Aristotle, Term Logic, and QUARC

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August 7, 2024

In: George Englebretsen (ed.) (2024), *New Directions in Term Logic*, London: College Publications, pp. 427–503
(https://www.collegepublications.co.uk/logic/?00054)

Abstract. Aristotle counts as the founder of formal logic. The logic he develops dominated until Frege and others introduced a new logic. This new logic is taken to be more powerful and better capable of capturing inference patterns. The new logic differs from Aristotelian logic in significant respects. It has been argued by Fred Sommers and Hanoch Ben-Yami that the new logic is not well equipped as a logic of natural language, and that a logic closer to Aristotle's is better suited for this task. Each of them developed their own formalism—Sommers in form of term logic, Ben-Yami in form of his Quantified Argument Calculus (QUARC). I discuss Aristotle's logic—a term logic—and attempt a comparison between Aristotelian logic and (i) the new logic, (ii) Sommers' term logic, and (iii) Ben-Yami's QUARC. I consider differences between the systems, and show how they are related to and diverge from the new logic.

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1 Introduction

It is widely agreed that Aristotle is the inventor of *formal* logic. The logic he develops remains the dominant one until Gottlob Frege introduces his logical language in form of the *Begriffsschrift* (1879). As one might expect, their logics and formal languages are strikingly different. Aristotle develops a *term logic*, i.e., a logic which concerns the relation between *terms*. Terms can be affirmed or denied of terms, and can be assigned different quantities.

Fregean languages, on the other hand, distinguish different elements, such as *predicates-symbols*, *individual-constants*, *variables*, and *logical symbols*.[1] This goes beyond the language of term logic in several respects. In particular, terms most closely correspond to certain predicate-symbols, but not every predicate-symbol can easily be considered to be a term. Moreover, the Fregean language knows *quantifiers* which directly indicate something like the quantity in question, whereas Aristotle's term logic does not include them.[2]

Fregean languages are a success-story. Ever since their introduction, they almost completely superseded term languages. The power and flexibility of Fregean languages made the term approach pretty much obsolete—which is also one of the main reasons to prefer Fregean languages. This, however, does not mean that there is *no* competition; and it is the competition that we are interested in here.

Two of the competitors are Fred Sommers's so-called *Term Functor Logic* (TFL) and Hanoch Ben-Yami's so-called *QUantified ARgument Calculus* (QUARC). Both Sommers and Ben-Yami point towards Aristotelian logic as a potential ally, and as a reason to reject Fregean languages. This is why I consider Aristotle's approach as a base for both TFL and QUARC. Moreover, as both systems attempt to replace the Fregean approach, it is necessary to compare them to it. Overall, we are interested in a somewhat four-fold comparison between Aristotle's logic, the Fregean approach, TFL, and QUARC.

This paper is structured as follows. Section 2 discusses Aristotle's logic. Section 3 provides a generic picture of the Fregean approach as currently understood, but in a form more suitable for our purposes. Section 4 introduces Ben-Yami's QUARC and Section 5 Sommers's TFL; Section 6 compares the systemssystems, though I also compare the approaches within the previous sections. Section 7 concludes the paper.

[1] Frege (1879) introduces what we now consider a *second-order* language; however, only the first-order fragment is relevant here.

[2] Note that this does *not* mean that Aristotle and Aristotle's logic do not know *quantification*.

2 Aristotle's Logic

In Raab (2018), I reconstruct Aristotle's assertoric logic in a subsystem of QUARC, and show that the reconstruction is very close to the original text. The target of the reconstruction is only the first few chapters of Aristotle's *Prior Analytics* (viz., APr A1–6), but I suggest how to introduce complex terms (2018: §3.5) which are not to be found in those chapters. This original extension is the relevant one for our purposes, and there is some textual evidence that that's the version Aristotle had in mind (see Section 2.4). We encounter the formalism in Section 2.9.

To arrive there, I don't just consider Aristotle's *Prior Analytics*, but the whole so-called *Organon*. One question to be asked (but, unfortunately, not really answered) is why Aristotle developed a *term* logic. Another question is what counts as a *term* to begin with. In order to answer these questions, I reconstruct parts of the *Organon*, though I cannot discuss every aspect.

In the following, I put the quotations of cited passages—including the Greek text—into footnotes (and I'd suggest ignoring them for the most part).[3]

2.1 Ti Kata Tinos

The general picture is something like this. In *On Interpretation*, Aristotle distinguishes between words (ὄνομα/onoma)[4] and verbs (ῥῆμα/rhêma) (Int 1, 16a1)[5], both of which can then be considered to be *terms* (ὅρος/horos) (Int 3, 16b19f.[6], APr A1, 24b16[7]). Terms on their own do not constitute a sentence and are neither true nor false, yet they are mean-

[3]I follow the following translations and Greek texts (though streamline the translation of the technical terms etc.): *Categories* (Cat) and *On Interpretation* (Int): J. L. Ackrill's translation as printed in Aristotle (1963), Greek taken from Aristotelis (1949); *Topics* (Top): R. Smith's translation of Books A and H as printed in Aristotle (1997), all other books by W. A. Pickard-Cambridge as printed in Barnes (1995), Greek taken from Aristotelis (1958); *Sophistical Refutations* (SE): W. A. Pickard-Cambridge's tanslation as printed in Barnes (1995), Greek taken from Aristotelis (1958); *Prior Analytics* (APr): G. Striker's translation of Book A as printed in Aristotle (2009), A. J. Jenkinson's translation of Book B as printed in Barnes (1995), Greek taken from Aristotelis (1964); *Posterior Analytics* (APo): J. Barnes's translation as printed in Aristotle (1993a), Greek taken from Aristotelis (1964); *Metaphysics* (Met): W. D. Ross's translation of Book B as printed in Barnes (1995), C. Kirwan's translation of Book Γ as printed in Aristotle (1993b), D. Bostock's translation of Book Z as printed in Aristotle (1994), Greek taken from Aristotelis (1957).

[4]The literal translation is 'name', but what's meant is something like 'word'.

[5]"First we must settle what a word is and what a verb is [Πρῶτον δεῖ θέσθαι τί ὄνομα καὶ τί ῥῆμα]".

[6]"When uttered just by itself a verb is a word and signifies something [αὐτὰ μὲν οὖν καθ᾽ αὑτὰ λεγόμενα τὰ ῥήματα ὀνόματά ἐστι καὶ σημαίνει τι]".

[7]"I call a term that into which a premiss is resolved [Ὅρον δὲ καλῶ εἰς ὃν διαλύεται ἡ πρότασις]".

ingful (Int 1, 16a13–16[8]); a sentence (λόγος/logos)[9] is constituted by the combination of a word and a verb, i.e., by combining appropriate terms. However, not every sentence is significant, i.e., true or false (Int 4, 17a3f.[10]), but every significant sentence must include a verb (Int 5, 17a9f.[11], Int 10, 19b12[12]).[13]

More importantly, a simple sentence *affirms something of something* (τὶ κατὰ τινός/ti kata tinos) or *denies something of something* (τὶ ἀπὸ τινός/ti apo tinos) (Int 5, 17a20f.[14]; see also, e.g., APo A2, 72a13f.[15])—a structure also appearing in Aristotle's *Metaphysics* (e.g., Met Z17, 1041a20–23[16]).[17] Aristotle also speaks of 'compounded' sentences (Int 5, 17a21f.[18]), though it does not appear that he is concerned with them again throughout the *Organon* (with, maybe, a few exceptions; see below).

A sentence is made up of terms which signify something (Int 4, 16b26f.[19], Int 6, 17a25f.[20]), but, as the 'ti kata/apo tinos' suggests,

[8]"Thus words and verbs by themselves—for instance 'man' or 'white' when nothing further is added—are like the thoughts that are without combination and separation; for so far they are neither true nor false [τὰ μὲν οὖν ὀνόματα αὐτὰ καὶ τὰ ῥήματα ἔοικε τῷ ἄνευ συνθέσεως καὶ διαιρέσεως νοήματι, οἷον τὸ ἄνθρωπος ἢ λευκόν, ὅταν μὴ προστεθῇ τι· οὔτε γὰρ ψεῦδος οὔτε ἀληθές πω]".

[9]In APr, Aristotle uses a different word; see below and cf. Kneale and Kneale (1962: 34f.).

[10]"There is not truth or falsity in all sentences: a prayer is a sentence but is neither true nor false [οὐκ ἐν ἅπασι δὲ ὑπάρχει, οἷον ἡ εὐχὴ λόγος μέν, ἀλλ᾽ οὔτ᾽ ἀληθὴς οὔτε ψευδής]."

[11]"Every statement-making sentence must contain a verb or an inflexion of a verb [ἀνάγκη δὲ πάντα λόγον ἀποφαντικὸν ἐκ ῥήματος εἶναι ἢ πτώσεως]".

[12]"Without a verb there will be no affirmation or negation [ἄνευ δὲ ῥήματος οὐδεμία κατάφασις οὐδ᾽ ἀπόφασις]".

[13]As has often been noted, Aristotle's writings are ambiguous as to whether claims are about *linguistic expressions* or about *things* expressed by those expressions (see, e.g., Kneale and Kneale 1962: §II.2). That's less of a problem in *De Interpretatione*, but certainly so in the *Categories*; I generally take Aristotle to be interested in *things*, not linguistic items.

[14]"Of these the one is a simple statement, affirming or denying something of something [τούτων δ᾽ ἡ μὲν ἁπλῆ ἐστιν ἀπόφανσις, οἷον τὶ κατὰ τινὸς ἢ τὶ ἀπὸ τινός]".

[15]"The part of a contradictory pair which says something of something is affirmation; the part which takes something from something is a denial [μόριον δ᾽ ἀντιφάσεως τὸ μὲν τὶ κατὰ τινὸς κατάφασις, τὸ δὲ τὶ ἀπὸ τινὸς ἀπόφασις]".

[16]"However, one could ask why a man is such a kind of animal. It is clear that this is not to ask why one who is a man is a man. So what one asks is why it is that one thing is affirmed of another [ζητήσειε δ᾽ ἄν τις διὰ τί ὁ ἄνθρωπός ἐστι ζῷον τοιονδί. τοῦτο μὲν τοίνυν δῆλον, ὅτι οὐ ζητεῖ διὰ τί ὅς ἐστιν ἄνθρωπος ἄνθρωπός ἐστιν· τὶ ἄρα κατά τινος ζητεῖ διὰ τί ὑπάρχει]".

[17]The 'ti kata tinos' is important enough to become the title of Tugendhat (1958/2003).

[18]"the other is compounded of simple statements and is a kind of composite sentence [ἡ δ᾽ ἐκ τούτων συγκειμένη, οἷον λόγος τις ἤδη σύνθετος]".

[19]"A sentence is a significant spoken sound some part of which is significant in separation [Λόγος δέ ἐστι φωνὴ σημαντική, ἧς τῶν μερῶν τι σημαντικόν ἐστι κεχωρισμένον]".

[20]"An affirmation is statement affirming something of something, a denial is a state-

terms need to be combined (via copula) in order to affirm or deny (Int 4, 16b28ff.[21]). Given this basic structure, we can distinguish between simple sentences which affirm or deny something of a subject, and complex sentences which are compounds of simple sentences (Int 5, 17a20ff.[22]). However, as far as I can tell, Aristotle does not mention compounded sentences again, and he does not specify modes of composition (though see Section 2.4).

The basic unit is a sentence which contains two terms, viz., a subject and a verb, where the verb is said/predicated of the subject. Given this basic unit, a few more distinctions are possible. Aristotle distinguishes between things (πράγματα/pragmata) which are universal (καθόλου/katholou) and those which are particular (καθ᾽ ἕκαστον/kath hekaston). Note right away that, in his *Prior Analytics*, Aristotle uses a different expression when referring to a kind of sentence, viz., 'ἐν μέρει' (en merei) (e.g., APr A1, 24a17[23]), which is also translated as 'particular', though a more literal translation would be 'in part'.

The distinction that Aristotle draws is between universal and particular things. He calls 'things' like *human being* 'universal', and 'things' like *Callias* or *Socrates* 'particular'. The distinction is drawn by considering what something can be said of: universal things can be said of several things, particulars cannot (Int 7, 17a38–b1[24])—more on that in Section 2.5.

2.2 Universals and Universally

Both universal and particular things can be the subject of sentences so that things can be said of them (Int 7, 17b1ff.[25])—and, in the case of universal things, that in either of two ways, viz., universally (καθόλου

ment denying something of something [κατάφασις δέ ἐστιν ἀπόφανσις τινὸς κατὰ τινός, ἀπόφασις δέ ἐστιν ἀπόφανσις τινὸς ἀπὸ τινός]."

[21] "I mean that 'animal', for instance, signifies something, but not that it is or is not (though it will be an affirmation or denial if something is added) [λέγω δέ, οἷον ἄνθρωπος σημαίνει τι, ἀλλ᾽ οὐχ ὅτι ἔστιν ἢ οὐκ ἔστιν (ἀλλ᾽ ἔσται κατάφασις ἢ ἀπόφασις ἐάν τι προστεθῇ)]".

[22] "Of these the one is a simple statement, affirming or denying something of something, the other is compounded of simple statements and is a kind of composite sentence [τούτων δ᾽ ἡ μὲν ἁπλῆ ἐστὶν ἀπόφανσις, οἷον τὶ κατὰ τινὸς ἢ τὶ ἀπὸ τινός, ἡ δ᾽ ἐκ τούτων συγκειμένη, οἷον λόγος τις ἤδη σύνθετος]."

[23] ". . . and this is either universal or particular or indeterminate [οὗτος δὲ ἢ καθόλου ἢ ἐν μέρει ἢ ἀδιόριστος]."

[24] "Now of actual things some are universal, others particular (I call universal that which is by its nature predicated of a number of things, and particular that which is not; man, for instance, is a universal, Callias an particular [Ἐπεὶ δέ ἐστι τὰ μὲν καθόλου τῶν πραγμάτων τὰ δὲ καθ᾽ ἕκαστον, – λέγω δὲ καθόλου μὲν ὃ ἐπὶ πλειόνων πέφυκε κατηγορεῖσθαι, καθ᾽ ἕκαστον δὲ ὃ μή, οἷον ἄνθρωπος μὲν τῶν καθόλου Καλλίας δὲ τῶν καθ᾽ ἕκαστον]".

[25] "So it must sometimes be of a universal that one states that something holds or does not, sometimes of an particular [ἀνάγκη δ᾽ ἀποφαίνεσθαι ὡς ὑπάρχει τι ἢ μή, ὁτὲ μὲν τῶν καθόλου τινί, ὁτὲ δὲ τῶν καθ᾽ ἕκαστον]".

ἀποφαίνηται/katholou apophainêtai) or not (Int 7, 17b3ff.[26]). Examples of something being said universally of a universal are 'every human being is white' and 'no human being is white' (Int 7, 17b5f.[27]). It is of a universal thing, because 'human being' signifies one; and it is said universally, because it is said of every/none of those things.[28] The first of these two sentences counts as affirming something of something (ti kata tinos), whereas the latter as denying something of something (ti apo tinos) as the mode of predication changes, though the latter is not the negation of the former (see Section 2.3).

Something is said of a universal *not universally* when the subject is a universal thing, but the predication is not universally. The examples Aristotle provides are 'human being is white' and 'human being is not white' (Int 7, 17b8ff.[29]). The examples are of universals as 'human being' signifies a universal thing. However, the predications are not universal, because of the quantity of the subject. Regarding this, Aristotle also insists: '"every" does not signify the universal but that it is taken universally' (Int 7, 17b11f.[30], cf. also Int 10, 20a9f.[31]). This indicates where the quantity is meant to be applied to. Aristotle rejects that sentences such as 'every human being is every animal' (Int 7, 17b15f.[32]) can ever be true (Int 7, 17b12–15[33]). The quantity is meant to indicate of 'how much' of the subject-term the predicate-term is said.

It is less clear how Aristotle thinks about subjects which are particulars. He affirms that the sentences 'Socrates is white' and 'Socrates is not white' are *contradictories* (Int 7, 17b26–29[34]), but he does not mention

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[26]"Now if one states universally of a universal that something holds or does not [ἐὰν μὲν οὖν καθόλου ἀποφαίνηται ἐπὶ τοῦ καθόλου ὅτι ὑπάρχει ἢ μή]".

[27]"examples of what I mean by 'stating universally of a universal' are 'every man is white' and 'no man is white' [λέγω δὲ ἐπὶ τοῦ καθόλου ἀποφαίνεσθαι καθόλου, οἷον πᾶς ἄνθρωπος λευκός, οὐδεὶς ἄνθρωπος λευκός]".

[28]Note that the 'no human being is white' can actually be rendered differently, making the universal character explicit: 'every human being is not white'. In this formulation, it is clear that something is predicated universally—and that's the more appropriate way to understand it in the general subject/predicate structure together with the quantity and positive/negative copula involved; in the example sentence, it is *every human being* of whom *white* is *not* said, combining universal quantity with 'negative' predication, i.e., denial (ti apo tinos).

[29]"Examples of what I mean by 'stating of a universal not universally' are 'a human being is white' and 'a human being is not white' [λέγω δὲ τὸ μὴ καθόλου ἀποφαίνεσθαι ἐπὶ τῶν καθόλου, οἷον ἔστι λευκὸς ἄνθρωπος, οὐκ ἔστι λευκὸς ἄνθρωπος]".

[30]"τὸ γὰρ πᾶς οὐ τὸ καθόλου σημαίνει ἀλλ' ὅτι καθόλου".

[31]'For "every" does not signify a universal, but that it is taken universally [τὸ γὰρ πᾶς οὐ τὸ καθόλου σημαίνει, ἀλλ' ὅτι καθόλου]'.

