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We are delighted to include in this issue of *MIND* the winning entry for the third *MIND* Graduate Essay Prize, by Fabian Pregel. This is followed by another entry, by Óscar Antonio Monroy Pérez, which was also deemed worthy of publication. We offer both authors our warmest congratulations. The topic for this prize was 'The Philosophy of Logic'.

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Joint Editors

Neo-Logicism and Gödelian Incompleteness

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There is a long-standing gap in the literature as to whether Gödelian incompleteness constitutes a challenge for Neo-Logicism, and if so how serious it is. In this paper, I articulate and address the challenge in detail. The Neo-Logicist project is to demonstrate the analyticity of arithmetic by deriving all its truths from logical principles and suitable definitions. The specific concern raised by Gödel's first incompleteness theorem is that no single sound system of logic syntactically implies all arithmetical truths. I set out some responses that initially seem appealing and explain why they are not compelling. The upshot is that Neo-Logicism either offers an epistemic route only to *some* truths of arithmetic; or that it has to move from a syntactic to a semantic notion of logical consequence, which risks undermining its epistemic goals. I conclude by considering Crispin Wright's recent attempt to address Gödelian incompleteness, which I argue is not satisfactory.

1 Introduction

There is a historic gap in the literature as to whether Gödelian incompleteness constitutes a meaningful challenge for Neo-Logicism. The topic has, however, been picked up in a recent chapter by Crispin Wright (Wright 2020). In this article, I will evaluate Wright's proposal and seek to start filling this gap more generally.

To define the principal terms: *Gödelian incompleteness* is the phenomenon that, for any consistent, recursively enumerable axiom

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system sufficiently strong to derive certain parts of arithmetic (Robinson Arithmetic), a sentence in the language of arithmetic exists such that neither the sentence nor its negation is formally provable from the axiom system. *Neo-Logicism* is a school of thought that aims, in a qualified sense, to revive the Fregean project of showing the analyticity of arithmetic.

The structure of this paper is as follows. In section 2, I motivate the philosophical challenge Gödelian incompleteness poses for Neo-Logicism. In section 3, I review existing literature on the topic from the Neo-Logicist 'canon'. In section 4, I then explore potential initial responses to the challenge from incompleteness. I argue that there is no 'easy' solution for the Neo-Logicist. Finally, in section 5, I consider a recent proposal by Crispin Wright and argue that his suggestion of how to resolve the challenge is not satisfactory.

2 The challenge

2.1 Historical context

Gödel's incompleteness theorems have historically been viewed as a major challenge for classical Logicism. A few examples: Tennant states in the Stanford Encyclopedia entry on Logicism and Neo-Logicism that 'Logicist doctrines were espoused [...] until around 1930, at which point logicism went into decline, largely because of the discovery of Gödelian incompleteness, and the ascendancy of Zermelo-Fraenkel set theory' (Tennant 2017). Wright equally notes that 'most philosophers of mathematics would [...] cite [...] the incompleteness theorems of Gödel, as decisive for Logicism's failure' (Wright 1983, p. xxvi). Musgrave similarly states that the incompleteness theorems 'drive a further nail in the coffin' of Logicism (Musgrave 1977, p. 124). Finally, Henkin notes that, in light of incompleteness, 'it appears that one of the basic elements on which Russell rested his thesis that all of mathematics could be reduced to logic must be withdrawn and reconsidered' (Henkin 1962, p. 790).

In what sense then does incompleteness threaten classical Logicism and the Neo-Logicist project? Frege had maintained 'that all arithmetical propositions can be derived from definitions alone using purely logical means, and consequently that they must be derived in this way' (Frege 1885, p. 114). The specific concern for Logicism is that, due to Gödel's first incompleteness theorem, this appears impossible. Reducing the axioms of arithmetic to logic will not yield an axiom system in which one can (syntactically) derive all truths of arithmetic. Logicism's slogan that 'arithmetic is part of logic' thus seemingly does not redeem its promise (Parsons 1965, p. 183).

2.2 Incompleteness in the Neo-Logicist system

This challenge appears to carry over directly to Neo-Logicism. Neo-Logicism is usually formalised in a second-order language with the *N* operator. *N* maps second-order monadic relations (concepts) to first-order objects.

Definition 1 (Theory FA). Let FA (Frege Arithmetic) be the theory containing Hume's Principle as the sole non-logical axiom (Wright 1983, p. 158):

$$(\mathrm{N}^{=}/\mathrm{HP}) \quad (\forall F)\,(\forall G)\,[[(\mathrm{N}x : Fx) = (\mathrm{N}x : Gx)] \leftrightarrow (\exists R)\,(Fx1\text{–}1_R Gx)] \quad (1)$$

where $1\text{–}1_R$ denotes the existence of a one-to-one correspondence between the objects falling under the concepts *F* and *G*.

To connect Frege Arithmetic with ordinary arithmetic, let us introduce a bridge theory FD (Frege's Definitions) with suitable definitions for 0, the predecessor relation and the concept of being a natural number (Wright 1983, pp. 159–160). Let FA+FD denote the theory comprising the axioms of FA and the definitions of FD. Let $\mathcal{L}_{\mathrm{FA+FD}}$ denote the formal language of the theory FA+FD. Let us assume a reasonable second-order deductive system, such as Shapiro's second-order logical calculus D2 (Shapiro 1991, pp. 62–69). The second-order Peano axioms (PA2) can then be derived in the theory FA+FD (Frege's theorem). Therefore, by Gödel's first incompleteness theorem (and using Rosser's trick), there will be sentences σ in the language $\mathcal{L}_{\mathrm{FA+FD}}$ such that both $\mathrm{FA} + \mathrm{FD} \nvdash \sigma$ and $\mathrm{FA} + \mathrm{FD} \nvdash \neg\sigma$ (assuming FA+FD is consistent). At the same time, $\mathrm{FA} + \mathrm{FD} \vDash \sigma$ where $\vDash$ denotes consequence under full second-order semantics. Given that FA and FD are finitely axiomatisable in second-order logic, one can derive that $\nvdash \mathrm{FA} + \mathrm{FD} \rightarrow \sigma$ and $\vDash \mathrm{FA} + \mathrm{FD} \rightarrow \sigma$. The incompleteness thus arises from the second-order deductive system not proving certain validities of full second-order logic.

2.3 Three specific challenges

This technical result appears to bear directly on the philosophical appeal of the Neo-Logicist programme. Part of the Neo-Logicist programme's attraction is that the second-order Peano axioms can be derived from Hume's Principle and Frege's Definitions (Frege's Theorem) while avoiding one of the main challenges to classical Logicism—Russell's paradox. Yet, it appears the Neo-Logicist needs to also address another challenge to classical Logicism, namely that any axiom system containing the second-order Peano Axioms will be incomplete. Given the incompleteness of FA+FD, one would expect justification as to why deriving the second-order Peano axioms can provide an appealing route to the analyticity of arithmetic. We can formulate three specific challenges for the Neo-Logicist to tackle:

Challenge 1 (Analytic-synthetic status): Are the formulas not provable in D2 from FA+FD synthetic? Or are those formulas also analytic, but for a different reason than those provable from FA+FD in D2?
Challenge 2 (Epistemic access): Does the Neo-Logicist have to offer a separate epistemology for the formulas not provable in D2 from FA+FD? If so, is a bifurcation of our epistemology an attractive position?
Challenge 3 (Deductive system dependency): Given that the choice of second-order deductive systems, such as D2, affects exactly which formulas are provable from FA+FD (since the unprovable Gödel sentence encodes the rules of a given deductive system), does the resulting picture of arithmetic not have an undesirable dependency on the choice of second-order deductive system?

