



Purifying applied mathematics and applying pure mathematics: how a late Wittgensteinian perspective sheds light onto the dichotomy

José Antonio Pérez-Escobar¹ · Deniz Sarikaya² 

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Abstract

In this work we argue that there is no strong demarcation between pure and applied mathematics. We show this first by stressing non-deductive components within pure mathematics, like axiomatization and theory-building in general. We also stress the “purer” components of applied mathematics, like the theory of the models that are concerned with practical purposes. We further show that some mathematical theories can be viewed through either a pure or applied lens. These different lenses are tied to different communities, which endorse different evaluative standards for theories. We evaluate the distinction between pure and applied mathematics from a late Wittgensteinian perspective. We note that the classical exegesis of the later Wittgenstein’s philosophy of mathematics, due to Maddy, leads to a clear-cut but misguided demarcation. We then turn our attention to a more niche interpretation of Wittgenstein by Dawson, which captures aspects of the aforementioned distinction more accurately. Building on this newer, maverick interpretation of the later Wittgenstein’s philosophy of mathematics, and endorsing an extended notion of meaning as use which includes social, mundane uses, we elaborate a fuzzy, but more realistic, demarcation. This demarcation, relying on family resemblance, is based on how direct and intended technical applications are, the kind of evaluative standards featured, and the range of rhetorical purposes at stake.

Keywords Later Wittgenstein · Theory choice · Meaning as use · Meaning of mathematics · Embodied mathematics · Philosophy of applied mathematics · Philosophy of mathematics

José Antonio Pérez-Escobar and Deniz Sarikaya contributed equally to this work.

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Guest Editors: Matthew W. Parker, Davide Rizza

✉ Deniz Sarikaya
deniz.sarikaya@uni-hamburg.de

Extended author information available on the last page of the article

1 Introduction

The pure/applied distinction in mathematics is often taken for granted without much thought put into it. One of the few philosophers that have engaged with this dichotomy is Wittgenstein, especially the later Wittgenstein. The later Wittgenstein's work on philosophy of mathematics, despite its numerous and original insights, has received little attention, and hence, interpretations and applications of this work are still lacking or in process. We believe that one of the potential applications of this work is the building of a demarcation between pure and applied mathematics.

We argue that at the base of the pure/applied mathematics dichotomy, which mainly describes sociologically distinct groups, lie different community-specific priorities (regarding evaluative standards, directness of application and rhetoric) instead of a deeper metaphysical distinction between two putative realms. We show that Maddy's exegesis of the later Wittgenstein's philosophy of mathematics (Maddy, 1993) fails to capture such practice-based differences, and hence is not useful for our main aim, namely, to build a demarcation between pure mathematics and applied mathematics. However, another, maverick exegesis of the later Wittgenstein, recently proposed by Dawson (2014) and which has not received much attention yet, may be more appropriate for our aim. While this is not an exegetical work, given the principle of charity, we will endorse this maverick exegesis, further build on it, and apply it to our philosophical endeavor.

More specifically, we first draw from a classical, revisionary interpretation of Wittgenstein's late philosophy in section 2 which, we argue, cannot account for practices in pure mathematics.¹ Instead, we propose a sociological analysis of practices based on the Wittgensteinian notion of meaning as use in section 3. We acknowledge a sociological division between pure mathematics and applied mathematics communities based on putatively extra-mathematical factors: there are different prizes, journals, chairs, departments, and so forth. In line with late Wittgensteinian philosophy, we believe that this sociological distinction is ultimately rooted in the different ways in which each community *uses* mathematics. Therefore, we analyze their uses of mathematics to understand what is meant when one talks about applied mathematics or pure mathematics. We note that the same mathematics may be used differently in different contexts, and hence, "pureness" is not a property of the symbols or diagrams that comprise mathematics. By family resemblance, another key concept from Wittgenstein's late philosophy some of these uses can be considered "pure" and others can be considered "applied", but these uses, instead of being strongly demarcated as in section 2, share many traits.

In section 4, we discuss two aspects of pure mathematics which show that it is in fact grounded in the real world, in contrast to the tenets of Maddy's exegesis of the later Wittgenstein. These aspects are 1) the role of abstraction or mathematization as an iterative process from the real world to pure mathematics (what we call the

¹ This classical interpretation is revisionary in the sense that Maddy's Wittgenstein considers that pure mathematics, due to being meaningless, should be revised.

neo-Aristotelian view) and 2) the idea that meaning from our everyday experience is transferred to the realm of pure mathematics (called embodied mathematics).

In section 5, we note how the considerations in sections 3 and 4 are compatible with and able to be philosophically framed by another, maverick interpretation of Wittgenstein's late philosophy of mathematics, one that accepts meaning in pure mathematics (Dawson, 2014). Finally, in section 6, we build on this interpretation by extending meaning as use sociologically, and further elaborate on the fuzzy demarcation of pure and applied mathematics.

2 A classical reading of Wittgenstein framing the division pure/applied math, and the tension between them

In this section we elaborate on the revisionary reading of the later Wittgenstein, exemplified by Maddy's reading. As we will see, Maddy's Wittgenstein holds that only applied mathematics is actually meaningful and that pure mathematics in contrast is a meaningless sign game.

2.1 Maddy's interpretation of the later Wittgenstein

Maddy provides an interpretation of Wittgenstein which is very constraining on the meaningfulness of pure mathematics. According to her, Wittgenstein rejects meaning in pure mathematics and considers it a mere language game, since it has no direct applications, and therefore, no uses (it is "language on holiday"). Given that for the later Wittgenstein it is the use of propositions which endows them with meaning, pure mathematics would be meaningless. According to her interpretation, pure mathematics has no direct applications, but rather illusory, imaginary applications due to bad prose (that is, what mathematicians say about what they do, in contrast to what they do).² For this reason, pure mathematics should be pruned away. Applied mathematics may have bad prose too, but it has direct applications, so we have a reason to keep it despite misleading philosophical reflections on the symbols. Furthermore, this critical difference is where, in Maddy's eyes, the difference between pure and applied mathematics lies for Wittgenstein. Let us take a look at two quotes from Wittgenstein's *Remarks on the Foundations of Mathematics* (Wittgenstein, 1978; RFM) that she showcases to support her interpretation:

I want to say: it is essential to mathematics that its signs are also employed in mufti. ... It is the use outside mathematics, and so the meaning of the signs, that makes the sign-game into mathematics (RFM: V 2).