[32]"ἔστι πᾶς ἄνθρωπος πᾶν ζῷον".

[33]"It is not true to predicate a universal universally of a subject, for there cannot be an affirmation in which a universal is predicated universally of a subject [ἐπὶ δὲ τοῦ κατηγορουμένου τὸ καθόλου κατηγορεῖν καθόλου οὐκ ἔστιν ἀληθές· οὐδεμία γὰρ κατάφασις ἔσται, ἐν ᾗ τοῦ κατηγορουμένου καθόλου τὸ καθόλου κατηγορηθήσεται]".

[34]'Of contradictory statements about a universal taken universally it is necessary for one or the other to be true or false; similarly if they are about particulars,

anything like a quantity in such cases. Indeed, such sentences only occur very sparingly and do not get a proper discussion (see also Section 2.6).

2.3 Affirmation, Denial, and Truth

What Aristotle tells us, though, is how affirmation and denial are related:

> the denial must deny the same things as the affirmation affirmed, and of the same thing, whether an individual or a universal (taken either universally or not universally). (Int 7, 17b38–18a1[35])

A sentence is only then a denial of another sentence if the terms are the same; the denial of ‘every human being is white’ is ‘not every human being is white’,[36] i.e., we keep the terms as they are, and, in a sense, we also keep the quantity, though the negation acts on it. Aristotle does not discuss the complex case, but only suggests the following sentences as examples: ‘Socrates is white’ has as denial ‘Socrates is not white’ (Int 7, 18a2f.[37]). The correct denial of the more complex sentences is arrived at after further discussion (see, e.g., Int 10, 19b14–18[38]).

The underlying idea is still that of *ti kata tinos*: saying something of something. ‘A kata B’ has as its denial ‘A apo B’; the terms remain the same. Aristotle does not specify the denial of ‘A apo B’, though we can take the ‘A kata B’ as its denial, assuming the only options to be *ti kata tinos* and *ti apo tinos*.

Given this picture, Aristotle suggests when sentences are true and false:

> For it is true to say that it is white or is not white, it is necessary for it to be white or not white; and if it is white or is not white, then it was true to say or deny this. If it is not the case it is false, if it is false it is not the case. (Int 9, 18a39–b3[39])

e.g. “Socrates is white” and “Socrates is not white”. [ὅσαι μὲν οὖν ἀντιφάσεις τῶν καθόλου εἰσὶ καθόλου, ἀνάγκη τὴν ἑτέραν ἀληθῆ εἶναι ἢ ψευδῆ, καὶ ὅσαι ἐπὶ τῶν καθ᾽ ἕκαστα, οἷον ἔστι Σωκράτης λευκός – οὐκ ἔστι Σωκράτης λευκός]’.

[35]“τὸ γὰρ αὐτὸ δεῖ ἀποφῆσαι τὴν ἀπόφασιν ὅπερ κατέφησεν ἡ κατάφασις, καὶ ἀπὸ τοῦ αὐτοῦ, ἢ τῶν καθ᾽ ἕκαστά τινος ἢ ἀπὸ τῶν καθόλου τινός, ἢ ὡς καθόλου ἢ ὡς μὴ καθόλου”.

[36]Note that that’s technically not correct, as Aristotle does not recognize *sentence negation*; nevertheless, for present purposes, I put it like this.

[37]“λέγω δὲ οἷον ἔστι Σωκράτης λευκός – οὐκ ἔστι Σωκράτης λευκός”.

[38]“So a first affirmation and denial are: ‘a man is’, ‘a man is not’; then, ‘a not-man is’, ‘a not-man is not’; and again, ‘every man is’, ‘every man is not’, ‘every not-man is’, ‘every not-man is not’ [ὥστε πρώτη κατάφασις καὶ ἀπόφασις τὸ ἔστιν ἄνθρωπος – οὐκ ἔστιν ἄνθρωπος, εἶτα ἔστιν οὐκ ἄνθρωπος – οὐκ ἔστιν οὐκ ἄνθρωπος, πάλιν ἔστι πᾶς ἄνθρωπος – οὐκ ἔστι πᾶς ἄνθρωπος, ἔστι πᾶς οὐκ ἄνθρωπος – οὐκ ἔστι πᾶς οὐκ ἄνθρωπος]”.

[39]“εἰ γὰρ ἀληθὲς εἰπεῖν ὅτι λευκὸν ἢ οὐ λευκόν ἐστιν, ἀνάγκη εἶναι λευκὸν ἢ οὐ λευκόν, καὶ εἰ ἔστι λευκὸν ἢ οὐ λευκόν, ἀληθὲς ἦν φάναι ἢ ἀποφάναι· καὶ εἰ μὴ ὑπάρχει, ψεύδεται, καὶ εἰ ψεύδετεαι, οὐχ ὑπάρχει”.

This understanding of truth is pretty much the same as that in his *Metaphysics* (Met Γ7, 1011b25ff.[40]). The general idea is that if we have a sentence, there are two terms involved, and one term is affirmed/denied of the other. Now, a sentence is true, if what is said actually obtains, and it is false if not. Moreover, under certain conditions, if a sentence is false, its denial is true—since the denial keeps the terms etc. intact, and similarly the other way around. Also, if B is A, it is true to make a corresponding claim (‘A kata B’), and false to assert the corresponding denial (‘A apo B’); and if B is not A, it is true to deny that B is A (‘A apo B’), and false to affirm it (‘A kata B’).

2.4 Complex Terms

We can also note that, in *On Interpretation*, Aristotle allows *negated terms*, i.e., it is not only sentences which are denials, but we can have affirmations involving negated terms. One of the examples is ‘not-human being’ (e.g., Int 10, 19b37[41]); another is a negated verb: ‘not-just’ (Int 10, 19b28[42]). Thus, we can form affirmations out of negated terms: every non-human being is not-just. (Cf. also, e.g., Top E6, 136a33f.[43])

Furthermore, Aristotle also does not exclude the possibility of further *complex terms*. His standard example is ‘cloak’ (ἱμάτιον/himation) as word for something more complex (an example also occurring at Met Z4, 1029b25–28[44]). For example, Aristotle suggests to introduce the term ‘cloak’ for the complex ‘horse and man’, though he denies a certain unity to sentences containing such terms; he rather thinks they are equivalent to compounded sentences (Int 8, 18a19–23[45]).

[40] “This will be plain if we first define what truth and falsehood are: for to say that that which is is not or that which is not is, is a falsehood; and to say that that which is is and that which is not is not, is true; so that, also, he who says that a thing is or not will have the truth or be in error [δῆλον δὲ πρῶτον μὲν ὁρισαμένοις τί τὸ ἀληθὲς καὶ ψεῦδος. τὸ μὲν γὰρ λέγειν τὸ ὂν μὴ εἶναι ἢ τὸ μὴ ὂν εἶναι ψεῦδος, τὸ δὲ τὸ ὂν εἶναι καὶ τὸ μὴ ὂν μὴ εἶναι ἀληθές, ὥστε καὶ ὁ λέγων εἶναι ἢ μὴ ἀληθεύσει ἢ ψεύσεται]”.

[41] “τὸ οὐκ ἄνθρωπος”.

[42] “οὐ δίκαιος”.

[43] “Thus (e.g.) inasmuch as animate is a property of living creature, animate will not be a property of not-living creature [οἷον ἐπεὶ τοῦ ζῴου ἴδιον τὸ ἔμψυχον, οὐκ ἂν εἴη τοῦ μὴ ζῴου ἴδιον τὸ ἔμψυχον]”.

[44] “We must see, therefore, whether there is a formula of what being is for each of these compounds, and whether these too have a what-being-is, e.g. a white man. Suppose ‘cloak’ to be a word for this [σκεπτέον ἆρ᾽ ἔστι λόγος τοῦ τί ἦν εἶναι ἑκάστῳ αὐτῶν, καὶ ὑπάρχει καὶ τούτοις τὸ τί ἦν εἶναι, οἷον λευκῷ ἀνθρώπῳ [τί ἦν λευκῷ ἀνθρώπῳ]. ἔστω δὴ ὄνομα αὐτῷ ἱμάτιον]”.

[45] “Suppose, for example, that one gave the word ‘cloak’ to horse and man; ‘a cloak is white’ would not be a single affirmation. For to say this is no different from saying ‘a horse and a man is white’, and this is no different from saying ‘a horse is white and a man is white’ [οἷον εἴ τις θεῖτο ὄνομα ἱμάτιον ἵππῳ καὶ ἀνθρώπῳ, τὸ ἔστιν ἱμάτιον λευκόν, αὕτη οὐ μία κατάφασις [οὐδὲ ἀπόφασις μία]· οὐδὲν γὰρ διαφέρει τοῦτο εἰπεῖν ἢ ἔστιν ἵππος καὶ ἄνθρωπος λευκός, τοῦτο δ᾽ οὐδὲν διαφέρει τοῦ εἰπεῖν ἔστιν ἵππος λευκὸς καὶ ἔστιν ἄνθρωπος λευκός]”.

Aristotle does not say much more about these complexes, though he does say more about the relationship of sentences involving negated terms and denials:

> ‘No human being is just’ follows from ‘every human being is not-just’, while the contradictory of this, ‘not every human being is not-just’, follows from ‘some human being is just’[.] (Int 10, 20a20–23[46])[47]

Thus, if the predicate-term is negated, the former sentence implies a denial with unnegated predicate-term; and the positive sentence, likewise, implies a denial with negated predicate-term.

Aristotle also suggests the following:

> ‘every not-man is not-just’ signifies the same as ‘no not-man is just’. (Int 10, 20a39f.[48])[49]

This suggests the equivalence of denial and affirmation with negated predicate-terms, though one direction is problematic (see n. 49).

2.5 Categories of Terms

Potentially moving away from Aristotle’s *formal logic*, let us consider his *Categories* which categorizes the terms. Aristotle notes that terms can be said in or without combination (Cat 2, 1a16f.[50]), and it is the classification of terms without combination—that is: the terms, not sentences resulting from their combination—that he is interested in.

The categorization is based on two concepts:

(i) being said of a subject, and (ii) being in a subject.

Applying these concepts gives rise to a four-fold categorization:

(1) being said of a subject and being in a subject ((i) and (ii)),[51]

[46]“ἀκολουθοῦσι δ᾽ αὗται, τῇ μὲν πᾶς ἐστὶν ἄνθρωπος οὐ δίκαιος ἡ οὐδείς ἐστιν ἄνθρωπος δίκαιος, τῇ δὲ ἔστι τις δίκαιος ἄνθρωπος ἡ ἀντικειμένη ὅτι οὐ πᾶς ἐστὶν ἄνθρωπος οὐ δίκαιος”.

[47]These are captured by one of the semantics in Section 2.9; see Theorem 16 (p. 28). The former claim is an instance of (1) (where $\|\overline{\overline{B}}\|_{\mathfrak{M}_{\mathsf{A}}} = \|B\|_{\mathfrak{M}_{\mathsf{A}}}$), the latter of (3).

[48]“τὸ δὲ πᾶς οὐ δίκαιος οὐκ ἄνθρωπος τῷ οὐδεὶς δίκαιος οὐκ ἄνθρωπος ταὐτὸν σημαίνει”.

[49]Only one direction holds in one of the semantics of Section 2.9, the other not; see Theorem 16 (2). The other semantics validates both directions, but clashes with different claims of Aristotle; see n. 47.

[50]“Of things that are said, some involve combination while others are said without combination [Τῶν λεγομένων τὰ μὲν κατὰ συμπλοκὴν λέγεται, τὰ δὲ ἄνευ συμπλοκῆς].”

[51]For example: “knowledge is in a subject, the soul, and is also said of a subject, knowledge-of-grammar [ἡ ἐπιστήμη ἐν ὑποκειμένῳ μέν ἐστι τῇ ψυχῇ, καθ᾽ ὑποκειμένου δὲ λέγεται τῆς γραμματικῆς]” (Cat 2, 1b1ff.).

(2) being said of a subject, but not being in a subject ((i) and not-(ii)),[52]

(3) not being said of a subject, but being in a subject (not-(i) and (ii)),[53] and

(4) neither being said of a subject nor being in a subject (not-(i) and not-(ii)).[54]

Important for our purposes is what distinguishes (4) from (1)–(3): only *particulars* are neither said of a subject, nor in a subject, i.e., *particulars cannot be predicated* (cf. Met Z3, 1028b33–37[55]). This distinguished feature of particulars is why, in the *Categories*, Aristotle calls them 'substance' *in the strictest and primary sense* (Cat 5, 2a11–14[56]). On the other hand, the kinds and genera of primary substances are secondary substances (Cat 5, 2a14ff.[57]), and, as they are instances of (2), they can be predicated.

2.6 Particulars and Syllogistic

The immediate relevance for us is that *particulars do not occur as terms in Aristotle's syllogistic*—and particulars are not the only examples of such terms. There is a certain symmetry. In his *Prior Analytics*, Aristotle suggests a term hierarchy. At the bottom of the hierarchy, there are terms—particulars (καθ᾽ ἕκαστα/kath hekasta)—which cannot be predicated:

[52]For example: "human being is said of a subject, the particular human being, but is not in any subject [οἷον ἄνθρωπος καθ᾽ ὑποκειμένου μὲν λέγεται τοῦ τινὸς ἀνθρώπου, ἐν ὑποκειμένῳ δὲ οὐδενί ἐστιν]" (Cat 2, 1a21f.).

[53]For example: "the particular knowledge-of-grammar is in a subject, the soul, but is not said of any subject [ἡ τὶς γραμματικὴ ἐν ὑποκειμένῳ μέν ἐστι τῇ ψυχῇ, καθ᾽ ὑποκειμένου δὲ οὐδενὸς λέγεται]" (Cat 2, 1a25ff.).

[54]For example: "the particular human being or particular horse [ὁ τὶς ἀνθρωπος ἢ ὁ τὶς ἵππος]" (Cat 2, 1b4f.).

[55]"Of the several ways in which substance is spoken of, there are at any rate four which are the most important: the substance of a thing seems to be what being is for that thing, and its universal and its genus, and fourthly the subject. The subject is that of which other things are predicated while it itself is predicated of nothing further [Λέγεται δ᾽ ἡ οὐσία, εἰ μὴ πλεοναχῶς, ἀλλ᾽ ἐν τέτταρσί γε μάλιστα· καὶ γὰρ τὸ τί ἦν εἶναι καὶ τὸ καθόλου καὶ τὸ γένος οὐσία δοκεῖ εἶναι ἑκάστου, καὶ τέταρτον τούτων τὸ ὑποκείμενον. τὸ δ᾽ ὑποκείμενόν ἐστι καθ᾽ οὗ τὸ ἄλλα λέγεται, ἐκεῖνο δὲ αὐτὸ μηκέτι κατ᾽ ἄλλου]".

[56]"A substance—that which is called a substance most strictly, primarily, and most of all—is that which is neither said of a subject nor in a subject, e.g. the particular man or the particular horse [Οὐσία δέ ἐστιν ἡ κυριώτατά τε καὶ πρώτως καὶ μάλιστα λεγομένη, ἣ μήτε καθ᾽ ὑποκειμένου τινὸς λέγεται μήτε ἐν ὑποκειμένῳ τινί ἐστιν, οἷον ὁ τὶς ἄνθρωπος ἢ ὁ τὶς ἵππος]".

[57]"The species in which the things primarily called substances are, are called *secondary substances*, as also are the genera of these species [δεύτεραι δὲ οὐσίαι λέγονται, ἐν οἷς εἴδεσιν αἱ πρώτως οὐσίαι λεγόμεναι ὑπάρχουσιν, ταῦτά τε καὶ τὰ τῶν εἰδῶν τούτων γένη]".

> That some things are by nature such as to be said of nothing else is clear, for more or less every perceptible thing is such as not to be predicated of anything except accidentally—for we do sometimes say that the white thing there is Socrates, or that what is approaching is Callias. (APr A27, 43a32–36[58])

Aristotle even insists that

> of all the things there are, some are such that they cannot be predicated truly and universally of anything else (for instance, Cleon or Callias, that is, what is particular and perceptible)[.] (APr A27, 43a25ff.[59])

Taken together, it seems as if Aristotle is saying that particulars—and pretty much all perceptible things—cannot be predicated. The latter passage just suggests that they cannot be predicated 'truly and universally', but the former suggests something stronger.

This also suggests that Aristotle does not seem to consider identity statements such as 'Socrates is Callias' or even 'Socrates is Socrates'. Whatever the reason, Aristotle does not consider something like 'is Socrates' or just 'Socrates' as a predicate-term.

This situation is mirrored at the top of the hierarchy. Starting from the particular, we reach another limit:

> But that one also comes to a halt if one goes upwards, we will explain later [at APo A22, 83b24–31[60]]; for the moment let this be assumed. (APr A27, 43a36f.[61])

Both ends of the hierarchy consist of terms which are not the target of Aristotle's syllogistic. Aristotle is explicit (my translation):

[58] "ὅτι μὲν οὖν ἔνια τῶν ὄντων κατ᾽ οὐδενὸς πέφυκε λέγεσθαι, δῆλον· τῶν γὰρ αἰσθητῶν σχεδὸν ἕκαστόν ἐστι τοιοῦτον ὥστε μὴ κατηγορεῖσθαι κατὰ μηδενός, πλὴν ὡς κατὰ συμβεβηκός· φαμὲν γάρ ποτε τὸ λευκὸν ἐκεῖνο Σωκράτην εἶναι καὶ τὸ προσιὸν Καλλίαν".

[59] "Ἁπάντων δὴ τῶν ὄντων τὰ μέν ἐστι τοιαῦτα ὥστε κατὰ μηδενὸς ἄλλου κατηγορεῖσθαι ἀληθῶς καθόλου (οἷον Κλέων καὶ Καλλίας καὶ τὸ καθ᾽ ἕκαστον καὶ αἰσθητόν)".