We will return to these challenges as we consider possible responses by the Neo-Logicist.

3 The Neo-Logicist canon

In this section, I will discuss existing commentary on the challenge from incompleteness in the Neo-Logicist literature dating before 2007. I will refer to those texts as the 'canon'. The canon principally comprises Wright's 1983 *Frege's Conception of Numbers as Objects* and the 2001 essay collection *The Reason's Proper Study*. In section 5, I will then address two more recent works by Wright: the chapter 'On Quantifying into Predicate Position' (2007) and in particular several pages in the Festschrift *Logic, Language, and Mathematics* (2020).

In the canon, primary and secondary literature on the challenge from incompleteness are scarce. For example, at the end of *The Reason's Proper Study*, Wright and Hale list eighteen problems for Neo-Logicism that the authors regard as the most pressing (Hale and Wright 2001, pp. 421–436). The challenge from incompleteness does not make this list. In the canon of Neo-Logicism, Wright and Hale address the question only on two pages of *Frege's Conception of Numbers as Objects* and one footnote in *The Reason's Proper Study* (Wright 1983, pp. xiv, 131; Hale and Wright 2001, pp. 4–5). Here, Wright and Hale adopt two stances:

(WH1) Neo-Logicism invokes second-order logic independently of considerations of incompleteness, for example in the quantification over concepts in Hume's Principle. Second-order logic is known to have no complete deductive system (with respect to standard semantics). Thus, Gödel's incompleteness result 'creates no special problem' and has 'no specific bearing' on the Neo-Logicist project (Hale and Wright 2001, p. 5);

(WH2) Neo-Logicism's ambition should be the more modest claim of 'deriving all the fundamental laws of arithmetic' rather than 'deriving all laws of arithmetic' to 'avoid an obvious clash with Gödel's first incompleteness theorem' (Hale and Wright 2001, p. 4).

At first glance, stances (WH1) and (WH2) seem to be in tension: (WH2) admits that not all laws of arithmetic in Neo-Logicism are derivable as a direct consequence of incompleteness, yet (WH1) maintains that incompleteness has no specific bearing on Neo-Logicism.

In the following, I will try to illustrate that this tension (matters versus does not matter) is symptomatic of a broader internal tension in the historic work of Wright and Hale around Gödel's incompleteness theorems.

Consider the articulation of the ambition of the Neo-Logicist project in the following three statements. In *Frege's Conception of Numbers as Objects*, Wright defines the ambition as to 'deduce appropriate statements of the *fundamental* truths of number-theory, in particular the Peano axioms, in an appropriate system of higher-order logic' (Wright 1983, p. 153, emphasis added). His stressing of *fundamental* ties in with (WH2), that is, with 'deriving all the fundamental laws of arithmetic' to 'avoid an obvious clash with Gödel's first incompleteness theorem' (Hale and Wright 2001, p. 4).

Contrast this *fundamental truths*-ambition, however, with two separate statements of the goal of the Neo-Logicist project by the same author. According to the first, Neo-Logicism aims to defend the assertion that Frege was right in maintaining that 'the *truths* of Arithmetic are analytic, by which he meant that they are *all provable* on the basis of general *logical laws* together with suitable definitions' (Hale and Wright 2001, p. 1, emphasis added).[1] Separately, Wright and Hale also state that 'if such an explanatory principle [HP], in company with "implicit definitions" generally, can be regarded as analytic, then that should suffice at least to *demonstrate the analyticity of arithmetic*' (Hale and Wright 2001, p. 279, emphasis added).

For a further illustration of the tension, consider Wright and Hale's response as to whether the laws of arithmetic are meant to follow from Hume's Principle by syntactic or semantic means. Wright and Hale explicitly state that the goal is 'provability' by 'logical laws', that is, syntactic consequence (Hale and Wright 2001, p. 1). However, compare this affirmation of syntactic consequence with another statement by Wright

[1] Wright and Hale say that Frege was 'substantially' right on this. Right after, they clarify what the 'substantially' qualifier is referring to: 'Where neo-Fregeanism principally differs from Frege is in its taking a more optimistic view than Frege himself came to hold of the prospects for the kind of contextual explanation of the fundamental concepts of arithmetic and analysis—the concepts of cardinal number and real number—which he considered and rejected in the central sections (§ § 60–8) of *Grundlagen*' (Hale and Wright 2001, p. 1).

in *Frege's Conception of Numbers as Objects:* according to Wright, the intended notion of consequence is 'evidently constrained' to be 'semantic' (Wright 1983, p. xiii). In particular, Wright posits, 'the incompleteness of number theory' requires that the Neo-Logicist does not 'tie the notion of an intuitively correctly arithmetical proof to any particular syntactic characterisation' (Wright 1983, p. xiv).

If there was only a single instance of the tension, one could be tempted to dismiss it as mere rash writing. Given their multitude, however, first in the attitude taken to Gödelian incompleteness (matters versus does not matter), second in the ambition of the project (deriving all truths versus just the fundamental truths), and third in the notion of consequence (semantic versus syntactic), I would suggest that these tensions point to a deeper lack of clarity in the Neo-Logicist's work and ambition. The tension arises from Wright and Hale moving back and forth between two unpleasant options that Gödelian incompleteness seems to leave the Neo-Logicist with: either to maintain that the goal of Neo-Logicism is to derive all truths of arithmetic, in which case the Neo-Logicist has to resort to a non-syntactic (for example, semantic) consequence relation; or to employ the more tractable notion of syntactic provability and give up the idea of proving all truths of arithmetic.

That such a tension has emerged is not surprising if we consider the overall philosophical ambition of the Logicist project, and Neo-Logicism's attempt to revive it. Classical Logicism can be seen as an attempt to provide a compelling explanation of the metaphysics and epistemology of mathematics: according to the Logicist, mathematical concepts are really logical concepts, theorems of arithmetic can be derived by purely logical laws, and so our ability to reason logically allows us access to these (analytical) truths (Parsons 1965, p. 183). Thus, the Neo-Logicist seemingly desires two incompatible things: *provability as the consequence relation* to yield an attractive epistemology, and *completeness* to ensure that this attractive epistemology pertains to all of arithmetic.

4 Possible ways out of the dilemma for the Neo-Logicist

4.1 Possible options

How is the Neo-Logicist to respond to these challenges? What is the correct ambition of the Neo-Logicist project—to offer an epistemological route to the analyticity of all theorems of arithmetic, or only to the analyticity of the fundamental ones? And is the relevant consequence relation syntactic or semantic? In this section, I will consider initial ways the Neo-Logicist may seek to escape these tensions. Here, I will argue that there is no 'easy' solution for the Neo-Logicist. To succeed with the argument that there is no 'easy' solution, I will need to consider a range of possible replies for the Neo-Logicist, in order to demonstrate that each of the replies poses challenges that require substantial further consideration.

To start off, as noted before, the tension seems to arise from Neo-Logicism moving back and forth between two responses to Gödelian incompleteness:

Uphold Syntactic Consequence Maintain that syntactic consequence is the relevant consequence relation for Neo-Logicism and give up on deriving all truths of arithmetic from FA;
Uphold Completeness Discard syntactic consequence as the relevant consequence relation for Neo-Logicism and instead endorse a different consequence relation.

Both *Uphold Syntactic Consequence* and *Uphold Completeness* could potentially be fruitful responses. To argue that there is no 'easy' solution for the Neo-Logicist, I will thus have to consider both options. What is clear in any case is that, given the tensions in the Neo-Logicist canon highlighted in section 3, it will be useful to clarify the ambition and consequence relation of the Neo-Logicist project.

