Purism: The inconceivability of inconsistency within space as the basis of logic

Primus

ABSTRACT: *I propose that an irreducible property of physical space — consistency — is the origin of logic. I propose that an inconsistent space is inconceivable and that this inconceivability can be recognized as the force behind logical propositions. The implications of this argument are briefly explored and then applied to address two paradoxes: Zeno of Elea's paradox regarding the race between Achilles and the Tortoise, and Lewis Carroll's paradox regarding the Tortoise's conversation with Achilles after the race. I conclude that Achilles would have won on both accounts, in the race and the argument, by invoking the consistency of space as the foundation of both movement across space and logical argument.*

Introduction

Theorists continue to debate the epistemology of logic. Logic has been attributed to various forms of intuition (Kripke 1981; Frege 1997; Kant 1997); to habit (Wittgenstein 1922); to convention (Carnap 1934; Quine 1976); to the product of linguistic efficiency (Ayer 1946); and to a form of empiricism (Mill 1959; Putnam 1972; Quine 1951). Others (Aristotle Metaphysics; Frege 1997; Dummett 1991a, 1991b; Engel 2007)[1] have questioned whether epistemic justification of basic logical principles is necessary or even possible. This article does not attempt to better the existing comprehensive reviews of this subject, instead directing the reader to the reviews by Romina Padro (2015), Penelope Maddy (2012), and Michael Dummett (1991a). It will however attempt to add to this discussion by proposing that the ultimate source of logic has a physical basis in the material of physical space. I propose that each point of space possesses one property, consistency, and that the fabric of space is absolutely consistent by its nature. I observe that, given appropriate consideration, it is inconceivable to imagine otherwise.

Thus, I argue that consideration of the consistency of physical space — or more specifically, the inconceivability of its inconsistency — serves as a force which may compel people to accept the truth of logic under specific conditions. More specifically, this argument involves three propositions:

1. We cannot conceive of inconsistency (difference) within an individual point of space — e.g. that a point of space between two different entities, A and B, could embody both of their different natures — though we can conceive that difference occurs across multiple points of space — e.g. that entity A could be different from entity B. An implication of this proposition, which I need not explore in this article, is that space can conceivably be manipulated into different 'forms' (e.g. entities A and B) while itself remaining consistent (uniform) in its fabric;

2. Any proposition may, explicitly or implicitly, represent physical space: either representation of the fabric of space itself — e.g. point A of space is equal in nature to point B of space — or representation of forms of space — e.g. form A, an apple, is not equal in nature to form B, a banana;

3. When a proposition does or could refer to space itself — that is, a quantity of the fabric of space — and when the symbols purported by the proposition do not depict an inconsistency in space, and are hypothetically assumed to be empirically true to physical reality, we cannot conceive of discrepancy within such proposition, or between it and physical reality.

   More specifically, if

3a. the symbols within a proposition explicitly represent, or could (due to their ambiguous nature) implicitly represent, quantities of individual points of the fabric of space of the same size (e.g. the symbol '1' in the abstract equation 1 = 1 could implicitly represent one point of space because the nature of what '1' represents is unspecified);

and, if

3b. the use of symbolism within the proposition appears to be congruent on the basis that they would not appear to necessitate an inconsistency within space should the proposition be true (e.g. the symbols of the equation 1 = 1 appear congruent with each other because they do not appear to describe an inconsistency in space; one point of space can be, and is, conceivably equal to itself. However, the symbols of the equation 1 = 2 appear incongruent because they would require an inconsistency within space should it be true; one point of space would need to, in some way and for some time, equal two points of space);

and, if

3c. it is hypothetically accepted that each of the symbols within a proposition are as they appear, and do accurately represent physical reality (e.g. if it is granted, for argument's sake, that the symbols '1' in the above proposition each directly represent, neither zero, nor two, nor one and one tenth, but exactly one point of real space, each of equal size);

then

3d. we cannot conceive of incongruency within the symbols of such a proposition, nor within the proposition itself. The inconceivability of incongruency within the proposition and its symbols means that we are forced to grant truth to the proposition when we consider it across space (i.e. its truth is the same to all observers who consider the proposition) and time (i.e. its truth will be the same when observers reconsider the proposition on a later occasion);

and

3e. it is the act of consideration, leading to the inconceivability of any alternative conclusion, which generates the force which may be intuitively recognized as logic. This force can be used to grant the notion of 'truth' upon propositions which satisfy conditions 3a–c above.

Whether readers agree that the force I am describe is the phenomenon of logic is inconsequential to the aim of this article. My aim is to make readers recognize the force that the inconceivability of an inconsistent space brings, irrespective of what it is termed. The enclosed argument is one of philosophy, not mathematics or physics. Vitally, in this article I avoid establishing principles of logic and then using them to argue for the origin of logic, as many have wisely warned against (Quine 1976; Frege 1997; Dummett 1991a, 1991b; Engel 2007)[2]. Rather, I am arguing that the notion of logic stems from a physical property whose absence is inconceivable to observers, rather than a set of (conceived) principles. The force of my argument originates from the inconceivability of inconsistency within any point of space. Noting that my assertion about the nature of the fabric of physical space is a thought experiment regarding an entity which is not directly observable, I apply this framework to two theoretical challenges: Zeno of Elea's paradox, involving the race between Achilles and the Tortoise (Salmon 1999; Makin 1998; Grunbaum 1968), and Lewis Carroll's problem relating to the Tortoise's challenge to Achilles after the race (1895)[3]. The present notion of consistent space solves these paradoxes and could conceivably serve as the epistemological basis for the truth of logic.

This article consists of two parts. The first elaborates on each of the three propositions above. The second attempts to illustrate how a consistent space would solve two famous paradoxes involving Achilles and the Tortoise. A consistent space mandates that Achilles and the Tortoise will continue to move *across* points of space during their physical race; they cannot shrink in size and race *within* points of space, as Zeno's Paradox primes the reader to imagine. Furthermore, and serving as the main point of this article, the inconceivability of the inconsistency of space is used as a force in Achilles and the Tortoise's subsequent theoretical argument.

Part 1

I will begin by detailing some basic definitions relating to the aforementioned propositions. To reduce the potential for conceptual disagreement, I use the broadest conceivable definitions.

Truth/True

The use of the term 'true' refers to a condition wholly congruent with physical reality; that is, the reality that is independent of the frames of reference of observers. This reality need not be observed or verified because its nature is hypothetically agreed upon in the body of each inference.

Inconsistency

I define inconsistency using the broadest conceivable definition: *difference*. If an entity changes (varies or is limited) in any way, across space or time, then such difference represents an inconsistency. Conversely, the absence of any difference or change, across space or time, is consistency. Anything but absolute uniformity across space and time fails to meet the definition criteria of consistency, and is thus inconsistency for the purposes of this argument. This ultra-broad definition encapsulates existing, narrower definitions of inconsistency (see Schick 1966; Napoli 1975; Coons 1987; Gooch 2007; Carnielli & Coniglio 2016). For example, Coons explicitly considers the above definition before narrowing his scope to account for inconsistency within daily life:

> The task of defining inconsistency at first appears too easy. Taken literally, [Aristotle's notion that like cases should be treated alike] forbids mere unalikeness of treatment, thereby suggesting that difference itself is inconsistency. By common understanding this is obviously too broad (Coons p. 66, 1987).

In a world where we observe many differences across space (e.g. blue skies, red apples), and where everything appears to change across time (e.g. blue skies become grey skies), it is understandable that the authors referenced above would attempt to define narrower notions of inconsistency that are applicable to daily life: Coons (1987) uses the example of a judge passing different sentences for the same offence committed under similar conditions. However, because the present argument is limited to the notion that the fabric of space itself (and only space itself) is absolutely consistent, and that any difference within it is inconceivable, narrower notions of inconsistency need not be entertained here[4].

Point

I adopt de Laguna's (1922, Def. XI, p. 454) definition of a point in relation to a set of physical solids: "[a]n abstractive element in which no other abstractive element lies". He clarifies that a point can be best conceived as an "irreducible individual" (de Laguna 1922, p. 454). De Laguna's definition is substantively similar to other 'atomic' definitions[5] in which a point is considered to be the most elementary entity within a given condition (Clark 1985).

Notably, the phrase "which no other element lies within, and which cannot be reduced" only applies (across space) for the duration of conception, not across time. That is, one can later conceive smaller points within former conceived points, noting that the former, by definition, are no longer points at the moment that smaller entities are conceived within them.

Space

By 'space' I refer to physical space in the broadest possible terms. This includes vacuous space of any ontological construction — from 'spacetime' to 'absolute space' (Huggett 1999) — to the space of 'solids' (de Laguna 1922). Reference to space as a 'fabric' throughout this article (e.g. 'the fabric of space') indicates that I am referring to space itself (as a uniform material which I argue comprises all forms) rather than specific spatial forms possessing temporal, conditional properties, such as movement.

Points of space

The construct of space as a physicality logically requires that a point has a finite size and occupies a finite space at any moment. The quantity of space across a point of space — the size of a point of space — is perhaps largely inconsequential except for determining the quantity of points between any two points (Weyl 1949). As per Hermann Weyl's (1949) size thesis[6], there are only a finite quantity of points between any two points when the aforementioned points are the same size (see Figure 1).