(Pure mathematics is) ... a piece of mathematical architecture which hangs in the air, and looks as if it were, let us say, an architrave, but not supported by anything and supporting nothing. (RFM: II 35)

² The question of the prose of mathematics has been systematically studied, for instance, in Kant & Sarikaya (2020), in the context of narratives about mathematical practice.

Regarding the first quote, Maddy takes “*in mufti*” to mean “directly applicable” (Dawson, 2014). Thus, according to her reading of Wittgenstein, mathematics without a direct application is just an accumulation of meaningless signs. Via the second quote, it is further clarified that mathematics without an application has nothing to do with the world.

The question remains: how good is this interpretation to distinguish or relate pure and applied mathematics? First let us develop some consequences of endorsing this classical reading.

If pure mathematics is detached from the world, has no uses, does not sustain anything and nothing sustains them... how does it come to be, how does it grow? We can at least take this much: according to this view, pure mathematics and applied mathematics must be grounded in different grounds, and the ground of applied mathematics is its applications. But what about pure mathematics? A common picture is that it proceeds by deduction. Deduction needs not be grounded in anything in the world but its own foundations and rules of inference,³ and therefore this picture is compatible with the idea that pure mathematics is not sustained nor sustains anything. Applied mathematics on the other hand would need to be mapped to the world in order to yield successful applications. This would also explain the tension between pure and applied mathematics in terms of opposite derivation procedures: one proceeds by sheer deduction, while the other needs to accommodate non-deductive inferences and empirical observations characteristic of scientific practices, which leads to disagreements on what counts as “good mathematics” and what mathematical rigor is. However, we do not think this difference strictly holds so that it explains the pure/applied mathematics division. Moreover, the very idea of formalist deduction is misleading from a late Wittgensteinian point of view. Maddy’s interpretation is problematic in as much as it has to account for practices in pure mathematics without invoking formalist deduction (which goes against Wittgenstein’s anti-formalist late philosophy), and disregards potential uses and applications (the interpretation assumes there are none). As discussed in the next section, we believe that the distinction between pure and applied mathematics is better understood from a sociological point of view, one that takes into account uses and applications. Maddy’s Wittgenstein seems not to take this path, and yields an oversimplified demarcation which is not representative of mathematical practices. Therefore, Maddy’s exegesis is not an adequate tool for the purpose of demarcating pure and applied mathematics.

³ By “own foundations”, we mean elements which, according to platonist or formalist views, lead to “written in stone” mathematics, and do not depend on applications or other constraints on practices. For example, according to formalist views, if mathematics is fully constrained by deductive principles, other facts about the world (either physical or sociological) do not play a part in mathematical practices: what can be done in pure mathematics would, in such a view, be predetermined by logical constraints. A famous attempt to provide such mathematical foundations is Hilbert’s program. Views like these, in turn, account for an emphasis on the study of mathematical foundations, and the narrow focus that philosophy of mathematics has had for decades.

3 Bridging pure and applied mathematics

In the following section we will distil some properties of applied mathematics and stress how pure mathematics is more similar to applied mathematics than one might expect at first glance, and not so clear-cut as Maddy's Wittgenstein purported it to be.

A key notion in the later Wittgenstein's thought is meaning as use. This idea is present in his Blue and Brown Books (Wittgenstein, 1958; BB) and ubiquitous in the Lectures on the Foundations of Mathematics (Wittgenstein, 1976; LFM) and RFM. According to this idea, and in the context of mathematics, symbol arrays and diagrams do not have any intrinsic and univocal meaning; they are not stable representations of other kinds of phenomena. Instead, the meaning of symbol arrays and diagrams resides in how they are used in a broad sense, what techniques they enable, what role they play in practices, and so forth.⁴ Because of the broad scope of meaning as use and the character of practices where mathematics play a part, one cannot avoid a sociological analysis of mathematics in order to understand it.

3.1 Bridging from the applied perspective

A naïve definition of applied mathematics is actually well motivated by its very name: Applied mathematics comprises mathematical methods for other fields. However, this definition is problematic. It is too broad and too narrow at the same time. It is too broad as some uses of mathematical methods in other fields do not appear to be applied mathematics. Every physicist, engineers, most chemists, and many biologists use mathematical tools. Even more, psychologists and sociologists using quantitative methods would be considered applied mathematicians as well. Even mathematicians using mathematical tools to solve real world issues – since there are unintended applications of “pure” problems, that is, applications which were not intended or expected when the theoretical work was articulated – would not consider themselves to be applied mathematicians. Famous examples are number theoretic considerations concerning prime numbers that turned out to be useful for cryptography, and group theory being developed long before the physical theories that use it nowadays. For a broader, very recent discussion on intended vs unintended applications, we refer the reader to Molinini (2019), who in turn refers to the distinction as “when mathematics is introduced and developed in the context of a particular scientific application” vs “when mathematics is used in the context of a particular scientific application but it has been developed independently from that application”.

⁴ An anonymous reviewer raised the interesting comment that the sociological division of pure and applied mathematicians could be well understood from Wittgenstein's notion of language games: the division reflects different games being played. We believe that grounding this distinction in the notion of meaning as use entails this consideration: the analysis of the different uses of mathematical symbols as a way to ascertain their meaning rests on the premise that there are different games, featuring different rules, being played. In our account, “use” includes the application of different evaluative standards to mathematical symbols by the different communities. As we will see, some of these standards and their derived uses of mathematical symbols, sharing family resemblance, can be more or less smoothly grouped as “pure” or “applied”.