[60] "Thus one thing will not be said to hold of one thing either in the upward or in the downward direction: the incidentals are said of items in the substance of each thing, and these latter are not infinite; and in the upward direction there are both these items and the incidentals, neither of which are infinite. There must therefore be some term of which something is predicated primitively, and something else of this; and this must come to a stop, and there must be items which are no longer predicated of anything prior and of which nothing else prior is predicated. [οὔτ᾽ εἰς τὸ ἄνω ἄρα ἓν καθ᾽ ἑνὸς οὔτ᾽ εἰς τὸ κάτω ὑπάρχειν λεχθήσεται. καθ᾽ ὧν μὲν γὰρ λέγεται τὰ συμβεβηκότα, ὅσα ἐν τῇ οὐσίᾳ ἑκάστου, ταῦτα δὲ οὐκ ἄπειρα· ἄνω δὲ ταῦτά τε καὶ τὰ συμβεβηκότα, ἀμφότερα οὐκ ἄπειρα. ἀνάγκη ἄρα εἶναί τι οὗ πρῶτόν τι κατηγορεῖται καὶ τούτου ἄλλο, καὶ τοῦτο ἵστασθαι καὶ εἶναί τι ὃ οὐκέτι οὔτε κατ᾽ ἄλλου προτέρου οὔτε κατ᾽ ἐκείνου ἄλλο πρότερον κατηγορεῖται]".

[61] "ὅτι δὲ καὶ ἐπὶ τὸ ἄνω πορευομένοις ἵσταταί ποτε, πάλιν ἐροῦμεν· νῦν δ᾽ ἔστω τοῦτο κείμενον".

> Clearly, the things inbetween admit of both (for they can be predicated of others and others of them). And more or less the arguments and investigations are especially about them. (APr A27, 43a40–43[62])

Note that there are two occurrences of 'σχεδόν' ('schedon'), viz., at APr A27, 43a33 and at APr A27, 43a42, which have been translated as 'more or less'; they suggest the possibility of exceptions. For the former occurrence, the exception is already made explicit. Regarding the latter, Aristotle does not indicate what the exception is meant to be.[63]

Without the exceptions, the admissible terms for Aristotle's syllogistic are those that (i) can be predicated of other terms *and* (ii) have other terms predicated of them (APr A27, 43a41f.). This also makes sense once we consider the conversion rules (Sections 2.7–2.9). But Aristotle sometimes seemingly uses individuals in syllogisms. As far as I can tell, there are only three passages of this sort (in the *Organon*); let me quote the first in full (the second and third in footnotes 65 and 66, respectively):

> For example, if A is said of B and B of C—one might think that when the terms are so related, there is a syllogism, but in fact nothing necessary comes about, nor a syllogism. For let A designate always being, B, thinkable Aristomenes, and C, Aristomenes. Clearly it is true that A belongs to B, for Aristomenes is always thinkable. And it is also true that B belongs to C, for Aristomenes is a thinkable Aristomenes. But A does not belong to C, since Aristomenes is perishable. For no syllogism resulted from terms related in this way; rather, the premiss AB should have been taken as universal. But this is false—to claim that every thinkable Aristomenes always is, given that Aristomenes is perishable. (APr A33, 47b18–29[64]; see also APr A33, 47b29–37[65] and APr B27,

[62]"τὰ δὲ μεταξὺ δῆλον ὡς ἀμφοτέρως ἐνδέχεται (καὶ γὰρ αὐτὰ κατ᾽ ἄλλων καὶ ἄλλα κατὰ τούτων λεχθήσεται)· καὶ σχεδὸν οἱ λόγοι καὶ αἱ σκέψεις εἰσὶ μάλιστα περὶ τούτων".

[63]One suggestion here would be the term 'being'. Aristotle does not think that *being* forms a genus (see, e.g., Met B3, 998b22 ["But it is not possible that either unity or being should be a genus of things (οὐχ οἷόν τε δὲ τῶν ὄντων ἓν εἶναι γένος οὔτε τὸ ἓν οὔτε τὸ ὄν)"]), so maybe it can be said of everything else, but nothing of it.

[64]"οἷον εἰ τὸ Α κατὰ τοῦ Β λέγεται καὶ τὸ Β κατὰ τοῦ Γ· δόξειε γὰρ ἂν οὕτως ἐχόντων τῶν ὅρων εἶναι συλλογισμός, οὐ γίνεται δ᾽ οὔτ᾽ ἀναγκαῖον οὐδὲν οὔτε συλλογισμός. ἔστω γὰρ ἐφ᾽ ᾧ Α τὸ ἀεὶ εἶναι, ἐφ᾽ ᾧ δὲ Β διανοητὸς Ἀριστομένης, τὸ δ᾽ ἐφ᾽ ᾧ Γ Ἀριστομένης. ἀληθὲς δὴ τὸ Α τῷ Β ὑπάρχειν· ἀεὶ γάρ ἐστι διανοητὸς Ἀριστομένης. ἀλλὰ καὶ τὸ Β τῷ Γ· ὁ γὰρ Ἀριστομένης ἐστὶ διανοητὸς Ἀριστομένης. τὸ δ᾽ Α τῷ Γ οὐχ ὑπάρχει· φθαρτὸς γάρ ἐστιν ὁ Ἀριστομένης. οὐ γὰρ ἐγίνετο συλλογισμὸς οὕτως ἐχόντων τῶν ὅρων, ἀλλ᾽ ἔδει καθόλου τὴν Α Β ληφθῆναι πρότασιν. τοῦτο δὲ ψεῦδος, τὸ ἀξιοῦν πάντα τὸν διανοητὸν Ἀριστομένην ἀεὶ εἶναι, φθαρτοῦ ὄντος Ἀριστομένους."

[65]"Again, let C designate Miccalus, B educated Miccalus, and A, perishing tomorrow. Clearly it is true to predicate B of C, for Miccalus is an educated Miccalus. And also A of B, for an educated Miccalus might perish tomorrow. But to predicate A of C is false. Indeed, this is the same mistake as before, for it is not universally true that any educated Miccalus will perish tomorrow; but when this was not

70a16–20[66])

The general point Aristotle makes in the first two passages is that there is a certain danger when it comes to modal syllogisms involving necessity (APr A33, 47b15–18[67])—hence the modal flavour of these passages. They also involve syllogisms from the first figure, so no conversion occurs. The particular only occurs in subject-term position. It is not clear which mood of the first figure is concerned, and Aristotle's remark that 'the premiss AB should have been taken as universal' does not concern the particular Aristomenes who seems to be chosen just to illustrate the modal point.

The third passage involves a third-figure syllogism whose proofs all rely on conversion. However, the discussion is about *enthymemes* (ἐνθύμημα/enthymêma) which stem from the probable (εἰκός/eikos)—where "the probable is a reputable statement" (APr B27, 70a3f.[68]). Enthymemes are syllogisms involving the probable (APr B27, 70a10[69]), and so might be considered to not entirely fit the discussion of the syllogisms as developed in the first chapters of the *Prior Analytics*.[70]

In his *Posterior Analytics*, Aristotle seems to confirm the point that particulars are not said of anything (APo A1, 71a23f.[71]). Moreover, when he explains what 'in itself' (καθ᾽ αὑτά/kath hauta) means, Aristotle reiterates that there are things which are not said of anything else (APo A4, 73b5–10[72]), and he insists that "every term is always universal" (APo B13,

assumed, there was no syllogism. [πάλιν ἔστω τὸ μὲν ἐφ᾽ ᾧ Γ Μίκκαλος, τὸ δ᾽ ἐφ᾽ ᾧ Β μουσικὸς Μίκκαλος, ἐφ᾽ ᾧ δὲ τὸ Α τὸ φθείρεσθαι αὔριον. ἀληθὲς δὴ τὸ Β τοῦ Γ κατηγορεῖν· ὁ γὰρ Μίκκαλός ἐστι μουσικὸς Μίκκαλος. ἀλλὰ καὶ τὸ Α τοῦ Β· φθείροιτο γὰρ ἂν αὔριον μουσικὸς Μίκκαλος. τὸ δέ γε Α τοῦ Γ ψεῦδος. τοῦτο δὴ ταὐτόν ἐστι τῷ πρότερον· οὐ γὰρ ἀληθὲς καθόλου, Μίκκαλος μουσικὸς ὅτι φθείρεται αὔριον· τούτου δὲ μὴ ληφθέντος οὐκ ἦν συλλογισμός]."

[66] "The proof that wise men are good, since Pittacus is good, comes through the last figure. Let A stand for good, B for wise men, C for Pittacus. It is true then to predicate both A and B of C—only men do not say the latter, because they know it, though they state the former. [τὸ δ᾽ ὅτι οἱ σοφοὶ σπουδαῖοι, Πιττακὸς γὰρ σπουδαῖος, διὰ τοῦ ἐσχάτου. ἐφ᾽ ᾧ Α τὸ σπουδαῖον, ἐφ᾽ ᾧ Β οἱ σοφοί, ἐφ᾽ ᾧ Γ Πιττακός. ἀληθὲς δὴ καὶ τὸ Α καὶ τὸ Β τοῦ Γ κατηγορῆσαι· πλὴν τὸ μὲν οὐ λέγουσι διὰ τὸ εἰδέναι, τὸ δὲ λαμβάνουσιν]".

[67] "It often happens that we are deceived about syllogisms because of the necessity, as we said before. But sometimes it is due to the similarity in the position of terms. This must not escape our notice. [Πολλάκις μὲν οὖν ἀπατᾶσθαι συμβαίνει περὶ τοὺς συλλογισμοὺς διὰ τὸ ἀναγκαῖον, ὥσπερ εἴρηται πρότερον, ἐνίοτε δὲ παρὰ τὴν ὁμοιότητα τῆς τῶν ὅρων θέσεως· ὅπερ οὐ χρὴ λανθάνειν ἡμᾶς]."

[68] "τὸ μὲν εἰκός ἐστι πρότασις ἔνδοξος".

[69] "An enthymeme is a syllogism starting from probabilities or signs [Ἐνθύμημα δὲ ἐστὶ συλλογισμὸς ἐξ εἰκότων ἢ σημείων]". I'm leaving out the 'sign' in the discussion.

[70] In particular, the inference seems rather to be an *induction* than a deduction, inferring from a particular case to a general one.

[71] "this occurs when the items are in fact particulars and are not said of any underlying subject [ὅσα ἤδη τῶν καθ᾽ ἕκαστα τυγχάνει ὄντα καὶ μὴ καθ᾽ ὑποκειμένου τινός]."

[72] "Again, certain items are not said of some other underlying subject: e.g. whereas what is walking is something different walking (and similarly for what is white), substances, i.e. whatever means this so-and-so, are not just what they are in virtue

97b25[73]). Since terms corresponding to particulars are not universal—as particulars are exactly those things which aren't universal (Int 7, 17a38–b1)—relevant terms are not those of particulars.

2.7 The Syllogistic

With these preliminaries out of the way, let's consider the syllogistic. Aristotle develops it in his *Prior Analytics*; our focus is the assertoric part. Aristotle starts by suggesting which notions need to be introduced: *sentence/premiss* (πρότασις/protasis), *term* (ὅρος/horos), *syllogism* (συλλογισμός/syllogismos), *this (not) being in that as in a whole* (τὸ ἐν ὅλῳ (μὴ) εἶναι τόδε τῷδε/to en holô (mê) einai tode tôde), *predicate of all* (κατὰ παντὸς κατηγορεῖσθαι/kata pantos katêgoreisthai), and *predicate of none* (κατὰ μηδενὸς κατηγορεῖσθαι/kata mêdenos katêgoreisthai) (APr A1, 24a11–15[74]).

Syllogisms consist of sentences/premisses, and Aristotle defines sentences/premisses as *affirming or denying something of something* (APr A1, 24a16f.[75])—bringing back the *ti kata/apo tinos* structure. Both the 'ti' and the 'tinos', i.e., the predicate and the subject, respectively, are *terms*, as sentences/premisses are resolved in terms which are combined by a positive or negative copula (APr A1, 24b16ff.[76]).

There are different ways of affirming/denying something of something, viz., *universally* (καθόλου/katholou), *particularly* (ἐν μέρει/en merei), and *indeterminately* (ἀδιόριστος/adioristos) (APr A1, 24a17[77]). As noted in Section 2.1, the quantity is put as 'ἐν μέρει' (en merei), which can be translated as 'in part'. This contrasts with the universal predication which does not just predicate 'in part', but universally. Aristotle characterizes these as follows:

of being something different. Well, items which are not said of an underlying subject I call things in themselves, and those which are said of an underlying subject I call incidental. [ἔτι ὃ μὴ καθ᾽ ὑποκειμένου λέγεται ἄλλου τινός, οἷον τὸ βαδίζον ἕτερόν τι ὂν βαδίζον ἐστὶ καὶ τὸ λευκὸν ⟨λευκόν⟩, ἡ δ᾽ οὐσία, καὶ ὅσα τόδε τι σημαίνει, οὐχ ἕτερόν τι ὄντα ἐστὶν ὅπερ ἐστίν. τὰ μὲν δὴ μὴ καθ᾽ ὑποκειμένου καθ᾽ αὑτὰ λέγω, τὰ δὲ καθ᾽ ὑποκειμένου συμβεβηκότα]."

[73]"αἰεὶ δ᾽ ἐστί πᾶς ὅρος καθόλου".

[74]"Then, to define what is a premiss, what is a term, and what a syllogism, and which kind of syllogism is perfect and which imperfect. After that, what it is for this to be or not to be in that as in a whole, and what we mean by 'to be predicated of all' or 'of none' [εἶτα διορίσαι τί ἐστι πρότασις καὶ τί ὅρος καὶ τί συλλογισμός, καὶ ποῖος τέλειος καὶ ποῖος ἀτελής, μετὰ δὲ ταῦτα τί τὸ ἐν ὅλῳ εἶναι ἢ μὴ εἶναι τόδε τῷδε, καὶ τί λέγομεν τὸ κατὰ παντὸς ἢ μηδενὸς κατηγορεῖσθαι]."

[75]"A premiss, then, is a sentence that affirms or denies something of something [Πρότασις μὲν οὖν ἐστὶ λόγος καταφατικὸς ἢ ἀποφατικός τινος κατά τινος]".

[76]"I call a term that into which a premiss is resolved, that is, what is predicated and what it is predicated of, with the addition of 'to be'or 'not to be' [Ὅρον δὲ καλῶ εἰς ὃν διαλύεται ἡ πρότασις, οἷον τό τε κατηγορούμενον καὶ τὸ καθ᾽ οὗ κατηγορεῖται, προστιθεμένου [ἢ διαιρουμένου] τοῦ εἶναι ἢ μὴ εἶναι]".

[77]"and this is either universal or particular or indeterminate [οὗτος δὲ ἢ καθόλου ἢ ἐν μέρει ἢ ἀδιόριστος]".

> By 'universal' I mean belonging to all or to none of something; by 'particular', belonging to some or not to some, or not to all; by 'indeterminate', belonging without universality or particularity, as in 'of contraries there is a single science' or 'pleasure is not a good'. (APr A1, 24a18–22[78])

The universal affirmation and denial say something of all of the subject; the universal affirmation/denial says of all of the subject that a term applies/does not apply to it. The particular affirmation/denial says only of *part of* the subject (hence, the ἐν μέρει/en merei-phrasing) that a term does/does not apply to it. The 'indeterminate' case just does not indicate whether all or only part of the subject is meant; it doesn't play much of a role for us.

Given that terms built up sentences/premisses which say something of something, a *syllogism* is

> an argument in which, certain things being posited, something other than what was laid down results by necessity because these things are so. (APr A1, 24b18ff.[79])

Put differently, a syllogism is a valid argument (Read ms: §1), which is not trivial, i.e., something *new* has to be concluded (cf. SE 1, 164b27–165a2[80]).

Aristotle makes it clear that there must be a logical relationship between the sentences/premisses in order for a syllogism to obtain, a relationship that concerns the terms constituting the premisses (APr A1, 24b20ff.[81]).

The relationship Aristotle singles out is *being in another as in a whole* which he explains as follows: A *is in* B *as in a whole* iff B is predicated of all of A. Moreover, he explains: B *is predicated of all of* A iff there is no A that is not B (i.e., all A are B). Similarly, B *is predicated of none of* A iff there is no A that is B (i.e., no A are B) (APr A1, 24b26–30[82]).

[78]"λέγω δὲ καθόλου μὲν τὸ παντὶ ἢ μηδενὶ ὑπάρχειν, ἐν μέρει δὲ τὸ τινὶ ἢ μὴ τινὶ ἢ μὴ παντὶ ὑπάρχειν, ἀδιόριστον δὲ τὸ ὑπάρχειν ἢ μὴ ὑπάρχειν ἄνευ τοῦ καθόλου ἢ κατὰ μέρος, οἷον τὸ τῶν ἐναντίων εἶναι τὴν αὐτὴν ἐπιστήμην ἢ τὸ τὴν ἡδονὴν μὴ εἶναι ἀγαθόν".

[79]"συλλογισμὸς δέ ἐστι λόγος ἐν ᾧ τεθέντων τινῶν ἕτερόν τι τῶν κειμένων ἐξ ἀνάγκης συμβαίνει τῷ ταῦτα εἶναι".

[80]"For a syllogism rests on certain statements such that they involve necessarily the assertion of something other than what has been stated, through what has been stated [ὁ μὲν γὰρ συλλογισμὸς ἐκ τινῶν ἐστι τεθέντων ὥστε λέγειν ἕτερον ἐξ ἀνάγκης τι τῶν κειμένων διὰ τῶν κειμένων]".

