The Reciprocal of The Butterfly Theorem

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In this paper, we present two proofs of the reciprocal butterfly theorem.

The statement of the butterfly theorem is:

Let us consider a chord $PQ$ of midpoint $M$ in the circle $\Omega(O)$. Through $M$, two other chords $AB$ and $CD$ are drawn, such that $A$ and $C$ are on the same side of $PQ$. We denote by $X$ and $U$ the intersection of $AD$ respectively $CB$ with $PQ$. Consequently, $XM = YM$.

For the proof of this theorem, see [1].

The reciprocal of the butterfly theorem has the following statement:

In the circle $\Omega(O)$, let us consider the chords $PQ$, $AB$ and $CD$ which are concurrent in the point $M \neq O$, such as the points $A$ and $C$ are on the same side of the line $PQ$. Let $X$ and $Y$ respectively be the intersections of the chord $PQ$ with $AD$ and $BC$ respectively. If $XM = YM$, then $M$ is the middle of the chord $PQ$.

Proof 1.
We construct the circumscribed circle of the isosceles triangle $BOD$ and denote by $E$ and $F$ the points where $AB$ and $CD$ cut again the circle (see Fig. 1).

The quadrilateral $DBEF$ being inscribed, we have that $\angle CDB \equiv \angle BEF$. But $\angle CDB \equiv \angle BAC$, therefore we obtain that $\angle BAC \equiv \angle BEF$, with the consequence $AC \parallel EF$ (1).

We denote by $N$ the second point of intersection of the circumscribed circles of the triangles $AXM$ and $CYM$.

The quadrilaterals $AXMN$ and $CYMN$ being inscribed, we have that $\angle XAM \equiv \angle XNM$ and $\angle YCM \equiv \angle YNM$. Because $\angle XAM \equiv \angle YCM$ ($ADBC$ being an inscribed quadrilateral), previous relations lead to $\angle XNM \equiv \angle YNM$. This relation, along with the condition from the hypothesis $XM=YM$, shows that, in the triangle $NXY$, $NM$ is both median and bisector, therefore this triangle is isosceles, and $NM \perp XY$ (2).

The relation (2) implies $m(\angle NCB)=90^0$ and $m(\angle NAX)=90^0$. But $m(\angle NCB)=m(\angle NCM)+m(\angle DCB)=90^0$.

On the other hand, $m(\angle DCB)+m(\angle OBD)=90^0$, because $m(\angle DCB)=\frac{1}{2}m(\angle DOB)$.

We also have that $m(\angle ODB)=m(\angle OFD)$, because the quadrilateral $FDOB$ is inscribed.

These relations lead to $\angle NCM \equiv \angle OFD$, which further implies $NC \parallel OF$ (3).
Analogously it is shown that \( NA \parallel OE \) (4).

Relations (1), (3) and (4) show that the triangles \( NAC \) and \( OEF \) have respectively parallel sides, therefore they are homothetic, the center of homothety being the point \( \{M\} = CF \cap AE \).

Then the homothetic points \( N \) and \( O \) are collinear with \( M \), having \( NM \perp PQ \), it follows as well that \( OM \perp PQ \), consequently \( M \) is the middle of the chord \( PQ \).

The relation (2) implies \( m(\overrightarrow{NCA}) = 90^\circ \) and \( m(\overrightarrow{NAX}) = 90^\circ \).

But \( m(\overrightarrow{NCA}) = m(\overrightarrow{NCM}) + m(\overrightarrow{DCB}) = 90^\circ \).

On the other hand, \( m(\overrightarrow{DCB}) + m(\overrightarrow{OBD}) = 90^\circ \), because \( m(\overrightarrow{DCB}) = \frac{1}{2} m(\overrightarrow{DOB}) \).

**Proof 2.**

Assuming the opposite, \( PM \neq QM \), therefore \( OM \) is not perpendicular on \( PQ \).

We construct the perpendicular in \( M \) on \( OM \) and denote by \( U \) and \( V \) its intersections with the circle \( \Omega(O) \).

We denote by \( R \) and \( S \) the intersections of the chord \( UV \) with \( AD \) and \( CB \) respectively (see Fig. 2).

![Figure 2](image)
Because $M$ is the middle of the chord $UV$, applying the butterfly theorem, we have that $MR=MS$.

We obtain that $\triangle MXR \equiv \triangle MYS$ (side-angle-side), and consequently $\angle XRM \equiv \angle YSM$, therefore $AD \parallel BC$.

The condition $AD \parallel BC$ leads to two possibilities for the quadrilateral $ADBC$. This can be an isosceles trapezoid if $AD \neq BC$, or rectangle if $AD = BC$.

We eliminate the possibility $ADBC$ - rectangle, because this rectangle would have the center $M$ and it should be that $M = O$.

Let us consider $ADBC$ - isosceles trapezoid with $AD$ the small base. In this case, we observe that $M$ - the intersection of the diagonals of the trapezoid, and $O$ are on the axis of symmetry of the trapezoid, and $UV \perp OM$ contradicts the fact that the points $A$ and $C$ must be on the same side of the right $UV$.

The contradictions show that $M$ must be the middle of the chord $PQ$.

Bibliography