4.2 Uphold Completeness

4.2.1 *Initial motivation* For the first option, *Uphold Completeness*: to discard syntactic consequence as the relevant consequence relation, the Neo-Logicist could, for example, insist that it is instead 'semantic completeness' that matters. Semantic completeness is the property that, for every sentence in the language, either the sentence or its negation is a semantic consequence of the axioms. PA2 is indeed semantically complete, that is, for every sentence σ in the formal language $\mathcal{L}_{\mathrm{PA2}}$ of PA2 either PA2 $\vDash \sigma$ or PA2 $\vDash \neg\sigma$, where $\vDash$ denotes full second-order semantic consequence.

The semantic completeness of PA2 is a result of the categoricity of the second-order Peano axioms. Take any two models $\mathcal{M}$, $\mathcal{N}$ of PA2, that is, $\mathcal{M} \vDash$ PA2 and $\mathcal{N} \vDash$ PA2. Either $\mathcal{M} \vDash \sigma$ or $\mathcal{M} \nvDash \sigma$, in which case $\mathcal{M} \vDash \neg\sigma$ by definition of the satisfaction relation. Suppose $\mathcal{M} \vDash \sigma$. Then, by categoricity of PA2, also $\mathcal{N} \vDash \sigma$. Since $\mathcal{N}$ was an arbitrary model of PA2, σ is satisfied under all models of PA2. Thus, PA2 $\vDash \sigma$. The case for $\mathcal{M} \vDash \neg\sigma$ is analogous.

Given that PA2 is semantically complete, the Neo-Logicist could try to argue that it is semantic completeness that suffices for the Neo-Logicist project. Call this view *Semantic Neo-Logicism*. Stephen Read, for example, has argued that to construe Frege as in search of proof-theoretic completeness is to 'impose a later interpretation on the word "complete"' (Read 1997, p. 79). According to Read, classical Logicism is better understood as striving for categoricity (Read 1997, p. 79).

Semantic Neo-Logicism may initially seem an appealing option for the Neo-Logicist. After all, if analyticity is truth in virtue of meaning, then one may be tempted to view semantic consequence (that is, entailment in virtue of meaning) to be analyticity-preserving. Furthermore, perhaps logical consequence is best understood semantically in any case (Griffiths and Paseau 2022, p. 91).

To evaluate this response, we can employ the common distinction between a metaphysical and an epistemological understanding of analyticity (Boghossian 1996, pp. 363–365). Wright and Hale appear to use both conceptions of analyticity at times, claiming for example that 'anyone who understands these statements is in a position to recognize them as being true' (epistemological account) but also speaking of 'conceptual truths' (metaphysical account) (Hale and Wright 2001, p. 12). The dialectical strategy in this section will be to show that the metaphysical understanding of analyticity is not the correct conception of analyticity for the Neo-Logicist project and that, on the epistemological understanding of analyticity, full second-order semantic consequence is not analyticity-preserving.

4.2.2 Metaphysical conception

On the metaphysical understanding, a sentence is analytic if its truth depends solely upon the meanings of its constituent terms and how the sentence combines those terms (Boghossian 1996, pp. 363–365). There are, of course, serious doubts about the metaphysical sense of analyticity in general. As Boghossian puts it, 'the metaphysical notion is of dubious explanatory value, and possibly also of dubious coherence' (Boghossian 1996, pp. 364). Let us suppose for the moment that those doubts about analyticity in the metaphysical sense can be overcome. Still, there is good reason to think that analyticity in the metaphysical sense is not the correct conception of analyticity for the Neo-Logicist project. The concern is that, for the Logicist, the purpose of establishing the analyticity of arithmetic was always meant to 'do epistemological work'—our epistemic access to the truths of arithmetic is meant to be explained by the analyticity of Hume's Principle plus a consequence relation that preserves analyticity. Wright and Hale repeatedly emphasise their project's *epistemic* ambitions. For example, in his 1983 *Frege's Conception of Numbers as Objects*, Wright says:

> Frege's question is surely a good one: what is the ultimate, the so-to-speak *epistemologically canonical* source of our *knowledge* of number-theoretic statements? (Wright 1983, p. xxi, emphasis added)

The centrality of the project's epistemic aims continues into Wright and Hale's 2001 *The Reason's Proper Study* with:

> So, prima facie, the *philosophical significance of Frege's Theorem* cannot be less than this: that, at least as far as number theory is concerned, *the more extensive epistemological programme* which Frege hoped to accomplish in *Grundgezetse* [*sic*] is still a going concern. (Hale and Wright 2001, p. 280, emphasis added)

and:

> The two main components in Frege's mathematical philosophy were [...] the claims, respectively, that mathematics is a body of *knowledge* about independently existing objects, and that this *knowledge* may be *acquired* on the basis of general logical laws and suitable definitions. (Hale and Wright 2001, p. i, emphasis added)

I have provided three quotations to highlight the importance of the epistemic ambitions for Neo-Logicism.

Now suppose we had successfully established that all truths of arithmetic are analytic in the metaphysical sense because they are full second-order semantic consequences of HP. Then, we would still be left without an explanation of our epistemic access to these mathematical truths. Thus, adopting the metaphysical sense of analyticity to achieve completeness is at best a pyrrhic victory, for we would have made no progress towards Hale and Wright's stated primary goal of the Logicist project. The Neo-Logicist must have the epistemological understanding of analyticity in mind.

4.2.3 Epistemological conception

On the epistemological understanding, a sentence is analytic if its truth can be known merely by comprehending the meanings of its constituent terms and how the sentence combines those terms (Boghossian 1996, pp. 363–365). The epistemologically motivated concern about the Neo-Logicist adopting a semantic consequence relation is the following: the aim of the Neo-Logicist project is to establish the analyticity of arithmetic. To establish the analyticity of arithmetic, the Neo-Logicist argues that Hume's Principle qualifies as a contextual explanation and then derives the truths of arithmetic from Hume's Principle plus second-order logic (Hale and Wright 2001, pp. 1–2). Yet, if the derivation of the truths of arithmetic uses full second-order semantic consequence (that is, the existence of models satisfying certain sets of propositions), then to establish the analyticity of arithmetic from the analyticity of HP, one needs to proclaim that semantic consequence preserves analyticity. If analyticity is epistemologically understood, as we are supposing for the moment, then one needs to argue in particular that semantic consequences in second-order logic are (always) epistemically accessible to humans. This would be a drastic claim for the Neo-Logicist to make. Raatikainen, for example, points out that the set of full second-order logical truths is not just not recursively enumerable (that is, not $\sum_1^0$, in contrast to first-order logic), but in fact does not appear in any finite level of the Kleene hierarchy (that is, not $\sum_n^m$ for any finite m and n) (Raatikainen 2020, p. 83). It thus appears far more natural to suppose that proof, that is, syntactic consequence, captures accessibility to (human) reasoning and that, correspondingly, proof is what preserves analyticity in the epistemological sense.

