = 1,\ q7(p3) = 1,\ q7(p4) = 0$$

$$q8\colon V^A \to V;\ q8(p1) = 1,\ q8(p2) = 1,\ q8(p3) = 1,\ q8(p4) = 0$$

$$q9\colon V^A \to V;\ q9(p1) = 0,\ q9(p2) = 0,\ q9(p3) = 0,\ q9(p4) = 1$$

$$q10\colon V^A \to V;\ q10(p1) = 1,\ q10(p2) = 0,\ q10(p3) = 0,\ q10(p4) = 1$$

$$q11\colon V^A \to V;\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$$

$$q12\colon V^A \to V;\ q12(p1) = 1,\ q12(p2) = 1,\ q12(p3) = 0,\ q12(p4) = 1$$

$$q13\colon V^A \to V;\ q13(p1) = 0,\ q13(p2) = 0,\ q13(p3) = 1,\ q13(p4) = 1$$

$$q14\colon V^A \to V;\ q14(p1) = 1,\ q14(p2) = 0,\ q14(p3) = 1,\ q14(p4) = 1$$

$$q15\colon V^A \to V;\ q15(p1) = 0,\ q15(p2) = 1,\ q15(p3) = 1,\ q15(p4) = 1$$

$$q16\colon V^A \to V;\ q16(p1) = 1,\ q16(p2) = 1,\ q16(p3) = 1,\ q16(p4) = 1$$

Of these 16 functionals, there are only two V-generalized points of A corresponding to the two points of A = {a1, a2}. The two V-generalized points of A are

$$q11\colon V^A \to V;\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$$

and

$$q13\colon V^A \to V;\ q13(p1) = 0,\ q13(p2) = 0,\ q13(p3) = 1,\ q13(p4) = 1$$

i.e. they both satisfy

$$q\ (e \circ p) = e\ (q\ (p))$$

Let’s consider the functional

$$q11: V^A \to V;\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$$

In order for the functional *q11* to be a V-generalized point of A, it has to satisfy

$$q11\ (e \circ p) = e\ (q11\ (p))$$

for all four p: A → V and all four endomaps e: V → V. They are

$$e1: V \to V;\ e1(0) = 0,\ e1(1) = 0$$

$$e2: V \to V;\ e2(0) = 1,\ e2(1) = 0$$

$$e3: V \to V;\ e3(0) = 0,\ e3(1) = 1$$

$$e4: V \to V;\ e4(0) = 1,\ e4(1) = 1$$

So, the set of all endomaps of property type V is $V^V = \{e1, e2, e3, e4\}$.

Thus, there are 16 cases we have to evaluate to show that the functional

$$q11: V^A \to V;\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$$

is a V-generalized point of A.

Case 1: (*p1*, *e1*)

$$q11\ (e1 \circ p1) = e1\ (q11\ (p1))$$

LHS: $q11\ (e1 \circ p1) = q11\ (p1) = 0$

RHS: $e1\ (q11\ (p1)) = e1\ (0) = 0$

Case 2: (*p2*, *e1*)

$$q11\ (e1 \circ p2) = e1\ (q11\ (p2))$$

LHS: $q11\ (e1 \circ p2) = q11\ (p1) = 0$

RHS: $e1\ (q11\ (p2)) = e1\ (1) = 0$

Case 3: ($p3$, $e1$)

$$q11\ (e1 \circ p3) = e1\ (q11\ (p3))$$

LHS: $q11\ (e1 \circ p3) = q11\ (p1) = 0$

RHS: $e1\ (q11\ (p3)) = e1\ (0) = 0$

Case 4: ($p4$, $e1$)

$$q11\ (e1 \circ p4) = e1\ (q11\ (p4))$$

LHS: $q11\ (e1 \circ p4) = q11\ (p1) = 0$

RHS: $e1\ (q11\ (p4)) = e1\ (1) = 0$

$$q11: \mathrm{V}^{\mathrm{A}} \rightarrow \mathrm{V};\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$$