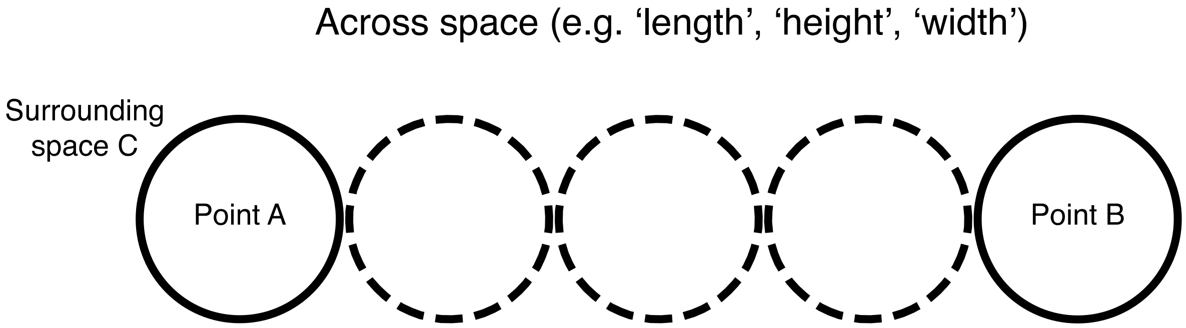

Figure 1. Points across space, spanning one of the three traditional Euclidean dimensions of space: length, width, height.

So whereas there are infinite points across the surrounding space C in Figure 1, there are finite points across the space between points A and B assuming that the size of the points is kept constant (Weyl 1949).

Points within space

Although each point of space occupies a finite quantity of space, *across* space, at any moment, there are potentially infinite other points of space that can be conceived *within* each point of space as the size of the points is reduced (Weyl 1949)[7]. Whilst Weyl (1949) would simply consider these to be points of (increasingly) smaller size co-existing within larger points, I claim that the space within space should be recognized as a dimension (e.g. 'zoom') in its own right, to be considered alongside the traditional Euclidean dimensions which extend across space (length, width, height). This concept is depicted in Figure 2.

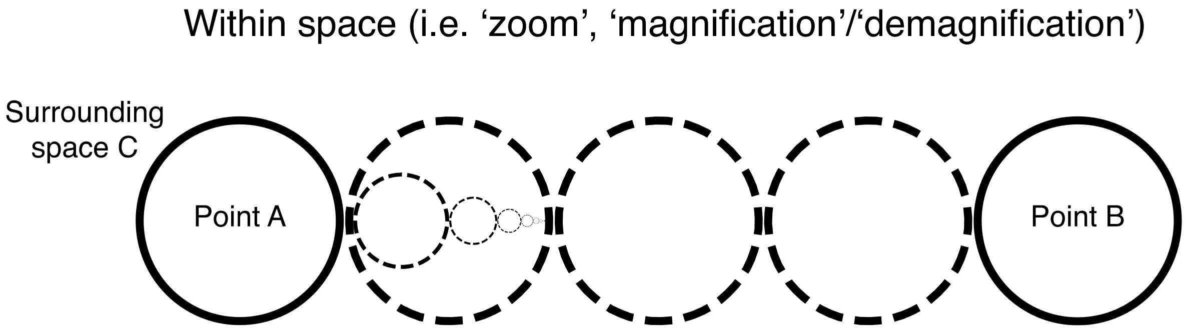

Figure 2. Points within space. Whereas there are finite points of a given size across the space between points A and B, there are conceivably infinite smaller points within each point of space (Weyl, 1949).

The relevance of considering dimensions 'within' and 'across' space will become apparent in the following argument regarding the inconceivability of inconsistent space. I will claim that we can imagine inconsistency (difference) *across* multiple points of space, but not *within* individual points of space.

Proposition 1. We cannot conceive of inconsistency (difference) within an individual point of space

I propose that the fabric of space can only be conceived to possess an unvarying and absolute consistency. The notion of a consistent space can also be expressed as the notion that difference *at* or *within* any individual point of space is impossible, or at least inconceivable: each individual point of the infinite points both across and within space is necessarily consistent and not permitting of difference (i.e. variation) within itself. I will explain this concept further and make a case that although variation cannot be imagined to occur within individual points of space, the differences we perceive in daily life (e.g. blue skies, red apples) could conceivably occur across multiple points of space. Put more simply, I am arguing that we cannot conceive of difference within the fabric of space, though we can conceive of difference within and between the various forms that each comprise multiple points of space; that we can conceive (and perceive) a difference between any point of an object, A, and points of surrounding space, B (e.g. person A is different and distinct from their environment B), though we cannot imagine any points of space (or, more specifically, any individual point(s) *within* space) that singularly embody the difference between the two entities (A and B).

The inconceivability of inconsistency within individual points of space

Before I get to the impossibility of conceiving difference within space, I should emphasize that the conception of difference across space necessarily involves the conception of multiple (i.e. two or more) points of space. This is because a difference, by definition, necessitates at least two points of conception (e.g. point A is different from point B). By contrast, attempts to conceive difference at points within space necessarily requires the imagination of difference at individual points of space. In this instance, one is attempting to imagine an *individual* point of (within) space — the point of difference between two or more entities. To visually highlight this, I refer the reader to Figure 3.

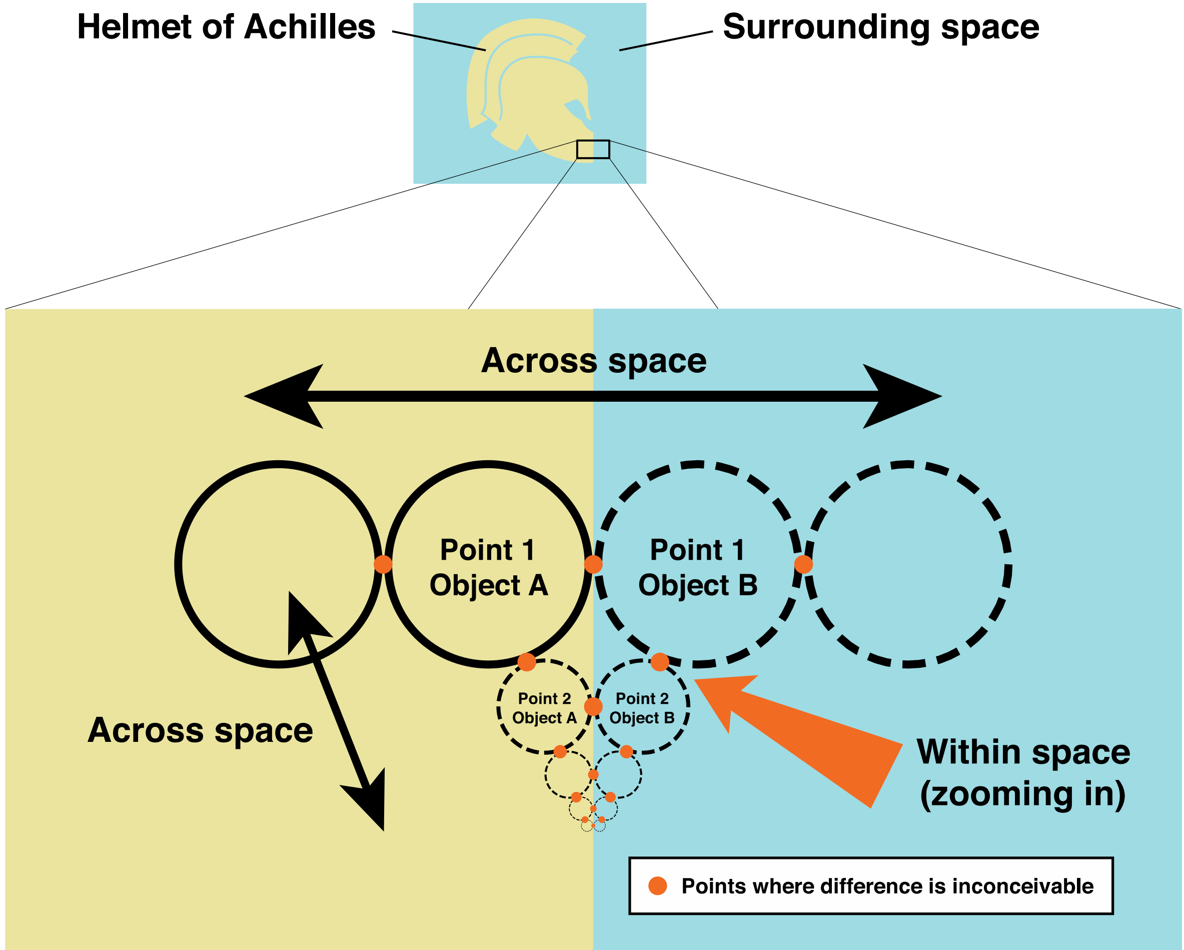

Figure 3. Individual or multiple points may be conceived across space, as depicted by black circles. The conception of difference across space necessarily involves the conception of multiple (>1) points, such as the pair of Point 1 of Object A and Point 1 of Object B. Individual or multiple points may also be conceived within space, as depicted by orange dots. Attempts to conceive difference within space necessarily occur at individual points of space.

Let us assume that Object A in Figure 3 is Achilles' helmet and that Object B is the atmosphere surrounding Achilles. Notably, the difference which is conceived (or perceived) across space in Figure 3 is necessarily occurring across multiple points. The reader should readily agree that they can conceive and perceive difference between (multiple) points across space, such as that a red apple differs from its blue sky; or that Point 1 of Object A — perhaps a sub-atomic particle of Achilles (let us call it '1A') — is different from, and separate from, Point 1 of Object B — perhaps an imaginary point of vacuum in the atmosphere surrounding Achilles (let us call it '1B'). Yet, I will argue, we cannot conceive a difference at individual points within space — colored orange in Figure 3. When attempting to imagine the point in which 1A (a particle of Achilles) contacts 1B (the surrounding atmosphere), or where 1A contacts 2A (e.g. a smaller particle), we are now conceptually 'zooming in' and attempting to imagine difference (>1 entity) *within individual* (1) points of space. We are no longer attempting to imagine difference *across multiple* (>1) points of space, as we are when we imagine that Achilles is different from his surroundings (e.g. that point A is different from point B).