According to Molinini, this connection is the basis for a “weak objectivity” account of mathematics, and the “reasonable effectiveness of mathematics”. There is also the whole community of statisticians, who would often consider their discipline a separate branch of mathematics (besides pure and applied).⁵

The definition is also too narrow for the following reason: there is theory building in applied mathematics in addition to the mere application of mathematical tools. Differential equations, approximation theory and functional analysis are typically closer to the applied side of things, while also entailing many theoretical results. Not all work in applied mathematics is useful for application: some work may be about the theory of mathematical techniques with real world applications, that is, about the “inner theory” of applied mathematics.

Therefore, not every mathematical theory with direct applications is applied mathematics, and some mathematical theories without direct applications are applied mathematics. It remains true, however, that the latter theoretical work is grounded and motivated by models used in practice. Therefore, the picture inspiring the simplistic definition of applied mathematics is in need of refinement. The main aspect of applied mathematics is not that it has applications but the conception and development of mathematical theories with applications in mind. As we will see, the key characteristic is that applied mathematics constitutes a limited number of mathematical theories/tools which cater to the possibilities of evaluating and developing (mathematical) models for physical systems and other real world applications.

Hence, the above definition sets those applied mathematicians who do genuine theory building apart from those who focus on applications based on such theories. The exact classification of these theories might not be well established but one could say that the courses in a university catalog give a rough estimate of this picture. As mentioned before, some theories (esp. those that are closer to statistics) do not count, at least sociologically, as applied mathematics.

The above definition also excludes pure fields that just happen to have applications. Pure mathematics is evaluated against inner mathematical criteria, like beauty, simplicity, unificatory power and so forth. But again, theories are both pure and applied. Thus a mathematical theory can be used/seen from a pure or applied perspective, even if some are much more often seen from one lens than from the other. These perspectives differ in the evaluation standards used, related to either application or inner mathematical criteria.

3.2 Bridging from the pure perspective: Axiomatization as modelling in pure mathematics

While it is true that pure mathematics can in principle start from arbitrary axioms, this does not reflect the actual research practice. There is a large modelling-like part when building pure theories. A mathematical theory usually starts from informal notions; it might not be entirely clear how to define the structures mathematicians

⁵ Some would even say that statisticians are not mathematicians at all, but this is beyond the scope of this paper.

talk about and there might be different competing notions or clarifications of those. Older disciplines are usually more settled and work in concrete axiomatic settings. A short example discussed by Mancosu (2009) is the following: It is not straightforward to teach undergraduates that bijections are the way to measure sizes in infinite domains. There is a reason for this: there are fundamental conflicts between intuitions when comparing sizes infinitely, so the following principles seem very plausible ad hoc.

Hume-cantor intuition If there is a bijection between two sets, they are the same size.

Part-whole intuition If a set A is a real subset of set B, then quantity B is larger than quantity A.

Both intuitions quickly lead to the so-called Galileo paradox: the square numbers are a real subset of the natural numbers, so the natural numbers (after the part-whole intuition) are larger than the natural numbers. Now the function

$$f : x \rightarrow x^2$$

is a bijection between square numbers and natural numbers, so both are sets (according to Hume-Cantor intuition) of the same size. We have the following statements, the conjunction of which is paradoxical: 1) The natural numbers are larger than the square numbers and 2) the natural numbers are the same size as the square numbers. Today's resolution resides mostly in that we see part-whole intuition as misleading in infinity. In fact, it is possible through so-called Euclidean Theories of Size to save the part-whole intuition (cf Benci et al., 2006).

Here decisions are made in a way we would not find in applied endeavours. Both theories are consistent and the adaption of one of those to be *the* theory of infinite sizes, is partly arbitrary.

3.3 The context matters. An illustrative example: The Dirac delta function

A pertinent third point is that the same piece of mathematics can be treated differently depending on the context: the same mathematics can be pure or applied. We will illustrate this with the following example. The Dirac delta function (δ) has been subject of controversies due to its lack of mathematical rigor, which culminated in substantial efforts towards its rigorization. The mathematical problem of the Dirac delta function is well exposed by Davey (2003, 443-449). There are two key ingredients: First, a basic intuition is that δ is a function that is $+\infty$ exactly at 0 and everywhere else 0. A – by today's standards – mathematically trained person would argue that this is no function (at least with the usual domain and target being the rational numbers), since ∞ is no real or complex number. But let us leave these considerations aside for a moment and try to work with this strange mathematical entity. The second ingredient is that when we integrate over the

whole real numbers the product of the delta function with any function $f(x)$, only its value at position 0, namely $f(0)$, plays a role, or in a formula:

$$\int_{-\infty}^{+\infty} f(x)\delta(x) \, dx = f(0),$$

for L2 functions, i.e. functions for which in turn it holds that:

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx$$

is finite and which play an important role in physics and engineering. Using the constant function assigning all values to 1, this leads to:

$$\int_{-\infty}^{+\infty} \delta(x) \, dx = 1.$$

Again, we could reflexively object that a function that is only non-zero at a Lebesgue zero set (in this case at one point) cannot have a non-zero integral. While Lebesgue zero can be made precise with some measurement theory, this possibility remains uncertain in the case at stake, as it is a rather small set. And yet, the point is that the above treatment of this strange mathematical entity translates to important contributions to physics, as presented by Dirac in his *Principles of Quantum Mechanics*, reprinted as Dirac (1981).

However, strictly speaking, there is no such function $D(x)$. Let us say that we set L2 functions f and g such that $f(x) = g(x)$ everywhere except at point 0, that is, $f(0) \neq g(0)$. Then, $f(x)\delta(x) = g(x)\delta(x)$ almost everywhere; a change of a function at a Lebesgue zero set (or at one point in this case) cannot change its integral, thus we get:

$$\int_{-\infty}^{+\infty} f(x)\delta(x) \, dx = \int_{-\infty}^{+\infty} g(x)\delta(x) \, dx$$

Nevertheless, if the left-hand side of the equation is $f(0)$, the right-hand side contradicts $f(0) \neq g(0)$. Therefore, the delta function can introduce contradictions in physics.