Case 5: ($p1$, $e2$)

$$q11\ (e2 \circ p1) = e2\ (q11\ (p1))$$

LHS: $q11\ (e2 \circ p1) = q11\ (p4) = 1$

RHS: $e2\ (q11\ (p1)) = e2\ (0) = 1$

Case 6: ($p2$, $e2$)

$$q11\ (e2 \circ p2) = e2\ (q11\ (p2))$$

LHS: $q11\ (e2 \circ p2) = q11\ (p3) = 0$

RHS: $e2\ (q11\ (p2)) = e2\ (1) = 0$

Case 7: ($p3$, $e2$)

$$q11\ (e2 \circ p3) = e2\ (q11\ (p3))$$

LHS: $q11\ (e2 \circ p3) = q11\ (p2) = 1$

RHS: $e2\ (q11\ (p3)) = e2\ (0) = 1$

Case 8: ($p4$, $e2$)

$$q11\ (e2 \circ p4) = e2\ (q11\ (p4))$$

LHS: $q11\ (e2 \circ p4) = q11\ (p1) = 0$

RHS: $e2\ (q11\ (p4)) = e2\ (1) = 0$

$$q11: \mathrm{V}^{\mathrm{A}} \rightarrow \mathrm{V};\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$$

Case 9: ($p1$, $e3$)

$$q11\ (e3 \circ p1) = e3\ (q11\ (p1))$$

LHS: $q11\ (e3 \circ p1) = q11\ (p1) = 0$

RHS: $e3\ (q11\ (p1)) = e3\ (0) = 0$

Case 10: ($p2$, $e3$)

$$q11\ (e3 \circ p2) = e3\ (q11\ (p2))$$

LHS: $q11\ (e3 \circ p2) = q11\ (p2) = 1$

RHS: $e3\ (q11\ (p2)) = e3\ (1) = 1$

Case 11: $(p3, e3)$

$$q11\ (e3 \circ p3) = e3\ (q11\ (p3))$$

LHS: $q11\ (e3 \circ p3) = q11\ (p3) = 0$

RHS: $e3\ (q11\ (p3)) = e3\ (0) = 0$

Case 12: $(p4, e3)$

$$q11\ (e3 \circ p4) = e3\ (q11\ (p4))$$

LHS: $q11\ (e3 \circ p4) = q11\ (p4) = 1$

RHS: $e3\ (q11\ (p4)) = e3\ (1) = 1$

$$q11\colon \mathrm{V}^{\mathrm{A}} \rightarrow \mathrm{V};\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$$

Case 13: $(p1, e4)$

$$q11\ (e4 \circ p1) = e4\ (q11\ (p1))$$

LHS: $q11\ (e4 \circ p1) = q11\ (p4) = 1$

RHS: $e4\ (q11\ (p1)) = e4\ (0) = 1$

Case 14: $(p2, e4)$

$$q11\ (e4 \circ p2) = e4\ (q11\ (p2))$$

LHS: $q11\ (e4 \circ p2) = q11\ (p4) = 1$

RHS: $e4\ (q11\ (p2)) = e4\ (1) = 1$

Case 15: $(p3, e4)$

$$q11\ (e4 \circ p3) = e4\ (q11\ (p3))$$

LHS: $q11\ (e4 \circ p3) = q11\ (p4) = 1$

RHS: $e4\ (q11\ (p3)) = e4\ (0) = 1$

Case 16: $(p4, e4)$

$$q11\ (e4 \circ p4) = e4\ (q11\ (p4))$$

LHS: $q11\ (e4 \circ p4) = q11\ (p4) = 1$

RHS: $e4\ (q11\ (p4)) = e4\ (1) = 1$

Thus, the functional

$$q11\colon \mathrm{V}^{\mathrm{A}} \rightarrow \mathrm{V};\ q11(p1) = 0,\ q11(p2) = 1,\ q11(p3) = 0,\ q11(p4) = 1$$

is a V-generalized point of A. Along similar lines, we can show that the functional

$$q13\colon \mathrm{V}^{\mathrm{A}} \rightarrow \mathrm{V};\ q13(p1) = 0,\ q13(p2) = 0,\ q13(p3) = 1,\ q13(p4) = 1$$

is another V-generalized point of A. These two perceived elements (generalized points) correspond to the two points

$$a1\colon \mathbf{1} \to \mathrm{A}$$

$$a2\colon \mathbf{1} \to \mathrm{A}$$

of the set A = {a1, a2}.