A possible response by the Neo-Logicist is to say that while second-order semantic consequence in general may not be analyticity-preserving in the epistemological sense, what is under consideration here is far narrower. What we are concerned with is whether a formula σ is a semantic consequence of PA2. Since PA2 is categorical, all the models of PA2 are isomorphic, and so σ is true in one model of PA2 if and only if σ is true in all models of PA2. Thus, we do not need to fathom all possible models but merely conduct model-checking, that is, evaluate whether a sentence σ is true under some particular model of PA2. Unfortunately, however, even whether a sentence σ is satisfied under a particular model of PA2 is not (in general) decidable, for if it was, then a computer could enumerate all true sentences of arithmetic by enumerating all sentences of $\mathcal{L}_{\text{PA2}}$ and filtering out those that are satisfied by a model of PA2. Given sufficient memory and computing time, we could, for example, just evaluate whether Fermat's Last Theorem or the Goldbach Conjecture are satisfied by the standard model. Nonetheless, this response highlights an important clarification: the objection that second-order semantic consequence in general is not analyticity-preserving is aimed at the wrong target, for the Neo-Logicist only requires satisfaction under a specific model. However, in either case, the conclusion remains the same. On the epistemological understanding of analyticity, semantic consequence (of the axioms of PA2) is not analyticity-preserving.

4.2.4 Semantic Neo-Logicism

In light of the challenges with arguing that second-order semantic consequence preserves epistemological analyticity, the Neo-Logicist may want to revisit the metaphysical conception of analyticity and adopt *Semantic Neo-Logicism* after all.

As noted earlier, Wright and Hale's opus is explicit that epistemic aims are central to the Neo-Logicist project. *Semantic Neo-Logicism* leaves the Neo-Logicist without any progress towards those goals. That no progress has been made does not mean that no progress could be made. But the complexity concern (that the set of full second-order logical truths does not appear in any finite level of the Kleene hierarchy) suggests that the metaphysical conception of analyticity cannot deliver Neo-Logicism's stated epistemic aims either. Thus, would the project lose all of its attraction if *Semantic Neo-Logicism* was adopted?

One may hold that if the Neo-Logicist managed to establish that all arithmetical truths are analytical in the metaphysical sense, then that would still be a substantial philosophical achievement. While less attractive than originally advertised, such a Semantic Neo-Logicism would at least have the benefit of not being vulnerable to challenges from incompleteness.

The claim that the resulting account is both *attractive* and still *substantively Logicist* is somewhat vague. Neo-Logicism's 'attractive' epistemology has generally been seen as one of Neo-Logicism's main advantages (Shapiro and Weir 2000, p. 160). As we will see in §5, Wright in his 2020 book chapter also decided to pursue an inferentialist (that is, syntactic) rather than a semantic response to the challenge from incompleteness (Wright 2020, p. 325). At this point, I thus note that at a minimum *Semantic Neo-Logicism* would be a very substantial clarification of the Neo-Logicist project, and a different approach from the response Neo-Logicists have in fact pursued.

In summary, on either interpretation of analyticity, there is a corresponding argument to demonstrate that the Neo-Logicist making a success of *Uphold Completeness* via a semantic consequence relation is highly challenging.

4.3 *Uphold Syntactic Consequence*

4.3.1 *No reason to expect response* I will now move to the alternative option: maintain that syntactic consequence is the relevant consequence relation for Neo-Logicism, but abandon achieving an axiomatisation from which all truths of arithmetic are derivable. To successfully embrace *Uphold Syntactic Consequence*, the Neo-Logicist needs to argue that syntactic completeness is not as relevant for the Neo-Logicist project as it is made out to be. This position can be argued for from at least two distinct directions. Consider the following as a first elaboration of

Uphold Syntactic Consequence: Wright briefly alludes to the position that there is ‘no reason to expect’ number-theoretic truth to be completely recursively axiomatisable (Wright 1983, p. 131). Wright highlights that the same holds for second-order logical truth. Thus, perhaps—Wright’s position might be spelled out—we just need to accept as a fact of life that both number-theoretic truth and second-order logical truth cannot be recursively axiomatised. This position suggests that the whole demand for a complete recursive axiomatisation of arithmetic may have been misguided to begin with.

Such a response has a few advantages: first, as we saw in §2.2, the technical part of the response is correct. The incompleteness of second-order logic can be derived from the incompleteness of PA2: if G_{PA2} is the Gödel sentence for PA2 ($PA2 \nvdash_{D2} G_{PA2}$ but $PA2 \vDash G_{PA2}$), then $PA2 \rightarrow G_{PA2}$ is not derivable in D2 ($\nvdash_{D2} PA2 \rightarrow G_{PA2}$), but is still valid in full second-order logic ($\vDash PA2 \rightarrow G_{PA2}$).

Secondly, this response appears to still maintain Frege’s thesis that ‘arithmetic is part of logic’—it just so happens that both are incomplete, a fact not yet discovered at the time of Frege.

However, while the ‘no reason to expect’ response may be correct at a technical level, I would like to suggest that it is open to debate whether the response is sufficient to absolve the Neo-Logicist from any difficulties. One may object that this response simply begs the question. Sure enough, a challenger could say, maybe there was indeed ‘no reason to expect’ what turned out to be false, but the Neo-Logicist specifically is still in a precarious position. There are at least three reasons to suspect this.

First, simply stating that the expectation of completeness was misguided does not explain why the expectation had arisen, and whether giving up on the expectation is possible without abandoning Neo-Logicism altogether. Perhaps, after all, the entire Neo-Logicist project was founded upon misguided expectations?

At least historically, Gödel’s incompleteness results came as a surprise. The opening page of Gödel’s publication of his incompleteness theorems in fact starts with the observation that, given how advanced the systems of *Principia Mathematica* and ZFC are, one may be tempted to think they are complete—but, as Gödel set out to demonstrate, this is not so (Gödel 1931, p. 145). Neo-Logicist writing that speaks of the truths of arithmetic being ‘*all* provable on the basis of general logical laws together with suitable definitions’ (emphasis added) may be further evidence that the Logicist project had really been aiming at demonstrating the analyticity of all arithmetical statements, and that Neo-Logicism still has not entirely, and possibly cannot, distance itself from that ambition (Hale and Wright 2001, p. 1).

Second, Wright argues that second-order logic itself is incomplete, and that, therefore, 'no special problem' arises for the Neo-Logicist specifically. However, this argument does not address whether Neo-Logicism achieves its stated objectives. The incompleteness of second-order logic is sometimes taken as a point of departure for a debate on whether second-order logic 'really is logic' (Rossberg 2006, pp. 208–221). However, one could presumably consistently hold the position that second-order logic is indeed logic (for example, because there was no reason to expect the existence of a complete deductive system for second-order logic (Shapiro 2005, p. 774)) and still maintain that incompleteness means that Neo-Logicism has failed to achieve its own goal of demonstrating the analyticity of arithmetic.

Third, the aforementioned response raises the question of what account the Neo-Logicist is to provide of those semantic consequences of Frege Arithmetic not derivable in a given formal system, such as D2. The three challenges we raised in §2.3 remain unaddressed. Are the semantic consequences of Frege Arithmetic not derivable in a given formal system, such as D2, synthetic? Or also analytic, but for a different reason than the fundamental ones? If we conceive of Neo-Logicism as aiming to explain how we have access to mathematical truths, that is an epistemological project, it would appear that the Neo-Logicist is in the awkward position of having to offer a separate account of the metaphysics and epistemology of those non-deducible truths without raising questions about the Neo-Logicist account of the metaphysics and epistemology of the deducible theorems. Furthermore, given that the choice of second-order deductive system impacts exactly which formulas are provable from FA+FD, does the resulting picture of mathematics not have an undesirable dependency on the choice of second-order deductive system?

To me, the 'no reason to expect' answer is thus not compelling—perhaps the Neo-Logicist has a way out of the earlier described tension between the classical Logicist ambition and restrictions imposed by Gödelian incompleteness, but pretending that there was never an issue to begin with is unlikely to be the solution. What is needed, if the Neo-Logicist is to defend a syntactic approach, is a precise Neo-Logicist account of how the syntactic approach reconciles Gödelian incompleteness with the claimed analyticity of arithmetic.