What do physicists do about it? Something quite easy: if delta functions are kept inside integral signs spanning the whole real numbers, they pick out concrete values of that function. Physicists only use delta functions inside integral signs for this purpose. Why should they care about or run into the niche contradictions mentioned above? Davey (2003: fn 13) mentions how the rigorization of delta functions by Schwartz was of no significance for physicists, as they considered that the situation of delta functions “was already well under control”. This is the crux of the matter: it shows that physicists are sometimes perfectly happy to knowingly exploit contradictions, before and even when mathematically rigorous alternatives are known to be available.

We could elaborate a more encompassing story of the delta function, including whether it implies a fruitful abuse of notation, or whether it would make sense if interpreted slightly differently with measure or generalized functions. The latter are indeed parts of the history of the introduction of the delta function. For instance, its properties were justified by examples of families of functions “converging” into it (albeit in “unrigorous” ways). But then again, these divergences would miss the point.

A mathematical move is “forbidden” for things to work fine. What should we do then? Should we give primacy to mathematics, and then admit that this piece of mathematics is wrong and that our mathematical understanding of quantum mechanics therefore is flawed as well? Or should we preserve this piece of mathematics, keep it within integral signs at all times, and let things work perfectly fine? If we follow the considerations from the sections above, it is not legitimate to extrapolate mathematical rigor to science, especially if it harms science. The point is that this is a piece of applied mathematics, not pure mathematics, with different purposes, and where different rules apply. Here, we do not want to explore the consequences of applying certain logical rules, perform logically valid steps to manipulate symbols, and see what we get, but to understand a physical phenomenon and develop concrete techniques based on this understanding. The process of mapping mathematics and quantum mechanics served the second purpose and not the first, which is why not all accepted steps in pure mathematics are allowed, and in this example, not even contemplated in advance.

Wittgenstein acknowledges this demarcation between idealized deductivist mathematics and local mathematical techniques, and favors the latter. Consider the following quotes from the Lectures on the Foundations of Mathematics (LFM):

You might get $p \cdot \neg p$ by means of Frege’s system. If you can draw any conclusion you like from it, then that, as far as I can see, is all the trouble you can get into. And I would say, ‘Well then, just don’t draw any conclusions from a contradiction’. (LFM: 220)

Suppose I convince Rhees of the paradox of the Liar. And he says, “I lie, therefore I do not lie, therefore I lie and I do not lie, therefore we have a contradiction, therefore $2 \times 2 = 369$. Well, we should not call this ‘multiplication’; that is all” (LFM: 218)

Hence, he would probably call the delta function “a proposition of physics” rather than “a proposition of mathematics”. The former constitutes local techniques with direct applications, and therefore constitutes applied mathematics to which the deductivist notion of rigor does not apply. By extension, if something goes wrong with the technique, the problem lies not in mathematics, but in the technique itself (in this case, in the physical technique, which locally maps “rogue” mathematics to certain phenomena in a way that yields desired results). As we have seen, the physicist, aware or unaware of what is going on at a philosophical level, employs a similar strategy, which is embedded in his/her disciplinary tradition: certain mathematical moves are meaningless in their language.

Maddy’s Wittgenstein would say that applying the standards from pure mathematics in this context (perhaps in the form of pure deduction) does not make sense,

because we are dealing with the world, instead of “nothing”. Our take here also reaches the conclusion that pure mathematics should not overtake applied mathematics, but via another path: pure and applied math share similar principles (as shown in sections 3.1 and 3.2), both are “worldly” (next section will show how); the difference is one of context and purpose instead and these should not be confused. Pure mathematics does have a legitimate place in the world, but it should not overtake the place of applied mathematics.

As we see, the difference between applied and pure mathematics seems to be more nuanced than how it was purported to be by Maddy’s Wittgenstein in section 2 and the naïve definition offered at the beginning of section 3, since it does not reside in applications vs a lack thereof.

4 The embodied world and mathematics: Against the strong division of pure and applied mathematics

In this section we argue that pure and applied mathematics actually share a common starting point. This is relevant for grounding a non-revisionary reading of the later Wittgenstein (one that does not interpret Wittgenstein as prescribing revisions to pure mathematics) accepting mediated application, as we will see in section 5. This section further blurs the distinction between pure and applied mathematics.⁶ Once this similarity has been stressed, it will be argued that the main difference between pure and applied mathematics resides solely in evaluation standards and selection of theories. This entails that it is not right to talk about pure and applied areas of mathematics but only about pure and applied lenses.

We relate pure mathematics to the real world by noting: 1) the successive abstraction from real world problems and 2) the usage of intuitions from other domains (we call the former the *neo-Aristotelian* view and the latter the *metaphorical view*). Both approaches were called “embodied mathematics” by Lakoff and Núñez (2000) and Tall (1991, 2013). Both directions have been recently explored from different angles. For instance, it has recently been argued that input from the real world (mainly about mathematical practices, including uses of intuitions and metaphors) is essential to philosophize about mathematics (Kant et al., 2021). Moreover, it has been shown that mathematics is connected to intuitions about the real world, and accommodates them via representational ambiguity (Pérez-Escobar, 2020).

4.1 The neo-Aristotelian view

Aristotle saw mathematics as coming from the real world. There are several ways this may happen.⁷ There is the example of the bronze sphere: if we take away the “bronze” we are left with a sphere, a mathematical entity. Here we see that new

⁶ A more historically grounded idea along similar lines can be found in Maddy (2008).