Next, we give an example of a functional (one of the remaining 14 functionals of the total 16 functionals) which is not a generalized point. Consider the functional

$$q12\colon \mathrm{V}^{\mathrm{A}} \to \mathrm{V};\ q12(p1) = 1,\ q12(p2) = 1,\ q12(p3) = 0,\ q12(p4) = 1$$

In order to be a V-generalized point of A, the functional

$$q12\colon \mathrm{V}^{\mathrm{A}} \to \mathrm{V}$$

must satisfy

$$q12\ (e \circ p) = e\ (q12\ (p))$$

for all p in $\mathrm{V}^{\mathrm{A}} = \{p1, p2, p3, p4\}$ and for all e in $\mathrm{V}^{\mathrm{V}} = \{e1, e2, e3, e4\}$, i.e., for all 16 cases we evaluated earlier. Let's consider the case of $p = p1$ and $e = e1$. We have to check for the equality

$$q12\ (e1 \circ p1) = e1\ (q12\ (p1))$$

LHS: $q12\ (e1 \circ p1) = q12\ (p1) = 1$

RHS: $e1\ (q12\ (p1)) = e1\ (1) = 0$

Since LHS is not equal to RHS, the functional

$$q12\colon \mathrm{V}^{\mathrm{A}} \to \mathrm{V};\ q12(p1) = 1,\ q12(p2) = 1,\ q12(p3) = 0,\ q12(p4) = 1$$

is not a V-generalized point of A.

Now we spell out how each point of a set gives rise to a generalized point (perceived element). Consider a set A and a type V. Since generalized point is a functional

$$q: V^A \to V$$

we first construct a functional called evaluation functional for each element ‘a’ of the set A. Recollect that the elements of domain set V^A of the functional q are V-valued properties of A, i.e.

$$p: A \to V$$

and the functional q assigns to each p in V^A an element ‘v’ of the codomain set V of the functional q. An evaluation functional corresponding to an element ‘a’ of the set A is defined as

$$q_a: V^A \to V$$

with

$$q_a(p) = p(a)$$

Now we show that this evaluation functional is a generalized point, i.e. satisfies

$$q_a(e \circ p) = e\,(q_a(p))$$

where

$$e: V \to V$$

is an endomap on the type V of property.

LHS: $q_a(e \circ p) = (e \circ p)$ (a) $= e(p$(a))

RHS: e ($q_a(p)) = e(p$(a))

Thus, the evaluation functional corresponding to each point of a set is a generalized point (Lawvere and Rosebrugh, 2003, p. 150). We now give an example of this general result. Consider a set A = {a1, a2, a3} and a property type V = {0, 1}. The set V^A of all V-valued properties of A consists of 8 functions, i.e.

$$V^A = \{p1, p2, p3, p4, p5, p6, p7, p8\}$$

where

$$p1: A \rightarrow V;\ p1(a1) = 0,\ p1(a2) = 0,\ p1(a3) = 0$$

$$p2: A \rightarrow V;\ p2(a1) = 1,\ p2(a2) = 0,\ p2(a3) = 0$$

$$p3: A \rightarrow V;\ p3(a1) = 0,\ p3(a2) = 1,\ p3(a3) = 0$$

$$p4: A \rightarrow V;\ p4(a1) = 1,\ p4(a2) = 1,\ p4(a3) = 0$$

$$p5: A \rightarrow V;\ p5(a1) = 0,\ p5(a2) = 0,\ p5(a3) = 1$$

$$p6: A \rightarrow V;\ p6(a1) = 1,\ p6(a2) = 0,\ p6(a3) = 1$$

$$p7: A \rightarrow V;\ p7(a1) = 0,\ p7(a2) = 1,\ p7(a3) = 1$$

$$p8: A \rightarrow V;\ p8(a1) = 1,\ p8(a2) = 1,\ p8(a3) = 1$$

Let us now consider a point

$$a1\text{: } \mathbf{1} \rightarrow \mathrm{A}$$

and the corresponding evaluation functional

$$q_{a1}\text{: } \mathrm{V}^{\mathrm{A}} \rightarrow \mathrm{V}$$

defined as

$$q_{a1}(p) = p(\mathrm{a1})$$

for all p in V^{A}, i.e.