4.3.2 Fundamental truths response

Hence, as a second elaboration of *Uphold Syntactic Consequence*, let us revisit another idea mentioned by Wright and Hale: that at least the fundamental truths of arithmetic are provable from FA (WH2), and that, by implication, what is left unprovable is not fundamental. To maintain that at least the fundamental truths of arithmetic are derivable requires drawing a distinction between 'fundamental' and 'non-fundamental'. Such a distinction could seem *ad hoc*—where are we to draw the line?

Unfortunately, while Wright and Hale speak of 'fundamental truths of number theory' on multiple occasions, no explicit definition of 'fundamental' is provided. In one instance, Wright seems to assert that the Peano axioms are precisely the fundamental truths of number theory ('the fundamental truths of number theory, *that is*, the Peano axioms' (Wright 1983, p. 131, emphasis added)). In a different place, Wright speaks as if the fundamental truths contain the Peano axioms ('the fundamental truths of number theory, *in particular* the Peano axioms' (Wright 1983, p. 153, emphasis added)).

Thus, let us explore the possible options of how to interpret 'fundamental'. A first potential response is that precisely where the line between fundamental and non-fundamental arithmetical truths is drawn does not really matter, much as concepts such as 'heap' or 'sunny' do not have sharply defined boundaries, and yet these concepts are useful and not discredited by their vagueness. Furthermore, suppose we understand 'fundamental' to mean what is under consideration in typical arithmetical reasoning, a human discipline of thinking. In that case, one could not realistically expect to be able to draw a sharp boundary around it.

However, considering 'fundamental' as a vague predicate is problematic for the Neo-Logicist. At least along the current line of argument we are pursuing on behalf of the Neo-Logicist, formal provability is the property that guarantees analytical status. The Neo-Logicist will therefore want to show that formal provability in Frege Arithmetic pertains to (at least) all fundamental truths of arithmetic. Sometimes we can make precise judgements involving vague predicates—for example, one does not need to draw a boundary between 'tall' and 'not tall' to see that there are no tall babies. So, analogously to the baby case, one may wonder whether the deductive consequences of PA2 clearly contain the fundamental truths even if we cannot define 'fundamental' precisely. However, to show that the fundamental truths are all deductive consequences of Frege Arithmetic, the criterion for 'fundamental' will need to at least give us precise negative answers. For example, Con(PA2) is not a deductive consequence of PA2. But, from all that has been said, it is far from obvious that Con(PA2) is definitely not fundamental (as opposed to it being vague whether Con(PA2) is fundamental).

A second possible interpretation of 'fundamental' is that, definitionally, fundamental truths are precisely the syntactic consequences of the second-order Peano axioms under some preferred deductive system. This seems to be what Wright had in mind when he wrote 'the fundamental truths of number theory, *that is*, the Peano axioms'. Let us assume this definition of fundamental truths. The second-order Peano axioms are among the syntactic consequences of Frege Arithmetic (by Frege's Theorem) and syntactic consequence is transitive. Thus, the fundamental truths of arithmetic are among the syntactic consequences of Frege Arithmetic. If Hume's Principle is analytic, and analyticity is closed under syntactic consequence, then on this understanding of 'fundamental', we would therefore have a valid argument for the Neo-Logicist thesis that the fundamental truths of arithmetic are analytic.

Yet two questions arise. First, why draw the line between the fundamental and non-fundamental truths exactly here? For example, why should the semantic consequences of Frege Arithmetic that are not derivable in a given deductive system not be fundamental? To this, the Neo-Logicist may respond that it just so happens that this is where the Neo-Logicist defines the boundary. However, such a response is not very satisfactory: if there is no more principled account of why the distinction is to be drawn here, then the aforementioned argument appears merely as backsolving: the definition of 'fundamental' is such as to render the Neo-Logicist thesis true. The Neo-Logicist thesis would then have an apparent strength that is misleading because the use of 'fundamental' might not coincide with our ordinary understanding of the term. For example, why is Con(PA2), an arithmetical truth expressible in the language of first-order Peano Arithmetic, not equally fundamental? What sense does it make to speak of 'the fundamental theorems' if we know these theorems are not the complete foundations?

The second question that emerges is, even if the distinction between fundamental and non-fundamental is drawn here, what account does the Neo-Logicist have to provide of the semantic consequences of Frege Arithmetic that are not derivable in a given deductive system such as D2? For example, are the semantic consequences not derivable in a given deductive system analytic or synthetic?

At this point, Isaacson's Thesis may seem helpful. Isaacson maintains that it is not historical coincidence that the first-order Peano Axioms (PA1) have become the default choice as axiom system (Isaacson 1987, p. 147). Instead, Isaacson advances the following thesis:

Isaacson's Thesis (IT) PA1 'consists of those truths which can be perceived as such directly from the purely arithmetical content of a categorical conceptual analysis of the notion of natural number' (Isaacson 1987, p. 147). Truths expressible in the (first-order) language of arithmetic beyond what is provable from PA1 are such that 'there is no way by which their truth can be perceived in purely arithmetical terms' (Isaacson 1987, p. 147). Finally, PA1 'occupies an intrinsic, conceptually well-defined region of arithmetical truth' and 'may be seen as complete for finite mathematics' (Isaacson 1987, pp. 147–148).

The Neo-Logicist canon does not mention IT. However, the hope for Neo-Logicism employing IT could be to refute the allegation that Neo-Logicism sets up an artificial distinction between fundamental and non-fundamental truths. Instead, the defence suggests, the Neo-Logicist draws the distinction exactly where Isaacson posits our ability to perceive truth in purely arithmetical terms ends. To the extent that Neo-Logicism is an account of the epistemology of arithmetic, it should not be at all surprising that this is where the Neo-Logicist has to draw the line.

It would be beyond the scope of this article to assess the merit of Isaacson's Thesis. Instead, I would like to discuss one obstacle in re-purposing IT for Neo-Logicist aims as is: IT, as formulated earlier, pertains explicitly to PA1, whereas the Neo-Logicist was distinguishing fundamental and non-fundamental truths along the lines of syntactic versus semantic implications of PA2. For example, Isaacson maintained that Con(PA1), provable in PA2, contained 'hidden higher-order content' (Isaacson 1987, p. 154).

At least two responses are possible. First, to align his distinction with IT, Wright could redefine the fundamental truths to instead be the deductive consequences of the first-order Peano Axioms. However, redefining Wright's distinction would result in the second-order induction axiom of PA2 being no longer *fundamental*, but still *derivable* from Hume's Principle. This seems undesirable because the second-order induction axiom is central to Frege's proof that every natural number has a natural number as successor $(\forall x\,(\underline{\mathrm{Nat}}x \rightarrow \exists y\,(\underline{\mathrm{Nat}}y \wedge \mathrm{P}xy)))$, which is even one of the first-order Peano Axioms (Wright 1983, pp. 161–162).

An alternative response for the Neo-Logicist would be to point out that Isaacson accepts that our understanding of the concept of number goes beyond PA1. As Isaacson frames it, to the question whether 'PA1 is conceptually strong enough to analyse the concept of natural number', the answer 'must always be no' (Isaacson 1987, p. 154). Further, Isaacson clarifies that he is 'not claiming that PA1 could itself constitute an adequate conceptual basis for our understanding of the concept of natural number' (Isaacson 1987, p. 154). Instead, Isaacson is convinced that 'we can only arrive at such a system on the basis of some higher-order understanding' (Isaacson 1987, p. 154).