[81]"By 'because these things are so' I mean that it results through these, and by 'resulting through these' I mean that no term is required from outside for the necessity to come about [λέγω δὲ τῷ ταῦτα εἶναι τὸ διὰ ταῦτα συμβαίνειν, τὸ δὲ διὰ ταῦτα συμβαίνειν τὸ μηδενὸς ἔξωθεν ὅρου προσδεῖν πρὸς τὸ γενέσθαι τὸ ἀναγκαῖον]".

[82]"For one thing to be in another as in a whole is the same as for the other to be predicated of all of the first. We speak of 'being predicated of all' when nothing can be found of the subject of which the other will not be said and the same account holds for 'of none' [τὸ δὲ ἐν ὅλῳ εἶναι ἕτερον ἑτέρῳ καὶ τὸ κατὰ παντὸς κατηγορεῖσθαι

Overall there are four sentence-types, depending on quantity and mode. The quantity can either be universal or particular ('in part'), and the mode can be positive ('kata', affirming) or negative ('apo', denying). These account for the relation of the two terms involved in sentences. Given two terms A and B, we get a sentence AB whose predicate-term is A and whose subject-term is B. The sentence AB can be either

(**a**) universal-affirmative ("all B are A"; AaB), or

(**i**) particular-affirmative ("some B are A"; AiB), or

(**e**) universal-negative ("all B are not A"; AeB), or

(**o**) particular-negative ("some B are not A; AoB).

Aristotle calls sentence-types **e** and **o** *privative* (στερητικός/sterêtikos); he does not think of them as involving what we would understand as a *negation*. Indeed, he has different ways of referring to the same type. On the one hand, a sentence can be an *affirmation* (κατάφασις/kataphasis) and a *denial* (ἀπόφασις/apophasis) (and, derivatively, sentences can be affirmative (καταφατικός/kataphatikos) and negative (ἀποφατικός/apophatikos)), and he refers to the sub-types as *universal* and *particular*. On the other hand, he refers to the denials as *privative*; e.g., he speaks of the "universal privative premiss" (APr A2, 25a5f.[83]). The 'privative' applies to the copula—it suggests a negative copula—the 'universal' to the subject—indicating the quantity of the subject.

This second way singles out the subject (universally or particularly) and notes the privation, i.e., that a term does not apply to it. For example, some human beings are *not* healthy, i.e., *lack* health, and so health is privative to those human beings. The whole sentence is a denial (ti apo tinos), and the predicate-term is privative (apo), i.e., the subject lacks the corresponding property.

Since both constituents of a sentence are terms, there is a natural question as to their relationship. Aristotle notes that three of the sentence-types *convert* (ἀντιστρέφειν/antistrephein), viz., **a**, **i**, and **e**. Sentence-type **o**, however, does not.

Converting a sentence means interchanging the predicate-term and subject-term; the sentence AB converts to BA. Sentence-types **i** and **e** convert to the same sentence-type; sentence-type **a**, on the other hand, converts to an **i**-type sentence (APr A2, 25a5–13[84]). The conversions can be summarized as follows (symbolizing 'converts to' as '$\rightsquigarrow$'):

θατέρου θάτερον ταὐτόν ἐστιν. λέγομεν δὲ τὸ κατὰ παντὸς κατηγορεῖσθαι ὅταν μηδὲν ᾖ λαβεῖν [τοῦ ὑποκειμένου] καθ' οὗ θάτερον οὐ λεχθήσεται· καὶ τὸ κατὰ μηδενὸς ὡσαύτως]".

[83]"τὴν μὲν ἐν τῷ ὑπάρχειν καθόλου στερητικήν".

[84]"... it is necessary for the universal privative premiss of belonging to convert with respect to its terms. So, for instance, if no pleasure is a good, then neither will any good be a pleasure. And the positive premiss converts necessarily, though not universally, but to the particular; for instance, if every pleasure is a good, it is necessary that some good be also a pleasure. Of the particular premisses

(a-i-conv) $AaB \rightsquigarrow BiA$ (i-i-conv) $AiB \rightsquigarrow BiA$

(e-e-conv) $AeB \rightsquigarrow BeA$ (o-o✗conv) $AoB \not\rightsquigarrow BoA$

That Aristotle claims that these sentence-types convert—and that without any suggestion of a restriction in place—suggests that predicate- and subject-terms are on a par as worked out in Section 2.6. Suppose Aristotle allowed *particulars* into his syllogistic. Then sentences with such particulars cannot convert, since particulars can*not* play the role of predicates; the validity of the conversions rules out terms denoting particulars.

With all these preliminaries out of the way, Aristotle goes on to introduce three figures and to establish their syllogisms. The figures come about by considering the different roles three terms, A, B, C, can play. Sentences of the form 'AB' have 'B' as their subject-term and 'A' as their predicate-term. Since the syllogisms come about via the relation of the terms, one term has to occur in two premisses as to establish a relation between the other terms. The three figures encode exactly that.[85]

The first figure has one term—the so-called *middle* term (μέσον/meson)—occurring as predicate-term in one premise and subject-term in the other, i.e., the premises are AB and BC (APr A4, 25b35f.[86]). The conclusion concerns the other terms A and C—the so-called *extremes* (ἄκρα/akra) (APr A4, 25b36f.[87]).

The first-figure syllogisms are the following:

(Barbara) $\dfrac{AaB \qquad BaC}{AaC}$ (Celarent) $\dfrac{AeB \qquad BaC}{AeC}$

(Darii) $\dfrac{AaB \qquad BiC}{AiC}$ (Ferio) $\dfrac{AeB \qquad BiC}{AoC}$

The second figure has the middle term only as predicate-term (APr A5, 26b34–37[88]), and comprises the following syllogisms:

the affirmative necessarily converts to the particular, for if some pleasure is a good, then some good will also be a pleasure; but for the privative premiss this is not necessary. For it is not the case that, if man does not belong to some animal, then animal also does not belong to some man [τὴν μὲν ἐν τῷ ὑπάρχειν καθόλου στερητικὴν ἀνάγκη τοῖς ὅροις ἀντιστρέφειν, οἷον εἰ μηδεμία ἡδονὴ ἀγαθόν, οὐδ᾽ ἀγαθὸν οὐδὲν ἔσται ἡδονή· τὴν δὲ κατηγορικὴν ἀντιστρέφειν μὲν ἀναγκαῖον, οὐ μὴν καθόλου ἀλλ᾽ ἐν μέρει, οἷον εἰ πᾶσα ἡδονὴ ἀγαθόν, καὶ ἀγαθόν τι εἶναι ἡδονήν· τῶν δὲ ἐν μέρει τὴν μὲν καταφατικὴν ἀντιστρέφειν ἀνάγκη κατὰ μέρος (εἰ γὰρ ἡδονή τις ἀγαθόν, καὶ ἀγαθόν τι ἔσται ἡδονή), τὴν δὲ στερητικὴν οὐκ ἀναγκαῖον· (οὐ γὰρ εἰ ἄνθρωπος μὴ ὑπάρχει τινὶ ζῴῳ, καὶ ζῷον οὐχ ὑπάρχει τινὶ ἀνθρώπῳ]".

[85] I'm ignoring the fourth figure that Aristotle does not mention and which is not necessary to establish all the syllogisms.

[86] "I call 'middle' the term that is itself in another and in which there is also another—the one that also has the middle position [καλῶ δὲ μέσον μὲν ὃ καὶ αὐτὸ ἐν ἄλλῳ καὶ ἄλλο ἐν τούτῳ ἐστίν, ὃ καὶ τῇ θέσει γίνεται μέσον]".

[87] "Extremes are what is in another and that in which there is another [ἄκρα δὲ τὸ αὐτό τε ἐν ἄλλῳ ὂν καὶ ἐν ᾧ ἄλλο ἐστίν]".

[88] "When the same thing belongs to all of one and none of the other, or to all or none

(Cesare) $\dfrac{MeN \qquad MaX}{NeX}$ (Festino) $\dfrac{MeN \qquad MiX}{NoX}$

(Camestres) $\dfrac{MaN \qquad MeX}{NeX}$ (Baroco) $\dfrac{MaN \qquad MoX}{NoX}$

The third figure has the middle term only as subject-term (APr A6, 28a10–13[89]), and comprises the following syllogisms:

(Darapti) $\dfrac{PaS \qquad RaS}{PiR}$ (Datisi) $\dfrac{PaS \qquad RiS}{PiR}$

(Felapton) $\dfrac{PeS \qquad RaS}{PoR}$ (Bocardo) $\dfrac{PoS \qquad RaS}{PoR}$

(Disamis) $\dfrac{PiS \qquad RaS}{PiR}$ (Ferison) $\dfrac{PeS \qquad RiS}{PoR}$

Given the conversion rules, certain further conclusions can be drawn. For example, given (Barbara) with conclusion AaC, we can apply (**a**-**i**-conv) and infer CiA. Moreover, we can apply (**i**-**i**-conv) and infer AiC.

2.8 The Square of Opposition

Aristotle is taken to endorse the *square of opposition*, though he does not state it explicitly (Kneale and Kneale 1962: 56). The four vertexes of the square are labelled by the four sentence-types, and the relations between these types are captured by edges.

The possible relations are *contradictories*, *contraries*, *subcontraries*, and *subalternation*. Consider two sentences φ and ψ. They are *contradictories* iff exactly one of them is true; they are *contraries* iff they cannot both be true, but can both be false; they are *subcontraries* iff they cannot both be false, but can both be true; and ψ is a subaltern of φ iff φ implies ψ. For example, an **a**-type sentence has the corresponding **o**-type sentence as its contradictory, the corresponding **e**-type sentence as its contrary, and the corresponding **i**-type sentence as its subaltern. Figure 1 pictures the square.

Except for subalternation, the relations are *symmetrical*; only subalternation is directed. Moreover, given (**a**-**i**-conv) and (**i**-**i**-conv), we can account for the sentence-type **i** being the subaltern of sentence-type **a**: by (**a**-**i**-conv), AaB implies (converts to) BiA which, by (**i**-**i**-conv), implies (converts to) AiB.

of both other terms, I call this sort of figure the second. And in this figure I call middle the term that is predicated of both [῞Οταν δὲ τὸ αὐτὸ τῷ μὲν παντὶ τῷ δὲ μηδενὶ ὑπάρχῃ, ἢ ἑκατέρῳ παντὶ ἢ μηδενί, τὸ μὲν σχῆμα τὸ τοιοῦτον καλῶ δεύτερον, μέσον δὲ ἐν αὐτῷ λέγω τὸ κατηγορούμενον ἀμφοῖν]".

[89] "If one term belongs to all, another to none of the same thing, or both to all or both to none, I call this sort of figure the third. And in this figure I call the middle the term of which both the predicated terms are said [᾿Εὰν δὲ τῷ αὐτῷ τὸ μὲν παντὶ τὸ δὲ μηδενὶ ὑπάρχῃ ἢ ἄμφω παντὶ ἢ μηδενί, τό μὲν σχῆμα τὸ τοιοῦτον καλῶ τρίτον, μέσον δ᾿ ἐν αὐτῷ λέγω καθ᾿ οὗ ἄμφω τὰ κατηγορούμενα]".

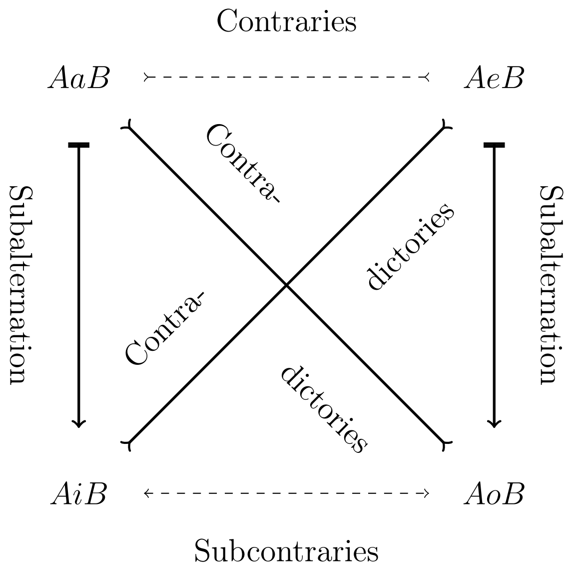

Figure 1: The Square of Opposition

Moreover, given the other relations, we can see that the **e**-type sentence has the **o**-type sentence as its subaltern. If *AeB* holds, then *AaB* cannot hold as its contrary. Thus, as exactly one of *AaB* and *AoB* has to be true, it follows that *AoB* must be true.

Aristotle provides the definitions for *contradictories* and *contraries* in *On Interpretation* (Int 7, 17b16–20[90], Int 7, 17b20–23[91], respectively), and he notes that contradictories cannot, but contraries can be ~~true~~ false together (Int 7, 17b23–29[92]). Moreover, in his *Topics*, Aristotle suggests the *subalternations* (Top B1, 109a3–6[93]).

[90]"I call an affirmation and a negation contradictory opposites when what one signifies universally the other signifies not universally, e.g. 'every man is white' and 'not every man is white', 'no man is white' and 'some man is white' [Ἀντικεῖσθαι μὲν οὖν κατάφασιν ἀποφάσει λέγω ἀντιφατικῶς τὴν τὸ καθόλου σημαίνουσαν τῷ αὐτῷ ὅτι οὐ καθόλου, οἷον πᾶς ἄνθρωπος λευκός – οὐ πᾶς ἄνθρωπος λευκός, οὐδεὶς ἄνθρωπος λευκός – ἔστι τις ἄνθρωπος λευκός]".

[91]"But I call the universal affirmation and the universal denial contrary opposites, e.g. 'every man is just' and 'no man is just' [ἐναντίως δὲ τὴν τοῦ καθόλου κατάφασιν καὶ τὴν τοῦ καθόλου ἀπόφασιν, οἷον πᾶς ἄνθρωπος δίκαιος – οὐδεὶς ἄνθρωπος δίκαιος]".

[92]"So these cannot be true together, but their opposites may be both true with respect to the same thing, e.g. 'not every man is white' and 'some man is white'. Of contradictory statements about a universal taken universally it is necessary for one or the other to be true or false; similarly if they are about particulars, e.g. 'Socrates is white' and 'Socrates is not white' [διὸ ταύτας μὲν οὐχ οἷόν τε ἅμα ἀληθεῖς εἶναι, τὰς δὲ ἀντικειμένας αὐταῖς ἐνδέχεται ἐπὶ τοῦ αὐτοῦ, οἷον οὐ πᾶς ἄνθρωπος λευκός, καὶ ἔστι τις ἄνθρωπος λευκός. ὅσαι μὲν οὖν ἀντιφάσεις τῶν καθόλου εἰσὶ καθόλου, ἀνάγκη τὴν ἑτέραν ἀληθῆ εἶναι ἢ ψευδῆ, καὶ ὅσαι ἐπὶ τῶν καθ᾽ ἕκαστα, οἷον ἔστι Σωκράτης λευκός – οὐκ ἔστι Σωκράτης λευκός]".

[93]"for when we have proved that a predicate belongs in every case, we shall also have proved that it belongs in some cases. Likewise, also, if we prove that it does not

Lastly, the *subcontraries* result from the established relations as well. For, in cases where both the **a**- and **e**-type sentence are false—as they can be as contraries—their contradictories must be true, i.e., the **o**-/**i**-type sentence is true as the contradictory of the **a**-/**e**-type sentence. Moreover, one of the **i**- and **o**-type sentence has to be true. Suppose that the **i**/**o**-type sentence is false. Then its contradictory **e**/**a**-type sentence is true which implies the corresponding **o**/**i**-type sentence. For the same reason, one of the **a**- and **e**-type sentence has to be false.

Aristotle is also aware that you can use the square to refute sentences. For, if you want to refute an **a**-type sentence, it suffices to establish the corresponding **o**-type sentence; and similarly with **e**- and **i**-type sentences (Top B3, 110a32–37[94]).

2.9 Formal Syllogistic

In order to have a comparison base, let me briefly introduce some formalism capturing Aristotle's syllogistic. The presentation is based on, but also differs from, Raab (2018) where more discussion and details can be found.

Definition 1 ***(The Language $\mathcal{L}_{\mathsf{A}}$)***
The *language of Aristotelian Syllogistic* ($\mathcal{L}_{\mathsf{A}}$) consists of the following:

- a countable set $\mathsf{STerm}_{\mathcal{L}_{\mathsf{A}}}$ of (*simple*) *terms*,
- the set of *logical symbols* including '$\neg$', '$-$', '$\wedge$', '$\vee$', '$\rightarrow$', '$\leftrightarrow$', '$\forall$', and '$\exists$', and
- the set of *auxiliary symbols* including '(' and ')'.

The '$-$'-symbol is used to distinguish term-negation from a negative copula.[95] The remaining symbols are to be understood as indicated below.

Since we allow complex terms, let us introduce them:

Definition 2 ***(Complex $\mathcal{L}_{\mathsf{A}}$-Terms)***
The *(full) set of terms* ($\mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$) is recursively defined as follows:

belong in any case, we shall also have proved that it does not belong in every case [δείξαντες γὰρ ὅτι παντὶ ὑπάρχει, καὶ ὅτι τινὶ ὑπάρχει δεδειχότες ἐσόμεθα: ὁμοίως δὲ κἂν ὅτι οὐδενὶ ὑπάρχει δείξωμεν, καὶ ὅτι οὐ παντὶ ὑπάρχει δεδειχότες ἐσομεθα]".