$$q_{a1}(p1) = p1(\mathrm{a1}) = 0$$

$$q_{a1}(p2) = p2(\mathrm{a1}) = 1$$

$$q_{a1}(p3) = p3(\mathrm{a1}) = 0$$

$$q_{a1}(p4) = p4(\mathrm{a1}) = 1$$

$$q_{a1}(p5) = p5(\mathrm{a1}) = 0$$

$$q_{a1}(p6) = p6(\mathrm{a1}) = 1$$

$$q_{a1}(p7) = p7(\mathrm{a1}) = 0$$

$$q_{a1}(p8) = p8(\mathrm{a1}) = 1$$

Now we have to show that this evaluation functional

$$q_{a1}\text{: } \mathrm{V}^{\mathrm{A}} \rightarrow \mathrm{V}$$

satisfies

$$q_{a1}\,(e \circ p) = e\,(q_{a1}\,(p))$$

for all elements (V-valued properties of A) p in $V^A = \{p1, p2, p3, p4, p5, p6, p7, p8\}$, and for all elements (endomaps of property type V) e in $V^V = \{e1, e2, e3, e4\}$ defined as

$$e1\colon V \to V;\ e1(0) = 0,\ e1(1) = 0$$

$$e2\colon V \to V;\ e2(0) = 1,\ e2(1) = 0$$

$$e3\colon V \to V;\ e3(0) = 0,\ e3(1) = 1$$

$$e4\colon V \to V;\ e4(0) = 1,\ e4(1) = 1$$

Thus we have to test for the equality

$$q_{a1}\,(e \circ p) = e\,(q_{a1}\,(p))$$

in 32 cases. They are:

Case 1: $(p1, e1)$

$$q_{a1}\,(e1 \circ p1) = e1\,(q_{a1}\,(p1))$$

LHS: $q_{a1}\,(e1 \circ p1) = q_{a1}\,(p1) = 0$

RHS: $e1\,(q_{a1}\,(p1)) = e1\,(0) = 0$

Case 2: $(p2, e1)$

$$q_{a1}\,(e1 \circ p2) = e1\,(q_{a1}\,(p2))$$

LHS: $q_{a1}\ (e1 \circ p2) = q_{a1}\ (p1) = 0$

RHS: $e1\ (q_{a1}\ (p2)) = e1\ (1) = 0$

Case 3: $(p3, e1)$

$$q_{a1}\ (e1 \circ p3) = e1\ (q_{a1}\ (p3))$$

LHS: $q_{a1}\ (e1 \circ p3) = q_{a1}\ (p1) = 0$

RHS: $e1\ (q_{a1}\ (p3)) = e1\ (0) = 0$

Case 4: $(p4, e1)$

$$q_{a1}\ (e1 \circ p4) = e1\ (q_{a1}\ (p4))$$

LHS: $q_{a1}\ (e1 \circ p4) = q_{a1}\ (p1) = 0$

RHS: $e1\ (q_{a1}\ (p4)) = e1\ (1) = 0$

Case 5: $(p5, e1)$

$$q_{a1}\ (e1 \circ p5) = e1\ (q_{a1}\ (p5))$$

LHS: $q_{a1}\ (e1 \circ p5) = q_{a1}\ (p1) = 0$

RHS: $e1\ (q_{a1}\ (p5)) = e1\ (0) = 0$

Case 6: $(p6, e1)$

$$q_{a1}\ (e1 \circ p6) = e1\ (q_{a1}\ (p6))$$

LHS: $q_{a1}\ (e1 \circ p6) = q_{a1}\ (p1) = 0$

RHS: $e1\ (q_{a1}\ (p6)) = e1\ (1) = 0$

Case 7: ($p7$, $e1$)

$$q_{a1}\ (e1 \circ p7) = e1\ (q_{a1}\ (p7))$$

LHS: $q_{a1}\ (e1 \circ p7) = q_{a1}\ (p1) = 0$

RHS: $e1\ (q_{a1}\ (p7)) = e1\ (0) = 0$

Case 8: ($p8$, $e1$)

$$q_{a1}\ (e1 \circ p8) = e1\ (q_{a1}\ (p8))$$

LHS: $q_{a1}\ (e1 \circ p8) = q_{a1}\ (p1) = 0$

RHS: $e1\ (q_{a1}\ (p8)) = e1\ (1) = 0$

Case 9: ($p1$, $e2$)

$$q_{a1}\ (e2 \circ p1) = e2\ (q_{a1}\ (p1))$$

LHS: $q_{a1}\ (e2 \circ p1) = q_{a1}\ (p8) = 1$

RHS: $e2\ (q_{a1}\ (p1)) = e2\ (0) = 1$

Case 10: ($p2$, $e2$)

$$q_{a1}\ (e2 \circ p2) = e2\ (q_{a1}\ (p2))$$

LHS: $q_{a1}\ (e2 \circ p2) = q_{a1}\ (p7) = 0$

RHS: $e2\ (q_{a1}\ (p2)) = e2\ (1) = 0$

Case 11: ($p3$, $e2$)