Thus, Isaascon's Thesis does not have to be in outright conflict with Neo-Logicism—the Neo-Logicist can maintain that what the Neo-Logicist is after is precisely our understanding of the *concept* of natural number. Wright and Hale argue, for example, that Hume's Principle is an explanation of the concept number (Hale and Wright 2001, p. 10). This explanation, by Isaacson's own account, is necessarily higher order (Isaacson 1987, p. 154).

Given we are in pursuit of a broader concept than just what Isaacson considered arithmetical truth, it thus seems open to the Neo-Logicist to formulate an expanded version of Isaacson's Thesis:

Neo-Logicist Isaacson's Thesis (IT2) The syntactic consequences of PA2 (with respect to a 'reasonable' deductive system, for example D2) are sound and complete with respect to the *concept* natural number.

With IT2, Wright's distinction between fundamental and non-fundamental truths can be given substance: the Neo-Logicist may claim that Frege Arithmetic, by proving the second-order Peano Axioms, is complete with respect to the concept natural number. Whether such a position is attractive is not immediately obvious—much like Isaacson's original thesis. Furthermore, a drawback compared to Isaacson's original thesis is that IT2 has to be relative to a particular deductive system because of the incompleteness of second-order deductive systems. In either case, if the Neo-Logicist was to adopt IT2, that would be both a substantial philosophical thesis and a helpful clarification of the Neo-Logicist project.

5 The recent Festschrift

5.1 Wright's Festschrift position

Besides the brief discussion of incompleteness in the 'canon' of Neo-Logicism summarised in §3, the topic is for the first time discussed in some detail in a chapter by Wright in a recent Festschrift (Wright 2020). In this section, I summarise Wright's position. In the subsequent two sections, I offer a critical analysis of Wright's argument.

At the outset, Wright distinguishes between a *Core logicist thesis,* which pertains solely to the deductive consequences of the

Dedekind-Peano axioms, and a *Supplementary thesis*, which pertains to all the truths of number theory. According to Wright, only the *Supplementary thesis* is under attack by the challenge from incompleteness (Wright 2020, p. 322).

Wright then suggests that the Neo-Logicist should regard logical consequence as deductive rather than semantic. However, for Wright, deductive consequence includes ascent to higher-order logics. This ascent, according to Wright, can be justified from an inferentialist perspective:

> Epistemologically, it would be a mistake for the inferentialist to think of higher-order quantifiers as coming in conceptually independent layers, with the meanings of the second-order quantifiers fixed by the second-order rules, the meanings of the third-order quantifiers fixed by the third-order rules, and so on. Rather, she should maintain that it is the *entire open series* of pairs of higher- and higher-order quantifier rules which collectively fix the meaning of quantification at each order: there are *single* concepts of higher-order universal and existential generalization, embracing all the orders, of which it is possible only to give a schematic, order-neutral statement. (Wright 2020, pp. 326–327, emphasis in original)

The correct notion of logical consequence, according to Wright, is therefore 'deductive consequence in the indefinitely extensible hierarchy of higher-order quantificational logics' (Wright 2020, p. 327).

Equipped with such a notion of logical consequence, Wright then maintains that:

> Third-order logic gives us the means to define a truth-predicate for second-order arithmetic and thereby to mimic rigorously in a formal third-order deduction the informal reasoning that justifies the conclusion that G [the Gödel sentence for the second-order Peano Axioms] holds good of any population of objects that satisfy 2PA [the second-order Peano Axioms]. (Wright 2020, p. 326)

According to Wright, this result can be extended to higher-order logics through continued ascent up the type hierarchy (Wright 2020, pp. 326–327). Based on this result, Wright goes on to argue that the Neo-Logicist can defend the Supplementary thesis as follows:

> (*) It remains open to a supporter of the Supplementary thesis to maintain that, for all Gödel's results have to say to the contrary, every validity of higher-order logic is provable using the deductive resources available at some (possibly transfinite) *n*th order. No semantic or model-theoretic conception of validity is needed. (Wright 2020, pp. 326–327)

Thus, provided we are willing to adopt 'deductive consequence in the indefinitely extensible hierarchy of higher-order quantificational logics', incompleteness disappears and is no longer a challenge for the Supplementary thesis.

Somewhat puzzlingly, on the same page, Wright also writes the following:

> All that said, it still has to be acknowledged, of course, that the proposal speaks only to the problem posed for a proponent of the Supplementary thesis by Gödel's incompleteness theorems. There is, so far as I am aware, no reason to think that for every arithmetical truth, ϕ, there is some nth-order quantificational logic such that ϕ is a deductive consequence of the Dedekind–Peano axioms in that logic. (Wright 2020, p. 327)

As an example, Wright states that 'Wiles's proof of Fermat's theorem, for example, has essential recourse to principles of algebraic geometry' (Wright 2020, p. 327). However, this comment is puzzling because in claim (*) earlier, Wright asserted that 'every validity of higher-order logic' becomes provable as we ascend the quantificational hierarchy, not just the Gödel sentences. As is well known, PA2 is finitely axiomatisable, so every truth of second-order Peano Arithmetic, including Fermat's last theorem, can be turned into a validity of second-order logic.

If Wright means to narrow the scope of (*) solely to Gödel sentences rather than 'every validity of higher-order logic', then this is, of course, a major revision of the claim just a few paragraphs earlier. Furthermore, his response to the challenge from incompleteness then appears open to an immediate objection: there are still truths of arithmetic that the Neo-Logicist is not in a position to prove using Hume's Principle, that is, Frege Arithmetic is still incomplete. Since only maintaining (*) as written has any prospect of addressing the challenge from incompleteness, which is seemingly Wright's purpose, I will in the following consider (*) as written.

5.2 Challenge: Core logicist thesis vs. Supplementary thesis

Wright's distinction between a *Core logicist thesis* and a *Supplementary thesis* clarifies what was already implicit in *Frege's Conception of Numbers as Objects*—namely that one way for the Neo-Logicist to respond is to focus just on the deductive consequences of the second-order Peano Axioms under some suitable deductive system. As noted in §4.3, Wright called these deductive consequences the 'fundamental truths of arithmetic' in *Frege's Conception*.

However, the terms 'Core logicist thesis, and 'Supplementary thesis' make the latter sound like a marginal issue—optional, perhaps a bonus.

As Wright puts it, 'The incompleteness theorems have no bearing on the Core thesis. But do they somehow scupper the Supplementary thesis?' (Wright 2020, pp. 321–322).

Of course, the Neo-Logicist is at liberty to simply define the Core logicist thesis as the claim that HP and its deductive consequences are analytic. Thus, in a narrow sense, the Core logicist thesis is indeed not affected by Gödel's incompleteness theorems. Yet this is effectively just relabelling the fundamental/non-fundamental truths distinction from *Frege's Conception of Numbers as Objects*. Therefore, the same three issues we encountered in sections 2.3 and 4.3 threaten the relevance of the distinction between a Core and a Supplemental logicist thesis. In particular, what account is the Neo-Logicist to offer of the analyticity status of the semantic consequences of HP that are not deductive consequences? Are they analytic as well, though for a different reason? Or synthetic? And how do we account for the fact that different possible choices of second-order deductive systems mean different formulas get categorised as 'core'?