Distinguishing between instances when one is conceptually 'zooming in' to imagine a singular, smaller point within space, as opposed to imagining difference between two or more points across space, is a subtle change of thought. And this distinction brings us to the inconceivability of imagining difference within individual points of space. Superficially, one might consider that object A in the example above and the space that surrounds it, B, are different from each other and are therefore two separate entities. However, if one more deeply considers A and B, it is impossible, with conceptual magnification — that is, conceptually 'zooming in' toward the (orange) points where A and B touch in Figure 3 — to imagine a single point where both remain as separate entities. This is because we must always imagine at least two points across the space embodying any difference — at least one point for each aspect (nature) of the difference — noting again that a difference is, by definition, a variation between at least two natures. The result is that, within any one point which is superficially thought to embody difference (i.e. two or more points) we are forced to choose to imagine either the singular property of the (one) point or the pluralistic nature of the difference (at least two points). If we conceptually zoom in and continue to imagine object A as separate from space B, and thus still consider them as two separate materials, this indicates that we are actually imagining at least two points of space, not one. We are attempting to conceive the singular point where A meets the surrounding 'nothing' of B — it must be a single point of space. Readers will find it impossible to imagine two entities (object A and the nothing of space, B) at a single point; they are attempting to imagine that two entities can literally equal one entity at the same moment: '2 = 1'. We can, however, imagine a property of uniformity (oneness, singularity) at every point of (within) space, and that this uniformity could (and subsequently, must) exist within the space which 'connects' any two points of difference across space.

The implication of the above — that we can only conceive of difference as occurring across multiple points, and never within an individual point of space — is that difference occurs continuously throughout space, and not as an inherent (i.e. absolute, fundamental, unconditional) property of any single point (i.e. the fabric of space). In other words, if space is consistent at each point, it is a necessary requirement that the differences we perceive and/or conceive on a daily basis (e.g. blue skies, red apples) exist as conditional, reducible states of space, rather than discrete, absolute properties. An appropriate analogy might be to consider forms (e.g. blue skies, red apples) as origami pieces, whose paper — space — does not tear or become creased; its forms can be created and reduced whereas the paper remains consistent. This, of course, is an argument for another article. The main point that I am making here is that it is reconcilable that we can perceive and conceive difference (inconsistency) across multiple points of space in our daily lives (e.g. blue skies, red apples), while still maintaining that difference is inconceivable at individual points of space itself.

The inconceivability of an inconsistent space has not been explicitly discussed in the literature. The inconceivability of this physicality supports the long-held notion that the absence of contradiction is a necessary requirement for truth. Aristotle (*Metaphysics*) was perhaps the first to note the fundamental importance of noncontradictory reasoning to all scientific assertions: "[A]ll men who are demonstrating anything refer back to [the Principle of Noncontradiction (PNC), that the same thing cannot simultaneously be and not be,] as an ultimate belief; for it is by nature the starting-point of all the other axioms as well" (Aristotle, *Metaphysics*, IV, 1003a). Aristotle argues that no proof is necessary or even possible for his principle:

> Some, indeed, demand to have the law [PNC] proved, but this is because they lack education; for it shows lack of education not to know of what we should require proof, and of what we should not. For it is quite impossible that everything should have a proof; the process would go on to infinity, so that even so there would be no proof (Aristotle, Metaphysics, IV, 1003a).

However, those who continue to search for a basis to the truth of logic would disagree with Aristotle. Without evidence of some type, a claim of principle holds no greater bearing than any other belief; it leaves the door open for others to proclaim, for example, the Principle of Religion: "God exists and we require no proof of this". I propose that the principle of noncontradiction can be validated based on the conceptual observation that difference is inconceivable within any point of space. In its most fundamental form, in accordance with our attempt described above to conceive difference at individual points of space, two points of space cannot at the same moment be equal to both two points of space and one point of space, if each are of equal size. This is an important realization. The lack of conceivability which I detail above is a notable 'force' or evidence. I will expand on this argument later: that when we continually consider a proposition and are essentially forced to continually arrive at a conclusion (due to the absence of conceivable alternatives), we might call such a conclusion 'truth' and such a force 'logic'.

Proposition 2. Any proposition may, explicitly or implicitly, be a representation of space: either a representation of the fabric of space or a representation of forms across space

Alfred Ayer (1946, p. 17) asks "what do the statements of logic describe?" His answer is: "they do not ... describe any facts at all". I respectfully disagree: they each may represent the consistent nature of space. There will, of course, be variation regarding whether a concept explicitly or implicitly represents space, and whether it represents a quantity of individual points within space (and thus, directly represents the fabric of space), or an entity whose properties exist as differences across multiple points of space (a 'form'[8]). For example, if one were to state 'Jim is physically bigger than Christine', this statement explicitly and literally refers to the quantity and size of the multiple points across space that Jim and Christine each consist of. If one were to more broadly state that Jim is 'the bigger person' of the two, then this statement might explicitly refer to the figurative natures of Jim and Christine's forms: Jim is perhaps more mature. Alternatively, when Alfred Whitehead and Bertrand Russell (1910) famously attempted to establish a mathematical proof for the equation '1 + 1 = 2,' their algebraic symbols (figures) could *implicitly* represent[9] — and, arguably, are fundamentally reliant[10] on — the consistency (uniformity) of individual points of the fabric of space: the notion that the quantity and nature of one individual point of space must[11] be exactly physically (literally) equivalent to the quantity and nature of any other point of space of the same size, across time and space.

Notably, points of space, each being conceivably consistent in nature, are physical entities that can conceivably be exactly equal to other physical entities. They are thus perhaps the only physical entities for which propositions asserting equivalency may literally be true. This literal, implicit representation does not exclude other, figurative uses for the same equation: my two one-dollar coins ('1 + 1') are (figuratively) equal ('=') to the quality of your two-dollar ('2') note, although they occupy different quantities of space. However, because the value of the currency is figurative — it is not directly derived from the consistency of space — their consistency is not necessarily universal across time and space. My two United States one-dollar coins ('1 + 1') are *not* equal ('≠') in quality to your two-dollar Australian coin ('2').

Furthermore, and more importantly for the purposes of my argument, because the nature of forms is conceivably inconsistent (i.e. variable; e.g. blue skies become grey skies), there is no certainty of equivalence between the size and quantity of the space they each occupy. Our empirical experience, of course, supports this. A gas may occupy less space when pressurized, even though it is still composed of the same quantity of solid points (e.g. atoms).

Proposition 3. When a proposition does or could refer to space itself — that is, a quantity of the fabric of space — and when the symbols purported by the proposition do not depict an inconsistency in space, and are hypothetically assumed to be empirically true to physical reality, we cannot conceive of discrepancy within such proposition, or between it and physical reality

Upon their consideration, the previous propositions combine to produce a notable force. More specifically:

a.1) when symbols within a proposition are explicitly chosen to literally represent a quantity of points of equal size along the fabric of space (e.g. let 'A' represent the quantity of points across space of which Jim consists, and let 'B' represent the quantity of points across space of which Christine consists);

or:

a.2) when the subject that is represented by a set of symbols within a proposition is unspecified (e.g. as per the abstract equation: '6 = 3 + 3'), and thus each symbol can implicitly, literally represent individual points of space (i.e. '6' may symbolize six points of space and '3' may symbolize three points of space);

and, if:

b) the symbolism within the proposition does not appear to describe an incongruency across space, defined as a state which would necessitate inconsistency within space should it be true

> (e.g. the symbolisms of the expressions 'A > B' and '6 = 3 + 3' appear congruent because they do not appear to require an inconsistency in space: it is conceivable that an entity ('A') may be

bigger ('>') than another entity ('B'), and it is conceivable that two, smaller quantities ('3') combined ('+') with each other could be equivalent ('=') to another, larger quantity which is twice the size of the smaller quantity ('6'), as such permits that one point of space must be equal to itself and each other point of the same size: '1 = 1'. However, the expressions 'A > B > A' and '6 = 3' each would require an inconsistency within space should they satisfy condition (a) above and be true; if so, one point of space must, at some point in time, equal two or more points of space);

and, if:

c) the conditions required of (a) and (b) are hypothetically assumed to be met — negating the possibility that one's empirical observations — the aforementioned 'appearances' of the symbols — are flawed (e.g. if we accept our observation that Jim is physically bigger than Christine is true to reality; that there really are two groups of three points of space: '3 + 3');

then:

d) we are forced to infer truth upon the proposition (e.g. that it is true that A is bigger than B, and that B is certainly smaller than A; that it is true that 6 equals 3 + 3);

and;

e) it is the *act* of consideration itself which provides the *force* that 'moves' the considerer to accept these truth(s), and it is the inconceivability of discrepancy within the proposition, and between it and physical reality, which prevents the considerer from 'moving' to other conclusions (such as that the proposition could be false).

It is possible to conceive how a proposition that does not satisfy the above conditions might be falsely accepted as being true or false. I can, for example, superficially conceive that the proposition 'A > B > A' might be true and that '6 = 3 + 3' might be false. This is perhaps due to my empirical experiences: Jim ('A') may be physically bigger ('>') than Christine ('B') whilst Christine may concurrently possess a 'bigger' ('>') personality than Jim ('A'); one coin can be worth the value of two coins and so it may be that six coins equals three coins ('6 = 3'). Perhaps, more simply, I might falsely accept a proposition because I can conceive each of its symbols (e.g. 'A,' 'B,' '=,' '>,' '6,' '3') individually, and so it is conceivable that they can each be listed one after each other without consequence (in the absence of further consideration of the value of these symbols and their relationship to each other). More broadly, without knowing the nature of any proposition, I can conceive that each could potentially be true, false, or partially true. It is conceivable that one could accept an invalid argument or dismiss a valid argument simply because truthfulness and falsity are conceivable properties of propositions in general[12]. Furthermore, even if A were to be hypothetically accepted as being bigger than B purely in physical terms, one might ask which physical law states that B cannot be simultaneously physically bigger than A? Why cannot six coins be physically equal to three coins of the exact same nature?