⁷ This includes: (1) abstraction or ‘removal’ (*aphaireseôs*) (2) Precision (*akribēia*) (3) As Separated (*hōs kekhōrismenon*) (4) *X Qua (hēi) Y* (5) Intelligible matter (*noëtikē hylē*)

aspects are added: being a *mathematical object* is one such new attribute.⁸ This line of thought can be found elsewhere: Husserl believed that what reaches our sense organs is always underdetermined and, in a sense, informed by our history. We actively structure what reaches our sense organs. Put differently, our *seeing* objects, our *perceiving* a world around us is the product of a complex activity of structuring sense-data. For instance, we grasp sense-data by incorporating them in a spatio-temporal structure. We see objects, not sense-data. Husserl recognizes two kinds of intuition: The first he simply calls “intuition”, the second “*Wesensschau*”. This second type of Husserlian intuition is the one that matters for our present purposes. It is the kind of intuition that, according to Husserl, is necessary to have a grasp of mathematical objects as well as of the *syncategorematical* parts of sentences, for instance of the connectives (“and”, “or”, “if ... then” between propositions).⁹ So to speak, the *Wesensschau* would give us the notion of sphere when we see that a wooden and a bronze sphere have something in common.

Such first steps towards mathematical objects can be criticized from a metaphysical perspective but it is hard to contest that they have a role in our formation of mathematical beliefs and are actually studied in great detail in mathematics education.

An important point here is that this process can be iterated. This line of thinking can also be found, for instance, in more modern math educational work. The founder of the RME-Tradition (Realistic Mathematical Education), Freudenthal, wrote:

In its first principles mathematics means mathematizing reality, and for most of its users this is the final aspect of mathematics, too. For a few ones this activity extends to mathematizing mathematics itself. The result can be a paper, a treatise, a textbook. A systematic textbook is a thing of beauty, a joy for its author, who knows the secret of its architecture and who has the right to be proud of it. (Freudenthal, 1968: 7)

If we have 5 beans and another 3 of them, we have 8 beans. We arrive at the same result when we start with 3 and add 5. We change the order of numbers and observe that 4 and 9 equals 9 and 4, that 10 and 2 equals 2 and 10 and so on. This entails that we get to understand commutativity.¹⁰ We also see that it seems that these rules do not depend on the kind of object that we consider. We can substitute beans for rocks or for pens and it works. But in chemistry we encounter a counterexample: we add 250 mL of water to 250 mL of ethyl alcohol and get 480 mL of liquid. Even worse: Pouring acid into water and pouring water into acid have very different outcomes! Thus, did we disprove arithmetics? Apparently not. We learn that our model of addition does not work in this context. This is an important step to understand our distinction between pure and applied mathematics. While this is only a toy model of addition, we are left with two ways to proceed:

⁸ See Mendell (2019, section 7)

⁹ See Lohmar (2017). For a more detailed account of Husserl’s Philosophy of Mathematics see f.i. Centre (2010).

¹⁰ One example that showcases the difference from basic arithmetic is to arrive at group theory from symmetries, as described by Heuer and Sarikaya (2019).

- (1) Giving up our demand on the mathematical theory of addition to model pouring liquids together and conceiving the remaining as a full theory
- (2) Considering the aforementioned inconsistency as a problem of the theory

So what must go, the use case or the theory? Generally, pure mathematicians take the first perspective and applied mathematicians take the second.¹¹

4.2 On the metaphorical view

Mathematical vocabulary is very flowery at times. As a mathematician, one talks about trees, spiders and forests, amoebas, good sets, or stable marriages. All this is partly done to quickly convey the intuitions behind a mathematical concept. To give an example: When we learn about filters we often also get the metaphor that elements of a filter are *so big that they contain nearly everything*. This helps a lot to remember the definition of a filter, namely:

If P is a set, partially ordered by (\leq) , then a subset F of P is called a *filter* on P iff

- (1) F is nonempty,
- (2) for every x, y in F , there is some element z in F such that $z \leq x$ and $z \leq y$, and
- (3) for every x in F and y in P , $x \leq y$ implies that y is in F , too

But all this follows from our intuitions of nearly everything: The non-emptiness is a basic property of reasonable notions of “nearly everything”, as “nothing” is not “nearly everything”. If two sets both contain nearly everything they must intersect nearly everywhere and (3) simply codes that a superset is bigger, thus cannot become small suddenly.

We could also try to grasp this concept by visualizing containers that are open from above and V shaped. We get (2) and (3) also visually (see Fig. 1), merely by the V-ish visualization of a filter.

This picture immediately clarifies that the intersection of two “large sets” is still large, and that a superset of a large set is large. Thus, we can project our basic understanding of notions, such as bigness or some special intuitions, into pure mathematics. But this is a dialectical process that can also go the other way round. As argued by Tao (2013, 2017), day-to-day practice is not purely rigorous and not purely pictorial either.¹² As a matter of fact, we tend to go by our intuition until we seem to reach problematic or implausible conclusions. Then we re-evaluate our findings rigorously. We train our intuitions while doing mathematics. Paradoxes are key moments here. The Galileo Paradox teaches us to be careful about the part-whole intuition in infinite contexts and teaches us to be cautious to trust some of our intuitions we have on infinite sizes. Our mental representation of the mathematical counts and counts a lot. The main part of good mentorship relations in mathematics is to convey

¹¹ This can also be done for mathematical concepts or any other different level. For instance as argued in (Sfard, 1991) calculus can be seen both from a physical perspective or a pure mathematics perspective.

¹² This has been described in a model of progress in mathematics by Heuer and Sarikaya (2021).



Fig. 1 Visualizations of filters

such intuitions to students that are often eliminated in the final output. Once those intuitions are internalized we can encounter them in mathematical contexts just like we can encounter familiar or prototypical situations in our everyday life. This consideration has led to a frame approach to mathematics; see Fisseni et al. (2019) and Carl et al. (2021).