$$q_{a1}\ (e2 \circ p3) = e2\ (q_{a1}\ (p3))$$

LHS: $q_{a1}\ (e2 \circ p3) = q_{a1}\ (p6) = 1$

RHS: $e2\ (q_{a1}\ (p3)) = e2\ (0) = 1$

Case 12: ($p4$, $e2$)

$$q_{a1}\ (e2 \circ p4) = e2\ (q_{a1}\ (p4))$$

LHS: $q_{a1}\ (e2 \circ p4) = q_{a1}\ (p5) = 0$

RHS: $e2\ (q_{a1}\ (p4)) = e2\ (1) = 0$

Case 13: ($p5$, $e2$)

$$q_{a1}\ (e2 \circ p5) = e2\ (q_{a1}\ (p5))$$

LHS: $q_{a1}\ (e2 \circ p5) = q_{a1}\ (p4) = 1$

RHS: $e2\ (q_{a1}\ (p5)) = e2\ (0) = 1$

Case 14: ($p6$, $e2$)

$$q_{a1}\ (e2 \circ p6) = e2\ (q_{a1}\ (p6))$$

LHS: $q_{a1}\ (e2 \circ p6) = q_{a1}\ (p3) = 0$

RHS: $e2\ (q_{a1}\ (p6)) = e2\ (1) = 0$

Case 15: ($p7$, $e2$)

$$q_{a1}\,(e2 \circ p7) = e2\,(q_{a1}\,(p7))$$

LHS: $q_{a1}\,(e2 \circ p7) = q_{a1}\,(p2) = 1$

RHS: $e2\,(q_{a1}\,(p7)) = e2\,(0) = 1$

Case 16: ($p8$, $e2$)

$$q_{a1}\,(e2 \circ p8) = e2\,(q_{a1}\,(p8))$$

LHS: $q_{a1}\,(e2 \circ p8) = q_{a1}\,(p1) = 0$

RHS: $e2\,(q_{a1}\,(p8)) = e2\,(1) = 0$

Case 17: ($p1$, $e3$)

$$q_{a1}\,(e3 \circ p1) = e3\,(q_{a1}\,(p1))$$

LHS: $q_{a1}\,(e3 \circ p1) = q_{a1}\,(p1) = 0$

RHS: $e3\,(q_{a1}\,(p1)) = e3\,(0) = 0$

Case 18: ($p2$, $e3$)

$$q_{a1}\,(e3 \circ p2) = e3\,(q_{a1}\,(p2))$$

LHS: $q_{a1}\,(e3 \circ p2) = q_{a1}\,(p2) = 1$

RHS: $e3\,(q_{a1}\,(p2)) = e3\,(1) = 1$

Case 19: ($p3$, $e3$)

$$q_{a1}\,(e3 \circ p3) = e3\,(q_{a1}\,(p3))$$

LHS: $q_{a1}\ (e3 \circ p3) = q_{a1}\ (p3) = 0$

RHS: $e3\ (q_{a1}\ (p3)) = e3\ (0) = 0$

Case 20: $(p4, e3)$

$$q_{a1}\ (e3 \circ p4) = e3\ (q_{a1}\ (p4))$$

LHS: $q_{a1}\ (e3 \circ p4) = q_{a1}\ (p4) = 1$

RHS: $e3\ (q_{a1}\ (p4)) = e3\ (1) = 1$

Case 21: $(p5, e3)$

$$q_{a1}\ (e3 \circ p5) = e3\ (q_{a1}\ (p5))$$

LHS: $q_{a1}\ (e3 \circ p5) = q_{a1}\ (p5) = 0$

RHS: $e3\ (q_{a1}\ (p5)) = e3\ (0) = 0$

Case 22: $(p6, e3)$

$$q_{a1}\ (e3 \circ p6) = e3\ (q_{a1}\ (p6))$$

LHS: $q_{a1}\ (e3 \circ p6) = q_{a1}\ (p6) = 1$

RHS: $e3\ (q_{a1}\ (p6)) = e3\ (1) = 1$

Case 23: $(p7, e3)$

$$q_{a1}\ (e3 \circ p7) = e3\ (q_{a1}\ (p7))$$

LHS: $q_{a1}\ (e3 \circ p7) = q_{a1}\ (p7) = 0$

RHS: $e3\ (q_{a1}\ (p7)) = e3\ (0) = 0$

Case 24: ($p8$, $e3$)