I am not claiming to provide a physical law in this article — lest I be accused of falling into the 'trap' of circular reasoning: using laws to validate laws (Quine 1976; Frege 1997; Dummett 1991a, 1991b; Engel 2007). Rather, the limit of this claim lies in the ability for conceivability, or lack thereof. I am claiming that A may be hypothetically accepted as being bigger than B on the basis of the quantities of space by which they each literally consist, and that we cannot conceive of inconsistency within such space, and thus, nor that A could conceivably be smaller than B. If A consists of 100 points of space and B consists of 95 points of space, each point of equal size, we cannot conceive of how B could possibly be bigger than A when we specifically consider the points of space themselves. We cannot conceive of how any of A's 100 points could be different (vary) within space from each other. The first (1) of A's 100 points of space, for example, could not somehow 'stretch' to occupy the space of two (2) points, nor could we conceive it that could somehow 'shrink' to become less than one (1) point (of uniform nature); either would necessitate the conception of a difference at a point of space. The same applies for each of the other 99 points of A, and each point of B. As per proposition 1, we cannot conceive of anything other than uniformity at any point of space. Of course, if the units within this example proposition were not points of space, but rather properties generated by differences existing across multiple points of space (e.g. mass: if A and B possessed masses of 100 kg and 95 kg, respectively), then we could conceive how B might be physically bigger than A. A's mass, for example, could be distributed with greater density than B, meaning that A could be smaller than B despite possessing greater mass.

Because we cannot conceive of discrepancies *within* individual points of space, we are forced to conceive that any discrepancies which are apparent or possible must occur *across* space: within or between the forms of the proposition. In other words, either the various symbols used within the proposition individually or collectively describe an inconsistency within space, or there is a discrepancy between the proposition's predicate (i.e. the empirical claims of its symbols) and the physical reality it claims to represent. As alluded to above, this may occur in a literal sense; for example, if Jim were not truly physically bigger than Christine, it may have merely appeared as such from the proposer's perspective, although ten other observers measure Jim to be physically smaller. If so, the inference that Jim is physically bigger than Christine could be considered untrue. Such an incongruency may also occur in a figurative sense within the proposition; for example, if Jim were, in the same moment, considered both 'bigger' (in terms of maturity — 'the bigger person') than Christine by one observer and 'smaller' by another. This latter discrepancy is possible if one's definition of 'bigger' is not literally tied to the consistent properties of individual points of space, but rather based on differences observed as occurring across multiple points of space (e.g. a quality of Jim's form — his personality — is different from Christine's form).

These examples reinforce the importance of the aforementioned three conditions, (a–c), which allow one to overcome the conceivable possibility of discrepancies *across* space. To reiterate, these conditions include: an appropriate selection of symbols which (explicitly or implicitly) literally represent quantities of individual points of space of equal size, whose combination within the proposition does not appear to describe or require an inconsistency within space should its predicate be considered true; and the hypothetical acceptance, for argument's sake, that these symbols accurately reflect reality (which negates the ever-present empirical limitations of observers).

The inconceivability of discrepancy within or across space forces us to accept the truth of inferences, such as *modus ponens*, upon each consideration. To be clear, this force, requiring one to accept the truth of propositions that satisfy the conditions (a–c) above, arises as a result of consideration which leads to an absence of conceivable alternatives. Engel (2007, p. 5) describes how a considerer may be "fully aware that [a proposition] is a logical truth or a valid rule of inference. But she fails to draw the conclusion because she does not take the rule or law as binding and as being able to move her to the appropriate conclusion". However, the considerer has already 'moved' by virtue of considering a proposition. The act of considering is a force, as is any other act. The relevant factor in terms of whether the proposition is accepted as being necessarily true depends whether the reader, after 'moving' to understand the predicate of an inference, is able to move onwards to any other (alternative) conclusion than that espoused by the predicate. If an inference appears incomplete (e.g. "A is bigger than") the reader will be able to 'move on' to other conclusions and not be bound by the statement's conclusion. After a few re-reads they may move on to the conclusion that it is not a complete inference and does not possess a predicate (except perhaps that A exists) nor a truth value. Alternatively, if the proposition does appear complete (e.g. "A is bigger than B") they may then continue to 'move' as they explore each of its symbols in combination with each other, looking for a discrepancy with empirical evidence (if applicable), or considering whether its symbols appear to express a predicate that requires an inconsistency in space. If there is any conceivable alternative to any aspect of the proposition (e.g. A may have only appeared bigger, rather than actually having been bigger, from the perspective of the inferrer), then conceivably a considerer can 'break free' from the predicate and 'move on' to other considerations. This may not necessarily mean that they conclude it to be 'false'; it may simply mean that on the basis that they are able to continue their consideration of alternatives, they are not bound to consider it true.

In this sense, in analogy with a static wall, a proposition does not actively apply force to 'move' actors in its environment, though its solid nature does provide a counter force to actors 'moving' across its path. Similarly, a proposition cannot affect someone who is not considering its predicate (moving along its path). The force of a robust proposition is therefore only applied during the process of its consideration. If the proposition is robustly tied to the nature of physical space, the lack of conceivable alternatives (places to escape from the predicate of the proposition) serves as an opposing force or 'roadblock'. This force blocks their ability to consider alternative conclusions to the proposed argument and 'traps' them at their only conceivable conclusion; however many times they consider the same proposition, they cannot escape from its truth. This force will be demonstrated below regarding Achilles' second victory.

Part 2: Achilles' two victories over the Tortoise

Achilles' first victory

'Achilles and the Tortoise' is perhaps one of the best-known paradoxes attributed to Zeno of Elea (c. 490–430 BC).[13] It involves the namesakes having a race, with the Tortoise given a head start over Achilles, and each moving concurrently after the race starts. The paradox suggests that in theory, Achilles, although much faster than the Tortoise, will never catch the Tortoise because he must always first reach the point that the Tortoise most recently occupied; no matter how slow the Tortoise's relative speed, it will have moved to another point by the time Achilles reaches its previous point, and so on ad infinitum. This outcome is of course counterintuitive to the notion that a faster runner will eventually catch and pass a slower runner if the race continues (Salmon 1999).

The problem in this paradox is generated in the mind of the observer because Zeno's paradox reframes the race into discrete intervals. The problem is therefore solved by realizing that Zeno's paradox does not work if each racer moves across a constant distance in each move sequence, and without shrinking in relative size, as racers necessarily do if their respective speeds and the time duration (interval) of each move sequence remains constant. In accordance with the movements of all objects that we encounter in daily life, Achilles and the Tortoise ran their race *across* the length of space. Achilles won due to his greater speed relative to that of the Tortoise, and because there are finite points for them to cross, as is the case for any entity travelling *across* space between two stationary points (see Figure 4).

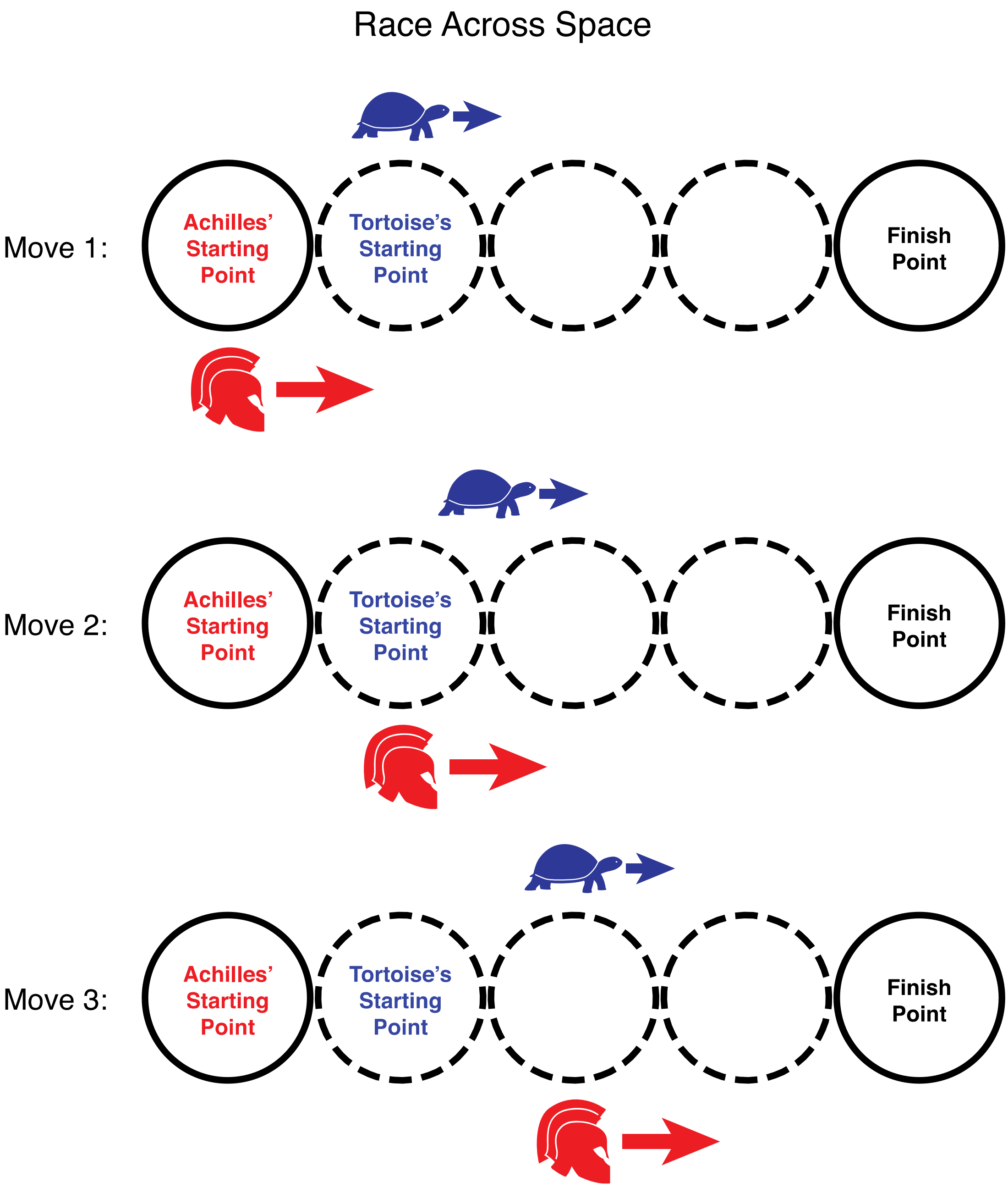

Figure 4. When Achilles races the Tortoise across space, because he possesses a faster relative speed, and because we assume constancy of speed across each move, and constancy of duration (i.e. equal intervals) of time across each move, Achilles reaches the Tortoise's starting point at the end of Move 1, reaches the Tortoise at the end of Move 2, and passes the Tortoise at the start of Move 3.