[94] "Of course, in refuting a statement there is no need to start the discussion by securing any admission, whether the attribute is said to belong to all or to none of something; for if we prove that in any case whatever the attribute does not belong, we shall have refuted the universal assertion of it, and likewise if we prove that it belongs even in a single case, we shall refute the universal denial of it [πλὴν ἀνασευκάζοντι μὲν οὐδὲν δεῖ ἐξ ὁμολογίας διαλέγεσθαι, οὔτ᾽ εἰ παντὶ οὔτ᾽ εἰ μηδενὶ ὑπάρχειν εἴρηται· ἐὰν γὰρ δείξωμεν ὅτι οὐχ ὑπάρχει ὁτῳοῦν, ἀνῃρηκότες ἐσόμεθα τὸ παντὶ ὑπάρχειν· ὁμοίως δὲ κἂν ἑνὶ δείξωμεν ὑπάρχον, ἀναιρήσομεν τὸ μηδενὶ ὑπάρχειν]".

[95] Including '$-$' differs from, but is equivalent to, the set-up in Raab (2018: §3.5).

(1) if $A \in \mathsf{STerm}_{\mathcal{L}_\mathsf{A}}$, then $A \in \mathsf{Term}_{\mathcal{L}_\mathsf{A}}$;

(2) if $A \in \mathsf{Term}_{\mathcal{L}_\mathsf{A}}$, then $\overline{A} \in \mathsf{Term}_{\mathcal{L}_\mathsf{A}}$;

(3) if $A, B \in \mathsf{Term}_{\mathcal{L}_\mathsf{A}}$, then, $(A \circ B) \in \mathsf{Term}_{\mathcal{L}_\mathsf{A}}$ $(\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\})$.

Given the language and the terms, we can define the formulas:

Definition 3 ***($\mathcal{L}_\mathsf{A}$-Formulas)***
The set of $\mathcal{L}_\mathsf{A}$-*formulas* ($\mathsf{Form}_{\mathcal{L}_\mathsf{A}}$) is defined as follows:

- If $A, B \in \mathsf{Term}_{\mathcal{L}_\mathsf{A}}$, then
  - $(\forall A)B \in \mathsf{Form}_{\mathcal{L}_\mathsf{A}}$
  - $(\forall A)\neg B \in \mathsf{Form}_{\mathcal{L}_\mathsf{A}}$
  - $(\exists A)B \in \mathsf{Form}_{\mathcal{L}_\mathsf{A}}$
  - $(\exists A)\neg B \in \mathsf{Form}_{\mathcal{L}_\mathsf{A}}$.

The formulas are to be read as follows: '$(\forall A)B$' as "all A are B" (BaA), '$(\exists A)B$' as "some A are B" (BiA), '$(\forall A)\neg B$' as "all A are not B" or "no A is B" (BeA), and '$(\exists A)\neg B$' as 'some A are not B" (BoA). Note that, according to Definition 2, complex terms are covered by Definition 3. For example, formulas of the form '$(\forall \overline{(A \wedge B)})\neg\overline{C}$' are allowed, and should correspondingly be read as "all not-(A-and-B) are not not-C" or "no not-(A-and-B) is not-C".

The absence/occurrence of a negation-symbol '¬' indicates whether the formula is affirming or denying, respectively; it represents the copula. According to Definition 3, at most one negation-symbol occurs in a formula. Negation does not act on sentences, and we need to ensure in a different way how the sentences are related with respect to affirmation and denial; as in Raab (2018), this achieved via positive ('+') and negative ('−') semantic clauses (Definitions 6–7).

The quantifier-symbols indicate the quantity, i.e., whether a sentence is universal (∀) or particular (∃) (or, whether the predication is *universally* or *particularly*)—and that's all they are doing: They are just a means to make explicit what kind of sentence is represented; instead of Aristotle's way of suggesting that, e.g., 'BA' is universal affirming/denying sentence, we directly depict it as '$(\forall A)B$'/'$(\forall A)\neg B$'.

Given this understanding, let me introduce a model-theoretic semantics. I provide two ways of doing so. One interpretation allows *empty* terms, i.e., the structure can assign the empty extension as interpretation of terms (I refer to it as 'the empty semantics'); the other interpretation forces the simple terms to be non-empty (shown to be inadequate below).

Definition 4 ***($\mathcal{L}_\mathsf{A}$-Model)***
Let $\mathcal{L}_\mathsf{A}$ be a language of Aristotelian Syllogistic. An $\mathcal{L}_\mathsf{A}$-*model* is a tuple $\mathfrak{M}_\mathsf{A} = \langle D, \|\cdot\|_{\mathfrak{M}_\mathsf{A}} \rangle$ such that

(1) D is a non-empty set (the *universe*)

(2) $\| \cdot \|_{\mathfrak{M}_{\mathsf{A}}}$ is an *interpretation-function of* $\mathfrak{M}_{\mathsf{A}}$ such that

   (a) if $A \in \mathsf{STerm}_{\mathcal{L}_{\mathsf{A}}}$, $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \subseteq D$;

   (b) if $A \in \mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$ is a complex term of the form '$\overline{B}$' for some $B \in \mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$, then $\|A\|_{\mathfrak{M}_{\mathsf{A}}} = D \setminus \|B\|_{\mathfrak{M}_{\mathsf{A}}}$;

   (c) if $A \in \mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$ is a complex term of the form '$(B \wedge C)$' for some $B, C \in \mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$, then $\|A\|_{\mathfrak{M}_{\mathsf{A}}} = \|B\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|C\|_{\mathfrak{M}_{\mathsf{A}}}$.

Since Definition 4 allows for empty terms, we have to specify that the domain D is non-empty. Negated terms are interpreted as the set-theoretic difference between the extension of a term and the domain. Complex terms are treated as expected; Definition 4 only specifies the clause for conjunctive terms ('$\wedge$'); the others are definable given clauses (2b)–(2c).

Definition 5 ***(NE-$\mathcal{L}_{\mathsf{A}}$-Model)***
Let $\mathcal{L}_{\mathsf{A}}$ be a language of Aristotelian Syllogistic. A *non-empty $\mathcal{L}_{\mathsf{A}}$-model* is a tuple $\mathfrak{M}_{\mathsf{ne}} = \langle D, \| \cdot \|_{\mathfrak{M}_{\mathsf{ne}}} \rangle$ such that

(1) D is a set (the *universe*);

(2) $\| \cdot \|_{\mathfrak{M}_{\mathsf{ne}}}$ is an *interpretation-function of* $\mathfrak{M}_{ne}$ such that

   (a) if $A \in \mathsf{STerm}_{\mathcal{L}_{\mathsf{A}}}$, $\emptyset \neq \|A\|_{\mathfrak{M}_{\mathsf{ne}}} \subseteq D$;

   (b) if $A \in \mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$ is a complex term of the form '$\overline{B}$' for some $B \in \mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$, then $\|A\|_{\mathfrak{M}_{\mathsf{ne}}} = D \setminus \|B\|_{\mathfrak{M}_{\mathsf{ne}}}$;

   (c) if $A \in \mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$ is a complex term of the form '$(B \wedge C)$' for some $B, C \in \mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$, then $\|A\|_{\mathfrak{M}_{\mathsf{ne}}} = \|B\|_{\mathfrak{M}_{\mathsf{ne}}} \cap \|C\|_{\mathfrak{M}_{\mathsf{ne}}}$.

In contrast to Definition 4, Definition 5 does not need to enforce the domain to be non-empty, as clause (2a) effectively takes care of it.[96] The remaining clauses are the same as in Definition 4.

Given the different models, we can introduce corresponding satisfaction relations. Since $\mathcal{L}_{\mathsf{A}}$ does not contain sentence-negation, we need to ensure that, for example, an **a**-type sentence has an **o**-type sentence as its contradictory by introducing positive ('$+$') and negative ('$-$') clauses. Moreover, since we take the validity of the square of opposition as a condition for any adequate satisfaction relation, we need to interpret the sentences accordingly. This results in different clauses for the **a**- and **o**-type sentences. (Note that, as Lemma 3.6 of Raab 2018 shows, the number of clauses is reducible to four.)

Definition 6 ***(Satisfaction $\models_{\mathsf{A}}$)***
Let the *satisfaction-relation* $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} \varphi$ for $\mathcal{L}_{\mathsf{A}}$-formulas φ and $\mathcal{L}_{\mathsf{A}}$-model $\mathfrak{M}_{\mathsf{A}}$ be defined as follows: Let $A, B \in \mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$, then:

[96] Note that Definition 1 does not enforce $\mathsf{STerm}_{\mathcal{L}_{\mathsf{A}}}$ to be non-empty. If $\mathsf{STerm}_{\mathcal{L}_{\mathsf{A}}} = \emptyset$, clause (2a) does still not produce a problem as the clause is then vacuous.

($\mathbf{a}_+$) $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\forall A)B$ iff $\|A\|_{\mathfrak{M}_\mathsf{A}} \cap \|B\|_{\mathfrak{M}_\mathsf{A}} = \|A\|_{\mathfrak{M}_\mathsf{A}}$ and $\|A\|_{\mathfrak{M}_\mathsf{A}} \neq \emptyset$;

($\mathbf{a}_-$) $\mathfrak{M}_\mathsf{A} \not\models_\mathsf{A} (\forall A)B$ iff $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\exists A)\neg B$.

($\mathbf{i}_+$) $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\exists A)B$ iff $\|A\|_{\mathfrak{M}_\mathsf{A}} \cap \|B\|_{\mathfrak{M}_\mathsf{A}} \neq \emptyset$.

($\mathbf{i}_-$) $\mathfrak{M}_\mathsf{A} \not\models_\mathsf{A} (\exists A)B$ iff $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\forall A)\neg B$.

($\mathbf{e}_+$) $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\forall A)\neg B$ iff $\|A\|_{\mathfrak{M}_\mathsf{A}} \cap \|B\|_{\mathfrak{M}_\mathsf{A}} = \emptyset$.

($\mathbf{e}_-$) $\mathfrak{M}_\mathsf{A} \not\models_\mathsf{A} (\forall A)\neg B$ iff $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\exists A)B$.

($\mathbf{o}_+$) $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\exists A)\neg B$ iff $\|A\|_{\mathfrak{M}_\mathsf{A}} \cap \|B\|_{\mathfrak{M}_\mathsf{A}} \neq \|A\|_{\mathfrak{M}_\mathsf{A}}$ or $\|A\|_{\mathfrak{M}_\mathsf{A}} = \emptyset$.

($\mathbf{o}_-$) $\mathfrak{M}_\mathsf{A} \not\models_\mathsf{A} (\exists A)\neg B$ iff $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\forall A)B$.

In order for the square of opposition to hold, we must ensure that an **a**-type sentences imply **i**-type sentences. The usual way to do so is by only allowing non-empty terms as in Definition 5, but Definition 4 allows for empty terms. Thus, a model $\mathfrak{M}_\mathsf{A}$ can only satisfy an **a**-type sentence if the term happens to be non-empty, i.e., if $\|A\|_{\mathfrak{M}_\mathsf{A}} = \emptyset$, no sentence of the form '$(\forall A)B$' can be satisfied. Since **a**-type sentences have **o**-type sentences as their contradictories, the satisfaction-clause ($\mathbf{o}_+$) needs to include the cases in which the subject-term is empty.

As the non-empty $\mathcal{L}_\mathsf{A}$-models $\mathfrak{M}_\mathsf{ne}$ don't allow non-empty terms, the clauses are simpler than those of Definition 6. However, as shown in Theorem 14, there is no fully general formal analogue of ($\underline{\mathbf{a}}$-$\underline{\mathbf{i}}$-conv) and so the square of opposition does not follow.

Definition 7 ***(NE-Satisfaction*** $\models_{\mathit{ne}}$***)***
Let the *non-empty satisfaction-relation* $\mathfrak{M}_{\mathit{ne}} \models_{\mathit{ne}} \varphi$ for $\mathcal{L}_\mathsf{A}$-formulas φ and non-empty $\mathcal{L}_\mathsf{A}$-model $\mathfrak{M}_\mathsf{ne}$ be defined as follows: Let $A, B \in \mathsf{Term}_{\mathcal{L}_\mathsf{A}}$, then:

($\mathbf{a}^\mathsf{ne}_+$) $\mathfrak{M}_\mathsf{ne} \models_\mathsf{ne} (\forall A)B$ iff $\|A\|_{\mathfrak{M}_\mathsf{ne}} \cap \|B\|_{\mathfrak{M}_\mathsf{ne}} = \|A\|_{\mathfrak{M}_\mathsf{ne}}$.

($\mathbf{a}^\mathsf{ne}_-$) $\mathfrak{M}_\mathsf{ne} \not\models_\mathsf{ne} (\forall A)B$ iff $\mathfrak{M}_\mathsf{ne} \models_\mathsf{ne} (\exists A)\neg B$.

($\mathbf{i}^\mathsf{ne}_+$) $\mathfrak{M}_\mathsf{ne} \models_\mathsf{ne} (\exists A)B$ iff $\|A\|_{\mathfrak{M}_\mathsf{ne}} \cap \|B\|_{\mathfrak{M}_\mathsf{ne}} \neq \emptyset$.

($\mathbf{i}^\mathsf{ne}_-$) $\mathfrak{M}_\mathsf{ne} \not\models_\mathsf{ne} (\exists A)B$ iff $\mathfrak{M}_\mathsf{ne} \models_\mathsf{ne} (\forall A)\neg B$.

($\mathbf{e}^\mathsf{ne}_+$) $\mathfrak{M}_\mathsf{ne} \models_\mathsf{ne} (\forall A)\neg B$ iff $\|A\|_{\mathfrak{M}_\mathsf{ne}} \cap \|B\|_{\mathfrak{M}_\mathsf{ne}} = \emptyset$.

($\mathbf{e}^\mathsf{ne}_-$) $\mathfrak{M}_\mathsf{ne} \not\models_\mathsf{ne} (\forall A)\neg B$ iff $\mathfrak{M}_\mathsf{ne} \models_\mathsf{ne} (\exists A)B$.

($\mathbf{o}^\mathsf{ne}_+$) $\mathfrak{M}_\mathsf{ne} \models_\mathsf{ne} (\exists A)\neg B$ iff $\|A\|_{\mathfrak{M}_\mathsf{ne}} \cap \|B\|_{\mathfrak{M}_\mathsf{ne}} \neq \|A\|_{\mathfrak{M}_\mathsf{ne}}$.

($\mathbf{o}^\mathsf{ne}_-$) $\mathfrak{M}_\mathsf{ne} \not\models_\mathsf{ne} (\exists A)\neg B$ iff $\mathfrak{M}_\mathsf{ne} \models_\mathsf{ne} (\forall A)B$.

Given a notion of satisfaction, we can introduce the usual notions:

Definition 8
Let $T \subseteq \mathsf{Form}_{\mathcal{L}_\mathsf{A}}$, $\Vdash \in \{\models_\mathsf{A}, \models_\mathsf{ne}\}$, and $\mathfrak{M}_\Vdash = \left\{ \begin{array}{lll} \mathfrak{M}_\mathsf{A} & \text{if } \Vdash \text{ is } \models_\mathsf{A} \\ \mathfrak{M}_\mathsf{ne} & \text{if } \Vdash \text{ is } \models_\mathsf{ne} \end{array} \right\}$.

(1) φ *is a logical consequence of* T ($T \Vdash \varphi$) iff for all $\mathcal{L}_\mathsf{A}$-models $\mathfrak{M}_\Vdash$, if $\mathfrak{M}_\Vdash \Vdash \psi$ for all $\psi \in T$, then $\mathfrak{M}_\Vdash \Vdash \varphi$.

If $T = \{\varphi_1, \ldots, \varphi_n\}$, we write '$\varphi_1, \ldots, \varphi_n \Vdash \varphi$' for '$\{\varphi_1, \ldots, \varphi_n\} \Vdash \varphi$'.

(2) φ *is logically valid* iff $\emptyset \Vdash \varphi$ ($\Vdash \varphi$).

(3) T *is satisfiable* iff there is an $\mathcal{L}_\mathsf{A}$-model $\mathfrak{M}_\Vdash$ such that $\mathfrak{M}_\Vdash \Vdash \varphi$ for all $\varphi \in T$.

Given these definitions, we can formulate some results. First, we can note that the empty $\mathcal{L}_\mathsf{A}$-models see sentence-types **a** and **e** as contraries:

Lemma 9 ***(Contraries)***
$(\forall A)B$ and $(\forall A)\neg B$ are contraries in $\mathcal{L}_\mathsf{A}$-models $\mathfrak{M}_\mathsf{A}$:

(1) $\{(\forall A)B, (\forall A)\neg B\}$ is *not* satisfiable;

(2) there are $\mathcal{L}_\mathsf{A}$-models $\mathfrak{M}_\mathsf{A}$ such that $\mathfrak{M}_\mathsf{A} \not\models_\mathsf{A} (\forall A)B$ and $\mathfrak{M}_\mathsf{A} \not\models_\mathsf{A} (\forall A)\neg B$.

Proof. Let $\mathfrak{M}_\mathsf{A}$ be an $\mathcal{L}_\mathsf{A}$-model.

(1): Suppose that $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\forall A)B$ and $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\forall A)\neg B$. Then, by ($\mathbf{a}_+$), $\|A\|_{\mathfrak{M}_\mathsf{A}} \cap \|B\|_{\mathfrak{M}_\mathsf{A}} = \|A\|_{\mathfrak{M}_\mathsf{A}} \neq \emptyset$, and, by ($\mathbf{e}_+$), $\|A\|_{\mathfrak{M}_\mathsf{A}} \cap \|B\|_{\mathfrak{M}_\mathsf{A}} = \emptyset$, a contradiction.

(2): Let $\|A\|_{\mathfrak{M}_\mathsf{A}} = \{a, b\}$ and $\|B\|_{\mathfrak{M}_\mathsf{A}} = \{a\}$. Then, $\|A\|_{\mathfrak{M}_\mathsf{A}} \cap \|B\|_{\mathfrak{M}_\mathsf{A}} \neq \emptyset$, i.e., by ($\mathbf{i}_+$), $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\exists A)B$. And, since $\|A\|_{\mathfrak{M}_\mathsf{A}} \cap \|B\|_{\mathfrak{M}_\mathsf{A}} \neq \|A\|_{\mathfrak{M}_\mathsf{A}}$, by ($\mathbf{o}_+$), $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\exists A)\neg B$. By ($\mathbf{e}_-$) and ($\mathbf{a}_-$), respectively, the result follows.