5 Readings of the later Wittgenstein's philosophy of mathematics: Meaning in pure mathematics

As we have seen so far, a demarcation of pure and applied math based on applications vs a lack thereof does not seem to hold. Because of this, and the fact that it is unclear that there is a section of mathematics guided by universal deductivist principles unconstrained by practical matters, we cannot draw upon Maddy's exegesis of the later Wittgenstein presented in section 2 to elaborate a demarcation between pure and applied mathematics. Pure mathematics is not strictly detached from the world in a sense that allows it to proceed and grow according to a platonically determined set of steps, nor grows in any other way that makes it "unsustained by the world". If anything, it is embedded in localities of its own, which may account for not being suited to deal with other localities (i.e. scientific localities), as we saw in section 3.3. For this reason, the claim that pure mathematics works in a way that sustains nothing and is sustained by nothing in the world, while applied mathematics comprise sets of directly applicable techniques (and therefore, is meaningful in the Wittgensteinian sense) does not hold and cannot be used for the demarcation.

However, a non-revisionary reading of the later Wittgenstein has recently been proposed by Dawson (2014). This reading is more in line with the intent of

demarcation presented in sections 3 and 4, since it does not preclude meaning in the Wittgensteinian sense in pure mathematics, and emphasizes its worldliness.¹³ Here we will explore this reading, put it in the context of recent considerations in philosophy of mathematics, and build on it to draw a sketch of a demarcation between pure and applied mathematics.

While this work does not aim to present a normative exegesis of the later Wittgenstein, here we will present the main points of divergence between Maddy's Wittgenstein and Dawson's Wittgenstein, with the aim of understanding the latter and its value for the demarcation between pure and applied mathematics. Dawson's critique of Maddy's reading stems from three basic points of divergence: 1) Maddy's Wittgenstein is at odds with Dawson's Wittgenstein's apology of non-revisionism: by "pruning mathematics" Dawson's Wittgenstein does not mean to revise mathematics, but to persuade the mathematicians to reflect on what they actually do (rather than on what they say they do, on their fantasies about mathematics, on prose) and change their focus. Concretely, Dawson's Wittgenstein hopes that mathematicians would do mathematics with application at the forefront of their motivation, without pretending to revise mathematics. 2) Maddy's Wittgenstein recognizes the grammar/prose distinction, but not in the case of pure mathematics, where he conflates grammar and prose: "Maddy takes Wittgenstein to criticize the Platonist ontology but leave the Platonist model of pure mathematical statements (as descriptions) untouched in the case of pure mathematics" (Dawson, 2014: 4138). This leads to Maddy's Wittgenstein's consideration that there is nothing meaningful to rescue from pure mathematics from a late Wittgensteinian perspective. 3) Dawson's Wittgenstein suggests that pure mathematics can be more than a mere language game and be meaningful via indirect use.

Maddy's Wittgenstein and Dawson's Wittgenstein mean something different by "*in mufiti*", from Wittgenstein's quote in section 2: Maddy's Wittgenstein means that mathematics needs a direct application, or else, it must be discarded (since it only has illusory applications sustained by prose). However, Dawson notes that this is questionable. For Dawson's Wittgenstein, pure mathematics may be connected to mathematics with direct applications. For example, it may constitute "bridging" techniques, like the commutative law of multiplication. Technique A and technique B may be present without intermediate mathematics (then a case could be made that the connecting mathematics may be superfluous), but technique B may come to life because of intermediate mathematics. In this case, the use of the connecting mathematics is creating a new technique, which in turn has direct applications. In this line, Dawson claims that "the transformation from applied to pure mathematics is being understood as a transformation from *application of instructions to rules formulated so as to allow derivations* (i.e. propositions). The people who lack pure mathematics do mathematics by moving from empirical statement to empirical statement without ever formulating the rules by

¹³ For this reason, one may appeal to the Principle of Charity and state that Dawson's interpretation is potentially appropriate because it contains wisdom relative to the demarcation that concerns us. However, this work is not concerned with evaluating which exegesis of the later Wittgenstein is correct.

which they make the transitions as propositions (formulating them instead only as instructions) [...]. Clearly Wittgenstein regards at least some of pure mathematics as quite easy to relate to applications in empirical propositions” (Dawson, 2014: 4144). Dawson substantiates this connective role of pure mathematics by alluding to several quotes from the RFM, of which, the more explicit is:

...the question “are there a hundred times as many marbles here as there?” is surely not a mathematical question. And the answer to it is not a mathematical proposition. A mathematical question would be: “are 170 marbles a hundred times as many as three marbles?” (And this is a question of pure, not of applied mathematics.) Now ought I to say that whoever teaches us to count etc. gives us new concepts; and also whoever uses such concepts to teach us pure mathematics? (RFM: 412)

But we think that there is an even more important point to make about Maddy’s Wittgenstein, which explicitly shows how this Wittgenstein is at odds with our account of pure mathematics as being as worldly as applied mathematics. As we saw in section 2, Maddy’s Wittgenstein claims that pure mathematics is “not supported by anything and supporting nothing”. Not only have we shown before that pure mathematics is “supported by something” (it comes from the world), but it is not the case either that it supports nothing (it may have indirect applications, and as we will see later, unintended and nontechnical applications).

Overall it seems that Maddy’s Wittgenstein and Dawson’s Wittgenstein have different things in mind when they speak of pure mathematics. If, as Maddy’s Wittgenstein maintains, pure mathematics is something “not supported by anything and supporting nothing”, then probably he is referring to something else than to that which is usually referred to as pure mathematics (possibly, a subset of pure mathematics). It is debatable that Wittgenstein thought this of all pure mathematics, as Dawson notes, or that he based the pure/applied distinction on this paragraph (as said before, maybe just a subset). In any case, we will not take exegetical sides in this work. Furthermore, this is a trivial definitional problem away from the point: much of pure mathematics is intimately related with applied mathematics and has uses. So far, we mentioned its indirect uses beyond mathematics itself, which connect it to actual techniques.