Assuming that Achilles and the Tortoise maintain constant speeds over time (as the paradox asserts), there is an implicit requirement for a shrinking of characteristics: the distance travelled per turn must shrink (and to maintain constant speed, so too must each racer). In the case of this paradox, where the speed is assumed to be constant, each would have to travel *within* space — in which case their sizes and distances traveled would need to decrease accordingly[14] — to ensure that neither ever reach their destination, nor that the former ever catches the latter (see Figure 5).

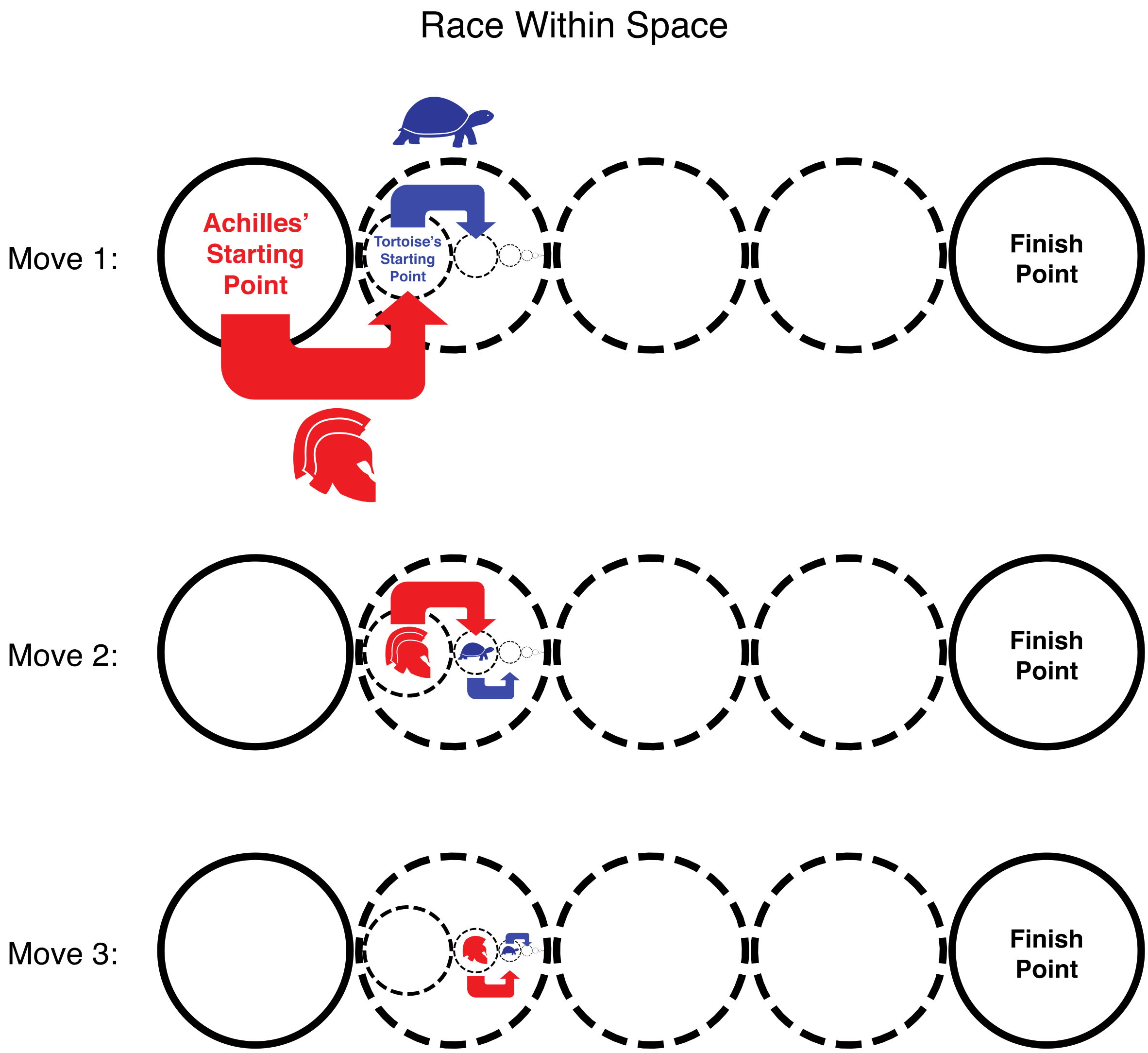

Figure 5. When racing the Tortoise within space, as Zeno's Paradox unwittingly demands of its racers, Achilles will never catch the Tortoise. Nor will either racer ever finish the race, whose finish point is reached by travelling across, not within, space.

Readers will note that the above race (within space) would be inconceivable according to the notion of the consistency of space described earlier. For example, assume that to complete his first movement, Achilles would need to consist of half that quantity of space at the end of his first movement compared to the quantity of space that he consisted of in his starting position. Such a movement would require that every two points of his multiple points of space become a single (1) point of space (Figure 6).

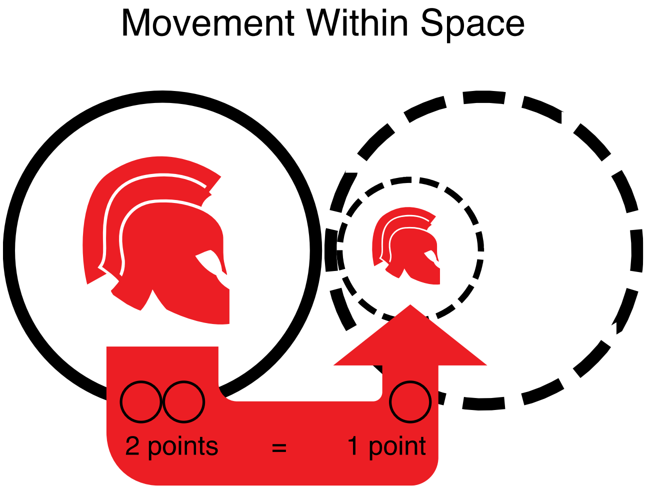

Figure 6. Achilles' movement within space may require him to consist of one point of space for every two points of the space that he originally consisted of.

Such a reduction in spatial quantity is unforeseeable in the context of Zeno's paradox. We cannot conceive how two points (2) of the space by which Achilles exists could become one (1) point of space *during* the race ('2 = 1').

To be clear, therefore, it is *not* the presence of Achilles and the Tortoise *statically* existing within any of the points of space depicted in Figure 5 that would create inconsistency. In any of the static positions depicted, Achilles and the Tortoise would be simply existing (within the original, larger points of space of their racecourse) as differences *across* multiple, albeit smaller, points of space[15]. As discussed in part 1, difference across (between) multiple points of space is conceivable — and, I argue, is common to all objects. Rather, it is Achilles and the Tortoise's *movement* within space — specifically, their movement into these positions from larger points across space — that is irreconcilable. The movement of Achilles and the Tortoise into positions within (and without) space, given the conditions of Zeno's paradox, is inconceivable because such movement would generate difference at individual points of space. Zeno is implicitly asking the reader to conceive that Achilles and the Tortoise could shrink in *size* whilst remaining composed of the same *quantity* of space throughout the race[16].

In summary, Achilles and the Tortoise would race *across* space in normal conditions, not *within* space as Zeno's paradox implicitly suggests. The latter would require each racer to continually shrink in terms of the quantity of the points of space by which they consist — thus also shrinking in size — to take infinitesimally smaller steps with each interval of time. If this were conceivable, Zeno's paradox would hold: a rearward racer traveling within space would never reach another point of the original racetrack, whether it be the 'finish line' or the starting point of the Tortoise (each of which are located across space from Achilles' starting point).

Achilles' second victory

After the race we enter the mind of the Tortoise: bitter, due to his recent physical loss, he seeks to gain an intellectual victory by taking Achilles along an infinite regression. He will ask Achilles to explain why the inferences of logic must be accepted as true. In doing so, the Tortoise believes he will demonstrate to Achilles that one cannot be forced to accept conclusions using logic because, he believes, logic does not bring its own self-contained truth. A statement which relies on logic for its proof — let us call it statement 1 — must also have at least one other subsequent statement — let us call this statement 2 — describing the rules and parameters whereby the former statement's rules and parameters are true[17]; and then this subsequent statement, statement 2, must have a statement — statement 3 — explaining its truth, and so on *ad infinitum*.[18]

To bring about this regression, the Tortoise plans to make three propositions to Achilles[19]. These would collectively form an assertion (inference) of truth, statement 1, about a triangle:

a) at least two sides of a symmetrical triangle must be equal in length;

b) this is a symmetrical triangle;
c) one side of this triangle must equal at least one other side in length.