The response we provide threatens to bear also on the Core logicist thesis. For example, if we reject the Supplementary thesis, that is, the semantic consequences of HP that are not deductive consequences are not analytic, then maintaining that the deductive consequences of HP are analytic appears to introduce a divergence in the epistemology of mathematics. Such a divergence should, at least *prima facie*, count as a reason against the Core logicist thesis. This discussion was taken up in detail in section 4.3—here I merely want to raise the question whether the distinction between a 'Core logicist thesis' and 'Supplementary thesis' is really as clean as the terminology suggests.

5.3 Challenge: Logical consequence under higher-order logic

The main aspect of Wright's response I would like to discuss is the notion of logical consequence he proposes, namely 'deductive consequence in the indefinitely extensible hierarchy of higher-order quantificational logics' (Wright 2020, p. 327). Wright's proposal in his *Festschrift* response is in a broadly inferentialist spirit—an inferentialism that Wright articulated at greater length in his 2007 paper *On Quantifying into Predicate Position*. In *On Quantifying into Predicate Position*, Wright observed that Frege 'seems to have conceived of quantification as such as an operation of pure logic, and in effect to have drawn no distinction between first-order, second-order and higher-order quantification in general' (Wright 2007, p. 150). According to Wright, Frege's 'insight into the nature of the conceptual resources properly regarded as logical' was later 'squandered' when Quine claimed that second-order logic was set theory in disguise (Wright 2007, p. 150).

Let us accept, for the sake of argument, Wright's inferentialism. Instead, we shall focus on whether cashing out this inferentialism as an indefinitely extensible hierarchy of higher-order quantificational logics will indeed resolve the challenge from incompleteness for the Neo-Logicist. Wright makes the following three key claims:

C1 Third-order logic 'gives us the means to define a truth-predicate for second-order arithmetic' (Wright 2020, p. 326);
C2 Third-order logic therefore allows us to 'mimic rigorously in a formal third-order deduction the informal reasoning that justifies the conclusion that G [the Gödel sentence for the second-order Peano Axioms] holds good of any population of objects that satisfy 2PA [the second-order Peano Axioms]' (Wright 2020, p. 326);
C3 'Every validity of higher-order logic is provable using the deductive resources available at some (possibly transfinite) *n*th order. No semantic or model-theoretic conception of validity is needed' (Wright 2020, pp. 326–327).

Let us go through these claims and evaluate their justification. Wright refers in his *Festschrift* response to two sections of Leivant's 1994 book chapter *Higher-order logic*. Given that Wright cites Leivant, I will use Leivant's presentation here. The relevant theorems from Leivant say the following:

Theorem (Leivant theorem 3.7.1). There is no second-order truth-definition for second-order sentences over $\mathcal{N}$ [the intended model], but there is a third-order truth-definition (Leivant 1994, p. 249);
Theorem (Leivant theorem 3.7.2). Let k≥1. Then there is a (k+1) order truth-definition for k order sentences over $\mathcal{N}$ (Leivant 1994, p. 250).

Wright glosses 3.7.1 as 'third-order logic gives us the means to define a truth-predicate for second-order arithmetic', and 3.7.2 as a generalisation of this result to higher-order logics. However, it is important to be clear that Leivant defines *being a truth-definition* semantically, and so both theorems 3.7.1 and 3.7.2 are semantic results. For example, theorem 3.7.2 says that there exists a formula τ of $k+1^{\text{th}}$-order with one free variable such that, for every sentence φ of at most k^{th}-order, we have:

$$\mathcal{N} \vDash \varphi \leftrightarrow \tau\left(\overline{\#\varphi}\right)$$

By contrast, Wright's claim C2 is about 'formal deduction', that is, syntactic. Given that in higher-order logics semantic consequence and provability do not generally coincide, one cannot immediately infer Wright's provability claim C2 from Leivant's semantic theorems. Nonetheless, related syntactic claims have been advanced. Gödel maintained the following in footnote 48a of his 1931 *On formally undecidable propositions of Principia mathematica and related systems I:*

> As will be shown in Part II of this paper, the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite (see Hilbert 1926, page 184), while in any formal system at most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added (for example, the type ω to the system P). An analogous situation prevails for the axiom system of set theory. (Gödel 1931, p. 181)

Unfortunately Gödel's remark is rather compressed, and Part II was of course never published. However, Gödel elaborated further on what he had in mind in his 1932 *Completeness and consistency:*

> If we imagine that the system Z is successively enlarged by the introduction of variables for classes of numbers, classes of classes of numbers, and so forth, together with the corresponding comprehension axioms, we obtain a sequence (continuable into the transfinite) of formal systems that satisfy the assumptions mentioned above, and it turns out that the consistency (ω-consistency) of any of those systems is provable in all subsequent systems. Also, the undecidable propositions constructed for the proof of Theorem 1 become decidable by the adjunction of higher types and the corresponding axioms; however, in the higher systems we can construct other undecidable propositions by the same procedure, and so forth. To be sure, all the propositions thus constructed are expressible in Z (hence are number-theoretic propositions); they are, however, not decidable in Z, but only in higher systems, for example, in that of analysis. (Gödel 1932, p. 237)

Thus, Gödel did indeed hold Wright's claims C1 and C2, that a suitable higher-order deductive system S′ can prove the Gödel sentence and consistency statement for a lower-order system S. Regarding C1, developing a truth theory for S in S′, the basic idea is a syntactic version of Leivant's work: define a 'truth' formula τ with one free variable for codes of formulas. Set up τ in such a way that τ simulates model-theoretic satisfaction with higher-order variables. The idea behind C2 is then to show that certain formulas not provable in S become provable in S′. For example, one can prove in S′ by induction that the statements provable in S are all 'true' in the sense of τ, and thus that S is consistent (Feferman 2006, p. 438). Of course, the details of this approach need to be carefully worked out. However, the details are not of importance for the following objections to Wright's conclusion C3. After all, if this approach to C1 and C2 cannot be made to work, then it would threaten Wright's argument.

The crux of Wright's response to incompleteness is thus C3, that every validity of higher-order logic is provable using the deductive resources available at some (possibly transfinite) *n*th order. While C1 and C2 'pave the way', only C3 provides the means to address the challenge from incompleteness. However, notice that Gödel did not go on from C1 and C2 to claim C3. Wright's C3 maintains that 'every validity of higher-order logic' becomes provable at some point in the type hierarchy. By contrast, in the first quote, Gödel merely states that 'the undecidable propositions constructed here become decidable', that is, the Gödel sentence and consistency statement. In the second quote, Gödel only says that 'the consistency (ω-consistency) of any of those systems is provable in all subsequent systems'.

First, it is important to be clear what exactly C3 is claiming. Suppose S is a second-order deductive system with the axiom set PA2. Consider a higher-order deductive system S′ that extends S. If S′ is of finite-order, then S′ will be a formal system to which Gödel's incompleteness theorems apply. Thus, there will be an undecidable Gödel sentence G for PA2 in S′. Moreover, as Gödel notes earlier, the arithmetisation of syntax allows us to express G as a sentence in the language of second-order Peano Arithmetic. Thus, G will be expressible in the language of S, but will not be provable from PA2 in the formal system S′. This is in line with the observation that there is no complete formal deductive system for second-order logic. In particular, we cannot construct a complete, consistent deductive system for second-order logic by ascending the type hierarchy.

However, this observation does not contradict Wright's claim C3 outright. C3 leaves open two possibilities: (i) for every validity ϕ of higher-order logic there is a *finite* order in which ϕ becomes provable, (ii) there are validities ϕ of higher-order logic which only become provable at *transfinite* orders, but every validity of higher-order logic becomes provable at some transfinite stage. These two possibilities are worth distinguishing because which of the two obtains may be of philosophical import in determining the significance of C3.