As we see, Dawson’s Wittgenstein presents a sort of “extended use” for pure mathematics, which includes intermediation of mathematical applications. The line of thought of Dawson’s Wittgenstein can be stretched to further assist in demarcating pure and applied mathematics. For example, it allows the incorporation of the fact that the connections between pure mathematics and the world can be evidenced accidentally, after the creation of the mathematical body in question. We saw in section 3.1 that the uses of (pure) mathematics can also be classified as to whether they are intended or unintended. What we take from this is that the fact that pure mathematics can unintentionally serve to solve problems in the sciences at some point just adds to the consideration that it has some connection to the world (as described in section 4), that it leads to concrete techniques, and that consequently it is meaningful in a Wittgensteinian sense. Dawson’s Wittgenstein

provides us with an appropriate framework to integrate phenomena described in sections 3 and 4 in a demarcation between pure and applied mathematics.

6 A three-layer meaning as use perspective of the pure/applied demarcation

After exposing the considerations raised by this non-revisionary reading of the later Wittgenstein's philosophy of mathematics and its relations to worldly pure mathematics and unintended applications, we believe it is possible to go even further in this direction and extend the notion of meaning as use. Meaning as use is a fuzzy notion, since what counts as "use" is debatable, even when it comes to interpreting a specific work (in this case, Wittgenstein's work).

The fuzziness of the notion of use is reflected in the tension between Maddy's and Dawson's readings of the later Wittgenstein's philosophy of mathematics. For Maddy's Wittgenstein, "use" means something along the lines of "direct extra-mathematical application", what we consider here a first layer of meaning as use. Dawson's Wittgenstein extends the meaning of this notion to include "indirect application" as well (for instance, "bridging" mathematical techniques), constituting a second layer of meaning as use (unintended applications in general, in the sense of Molinini (2019) may belong to this layer, as we argued in section 5).¹⁴ Moreover, Dawson's extension of meaning as use also includes aesthetic purposes which, we think, represents the first step of a shift from what modern readers may consider technical uses to what they may regard non-technical uses. In our view, there are reasons to ascribe an even more extended notion of use to the later Wittgenstein, one that builds in this direction and includes more mundane, less abstract/technical uses, which nonetheless are socially relevant: gaining the approval of colleagues and reputation, deciding where budget is allocated, what mathematics are worth developing, what scientific research is worth pursuing, with what to deal in conferences, making practical decisions, general persuasion, and so forth. If we accept this third layer of meaning as use, then not all uses of pure mathematics are indirect or unintended. While this layer applies in principle to both pure and applied mathematics, there is a difference of range. These uses may be more prominent or critical in pure mathematics inasmuch as the more technical uses are more prominent in applied mathematics (but this generalization does not apply to all cases). Perhaps more importantly, these uses apply in different communities and situations, among other reasons, because they are bound by different evaluative standards, as argued in section 3. This aspect of the demarcation presents some degree of fuzziness, just like the difference in evaluative standards for mathematical theories discussed in section 3, and is sustained by family resemblance.

This extension of meaning as use integrates more phenomena from the previous sections into a coherent philosophical framework and assists our demarcation.

¹⁴ The fact that Maddy and Dawson further specify their notion of use does not mean that it is not fuzzy anymore. This leads to a metawittgensteinian problem: the meaning of "use" depends on its use.

Besides its usefulness, there are other reasons that justify this extension, and which further clarify its relevance to the demarcation. First, even if we referred to the former list of examples as “more mundane, less abstract/technical”, we did so just to describe it according to the mainstream intuitions of contemporary readers, but late Wittgensteinian philosophy aims precisely at blurring the distinction between these categories (and for this reason, the three-layer division is only intended to appeal to the intuitions of mainstream readers). Consider the following quote from the LFM:

Why doesn't the contradiction work?—Does it make sense to ask this question? Can one just say, "Well, it doesn't work, and that's all"?

In giving a contradictory order, I may have wanted to produce a certain effect—to make you gape, say, or to paralyze you. One might say, "Well, if this effect is what is wanted, then it does work." (LFM: 175)

Hence, even math that – according to some – cannot “work”, like contradictions, may have “mundane” uses and work just fine.

Wittgenstein continues, becoming more explicit:

The point is that there is no sharp line between a regular use and an irregular or capricious use (LFM: 176).

Somewhat relatedly, other two reasons to support our extended meaning as use are briefly mentioned by Dawson to support his own extension by indicating that Wittgenstein took pure mathematics seriously: 1) One cannot just make the uses (the real meaning according to the later Wittgenstein) of mathematics disappear: “It is a strange mistake of some mathematicians to believe that something inside mathematics might be dropped because of a critique of the foundations. Some mathematicians have the right instinct: once we have calculated something it cannot drop out and disappear! And in fact, what is caused to disappear by such a critique are names and allusions that occur in the calculus, hence what I wish to call prose” (Waismann, 1979: 149). This “core” of mathematics, its uses, its impact in the world, just does not disappear by deleting the symbols today, which means that those symbols are meaningful and not just thin air. Our proposed extension is part of the impact of mathematics in the world, and hence it should be included in Dawson's Wittgenstein's defense of the meaning of pure mathematics. Therefore, mathematics can be meaningful with respect to their use in a way not contemplated either by Maddy's or Dawson's Wittgenstein.

2) Pure mathematics is more than a mere game as long as it is treated as if it is more than such (for instance, if we make practical decisions based on it). Wittgenstein exemplifies this with chess: chess may be just a game, but if we made war decisions based on a chess game, then it would be more than a game (Waismann, 1979: 103–105, 114, 163, 170). Hence, applications like these, based on what we take a language game to be so that it influences our actions beyond immediate actions (like moving a pawn, or writing mathematical symbols) or explicit narrative content (like that of religious texts) should be taken into account regarding meaning. Pure mathematics often is more than a game in

this sense: it is taken seriously. This “seriousness” applies equally to the domain of our proposed extension: if pure mathematics is not taken seriously, it does not, for instance, attract funding.