Unmoved to accept inference (c), the Tortoise believes that he could lead Achilles to add a second statement aimed at certifying the truth behind the inference:

Statement 2: If propositions (a) and (b) are both true, then (c) must be true.

Once this is accepted by Achilles, the Tortoise would remain unmoved to accept inference (c), and would thus demand a third statement from Achilles prior to accepting the truth of Statement 2, such as:

Statement 3: Statement 2 is true.

This expectation could then be repeated for a fourth statement, which the Tortoise would require of Achilles to validate Statement 3, such as:

Statement 4: Statement 3 is true.

And so on.[20]

In this respect, the Tortoise (incorrectly) believes that logic is purely a conceptual product — with no roots in physical reality — and thus requires infinite additive statements to certify the truth of previous propositions. The reality, as I argue in this paper, is that logic, although it is perceived and/or conceived of within the mind, may have physical origins. Just as the reader can perhaps superficially conceive '1 = 2', the Tortoise too could superficially discard the logic which mandates the necessity that at least two sides of Achilles' triangle are equal. With the above plan in mind, the Tortoise might exclaim: "The challenge I am making to you, Achilles, is that you cannot force me to accept a logical inference".

Achilles, not breaking a sweat from the race, and apparently unfazed by a talking tortoise, impatiently responds:
"Speak, Tortoise, and faster than you run; I have Trojans, not time, to kill".[21]
After hearing out the Tortoise, Achilles responds to the effect that, apart from perhaps surgically altering another's brain to implant ideas – a learning style that will not occur for another few thousand years – we cannot *force* anyone to reach any conclusion that must be considered.

"After all", Achilles continues, "the student must have the will or interest to know any truth, up until the point that a truth is known, and by 'known' I mean 'entertained as the only conceivable outcome, despite continued consideration of potential outcomes'. The only power a student has over his teacher is his ability to retire his tablets[22] (disengagement from learning), and this is a tactical victory. So, the question is: are you willing to consider my premise to its logical conclusion?"

"Yes", the Tortoise answers, "but this does not negate the infinite regression that will follow, you see…"

"I'm not finished, Tortoise", Achilles interrupts. "If you are willing to continue your consideration, and you reach a point where you cannot conceive any alternative but the conclusion you reach — then you are forced to accept a premise, and this is both the truth and force of logic. And when I say 'point', I mean a point of space. And as is the case for any point of space, difference *at* this point is inconceivable; that is, it cannot be imagined, despite continual consideration.

The regression that you imagine is only possible because you search for justification for my propositions *across* space — by adding further propositions" (see figure 7).

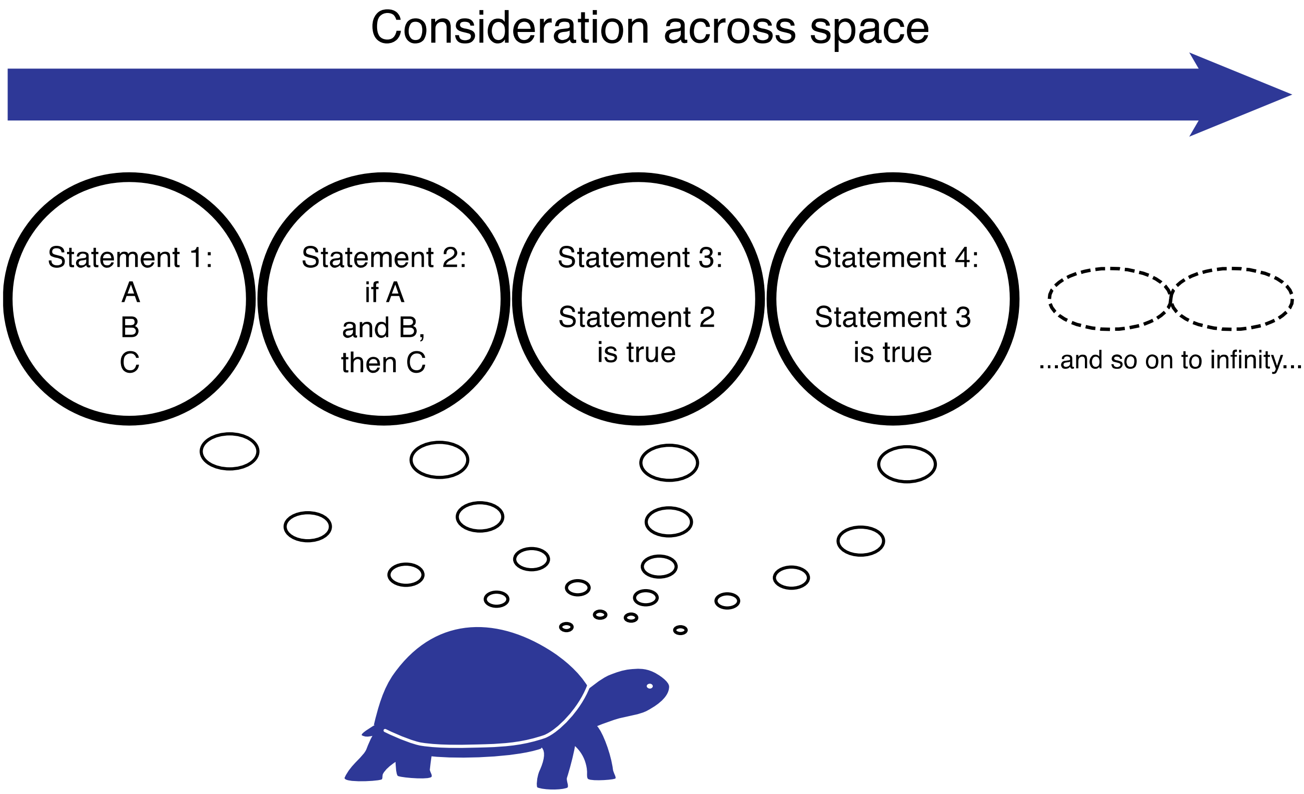

Figure 7. Consideration across space will not produce a conceptual force which forces the Tortoise to accept the truth of logical argument (e.g. statement 2 in the example above). Any forces that are created to grant truth to such argument (e.g. logical laws, such as statement 3 and 4) will require external and subsequent forces for their own validation, and so on...

Rather, the origin of the force of logical reasoning is to be found *within* space, as per figure 8:

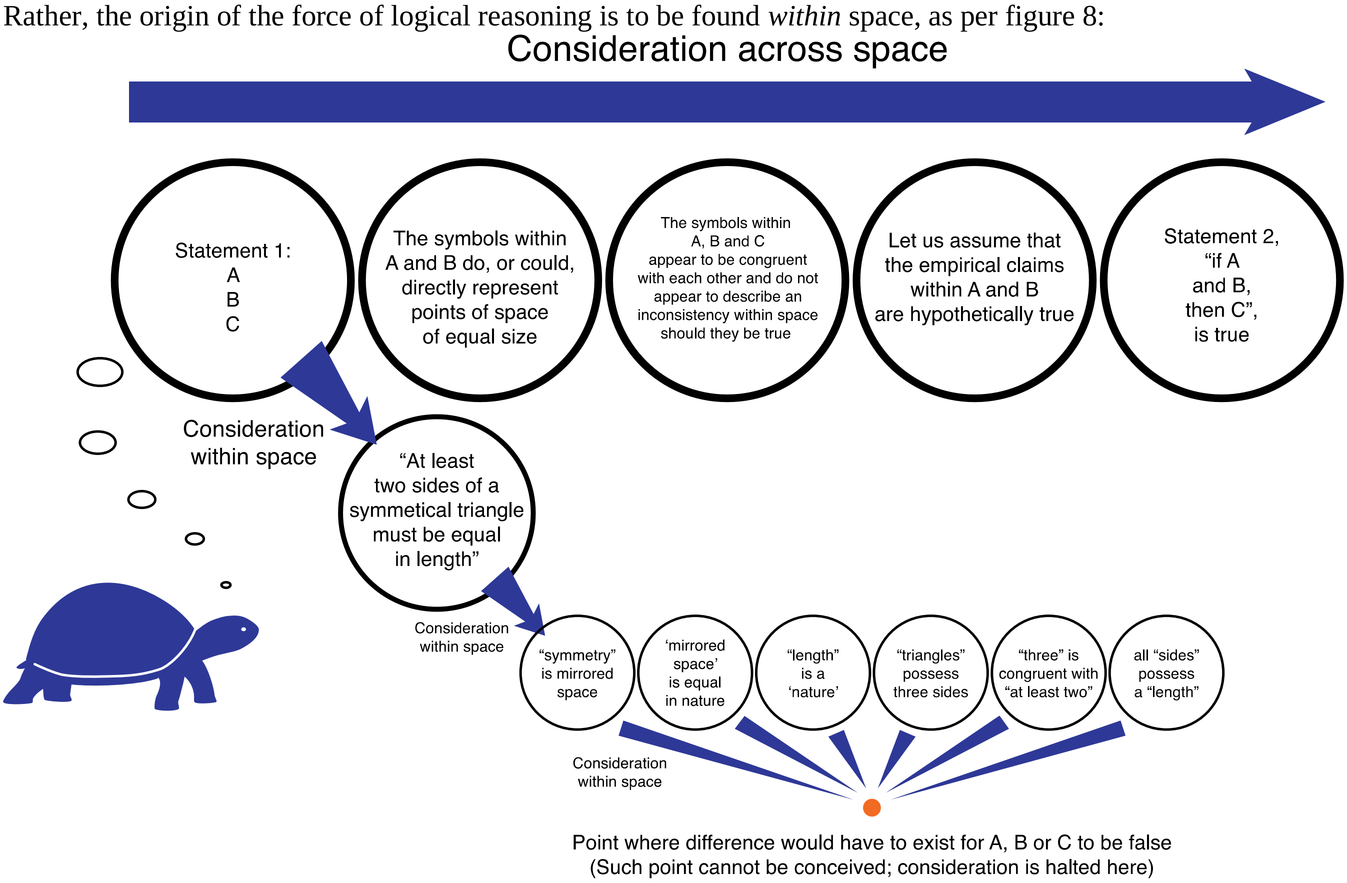

Figure 8. Providing that specific conditions of a statement are considered to be met (contained within Achilles' initial thought bubbles), the Tortoise can consider points within each of the symbols of Achilles' propositions (A–C in the figure above). An example of this might be the Tortoises' search for incongruence or ambiguity within the symbols (terms) that constitute Achilles' definition of "symmetrical", noting that 'symmetrical' is a key symbol within his proposition A: "At least two sides of a symmetrical triangle must be equal in length". If the Tortoise considers the symbols within each of these propositions deeply enough he may reach a (individual) point where difference (two or more points) would need to exist for such propositions to be false, however he will not be able to conceive such point.