□

Two characteristics of the semantics are the following:

Theorem 10
The following hold.

(1) $\not\models_\mathsf{A} (\exists A)A$

(2) $(\forall A)B \models_\mathsf{A} (\exists A)B$

(3) $\not\models_\mathsf{ne} (\exists A)A$

(4) $(\forall A)B \not\models_\mathsf{ne} (\exists A)B$

Proof. **(1):** Let $\mathfrak{M}_\mathsf{A}$ be an $\mathcal{L}_\mathsf{A}$-model such that $\|A\|_{\mathfrak{M}_\mathsf{A}} = \emptyset$. Then, $\|A\|_{\mathfrak{M}_\mathsf{A}} \cap \|A\|_{\mathfrak{M}_\mathsf{A}} = \emptyset$, i.e., by ($\mathbf{e}_+$), $\mathfrak{M}_\mathsf{A} \models_\mathsf{A} (\forall A)\neg A$. Thus, by ($\mathbf{i}_-$), $\mathfrak{M}_\mathsf{A} \not\models_\mathsf{A} (\exists A)A$.

(2): Let $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\forall A)B$. Then, by $(\mathbf{a}_+)$, $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{A}}} = \|A\|_{\mathfrak{M}_{\mathsf{A}}} \neq \emptyset$. Therefore, by $(\mathbf{i}_+)$, $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\exists A)B$.

(3): Let $\mathfrak{M}_{\mathsf{ne}}$ be a non-empty $\mathcal{L}_{\mathsf{A}}$-model such that $\|A\|_{\mathfrak{M}_{\mathsf{ne}}} = D$. Then, $\|\overline{A}\|_{\mathfrak{M}_{\mathsf{ne}}} = \emptyset$. Thus, $\|\overline{A}\|_{\mathfrak{M}_{\mathsf{ne}}} \cap \|\overline{A}\|_{\mathfrak{M}_{\mathsf{ne}}} = \emptyset$, so, by $(\mathbf{e}^{\mathsf{ne}}_+)$, $\mathfrak{M}_{\mathsf{ne}} \models_{\mathsf{ne}} (\forall\overline{A})\neg\overline{A}$. By $(\mathbf{i}^{\mathsf{ne}}_-)$, $\mathfrak{M}_{\mathsf{ne}} \not\models_{\mathsf{ne}} (\exists\overline{A})\overline{A}$.

(4): Consider the model in (3). Since $\|\overline{A}\|_{\mathfrak{M}_{\mathsf{ne}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{ne}}} = \|\overline{A}\|_{\mathfrak{M}_{\mathsf{ne}}}$, by $(\mathbf{a}^{\mathsf{ne}}_+)$, $\mathfrak{M}_{\mathsf{ne}} \models_{\mathsf{ne}} (\forall\overline{A})B$. However, since $\|\overline{A}\|_{\mathfrak{M}_{\mathsf{ne}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{ne}}} = \emptyset$, by $(\mathbf{e}^{\mathsf{ne}}_+)$, $\mathfrak{M}_{\mathsf{ne}} \models_{\mathsf{ne}} (\forall\overline{A})\neg B$, and so, by $(\mathbf{i}^{\mathsf{ne}}_-)$, $\mathfrak{M}_{\mathsf{ne}} \not\models_{\mathsf{ne}} (\exists\overline{A})B$.

□

Theorem 10 (3)–(4) imply that the non-empty semantics does not validate the square of oppostion; for example, sentence-types **a** and **e** fail to be contraries:

Corollary 11
In the non-empty semantics, $(\forall A)B$ and $(\forall A)\neg B$ are *not* contraries. In general, given a non-empty $\mathcal{L}_{\mathsf{A}}$-model $\mathfrak{M}_{\mathsf{ne}}$, if $\|A\|_{\mathfrak{M}_{\mathsf{ne}}} = \emptyset$, then $\{(\forall A)B, (\forall A)\neg B\}$ is satisfiable in $\mathfrak{M}_{\mathsf{ne}}$.

Proof. Consider the proof of Theorem 10 (4). The model $\mathfrak{M}_{\mathsf{ne}}$ is such that both $\mathfrak{M}_{\mathsf{ne}} \models_{\mathsf{ne}} (\forall\overline{A})B$ and $\mathfrak{M}_{\mathsf{ne}} \models_{\mathsf{ne}} (\forall\overline{A})\neg B$. □

The empty semantics has formal analogues of the conversions:

Theorem 12 ***(Conversion)***
The following conversions hold:

($\underline{\mathbf{a}}$-$\underline{\mathbf{i}}$-conv$^{\models_{\mathsf{A}}}$) $(\forall A)B \models_{\mathsf{A}} (\exists B)A$

($\underline{\mathbf{i}}$-$\underline{\mathbf{i}}$-conv$^{\models_{\mathsf{A}}}$) $(\exists A)B \models_{\mathsf{A}} (\exists B)A$

($\underline{\mathbf{e}}$-$\underline{\mathbf{e}}$-conv$^{\models_{\mathsf{A}}}$) $(\forall A)\neg B \models_{\mathsf{A}} (\forall B)\neg A$

The non-empty semantics only validates two such conversions:

Theorem 13 ***(NE-Conversion)***
The following conversions hold:

($\underline{\mathbf{i}}$-$\underline{\mathbf{i}}$-conv$^{\models_{\mathsf{ne}}}$) $(\exists A)B \models_{\mathsf{ne}} (\exists B)A$

($\underline{\mathbf{e}}$-$\underline{\mathbf{e}}$-conv$^{\models_{\mathsf{ne}}}$) $(\forall A)\neg B \models_{\mathsf{ne}} (\forall B)\neg A$

Proof. Let $\mathfrak{M}_{\mathsf{A}}$ be an $\mathcal{L}_{\mathsf{A}}$-model.

($\underline{\mathbf{a}}$-$\underline{\mathbf{i}}$-conv$^{\models_{\mathsf{A}}}$) Suppose that $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\forall A)B$. Then, by $(\mathbf{a}_+)$, $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{A}}} = \|A\|_{\mathfrak{M}_{\mathsf{A}}}$ and $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \neq \emptyset$. Thus, $\|B\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|A\|_{\mathfrak{M}_{\mathsf{A}}} = \|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{A}}} = \|A\|_{\mathfrak{M}_{\mathsf{A}}} \neq \emptyset$. So, by $(\mathbf{i}_+)$, $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\exists B)A$.

($\underline{\mathsf{i}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{A}}}$) Suppose that $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\exists A)B$. Then, by ($\mathbf{i}_{+}$), $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{A}}} \neq \emptyset$, i.e., $\|B\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|A\|_{\mathfrak{M}_{\mathsf{A}}} \neq \emptyset$ and so, by ($\mathbf{i}_{+}$), $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\exists B)A$.

($\underline{\mathsf{e}}$-$\underline{\mathsf{e}}$-conv$^{\models_{\mathsf{A}}}$) Suppose that $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\forall A)\neg B$. Then, by ($\mathbf{e}_{+}$), $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{A}}} = \emptyset$, so also $\|B\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|A\|_{\mathfrak{M}_{\mathsf{A}}} = \emptyset$, i.e., by ($\mathbf{e}_{+}$), $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\forall B)\neg A$.

($\underline{\mathsf{i}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{ne}}}$) and ($\underline{\mathsf{e}}$-$\underline{\mathsf{e}}$-conv$^{\models_{\mathsf{ne}}}$) are shown in the same way. □

The non-empty semantics does not validate the third conversion.

Theorem 14 ***(NE-Conversion-Failure)***
The formal analogue of ($\underline{\mathsf{a}}$-$\underline{\mathsf{i}}$-conv) fails for the non-empty semantics:

($\underline{\mathsf{a}}$-$\underline{\mathsf{i}}$$\not$-conv$^{\models_{\mathsf{ne}}}$) $(\forall A)B \not\models_{\mathsf{ne}} (\exists B)A$

Proof. Let $\mathfrak{M}_{\mathsf{ne}}$ be a non-empty $\mathcal{L}_{\mathsf{A}}$-model such that $\|C\|_{\mathfrak{M}_{\mathsf{ne}}} = D$. Then, by Definition 5 (2b), $\|\overline{C}\|_{\mathfrak{M}_{\mathsf{ne}}} = D \setminus \|C\|_{\mathfrak{M}_{\mathsf{ne}}} = D \setminus D = \emptyset$.

Now, let $\|A\|_{\mathfrak{M}_{\mathsf{ne}}} = \|\overline{C}\|_{\mathfrak{M}_{\mathsf{ne}}}$, and suppose that $\mathfrak{M}_{\mathsf{ne}} \models_{\mathsf{ne}} (\forall A)B$. Then, by ($\mathbf{a}_{+}^{\mathsf{ne}}$), $\|A\|_{\mathfrak{M}_{\mathsf{ne}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{ne}}} = \|A\|_{\mathfrak{M}_{\mathsf{ne}}}$, i.e., $\|A\|_{\mathfrak{M}_{\mathsf{ne}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{ne}}} = \emptyset$, so also $\|B\|_{\mathfrak{M}_{\mathsf{ne}}} \cap \|A\|_{\mathfrak{M}_{\mathsf{ne}}} = \emptyset$. By ($\mathbf{e}_{+}^{\mathsf{ne}}$), $\mathfrak{M}_{\mathsf{ne}} \models_{\mathsf{ne}} (\forall B)\neg A$. Thus, by ($\mathbf{i}_{-}^{\mathsf{ne}}$), $\mathfrak{M}_{\mathsf{ne}} \not\models_{\mathsf{ne}} (\exists B)A$.

Therefore, $(\forall A)B \not\models_{\mathsf{ne}} (\exists B)A$. □

All we get is a restricted version:

Theorem 15 ***(Restricted NE-a-i-Conversion)***
Let $\mathfrak{M}_{\mathsf{ne}}$ be a non-empty $\mathcal{L}_{\mathsf{A}}$-model. Then:

($\underline{\mathsf{a}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{ne}}} \upharpoonright \mathsf{ne}$) $(\exists A)A, (\forall A)B \models_{\mathsf{ne}} (\exists B)A$

I take this as evidence that Definitions 4 and 6 are the correct ones since the problem arises already with negative terms—which Aristotle explicitly discusses in his *Organon*, even though not in his *Prior Analytics*. Other complex terms might end up empty, too, even though the simpler terms are not. Let $\mathfrak{M}_{\mathsf{ne}}$ be a non-empty $\mathcal{L}_{\mathsf{A}}$-structure. Suppose that $\emptyset \subsetneqq \|A\|_{\mathfrak{M}_{\mathsf{ne}}} \subsetneqq D$. Then, $\emptyset \subsetneqq \|\overline{A}\|_{\mathfrak{M}_{\mathsf{ne}}} \subsetneqq D$. However, $\|(A \wedge \overline{A})\|_{\mathfrak{M}_{\mathsf{ne}}} = \emptyset$. Therefore, $\|(A \wedge \overline{A})\|_{\mathfrak{M}_{\mathsf{ne}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{ne}}} = \|(A \wedge \overline{A})\|_{\mathfrak{M}_{\mathsf{ne}}}$ since $\emptyset \cap \|B\|_{\mathfrak{M}_{\mathsf{ne}}} = \emptyset$. Thus, by ($\mathbf{a}_{+}^{\mathsf{ne}}$), $\mathfrak{M}_{\mathsf{ne}} \models_{\mathsf{ne}} (\forall(A \wedge \overline{A}))B$, but $\mathfrak{M}_{\mathsf{ne}} \not\models_{\mathsf{ne}} (\exists B)(A \wedge \overline{A})$. This also means again that $(\forall C)D \not\models_{\mathsf{ne}} (\exists C)D$.

Of course, one could still insist on the non-emptiness of terms. One option, though I don't take it to be particularly plausible, is to only allow terms which don't lead to empty ones. That, of course, rules out simultaneously having negated and conjunctive terms.[97] Another option is to do the same as in Definition 6, though then there is no reason to

[97]If we only consider negated terms, the option has some plausibility. Given the discussion of Section 2.6, assigning the whole domain as interpretation of a term might push us to the top of the term-hierarchy and, thus, to terms that Aristotle

assume that terms are non-empty to begin with. There might be different options available, but I take Definitions 4 and 6 to be the correct ones. Nevertheless, the semantics to be developed in the following sections are more like the non-empty one from Definitions 5 and 7.

Regarding the empty semantics, there is a difference between denials and affirmations modulo negated terms:

Theorem 16 ***(Negation)***
The following hold:

(1) $(\forall A)B \models_{\mathsf{A}} (\forall A)\neg\overline{B}$; (2) $(\forall A)\neg\overline{B} \not\models_{\mathsf{A}} (\forall A)B$;

(3) $(\exists A)B \models_{\mathsf{A}} (\exists A)\neg\overline{B}$; (4) $(\exists A)\neg\overline{B} \not\models_{\mathsf{A}} (\exists A)B$.

Proof. Let $\mathfrak{M}_{\mathsf{A}}$ be an $\mathcal{L}_{\mathsf{A}}$-model.

(1): Let $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\forall A)B$. By $(\mathbf{a}_+)$, $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{A}}} = \|A\|_{\mathfrak{M}_{\mathsf{A}}} \neq \emptyset$. By Definition 4 (2b) $\|\overline{B}\|_{\mathfrak{M}_{\mathsf{A}}} = D \setminus \|B\|_{\mathfrak{M}_{\mathsf{A}}}$, i.e., $\emptyset = \|B\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|\overline{B}\|_{\mathfrak{M}_{\mathsf{A}}}$. Therefore, $\emptyset = \|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|\overline{B}\|_{\mathfrak{M}_{\mathsf{A}}} = \|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|\overline{B}\|_{\mathfrak{M}_{\mathsf{A}}}$. Thus, by $(\mathbf{e}_+)$, $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\forall A)\neg\overline{B}$.

(2): Let $\|A\|_{\mathfrak{M}_{\mathsf{A}}} = \emptyset$. Then, $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|\overline{B}\|_{\mathfrak{M}_{\mathsf{A}}} = \emptyset$, i.e., by $(\mathbf{e}_+)$, $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\forall A)\neg\overline{B}$. Also, as $\|A\|_{\mathfrak{M}_{\mathsf{A}}} = \emptyset$, by $(\mathbf{o}_+)$, $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\exists A)\neg B$. Therefore, by $(\mathbf{a}_-)$, $\mathfrak{M}_{\mathsf{A}} \not\models_{\mathsf{A}} (\forall A)B$.

(3): Let $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\exists A)B$. By $(\mathbf{i}_+)$, $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{A}}} \neq \emptyset$. By Definition 4 (2b), $\|\overline{B}\|_{\mathfrak{M}_{\mathsf{A}}} = D \setminus \|B\|_{\mathfrak{M}_{\mathsf{A}}}$, so $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|\overline{B}\|_{\mathfrak{M}_{\mathsf{A}}} \neq \|A\|_{\mathfrak{M}_{\mathsf{A}}}$. Therefore, by $(\mathbf{o}_+)$, $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\exists A)\neg\overline{B}$.

(4): Let $\|A\|_{\mathfrak{M}_{\mathsf{A}}} = \emptyset$. Then, by $(\mathbf{o}_+)$, $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\exists A)\neg\overline{B}$. Also, $\|A\|_{\mathfrak{M}_{\mathsf{A}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{A}}} = \emptyset$, so, by $(\mathbf{e}_+)$, $\mathfrak{M}_{\mathsf{A}} \models_{\mathsf{A}} (\forall A)\neg B$. Thus, by $(\mathbf{i}_-)$, $\mathfrak{M}_{\mathsf{A}} \not\models_{\mathsf{A}} (\exists A)B$.

□

As shown in Lemma 3.29 of Raab (2018), the non-empty-semantics validates that sentence-types $\mathbf{e}/\mathbf{i}$ imply corresponding sentence-types $\mathbf{a}/\mathbf{i}$, i.e., $\mathfrak{M}_{\mathsf{ne}} \models_{\mathsf{ne}} (qA)\neg B$ iff $\mathfrak{M}_{\mathsf{ne}} \models_{\mathsf{ne}} (qA)\overline{B}$ $(q \in \{\forall, \exists\})$ whereas the empty semantics only validates the direction from sentence-type $\mathbf{a}/\mathbf{i}$ to sentence-type $\mathbf{e}/\mathbf{o}$, i.e., $(qA)\xcancel{B}\ \overline{B} \models_{\mathsf{A}} (qA)\neg\xcancel{\overline{B}}\ B$, but $(qA)\neg B \not\models_{\mathsf{A}} (qA)\overline{B}$.

dismisses as relevant for his syllogistic. Thus, if the only complex terms are negated terms, we might change Definition 5 (2b) to

(2) (b*) if $A \in \mathsf{STerm}_{\mathcal{L}_{\mathsf{A}}}$, $\emptyset \subsetneq \|A\|_{\mathfrak{M}_{\mathsf{ne}}} \subsetneq D$

which resolves the problem as for any $A \in \mathsf{Term}_{\mathcal{L}_{\mathsf{A}}}$, $\|\overline{A}\|_{\mathfrak{M}_{\mathsf{ne}}} \neq \emptyset$.

2.10 Identity

The syllogistic is lacking any treatment of *identity*. As suggested in Section 2.6, Aristotle does not consider something like 'is Socrates' to be a term; the *Organon* does not seem to include any identity claims.