There is a last, fourth reason to include such an extended domain as part of meaning as use. Imagine inventing a scary story just to dissuade someone from doing something: dissuasion is its use, even if the story is not only invented but also incoherent and includes logical fallacies. This story has no other evident use in a technical sense. Even if it is a technique in a sense, it is not a technique in another, more common sense: It may lack systematicity, it may be a one-time thing, or it could become a canonical story (a myth). Importantly, this is the kind of use that Wittgenstein had in mind for his own philosophy: the *Tractatus Logico-Philosophicus* (Wittgenstein, 2001; TLP) is self-refuting since it does not refer to anything in the world,¹⁵ and both it and his later philosophy make sense, according to Wittgenstein himself, not by virtue of introducing substantial changes in our conception of the world, but inasmuch as it persuades people to adhere to a more useful one (in his case, one that focused on the state of affairs of the world, applications, and little philosophical diversion).¹⁶ Pure mathematics can be used similarly and thus cannot but receive the same treatment. The problem comes when it has to be established what is useful, and what is not, but this is another issue, which falls outside the scope of this work.

Hence, even if pure mathematics had no indirect or unintended uses (in the sense meant by Dawson (2014) and Molinini (2019) respectively), it would still have uses in this “mundane” sense. This adds to the consideration that pure mathematics cannot be just “pruned away” on the ground that it is meaningless because it has no uses. Instead, it must enter as a participant in “what the most useful is” contests, evaluated by different communities, and compete against other mathematics.¹⁷ The defender of (a certain piece of) pure mathematics may want to make the case that taking a certain distance from immediate applications in a given direction may pay off in the long run for whatever purpose (e.g. in the form of future applications, or as a way to train cognitive capacities of mathematicians or students, or as a contribution to mathematics in the way that novels contribute to literature, or aesthetically, etc.), while the rivals may argue that this is only a waste of resources.

¹⁵ At the end of the TLP, Wittgenstein admits the following: “My propositions are elucidatory in this way: he who understands me finally recognizes them as senseless, when he has climbed out through them, on them, over them. (He must so to speak throw away the ladder, after he has climbed up on it.)” (TLP: 6.54, 89). Substantial secondary literature also supports the interpretation that the TLP is self-refuting (see, for instance, Diamond, 1988; Reid, 1998; Mualem, 2017)

¹⁶ “the whole point is that I must not have an opinion... I have no right to want you to say that mathematical propositions are rules of grammar. I only have the right to say to you, “Investigate whether mathematical propositions are not rules of expression, paradigms—propositions dependent on experience but made independent of it. Ask whether mathematical propositions are not made paradigms or objects of comparison in this way.” Paradigms and objects of comparison can only be called useful or useless”. (LFM: 55)

¹⁷ Kitcher (1981) presents a similar argument, concretely on when it is convenient to rigorize mathematics or leave it as it is, attending to practical considerations.

7 Conclusion

We have tried to make sense of and contribute to the demarcation of the pure/applied mathematics dichotomy from a late-Wittgensteinian point of view. First, in section 2, we note that Maddy's Wittgenstein's philosophy of mathematics establishes that pure mathematics is detached from the world and has no applications, while only applied math, which has direct applications, is to be kept. Because only mathematics with applications is meaningful, pure mathematics is meaningless. Hence, according to Maddy's Wittgenstein, the demarcation between pure and applied mathematics consists in that only the latter has applications. Although this reading leads to a very clear demarcation, consistent with the popular belief that pure mathematics is about theory and applied mathematics is about applications, we argue that it does not do justice to how actual mathematics works.

Second, in sections 3 and 4, we question this demarcation by showing that pure and applied mathematics both have theory and applications, and that both are sustained by the world. Pure and applied mathematicians – two sociologically demarcated groups – differ in their criteria to select the theories they study and how to evaluate them, but other than that there seems to be no major difference with respect to the character of the mathematics with which they are involved. We argue that, although perhaps more directly in the case of applied mathematics, both pure and applied mathematics are grounded in the real world to a certain extent. This is for two reasons: 1) pure concepts also develop from abstraction from real world knowledge and 2) we can project intuitions from other domains of discourse into the formal apparatus of pure mathematics, allowing us to manipulate symbols efficiently. This provides us with reasons to discard the clear demarcation from section 2, and take a first step towards a fuzzy demarcation: pure and applied mathematics feature different evaluative standards.

In section 5 we return to the later Wittgenstein to further elaborate on this fuzzy distinction. We note how a non-revisionary interpretation, based on an extended meaning as use notion, is in line with the worldly character of pure mathematics endorsed in sections 3 and 4, and notes that pure mathematics actually has applications, albeit indirect and sometimes unintended. Finally, in section 6, we extend this reading by arguing that pure mathematics is meaningful from a Wittgensteinian point of view also due to having non-technical uses, and such uses do not align perfectly with those of applied mathematics.

A related but yet open task would be to see whether our ideas make up for a challenging alternative to the classical exegeses of Wittgenstein: to what extent did Wittgenstein really conceive pure and applied mathematics this way? Future work will be concerned with this aim.

All in all, this extended notion of meaning as use contributes to the fuzzy demarcation based on family resemblance – substituting the clear-cut but misguided demarcation in section 2 – that we have been progressively sketching from section 3 until here. Pure and applied mathematics, while both are

anchored to the world: 1) have different evaluative standards, 2) the technical applications of pure mathematics are usually indirect or even unintended, and 3) they serve different mundane and rhetorical purposes. The difference between pure and applied mathematics may not admit a clear-cut demarcation, but at least, it is possible to identify relevant aspects for a demarcation based on family resemblance.

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Authors and Affiliations

José Antonio Pérez-Escobar¹ · Deniz Sarikaya² 

José Antonio Pérez-Escobar
jose.perez@gess.ethz.ch

¹ Chair for History and Philosophy of Mathematics, ETH Zurich, Zürich, Switzerland

² Philosophisches Seminar, Universität Hamburg, Hamburg, Germany