$$q_{a1}\ (e3 \circ p8) = e3\ (q_{a1}\ (p8))$$

LHS: $q_{a1}\ (e3 \circ p8) = q_{a1}\ (p8) = 1$

RHS: $e3\ (q_{a1}\ (p8)) = e3\ (1) = 1$

Case 25: ($p1$, $e4$)

$$q_{a1}\ (e4 \circ p1) = e4\ (q_{a1}\ (p1))$$

LHS: $q_{a1}\ (e4 \circ p1) = q_{a1}\ (p8) = 1$

RHS: $e4\ (q_{a1}\ (p1)) = e4\ (0) = 1$

Case 26: ($p2$, $e4$)

$$q_{a1}\ (e4 \circ p2) = e4\ (q_{a1}\ (p2))$$

LHS: $q_{a1}\ (e4 \circ p2) = q_{a1}\ (p8) = 1$

RHS: $e4\ (q_{a1}\ (p2)) = e4\ (1) = 1$

Case 27: ($p3$, $e4$)

$$q_{a1}\ (e4 \circ p3) = e4\ (q_{a1}\ (p3))$$

LHS: $q_{a1}\ (e4 \circ p3) = q_{a1}\ (p8) = 1$

RHS: $e4\ (q_{a1}\ (p3)) = e4\ (0) = 1$

Case 28: $(p4, e4)$

$$q_{a1}(e4 \circ p4) = e4(q_{a1}(p4))$$

LHS: $q_{a1}(e4 \circ p4) = q_{a1}(p8) = 1$

RHS: $e4(q_{a1}(p4)) = e4(1) = 1$

Case 29: $(p5, e4)$

$$q_{a1}(e4 \circ p5) = e4(q_{a1}(p5))$$

LHS: $q_{a1}(e4 \circ p5) = q_{a1}(p8) = 1$

RHS: $e4(q_{a1}(p5)) = e4(0) = 1$

Case 30: $(p6, e4)$

$$q_{a1}(e4 \circ p6) = e4(q_{a1}(p6))$$

LHS: $q_{a1}(e4 \circ p6) = q_{a1}(p8) = 1$

RHS: $e4(q_{a1}(p6)) = e4(1) = 1$

Case 31: $(p7, e4)$

$$q_{a1}(e4 \circ p7) = e4(q_{a1}(p7))$$

LHS: $q_{a1}(e4 \circ p7) = q_{a1}(p8) = 1$

RHS: $e4(q_{a1}(p7)) = e4(0) = 1$

Case 32: $(p8, e4)$

$$q_{a1}\ (e4 \circ p8) = e4\ (q_{a1}\ (p8))$$

LHS: $q_{a1}\ (e4 \circ p8) = q_{a1}\ (p8) = 1$

RHS: $e4\ (q_{a1}\ (p8)) = e4\ (1) = 1$

Thus the evaluation functional

$$q_{a1}\colon \mathrm{V}^{\mathrm{A}} \rightarrow \mathrm{V};\ q_{a1}(p1)=0,\ q_{a1}(p2)=1,\ q_{a1}(p3)=0,\ q_{a1}(p4)=1,\ q_{a1}(p5)=0,\ q_{a1}(p6)=1,\ q_{a1}(p7)=0,\ q_{a1}(p8)=1$$

satisfying

$$q_{a1}\ (e \circ p) = e\ (q_{a1}\ (p))$$

(for all p: A → V and for all e: V → V) is a V-generalized point of A corresponding to the point

$$a1\colon \mathbf{1} \rightarrow \mathrm{A}$$

Along the same lines, we can show that each of remaining two points of the set A = {a1, a2, a3} give rise to corresponding V-generalized points of A.

With property type V = **2** (two-element set), although there is a V-generalized point of A corresponding to each point of A, there can also be generalized points that do not correspond to any points of A, which we may call illusions (or ghost points). In order to obtain a 1-1 correspondence between points of any set A and V-generalized points of A, i.e. isomorphism between objects and perceived objects, we need a 3-element set as the property type V. One of our objectives is to calculate objects analogous to the 3-element set (in the category of sets) in categories that are reflective of reality such as the category of categories. We plan to approach this goal by calculating the basic types of knowing (objects analogous to 3-element set in the category of sets) in more structured categories such as dynamical systems, functions, graphs, and actions.

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