We now analyse these possibilities in turn. While Gödel sentences and consistency statements for a given order become provable at the next higher order, this does not rule out the possibility of constructing sentences that are not provable at any finite order. To do so, we just need to slightly modify the way the Gödel sentences are constructed.

The way the ordinary Gödel sentence is constructed is by arithmetising syntax and then using this arithmetisation to define a proof predicate $\mathrm{Prf}(v_1, v_2)$ in which v_1 is intended to be the code of, for example, a PA2-proof of the formula coded by v_2. Assuming the deductive system is extended to higher orders in the way Wright describes (for example, by introducing analogous quantifier introduction and elimination rules), there is nothing to stop us from defining an analogous three-place proof predicate $\mathrm{Prf}(v_1, v_2, v_3)$ in which the third variable v_3 is the order of the deductive system. So, for example, $\mathrm{Prf}(x, y, \overline{2})$ expresses that x codes a proof of the formula coded by y in a second-order deductive system. Rather than showing that $\mathrm{Prf}(v_1, v_2)$ expresses provability in (for example) a second-order deductive system, one would then need to show that $\mathrm{Prf}(v_1, v_2, v_3)$ expresses provability in the calculus coded by the third variable. Devising such a formula $\mathrm{Prf}(v_1, v_2, v_3)$ is straightforward provided each step up in the type hierarchy extends the deductive system along the same pattern in the way Wright envisages (for example, via new natural deduction rules for the incremental quantifiers). The idea is to take the usual definition of Prf and replace the part that checks whether a particular proof line is an instance of the second-order natural deduction rules with a check whether there exists a z^{th} order $(z \leq v_3)$ such that the proof line is an instance of the natural deduction rules of that z^{th} order. If v_3 is 0 or 1, let Prf reject all proofs.

Once Prf is defined, one can define provability in higher-order deductive systems in the analogous way: $\mathrm{Pr}(v_2, v_3) \leftrightarrow \exists x\, \mathrm{Prf}(x, v_2, v_3)$. We assume here without proof that Prf can be constructed such that Pr meets the Hilbert-Bernays conditions and is a Σ-formula. The intuition behind $\mathrm{Pr}(v_2, v_3)$ being a Σ-formula is that, as mentioned earlier, extending Prf to higher orders solely introduces existential quantifiers. We can then use the diagonal lemma to find a formula G such that:

$$\mathrm{PA2} \vdash_2 \mathrm{G} \leftrightarrow \forall k \neg \mathrm{Pr}(\ulcorner \mathrm{G} \urcorner,\ k) \tag{2}$$

where $\vdash_2$ indicates deduction in second-order logic. One can then show that G is not provable in any consistent, n^{th}-order deductive system, with n a positive integer, that extends PA2. To show this, consider the following:

1. Suppose (seeking a contradiction) that $\mathrm{PA2} \vdash_n \mathrm{G}$;
2. $\mathrm{PA2} \vdash_n \mathrm{Pr}(\ulcorner \mathrm{G} \urcorner, \overline{n})$, from 1 and assuming Pr fulfils the Hilbert-Bernays conditions (see earlier);
3. $\mathrm{PA2} \vdash_n \forall k \neg \mathrm{Pr}(\ulcorner \mathrm{G} \urcorner, k)$, from 1 and by (2);
4. $\mathrm{PA2} \vdash_n \neg \mathrm{Pr}(\ulcorner \mathrm{G} \urcorner, \overline{n})$, by instantiating 3, contradicting 2 given consistency.

However, we can see that PA2 ⊨ G:

1. PA2 ⊨ $\neg\mathrm{Pr}(\ulcorner G \urcorner, \bar{n})$ for any $n \geq 2$, by the usual argument for the truth of the Gödel sentence:

(a) Suppose (seeking a contradiction) that PA2 ⊨ $\mathrm{Pr}(\ulcorner G \urcorner, \bar{n})$;
(b) PA2 $\vdash_n \mathrm{Pr}(\ulcorner G \urcorner, \bar{n})$ by Σ-completeness of $\vdash_n$ and Pr being a Σ-formula (see earlier);
(c) PA2 $\vdash_n$ G, since $\mathrm{Pr}(\ulcorner G \urcorner, \bar{n})$ expresses provability of G in the n^{th}-order deductive system (see earlier) and (b);
(d) PA2 $\vdash_n \forall k \neg\mathrm{Pr}(\ulcorner G \urcorner, k)$, from (2) and (c);
(e) PA2 $\vdash_n \neg\mathrm{Pr}(\ulcorner G \urcorner, \bar{n})$, by instantiation of (d), contradicting (b) if the system is consistent.

2. PA2 ⊨ $\forall k\, \neg\mathrm{Pr}(\ulcorner G \urcorner, k)$, from 1, the definition of satisfaction for universal quantifiers, categoricity of PA2 and since Prf rejects any proofs if k is 0 or 1;
3. PA2 ⊨ G from (2) and 2.

Thus, possibility (i) of claim C3 is not borne out. This leaves open possibility (ii), that while there are validities ϕ of higher-order logic that only become provable at *transfinite* orders, every validity of higher-order logic becomes provable at some transfinite stage. However, I would like to suggest that there are two challenges with the Neo-Logicist adopting possibility (ii).

The first is that Wright's inferentialism does not provide sufficient justification to regard ascending to *transfinite* quantificational orders as properly logical. Let us grant the Neo-Logicist position that second-order logic is logic (any Quinean concerns notwithstanding). Furthermore, let us grant that there is a certain conceptual continuity in moving from a deductive system with n^{th} order quantifier introduction and elimination rules to a deductive system that also has $(n+1)^{\mathrm{th}}$ order quantifier introduction and elimination rules. It does not follow that adding, for example, ω-level quantifier introduction and elimination rules is still warranted by inferentialism. Second-order logic may not be set theory in sheep's clothing, but ω-order logic and beyond look suspiciously like it.[2]

The second concern is the following: suppose for the sake of argument that, by ascending the type hierarchy transfinitely many times, one

[2] Gödel had maintained that '[...] it turns out that this system [Zermelo-Fraenkel set theory] is [...] what becomes of the theory of types if certain superfluous restrictions are removed' (Gödel 1933, pp. 45–46). One of the three superfluous restrictions Gödel had in mind was the restriction to finite types. See Linnebo and Rayo 2012 for a discussion and qualified defence of Gödel's claim.

had indeed arrived at a complete (and consistent) system. Then, by contraposition of Gödel's first incompleteness theorem, the resultant system would no longer be recursive. But in giving up recursiveness, and thus computability (assuming the Church–Turing thesis), this Neo-Logicist response to Gödelian incompleteness threatens to undermine a key motivation for pursuing *Uphold Syntactic Consequence* (rather than the semantic approach) to begin with—namely, to maintain that the truths of arithmetic are analytic in the epistemological sense. Thus, neither interpretation (i) nor interpretation (ii) of C3 is attractive for the Neo-Logicist.

6 Conclusion

In this paper, I first argued that there is a long-standing gap in the literature as to whether Gödelian incompleteness constitutes a meaningful challenge for Neo-Logicism. I then defended the claim that many responses that may appear attractive initially are in fact not compatible with the Neo-Logicist's epistemically-motivated project of demonstrating the analyticity of arithmetic. Finally, I argued that Wright's recent proposal does not overcome the challenge from incompleteness.

I thus hope that, at a minimum, this investigation has helped highlight a few internal tensions in the Neo-Logicist project's ambition and methodology in order to facilitate clarifications.[3]

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