Yet, Aristotle formulates some principles to test for the (non-)identity of terms. Given two terms A and B, we can compare them with respect to other terms C.[98] One principle Aristotle suggests is that if A and B are identical, then if A is identical to C, B must also be identical to C (Top H1, 152a31f.[99]). Whereas finding differences breaks identities (cf. Top A18, 108b2ff.[100]), being identical to something else suffices for identity, i.e., if A is identical to C, and B is identical to C, then A and B are identical too (SE 6, 168b31f.[101]).

Aristotle does not say much more about this, and its not entirely clear of what sort of things he claims identity, though he seems to formulate a (more general) version of Leibniz's law:

> Speaking generally, one ought to be on the look-out for any discrepancy anywhere in any sort of predicate of each term, and in the things of which they are predicated. For all that is predicated of the one should be predicated also of the other, and of whatever the one is a predicate, the other should be a predicate as well. (Top H1, 152b25–29[102])

I put it in terms of *terms* above. It should be clear that there are no (explicit) principles to establish identities between terms, but the principles allow us to break some (see also Top H1, 152b34f.[103]). Suppose that we establish that $(\forall A)C$ and $(\exists B)\neg C$. Then A and B cannot be identical. Also, if we establish that $(qC)A$ and $(qC)\neg B$ ($q \in \{\forall, \exists\}$), then A and B cannot be identical. However, if A and B are identical, we can conclude from $(qC)A$ that $(qC)B$, as well as $(qB)C$ from $(qA)C$. As A and B are terms, they can occur both in subject- and predicate-position, and the principle Aristotle suggests is meant to check both options. A and B are

[98] Note that Aristotle does not speak about *terms*, but I take it to apply to them.

[99] "Again, look and see if, supposing the one to be the same as something, the other also is the same as it; for if they are not both the same as the same thing, clearly neither are they the same as one another [Πάλιν σκοπεῖν εἰ ᾧ θάτερον ταὐτόν, καὶ θάτερον· εἰ γὰρ μὴ ἀμφότερα τῷ αὐτῷ ταὐτά, δῆλον ὅτι οὐδ᾽ ἀλλήλοις]."

[100] "for when we have found any difference whatever between the things proposed, we shall have shown that they are not the same thing [εὑρόντες γὰρ διαφορὰν τῶν προκειμένων ὁποιανοῦν δεδειχότες ἐσόμεθα ὅτι οὐ ταὐτόν]".

[101] "for we claim that things that are the same as one and the same thing are also the same as one another [τὰ γὰρ ἑνὶ καὶ ταὐτῷ ταὐτὰ καὶ ἀλλήλοις ἀξιοῦμεν εἶναι ταὐτά·]".

[102] "Καθόλου δ᾽ εἰπεῖν ἐκ τῶν ὁπωσοῦν ἑκατέρου κατηγορουμένων καὶ ὧν ταῦτα κατηγορεῖται σκοπεῖν εἴ που διαφωνεῖ· ὅσα γὰρ θατέρου κατηγορεῖται, καὶ θατέρου κατηγορεῖσθαι δεῖ, καὶ ὧν θάτερον κατηγορεῖται, καὶ θάτερον κατηγορεῖσθαι δεῖ."

[103] "Moreover, see whether the one can exist without the other; for, if so, they will not be the same [Ἔτι εἰ δυνατὸν θάτερον ἄνευ θατέρου εἶναι· οὐ γὰρ ἂν εἴη ταὐτόν]".

only then identical if the same terms are predicated of them ($(qA)C$ and $(qB)C$) *and* they are predicated of the same terms ($(qC)A$ and $(qC)B$).

3 A Fregean Approach

3.1 Background

Aristotle's logical system remained the dominant system until Gottlob Frege developed his *Begriffsschrift* (1879). That does not mean, though, that the syllogistic did not undergo *any* changes at all. One notable change is the treatment of particulars as terms in a way analogous to other terms (see Parkinson's introduction in Leibniz 1966). Given the formalism from Section 2.9, a model $\mathfrak{M}_{\mathsf{A}}$ interprets such terms A as $\|A\|_{\mathfrak{M}_{\mathsf{A}}} = \{a\}$ for an $a \in D$. Thus, for any term B, $(\exists A)B \models_{\mathsf{A}} (\forall A)B$, i.e., the **i**-type sentence implies the **a**-type sentence. And, as already encoded in the square of opposition (Section 2.8), the latter also implies the former. For example, if $a \in D$ is Socrates and we blur the line between predicates and individuals, then 'some Socrates is human' implies 'every Socrates is human', and vice versa.[104]

However, Aristotelian syllogistic is limited in its expressive power. In particular, two limitations are generally pointed out, viz., Aristotelian syllogistic does not know *relational* terms, and, based on this, cannot deal with several quantificational phrases (see, e.g., Frege 1879, Carnap 1930/31/59, Russell 1946/2004: ch. 22, Kneale and Kneale 1962: 31, 487, and Link 2009: 10).

The main limitation is the syllogistic's restriction to terms which we can take to correspond to unary predicates so that it cannot account for *relations*. According to the *ti kata tinos*, the basic structure of sentences is subject-predicate. This means that the syllogistic cannot—in its current form—account for relational statements such as 'point a lies between point b and point c'. Frege overcomes this limitation by replacing the "concepts *subject* and *predicate* by *argument* and *function*" (1879: 7, his emphases). Of course, the most basic structure is still that of subject-predicate and is captured by a function applying to an argument—something that presupposes individual-constants that are not included in the syllogistic as presented above—but that immediately generalizes once the function is allowed to take more than one argument. Moreover, the subject-predicate structure is broken up once we consider sentences within the range of application of the syllogistic; for example, a sentence like 'all human beings are mortal' is not taken to have 'all human beings' or 'human beings' as its subject (depending on how one understands the quantity indicated by 'all'), but is analysed in terms for *quantifiers*, *variables*, *connectives*, and *functions* applying to *arguments*.

[104] As I've mostly treated terms as plural, it would be better to say 'some/every Socrates *are* human'.

The other limitation is what might be called *nested quantification* (aka *multiply general propositions*). The subject-predicate structure does not rely on quantifiers, but the quantity of its subject is somehow indicated; Aristotle specifies it explicitly by saying, e.g., 'let AB be a universal affirmative sentence', and the proposed formalism from Section 2.9 captures it by including a quantifier-symbol in front of the subject-term. Thus, the syllogistic can capture sentences like 'all human beings are mortal', but it lacks the means to express 'all human beings have someone they like' or 'some human beings like all human beings'. What's lacking is another way to even attach a quantity, and, as we have seen in Section 2.2, Aristotle does not think that sentences can be true if a quantity is assigned to more than the subject-term.

The Fregean approach with its function-argument analysis, on the other hand, has the means to assign quantities to several parts of sentences. Indeed, the *quantifiers* are treated as proper constituents of sentences. Only considering the first-order fragment, we can see that given arguments $a_1, \ldots, a_n$ and an n-ary function-symbol f, we can form a sentence '$f(a_1, \ldots, a_n)$' in which every argument-place allows to be quantified in. For example, '$\forall x_1 \exists x_2 f(x_1, x_2)$' is a sentence with nested quantifiers which can capture a sentence properly outside of Aristotelian syllogistic.

Overall, Frege captures a sentence like 'all human beings are mortal' as consisting of a *quantifier* ('$\forall$') binding a *variable* ('x') and acting on a *complex formula* with a *conditional* ('$\rightarrow$') as its main connective whose antecedent and consequent are *functions* applied to an *argument* ('$H(x)$', '$M(x)$'). None of these explicitly appears in the original sentence, and Frege is well aware that his formal language departs from ordinary language (1879: 6); he thinks he introduces a tool for "certain scientific purposes" (1879: 6), comparing it to the introduction of a microscope to better the human eye. Similarly, Carnap compares natural language to a "crude, primitive pocketknife" which is "useful for a hundred different purposes" (1963: 938), but not so much for specific purposes requiring greater precision. In this sense, we can—and will—understand the formal languages and their formalisms as *explications*.

One strength of the Fregean explication is that it allows for fairly simple solutions to the aforementioned limitations of Aristotelian syllogistic. For, as sentences are not forced to have subject-predicate structure, it is possible to allow relational predicates like 'x lies between y and z' ('$B(x, y, z)$'), and nest quantifiers. For example, we can render a sentence like 'every point a lies between some points b and c' as '$\forall x \exists y \exists z (B(x, y, z))$'.

3.2 A Formalism

To make the approach formally precise and to have a basis for comparison, let me introduce a convenient formalism (which more or less follows

Raab 2016: ch. 3). It should be clear that the following exposition is not following *Frege* in any detail, and is geared toward better comparison between the different formalisms to be introduced in the following and the one introduced in Section 2.9. Nevertheless, the following can rightly be claimed to expose, or explicate, a *Fregean* formalism.

The following exposition is not entirely standard, though it does not deviate much from a standard exposition. Insofar as it deviates, it is geared towards running in parallel with the exposition of the QUARC (Section 4.2). As I explain much of what's going on here already, I can present the QUARC-formalism succinctly while just pointing out the QUARC-specific features.

We start by specifying the vocabulary of a Fregean language.

Definition 17 ***(Fregean Language)***
A *Fregean language* ($\mathcal{L}_\mathsf{F}$) consists of the following:

- a countably infinite set $\mathsf{Var}_{\mathcal{L}_\mathsf{F}} = \{v_0, v_1, v_2, \dots\}$ of (*individual*-) *variables*,
- a countable set $\mathsf{Const}_{\mathcal{L}_\mathsf{F}} = \{c_0, c_1, c_2, \dots\}$ of (*individual*-)*constants*,
- for every $n > 0$, a countable set $\mathsf{Pred}^n_{\mathcal{L}_\mathsf{F}} = \{P^n_0, P^n_1, P^n_2, \dots\}$ of *n-ary predicate-symbols*,
- the set of *logical symbols* including '$\neg$', '$\wedge$', '$\vee$', '$\rightarrow$', '$\leftrightarrow$', '$=$', '$\forall$', and '$\exists$', and
- the set of *auxiliary symbols* including '(', ')', and ','.

The sets are assumed to be disjoint. Let $\mathsf{Pred}_{\mathcal{L}_\mathsf{F}} = \bigcup_{n>0} \mathsf{Pred}^n_{\mathcal{L}_\mathsf{F}}$.

The basic vocabulary of a Fregean language $\mathcal{L}_\mathsf{F}$ is fairly standard and extends the language of Aristotelian Syllogistic $\mathcal{L}_\mathsf{A}$ in several ways. Firstly, $\mathcal{L}_\mathsf{F}$ contains what can be taken to correspond to $\mathcal{L}_\mathsf{A}$-terms, but also predicate-symbols of any arity. It also contains individual-constants and individual-variables, making it a first-order language. Lastly, $\mathcal{L}_\mathsf{F}$ has an additional logical symbol, viz., '$=$', and it lacks the term-negation '$-$'. Even though the languages overlap significantly (as we can consider $\mathcal{L}_\mathsf{A}$ to be a sublanguage of $\mathcal{L}_\mathsf{F}$), the formation rules for $\mathcal{L}_\mathsf{F}$ are significantly different from those of $\mathcal{L}_\mathsf{A}$.

Definition 18 ***($\mathcal{L}_\mathsf{F}$-Formula)***
Let $\mathcal{L}_\mathsf{F}$ be a Fregean language. The set of $\mathcal{L}_\mathsf{F}$-*formulas* ($\mathsf{Form}_{\mathcal{L}_\mathsf{F}}$) is recursively defined by:

(1) given $n > 0$ $\mathcal{L}_\mathsf{F}$-constants $c_1, \dots, c_n$ and $P \in \mathsf{Pred}^n_{\mathcal{L}_\mathsf{F}}$, $P(c_1, \dots, c_n) \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$;

(2) given $\mathcal{L}_\mathsf{F}$-constants c_1 and c_2, $(c_1 = c_2) \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$;

(3) if $\varphi \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$, then $\neg\varphi \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$;

(4) if $\varphi, \psi \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$ and $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $(\varphi \circ \psi) \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$;

(5) if $\varphi(c) \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$, $x \in \mathsf{Var}_{\mathcal{L}_\mathsf{F}}$, and $q \in \{\forall, \exists\}$, then $qx\varphi[x/c] \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$.

Definition 18, in contrast to Definition 3, introduces a recursion to generate all the formulas. Clause (1) captures the function-argument structure, viz., the elements of $\mathsf{Const}_{\mathcal{L}_\mathsf{F}}$ are the arguments to the functions contained in Pred_F. Additionally, the symbol '=' figures as binary function/predicate. As usual, we can consider the formulas obtained by clauses (1)–(2) to be *atomic* and the constituents of *complex* formulas arrived by the remaining clauses.

Clause (5) allows for nested quantification for which clauses (1)–(2) provide the places. For example, for $P \in \mathsf{Pred}^2_{\mathcal{L}_\mathsf{F}}$ and $c_1, c_2 \in \mathsf{Const}_{\mathcal{L}_\mathsf{F}}$, clause (1) guarantees that $P(c_1, c_2) \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$. Applying clause (5) to it, $x \in \mathsf{Var}_{\mathcal{L}_\mathsf{F}}$ and $\forall$, leads to $\forall x\varphi(x, c_2) \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$. Applying it again to this, $y \in \mathsf{Var}_{\mathcal{L}_\mathsf{F}}$ and $\exists$, we get $\exists y\forall xP(x, y) \in \mathsf{Form}_{\mathcal{L}_\mathsf{F}}$.

Definition 18 is non-standard insofar as it does not allow for *open* formulas. Clause (5) is the only clause introducing *variables*, and those variables are bound. Because of this and in order to have a better comparable formalism, we understand quantification as *substitutional* and treat it accordingly below.

Given this increase in complexity, the interpretations of such Fregean languages have to be more complex, too, though the underlying model remains the same; we just make more use of it.

Definition 19 ***($\mathcal{L}_\mathsf{F}$-Model)***
Let $\mathcal{L}_\mathsf{F}$ be a Fregean language. A *model for* $\mathcal{L}_\mathsf{F}$ ($\mathcal{L}_\mathsf{F}$-model) is an ordered pair $\mathfrak{M}_\mathsf{F} = \langle D, \|\cdot\|_{\mathfrak{M}_\mathsf{F}}\rangle$ such that

(1) D is a set (the *domain* of $\mathfrak{M}_\mathsf{F}$);

(2) $\|\cdot\|_{\mathfrak{M}_\mathsf{F}}$ is an *interpretation-function of* $\mathfrak{M}_\mathsf{F}$ such that
  (a) if $c \in \mathsf{Const}_{\mathcal{L}_\mathsf{F}}$, then $\|c\|_{\mathfrak{M}_\mathsf{F}} \in D$;
  (b) if $P \in \mathsf{Pred}^1_{\mathcal{L}_\mathsf{F}}$, then $\emptyset \neq \|P\|_{\mathfrak{M}_\mathsf{F}} \subseteq D$;
  (c) if $n > 1$ and $P \in \mathsf{Pred}^n_{\mathcal{L}_\mathsf{F}}$, then $\|P\|_{\mathfrak{M}_\mathsf{F}} \subseteq D^n$.

Clause (2b) forces unary predicates to be assigned a non-empty extension. This has been done in order for a smoother comparison with QUARC. Moreover, in this way we also generate better comparability to (non-empty) $\mathcal{L}_\mathsf{A}$-models.

Moreover, as QUARC relies on *substitutional quantification*, we understand it similarly here. Hence, in order to correctly interpret the formulas involving quantification, we need to make sure that the interpretation does not rely on the specific choice of $\mathsf{Const}_{\mathcal{L}_\mathsf{F}}$. In order to do so, we first expand the underlying language (Definition 20), enriching it with further individual-constants, and then making sure that the interpretation keeps up (Definition 21). With these in place, we can specify when a model satisfies a formula (Definition 22).

Definition 20 ***($\mathcal{L}_{\mathsf{F}}$-A-Expansion)***
Let $\mathcal{L}_{\mathsf{F}}$ be a Fregean language. Let $\mathfrak{M}_{\mathsf{F}} = \langle D, \|\cdot\|_{\mathfrak{M}_{\mathsf{F}}} \rangle$ be an $\mathcal{L}_{\mathsf{F}}$-model. Let $A \subseteq D$. The *$\mathcal{L}_{\mathsf{F}}$-A-expansion of* $\mathcal{L}_{\mathsf{F}}$ is the language $\mathcal{L}'_{\mathsf{F}} \coloneqq \mathcal{L}_{\mathsf{F}} \cup \{c_a | a \in A\}$ where the c_a are new (individual-)constants not contained in $\mathcal{L}_{\mathsf{F}}$.

If $A = \{a\}$, we call $\mathcal{L}'_{\mathsf{F}}$ an $\mathcal{L}_{\mathsf{F}}$-a-expansion.

The idea is that we consider part of the domain of a model $\mathfrak{M}_{\mathsf{F}}$ and introduce new names for the elements of the chosen part. The new symbols need to be interpreted in the correct way too, which cannot be done in the original model $\mathfrak{M}_{\mathsf{F}}$ so that we have to expand it to $\mathfrak{M}'_{\mathsf{F}}$ in the following way.