"You see," Achilles continues, "consideration in the conceptual realm is like running in the physical realm: the runner, moving with force, will look for a path that is not blocked by an opposing force, such as a wall, or a very slowly moving Tortoise. The considerer, looking to break free of an opponent's proposition, will do the same".

The Tortoise replies: "What could this possibly have to do with our argument? I think that you're going off point…"

Achilles continues: "The person considering the truth of a proposition is analogous to the runner — they are actively moving along a conceptual path, looking for gaps to break free of the 'walls' of the argument which contain its truth. When there are gaps in the proposition — symbols which are not clearly defined, or which could or do contradict each other — the considerer is free to wander to other possibilities: that the argument may be incorrect, it may require an inconsistency in space to be true, it may describe an incongruency in terms of the nature of its forms, or it may simply be nonsensical. However, when there is nothing but walls and a solid floor, the considerer is contained within; they are then forced to accept the truth of the argument because they can conceive no alternative; no matter how many times they consider or run around in circles, their answer is certain and unchanging.

"So, do not forget that for as long as you consider my argument, you are conceptually running, looking for gaps so you can keep 'running'; you are moving with a force; and when you are surrounded and contained by walls that run across your path, and by a solid floor that prevents you from digging under the walls, you are forced, or moved, to accept the truth of logic."

Now enter the mind of Achilles. He plans to lead the Tortoise to realize the force generated when conditions are such that incongruency is inconceivable both within and across space.

Achilles plans to accept and retain the Tortoise's original statement, while also adding proposition (d):

a) at least two sides of a symmetrical triangle must be equal in length;
b) this is a symmetrical triangle;
c) one side of this triangle must equal at least one other side in length;
d) if propositions (a) and (b) are both true, then (c) must be true[23].

If we stay true to Carroll's (1895) version, the Tortoise accepts proposition (d) — "if (a) and (b) are true, then (c) must be true" — yet does not yet accept the concluding inference of proposition (c): "one side of this triangle must equal at least one other side in length". As mentioned, the Tortoise does not recognize the authority of logic and requires further certification: in this case, endless propositions (e), (f), and so on.

However, unlike in Carroll's version, Achilles' next proposition, not (e), but rather (a.1), would aim not to certify statement 1 (as the Tortoise imagined he would in the example above) but rather to clarify how (the congruency of) statement 1 (propositions (a–d) above) directly represents (the necessary consistency of) space[24]:

> a.1) "A body, a spatial configuration, is symmetric with respect to a given plane, *E*, if it is carried into itself by reflection in *E"* (Weyl, 1952, p. 4). (This proposition is expressed visually in Figure 9.)

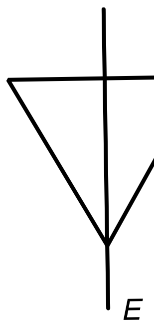

Figure 9. A body, symmetric to plane E.

Once this is accepted by the Tortoise, Achilles might next propose (a.2), aimed at clarifying the truth of statement (a.1):

> a.2) "Reflection in *E* is that mapping of space upon itself, *S*: *p* → *p′*, that carries the arbitrary point, *p*, into this its mirror image, *p′*, with respect to *E"* (Weyl, 1952, pp. 4–5).

To illustrate this, Achilles would ask the Tortoise to mark the following in the sand (thereby creating Figure 10)[25]:

> "Take any line, *L*, perpendicular to *E* and any point, *p*, on *L*: there exists one and only one point, *p′*, on *L* which has the same distance from *E* but lies on the other side. The point *p′* coincides with *p* only if *p* is on *E"* (Weyl, 1952, p. 4).

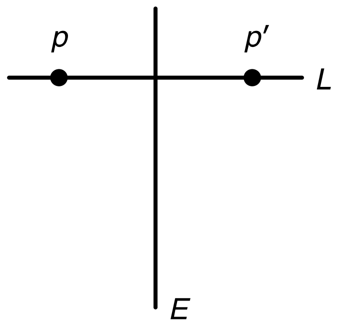

Figure 10. Point p, reflected in E, as p' (reproduced from Weyl, 1952, p. 4).

Achilles' next proposition (a.3) might be the following, as depicted in Figure 11:

> a.3.) "A mapping is defined whenever a rule is established by which every point *p* is associated with an image *p′*" (Weyl, 1952, p. 5).

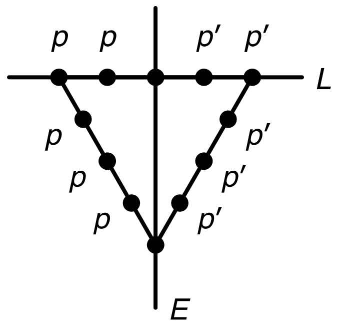

Figure 11. Twelve points of a triangle mapped by means of reflection through plane E.

Achilles, counting 36 stones on the ground that the Tortoise could use to completely map his triangle (see Figure 12), would make a further proposition:

> a.4) Let us call a linear collection of mapped points in contact with each other a 'side'. Points, *p*, of a(ny) side, *S*, are equal to points, *p′*, of its reflected side, *S′*, whereby equality occurs in terms of the arrangement (i.e. linearity and direct adjacency), quantity (13), and nature (size and consistency), of their respective points across space[26]. Let us define length as the measurement of the distance across space that a quantity of points extends to when they are arranged in a linear manner in contact with each other.

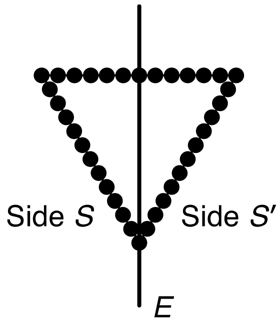

Figure 12. Completed mapping of a symmetrical triangle, with two defined sides, S and S′, whose respective points, p and p′, are each of equal arrangement (i.e. linear and directly adjacent), quantity (13), and nature (size and consistency), across space.

Following the above proposition, Achilles would ask the Tortoise to accept a reformulation of proposition (a), substituting the terms 'symmetrical' and 'length' with the aforementioned definitions:

> (a) at least two sides of a symmetrical triangle must be equal in length;

is revised as

> (a) at least two sides of a triangle whose points, *p*, of side, *S*, are equal to points, *p′*, of its reflected side, *S′* — whereby equality occurs in terms of the arrangement (i.e. the linearity and adjacency), quantity (13), and nature (size and consistency) of their respective points across space — must be equal in property (length) if measured according to the distance that each extends across space when their respective quantities (13) of points are arranged in a linear, adjacent manner.

Remembering that it is proposition (c) which the Tortoise has yet to accept, Achilles would condense the above revised (expanded) version of proposition (a), whilst integrating proposition (b) — that this is a symmetrical triangle — to more parsimoniously formulate propositions (a–c) as follows.

> (a–c) One **(1)** side **(*S*)** of this triangle, whose **(1)** side ***S***, **13** points long, is equal **(=)** in length to its reflected **(1)** side, ***S′***, **13** points long, must equal **(=)** at least one **(1)** other side **(*S′*)** in length.

I have added algebraic symbols and bolding to enhance the visibility of the internal congruency within Achilles' inference that two sides of the triangle are equal in length. This can be further represented as follows:

$$1S = ([1S = 13p] = [1S' = 13p]) = 1S'$$
$$= 1S = 13p = 1S'$$

We can see that there is equality between at least two sides of the triangle. But as discussed earlier, the congruency of symbols across Achilles' statement is not sufficient to provide an argument for the truth of logic. Vitally, the congruency within Achilles' statement is also congruent with the inevitably conceivable consistency within space; we know this because the quantities being measured in the above example are points of (the consistent fabric of) space themselves. We can directly link the above algebraic symbols to points of space because the Tortoise accepts Achilles' hypothetical conditions of (d), and thus, for the purposes of the argument, the triangle can be assumed to exist, occupying the points of space as depicted in Figure 12. Achilles is arguing that the 13 points of space comprising side *S* are equivalent (consistent) in every nature to each of the 13 points of side *S′*, as each point within side *S* and *S′* are consistent with each other. For any of these points to differ from each other would require a difference at a point within space, which is not conceivable, as discussed in part 1.

Of course, incongruency would be conceivable *across* space in this example had the Tortoise not accepted the hypothetical, (d). In the absence of the Tortoise's acceptance of (d), it could have been that the above triangle, despite being described as symmetrical, might have sides *S* and *S′* that actually measure 13 and 12 points long, respectively. If this were so, their arrangement across space might be expressed algebraically as follows.

$$1S = ([1S = 13p] = [1S' = 12p]) = 1S'$$
$$= 1S = 13p \neq 12p = 1S'$$

This incongruency, occurring across, but not within points of space, could be attributed to definitional incorrectness or an empirical failure by observers. That is, the triangle would generally be definitionally described as asymmetrical, not symmetrical, and so the observer who defined it as such would be mistaken, either because of definitional incongruencies with other observers (e.g. their respective definitions of symmetrical and asymmetrical are inverted) or because of empirical error (e.g. the observer measured the sides inaccurately). However, neither of these explanations are conceivable outcomes in the present case because the definition of symmetrical — requiring equality of *S* with *S′* — and the assertion that *S* and *S′* each measure 13 points long, have been included within proposition (a–c); the Tortoise accepts both these facts with his acceptance of the hypothetical (d). Because of this, an incongruency within Achilles' proposition — such as where sides *S* and *S′* are of equal length but respectively measure 13 and 12 points of space in length — would require an inconsistency *within* space. In accordance with part 1, we cannot imagine such occurring; we can only imagine a consistent space, and this is reinforced by the congruency of our daily empirical interactions (e.g. the continuation of unobserved entities once they are observed again, and cause and effect). So although we can superficially believe '13 = 12', it will come to the point — for example, where the 13th point must squeeze into the adjacent 12th point (2 = 1) so that side *S* is not longer than side *S′*, or where the 12th point must stretch (1 = 2) so that its side *S′* can equal the length of 13 points — that we must attempt to conceive how a difference (e.g. >1 point) could exist in the space of a single (1) point. As I have argued previously, we cannot imagine this without acknowledging that we are imagining two (2) points rather than one (1). This inconceivability, I offer, is the force behind all logic.

Let me be clear that it would not merely be the internal congruency of the proposition (across space) nor the acceptance of the hypothetical (d) that would seal the Tortoise's fate[27]. These factors must combine with the Tortoise's willingness to consider the proposition as deeply (within and across space) and for as long as necessary to concede that there is no conceivable 'escape' from its argument, and the fact that the units of measurement within this hypothetical — points *p′* and *p* — directly represent individual points of space, rather than forms of space (entities composed of difference expressed across multiple points of space). Had the sides *S* and *S′* been measured in terms of atoms (*a*, *a′*) or molecules (*m*, *m′*), for example, there would exist the possibility of variation across space. The natures of atoms and molecules may conceivably vary in their nature, in terms of size and internal (in)congruency. We can imagine, for example, that a molecule consisting of multiple points of space may be able to be condensed (with force, namely pressure). In such a case, it is conceivable that sides *S* and *S′* could respectively consist of 13 and 12 molecules and yet be equal in length (the difference could be absorbed by the differences in properties between various types of molecules and their atoms). So despite the Tortoise's acceptance of the hypothetical, which assures equality of either side of the triangle, a conceivable incongruency between the nature of either side of the triangle may serve as a sufficient gap for the Tortoise to conceptually break free and reject the truth of Achilles' logic (irrespective of whether such rejection is warranted). This might especially be the case had Achilles not specified equality *in length* as he did in the present version (Carroll's original version simply specifies "equal[ness]"[28]).

After consideration of the above, Achilles would ask the Tortoise:

"Having considered the space within and across my statement, where — by which conceivable incongruency — do you break free from its hold? If you choose to abandon the argument at this point you are still within its hold and will remain there until you reconsider it and break free".

We don't know how the Tortoise answered because in the end, Achilles does not take the Tortoise up on his offer to force him to believe proposition (c). Achilles knew that the Tortoise could refuse to consider his argument, but could not deny the force of its logic if he chose to engage and consider the argument.

"Let me be remembered for my glory in battle — as the greatest warrior that has ever existed — not through my conversation with a tortoise", he thought to himself. And this — pursuing his goals, rather than being diverted by the prospect lesser victories — was Achilles' second victory.

Conclusion

The difference we experience daily (e.g. blue skies, red apples), may conceivably occur *across* (multiple points of), but never *within* (a single point of), space. This, I have argued, is necessitated by the consistency of the fabric of space. Although this consistency cannot be directly observed — investigation of the fabric itself falls below the threshold of empirical observability — I have applied the notion of a consistent space to potentially reconcile two theoretical paradoxes involving Achilles and the Tortoise. When we consider the question of what logical expressions may represent in physical reality, we can, implicitly or explicitly, recognize that they represent the consistency of space; that one point of space is certainly consistent with each other, across space and time. From this basis, we can make other valid extrapolations. The consistency of space, I have argued, is the authority upon which a mathematician can be certain that one equals one ('1 = 1'), and is the basis for further extrapolation (e.g. the claim that '1 + 1 = 2'). It is similarly the authority by which a logistician can claim that Jim is physically bigger than Christine if Christine is physically smaller than Jim.

Wittgenstein argues that, at some point, our reasons for acting in accordance with rules must come to an end (Wittgenstein 1922; Kripke 1981). I argue that space is (literally) the point where deductive reasoning ends, or perhaps more accurately, begins. The consistency of space is conceivably both the origin and the force of all logic, and this, I argue, is the ultimate reason why we are forced to accept congruent notions when, and if, we consider them.

Notes

[1] These are but examples; Dummett (1991a) asserts that most philosophers have taken this view.
[2] Quine (1976) warns us against this approach in his response to Carnap's (1934) conventionalism (Engel 2007; Tennant 1986). This avoids what Dummet (1991a) calls a "proof-theoretic justification of the first grade" when we derive a given logical law from other laws (Engel 2007). Such justifications, Dummett argues, can only be "relative", due to the derivation relying on the validity of other laws (preventing its truth from being "universal"). Dummett argues that to have universal, non-relative justification, logical laws must be "self-justifying". He argues that laws are logical when they satisfy the requirement of "harmony" (Engel 2007), a notion not dissimilar from the notion of 'consistency' and 'congruency' discussed in the present work.
[3] Pascal Engel (2007) notes that Carroll's problem is generally described as a "paradox of inference" but questions whether it is a true paradox or merely an unsolved problem. Like Engel I adopt the latter view.
[4] I will, however, use the term 'incongruency' to describe general discrepancies, such as Coons' (1987) judge example.
[5] It aligns, for example, with Euclid's definition: "A point is that which has no part" (Euclid, *The elements*, Book 1, Def. 1.1.).

[6] The size thesis asserts that the distance between two points is given by the number of 'tiles' (points of the same size) between the two points.
[7] Notably, points that are conceived to possess points within themselves (for example see the second point from the left in Figure 2) are no longer points according to the definition that I have adopted.
[8] I use the term 'form,' though of course they might not be a form in the true sense of the word (see Primus, 2019, for a discussion on forms and materials).
[9] By 'represent' I am referring to the notion that '1' represents one point of space.
[10] By 'reliant on' I am referring to the requirement that each of the '1' symbols are one and the same at the beginning of the proof and that they retain this property of sameness or equivalence when the proof is completed; the proof would not be possible if their values were constantly changing.
[11] By 'must' I mean that the reader cannot conceive any alternative.
[12] Pascal Engel (2007) offers 'a lack of understanding,' of either the presented propositions or of 'rule of inference,' as a reason for the Tortoise rejecting Achilles' logical argument in Carroll's paradox (1895).
[13] Zeno's Paradoxes are thought to have been created to support Parmenides' doctrine that the belief in motion and change is unjustified, contrary to illusions of the senses; for a more detailed explanation of his paradoxes and for overviews of attempted solutions thus far see Salmon 1999; Makin 1998; Grunbaum 1968; Lynds 2003; Engel 2007; and Wieland 2013. Though I hold that the notion of the consistency of space can be applied to each of Zeno's paradoxes, this article will focus narrowly on the famous race between Achilles and the Tortoise.
[14] This is a requirement for a scenario which imagines infinite points between Achilles and the Tortoise.
[15] If the appropriate technology were used, the forms of Tortoise and Achilles could each theoretically be reconstructed, recreated, or replicated at points within (inside) or without (outside) the points of space of their original positions.
[16] Zeno's paradox does not, for example, mention that either racer is able to use advanced technology to continually recreate their forms using increasingly fewer points of space, thereby reducing the *quantity* of space from which they are composed as they proceed in the race.
[17] This statement, the Tortoise allows, could be contained within statement 1, if it is a justification for the truth of logical law.
[18] The only alternative to this, the Tortoise views, would be the contained, circular reasoning that has been widely cautioned against (Quine 1976; Frege 1997; Dummett 1991a, 1991b; Engel 2007).
[19] I prefer these propositions to the Tortoise's original (Carroll, 1895) propositions because the term 'equal to the same' is not clear to me in terms of a direct representation of space. The modification of proposition (a) from the original also makes it more obvious, I believe, that the answer in proving the truth of the Tortoise's logic lies in the dissection of (a), to clarify its relationship with the consistency of physical space, rather than through additional statements.
[20] I have rearranged the wording, but not the essence, of Carroll's (1895) original regression to highlight the absurdity of entering the regression, and how little proof each new statement contributes.
[21] I am attempting to restore Achilles' character to that depicted in Homer's *Iliad*.
[22] Whether stone or electronic; of course, there will be no need for tablets or teachers in the future.
[23] This is identical to the Tortoise's second statement above; I combine it into Achilles' first statement here for aesthetic purposes.
[24] More specifically, this clarification provides a definition of the term 'symmetrical' used in propositions (a) and (b), and how use of the term 'symmetrical' is consistent with the term 'equal' in propositions (a), and (c). Over the next few propositions I directly draw upon Weyl's (1952) established definition of symmetry.
[25] To quote the Tortoise: "[w]hatever Logic is good enough to tell me is worth writing down" (Carroll 1895).
[26] As per part 1, there are finite points across space between any two points, although infinite points within any point of space.
[27] This step only overcomes the inherent problems with drawing inferences from empirical observations.

[28] Carroll's original version: (A) Things that are equal to the same are equal to each other; (B) The two sides of this triangle are things that are equal to the same; therefore: (Z) The two sides of this triangle are equal to each other.

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