Definition 21 ***($\mathcal{L}_{\mathsf{F}}$-Model Expansion)***
Let $\mathcal{L}_{\mathsf{F}}$ be a Fregean language and $\mathfrak{M}_{\mathsf{F}} = \langle D, \|\cdot\|_{\mathfrak{M}_{\mathsf{F}}} \rangle$ an $\mathcal{L}_{\mathsf{F}}$-model. Let $A \subseteq D$, and $\mathcal{L}'_{\mathsf{F}}$ be an $\mathcal{L}_{\mathsf{F}}$-A-expansion. The *A-expansion of* $\mathfrak{M}_{\mathsf{F}}$ *to* $\mathcal{L}'_{\mathsf{F}}$ is the model $\mathfrak{M}'_{\mathsf{F}} = \langle D', \|\cdot\|_{\mathfrak{M}'_{\mathsf{F}}} \rangle$ such that

(1) $D' = D$;

(2) $\|\cdot\|_{\mathfrak{M}_{\mathsf{F}}} \subseteq \|\cdot\|_{\mathfrak{M}'_{\mathsf{F}}}$;

(3) $\|c_a\|_{\mathfrak{M}'_{\mathsf{F}}} = a \in A$ for every new symbol c_a.

The domains of the model $\mathfrak{M}_{\mathsf{F}}$ and its expansion $\mathfrak{M}'_{\mathsf{F}}$ are the same. The new constants are interpreted according to how they have been introduced. Since $A \subseteq D$ and $D' = D$, $A \subseteq D'$, and the new symbols c_a for $a \in A$ just provide names for the elements $a \in D$.

Lastly, the expanded interpretation-function $\|\cdot\|_{\mathfrak{M}'_{\mathsf{F}}}$ extends the interpretation-function $\|\cdot\|_{\mathfrak{M}_{\mathsf{F}}}$, i.e., it leaves unaltered the interpretations of the original model $\mathfrak{M}_{\mathsf{F}}$. In particular, suppose that $\|P\|_{\mathfrak{M}_{\mathsf{F}}} = \{a\}$ for $a \in D$, but there is no $c \in \mathsf{Const}_{\mathcal{L}_{\mathsf{F}}}$ such that $\|c\|_{\mathfrak{M}_{\mathsf{F}}} = a$. We can then expand the language to $\mathcal{L}'_{\mathsf{F}}$ to include $c_a \in \mathsf{Const}_{\mathcal{L}'_{\mathsf{F}}}$ without altering $\|P\|_{\mathfrak{M}_{\mathsf{F}}}$; all that the expansion does is give a name to a (potentially) unnamed object without altering the interpretation of the $P \in \mathsf{Pred}_{\mathcal{L}_{\mathsf{F}}}$.

With this machinery, we can define the corresponding satisfaction-relation. It also suffices to expand the language by *one* individual-constant at a time as we quantify over *all* such expansions so that no element of D gets missed.

Definition 22 ***(Satisfaction $\models_{\mathsf{F}}$)***
Let the *Fregean satisfaction-relation* $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \varphi$ for $\varphi \in \mathsf{Form}_{\mathcal{L}_{\mathsf{F}}}$ and $\mathcal{L}_{\mathsf{F}}$-model $\mathfrak{M}_{\mathsf{F}} = \langle D, \|\cdot\|_{\mathfrak{M}_{\mathsf{F}}} \rangle$ be recursively defined as follows:

(1) $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} P(c_1, \ldots, c_n)$ iff $\langle \|c_1\|_{\mathfrak{M}_{\mathsf{F}}}, \ldots, \|c_n\|_{\mathfrak{M}_{\mathsf{F}}} \rangle \in \|P\|_{\mathfrak{M}_{\mathsf{F}}}$;

(2) $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} c_1 = c_2$ iff $\|c_1\|_{\mathfrak{M}_{\mathsf{F}}} = \|c_2\|_{\mathfrak{M}_{\mathsf{F}}}$;

(3) $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \neg\varphi$ iff it is not the case that $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \varphi$ ($\mathfrak{M}_{\mathsf{F}} \not\models_{\mathsf{F}} \varphi$);

(4) $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \varphi \wedge \psi$ iff $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \varphi$ and $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \psi$;

(5) $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \exists x\varphi[x]$ iff for some a-expansion $\mathfrak{M}'_{\mathsf{F}}$ of $\mathfrak{M}_{\mathsf{F}}$, $\mathfrak{M}'_{\mathsf{F}} \models_{\mathsf{F}} \varphi[c_a/x]$;

(6) $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \forall x\varphi[x]$ iff for all a-expansions $\mathfrak{M}'_{\mathsf{F}}$ of $\mathfrak{M}_{\mathsf{F}}$, $\mathfrak{M}'_{\mathsf{F}} \models_{\mathsf{F}} \varphi[c_a/x]$.

The definition is mostly standard. Given clauses (3)–(4), we can define the clauses for the remaining connectives in the usual way. In contrast to *objectual quantification* which interprets the quantifiers via *variable assignments*, here the quantifiers are interpreted *substitutionally*; instead of considering all the possible values for the variables, the base model $\mathfrak{M}_{\mathsf{F}}$ satisfies a formula of the form '$\forall x\varphi$' if all expansions $\mathfrak{M}'_{\mathsf{F}}$ satisfy '$\varphi[c_a]$' where the new constants 'c_a' are substituted for the variable 'x'. By Definitions 20–21, every element of the domain D is considered so that the truth of '$\forall x\varphi$' does *not* depend on the particular choice of $\mathsf{Const}_{\mathcal{L}_{\mathsf{F}}}$.

We can define a corresponding notion of *logical consequence* analogous to Definition 8; just substitute '$\mathcal{L}_{\mathsf{F}}$' for '$\mathcal{L}_{\mathsf{A}}$', '$\mathfrak{M}_{\mathsf{F}}$' for '$\mathfrak{M}_{\Vdash}$', and '$\models_{\mathsf{F}}$' for '$\Vdash$'. With that at hand, one peculiarity of the above is the following.

Theorem 23
Let $P \in \mathsf{Pred}^1_{\mathcal{L}_{\mathsf{F}}}$. Then: $\models_{\mathsf{F}} \exists x P(x)$.

Proof. Let $P \in \mathsf{Pred}^1_{\mathcal{L}_{\mathsf{F}}}$. Let $\mathfrak{M}_{\mathsf{F}} = \langle D, \|\cdot\|_{\mathfrak{M}_{\mathsf{F}}}\rangle$ be an $\mathcal{L}_{\mathsf{F}}$-model. By Definition 19 (2b), $\emptyset \neq \|P\|_{\mathfrak{M}_{\mathsf{F}}} \subseteq D$. Let $a \in \|P\|_{\mathfrak{M}_{\mathsf{F}}}$. Let $\mathcal{L}'_{\mathsf{F}}$ be an $\mathcal{L}_{\mathsf{F}}$-a-expansion of $\mathcal{L}_{\mathsf{F}}$, and $\mathfrak{M}'_{\mathsf{F}}$ be an a-expansion of $\mathfrak{M}_{\mathsf{F}}$ to $\mathcal{L}'_{\mathsf{F}}$. By Definition 21 (2), $\|P\|_{\mathfrak{M}_{\mathsf{F}}} \subseteq \|P\|_{\mathfrak{M}'_{\mathsf{F}}}$ so that $a \in \|P\|_{\mathfrak{M}'_{\mathsf{F}}}$. By Definition 21 (3), $\|c_a\|_{\mathfrak{M}'_{\mathsf{F}}} = a \in \|P\|_{\mathfrak{M}'_{\mathsf{F}}}$. Thus, by Definition 22 (1), $\mathfrak{M}'_{\mathsf{F}} \models_{\mathsf{F}} P(c_a)$. Then, there is an a-expansion $\mathfrak{M}'_{\mathsf{F}}$ of $\mathfrak{M}_{\mathsf{F}}$, $\mathfrak{M}'_{\mathsf{F}} \models_{\mathsf{F}} P(c_a)$. Therefore, by Definition 22 (5), $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \exists x P(x)$. □

This also means that universal quantification implies the existential one.

Corollary 24
$\forall x P(x) \models_{\mathsf{F}} \exists x P(x)$.

We can also note that the quantifiers behave as expected.

Theorem 25
The following equivalences hold:

(1) $\models_{\mathsf{F}} \forall x\varphi \leftrightarrow \neg\exists x\neg\varphi$;

(2) $\models_{\mathsf{F}} \exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$;

(3) $\models_{\mathsf{F}} \neg\exists x\varphi \leftrightarrow \forall x\neg\varphi$;

(4) $\models_{\mathsf{F}} \neg\forall x\varphi \leftrightarrow \exists x\neg\varphi$.

Furthermore, because of the non-emptiness requirement in Definition 19 (2b), analogues of conversion hold.

Theorem 26 ***(Conversion)***
The following conversions hold:

($\underline{\mathsf{a}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{F}}}$) $\forall x(A(x) \rightarrow B(x)) \models_{\mathsf{F}} \exists x(B(x) \wedge A(x))$

($\underline{\mathsf{i}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{F}}}$) $\exists x(A(x) \wedge B(x)) \models_{\mathsf{F}} \exists x(B(x) \wedge A(x))$

($\underline{\mathsf{e}}$-$\underline{\mathsf{e}}$-conv$^{\models_{\mathsf{F}}}$) $\forall x(A(x) \rightarrow \neg B(x)) \models_{\mathsf{F}} \forall x(B(x) \rightarrow \neg A(x))$

Proof. I only show the interesting case.

($\underline{\mathsf{a}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{F}}}$): Let $\mathfrak{M}_{\mathsf{F}}$ be an $\mathcal{L}_{\mathsf{F}}$-model such that $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \forall x(A(x) \rightarrow B(x))$. Then, by Definition 22 (6), for all a-expansions $\mathfrak{M}'_{\mathsf{F}}$ of $\mathfrak{M}_{\mathsf{F}}$, $\mathfrak{M}'_{\mathsf{F}} \models_{\mathsf{F}} A(c_a) \rightarrow B(c_a)$. By Definition 19 (2b), $\emptyset \neq \|A\|_{\mathfrak{M}_{\mathsf{F}}}$. Let $a \in \|A\|_{\mathfrak{M}_{\mathsf{F}}}$. Then, for the a-expansion $\mathfrak{M}^*_{\mathsf{F}}$ of $\mathfrak{M}_{\mathsf{F}}$, $\mathfrak{M}^*_{\mathsf{F}} \models_{\mathsf{F}} A(c_a) \rightarrow B(c_a)$. By Definition 21 (2), $a \in \|A\|_{\mathfrak{M}_{\mathsf{F}}} \subseteq \|A\|_{\mathfrak{M}^*_{\mathsf{F}}}$, i.e., $a \in \|A\|_{\mathfrak{M}^*_{\mathsf{F}}}$. Thus, for the a-expansion $\mathfrak{M}^*_{\mathsf{F}}$ of $\mathfrak{M}_{\mathsf{F}}$, $\mathfrak{M}^*_{\mathsf{F}} \models_{\mathsf{F}} A(c_a)$. Also for the a-expansion $\mathfrak{M}^*_{\mathsf{F}}$ of $\mathfrak{M}_{\mathsf{F}}$, $\mathfrak{M}^*_{\mathsf{F}} \models_{\mathsf{F}} A(c_a) \rightarrow B(c_a)$. Therefore, for the a-expansion $\mathfrak{M}^*_{\mathsf{F}}$ of $\mathfrak{M}_{\mathsf{F}}$, $\mathfrak{M}^*_{\mathsf{F}} \models_{\mathsf{F}} B(c_a)$ and, so, for some a-expansion $\mathfrak{M}'_{\mathsf{F}}$ of $\mathfrak{M}_{\mathsf{F}}$, $\mathfrak{M}'_{\mathsf{F}} \models_{\mathsf{F}} B(c_a) \wedge A(c_a)$. By Definition 22 (5), $\mathfrak{M}_{\mathsf{F}} \models_{\mathsf{F}} \exists x(B(x) \wedge A(x))$.

□

This much suffices in terms of exposition of a Fregean language and its semantics. As we are only interested in semantics, there is no need to introduce a proof system.

4 Ben-Yami's QUARC

4.1 Background

In recent years, Hanoch Ben-Yami has introduced a novel logical system called the *QUantified ARgument Calculus* (QUARC). The underlying motivation is to find a formal system that captures more adequately the semantics of natural language.[105] Section 3.1 already suggests that Frege's main motivation is *not* to come up with a formal language to capture the semantics of natural language; however, the elegance and strength of his formal language surpassed anything else known and so was a natural candidate to be used outside its original intended range of application.

Ben-Yami introduces an early version of QUARC in his book *Logic & Natural Language* (2004)—which is the main focus of this brief exposition. Note, though, that certain of Ben-Yami's views have developed and changed since the book was published in 2004; my concern here is not to

[105] Hanoch prefers '*logic* of natural language' (personal communication). I stick to 'semantics': Where it is clear to me that natural language has a semantics (and, potentially, several), it is less clear to me that it has a *logic*. My views are not settled, but am inclined to deny that there is *the* logic of natural language.

paint an accurate picture of his current views (for some of those see Yin and Ben-Yami 2023), though I mention some in footnotes.

Since then, he published an article exposing QUARC (2014), and considered how it treats the Barcan formulas and necessary existence (2020a) as well as how QUARC compares to natural logic (2020b). There have also been discussions with respect to *generalized quantifiers* (Ben-Yami 2009, 2012, and Westerståhl 2012).

Moreover, QUARC's logical properties have been investigated. Lanzet and Ben-Yami (2004) provide an early assessment in model-theoretic terms, Raab (2016, ms) consider QUARC's relationship to classical logic and so does Lanzet (2017) in a three-valued setting. There are completeness results for the QUARC in different settings (e.g., Lanzet and Ben-Yami 2004, Raab 2016, Ben-Yami and Pavlović 2022), and treatments based on many-valued truth-valuational semantics (Yin and Ben-Yami 2023).

QUARC has also been investigated proof-theoretically (Pavlović 2017, Pavlović and Gratzl 2019a, 2019b, 2021a) as well as axiomatically (Pascucci 2023). Moreover, Pavlović and Gratzl also consider abstract forms of quantification within QUARC (2023a) and investigate into decidable fragments (2023b). Several further aspects of QUARC are currently investigated.

Ben-Yami (2004) rejects Fregean languages when investigating the semantics of natural language. He suggests two main reasons, viz., the treatment of *reference* and *quantification*.[106] I do not go into detail with all the subtleties, but focus on some general points.

Regarding reference, Ben-Yami notes that natural language contains *plural* referring expressions. Fregean languages, on the other hand, only allow *singular* reference. In the Fregean languages, this is achieved solely via the *variables* and *individual-constants*. Thus, as detailed in Section 3.1, a sentence like 'All human beings are mortal' is captured as '$\forall x(H(x) \rightarrow M(x))$', quantifying *singularly* over everything. However, the surface structure of the sentence sees 'all human beings' as the subject of the sentence and 'human beings' refers *plurally* to *human beings* while 'all' specifies the relevant quantity of what's being referred to.

Based on the treatment of reference as singular, Ben-Yami (2004: 2) also argues that Fregean languages misconstrue *predication* and *quantification* in natural language. Ben-Yami (2004: 8) suggests that Fregean languages understand singular terms to be the *sole* source of reference, and common nouns as logical predicates. Ben-Yami (2004: 8), on the other hand, argues that common nouns are used to refer to (pluralities of) particulars too. Given this understanding, he claims to arrive, among others, at a "radically different analysis of quantification" (2004: 12).

Ben-Yami's main point is that "quantification involves reference to a

[106]Hanoch's current views changed regarding *reference* which dropped out of the picture; indeed, he insists that the notion of reference is irrelevant for QUARC (personal communication).

plurality" (2004: 59). In the example sentence above, 'human beings' refers to a plurality of *human beings*, and the quantifier 'all' specifies how much of that plurality is relevant, i.e., a "quantifier is attached to a noun that is used to refer to a plurality" (2004: 59f.) and these elements together "form a noun phrase" (2004: 60). Such noun phrases—called *quantified arguments*—can function as subjects of sentences; they can be put in the argument places of predicates. In the example sentence, 'mortal' is predicated of 'all human beings', i.e., the quantified argument 'all human beings' is put into the argument-slot of the predicate 'mortal'. Ben-Yami (2004: 62) claims that this is in agreement with Aristotle's understanding of predication.

One topic of concern for Ben-Yami is that of the *expressive power* of systems. Ben-Yami (2004: 78) notes that Aristotelian logic is *not* expressive enough to handle relations and nested quantification. His goal is to develop a system that is able to handle these, and suggests that "[a]ny alternative logic should have comparable power" (2004: 78) to Fregean logic with its predicate calculus. The QUARC is meant to have that.

In order to achieve that, QUARC needs a device to establish *cross-reference*; it captures it by the incorporation of *anaphora*. Moreover, natural language contains *active* and *passive* constructions, and different ways of negating, viz., negating a whole sentence ('*it is not the case that* Socrates is mortal') and *negative predication* ('Socrates *is not* mortal'); all this is incorporated in the QUARC too. The QUARC also includes *identity*, though it treats it slightly different from the way it is in Fregean languages, as predication is understood differently (Ben-Yami 2004: 142). All these elements are incorporated in the formalism below.

4.2 A Formalism

Let me make the QUARC formalism precise. I generally follow the exposition of Section 3.2, and comment only on the QUARC-specific details of the formalism.

First, again, let's specify the underlying vocabulary.

Definition 27 ***(QUARC-Language)***
A *QUARC-language* ($\mathcal{L}_{\mathsf{Q}}$) consists of the following:

- a countably infinite set $\mathsf{Ana}_{\mathcal{L}_{\mathsf{Q}}} = \{\alpha_0, \alpha_1, \alpha_2, \dots\}$ of *anaphors*,
- a countable set $\mathsf{SA}_{\mathcal{L}_{\mathsf{Q}}} = \{s_0, s_1, s_2, \dots\}$ of *singular arguments*,
- for every $n > 0$, a countable set $\mathsf{Pred}^n_{\mathcal{L}_{\mathsf{Q}}} = \{P_0^{1,\dots,n}, P_1^{1,\dots,n}, P_2^{1,\dots,n}, \dots\}$ of *n-ary predicate-symbols*,
- for every $n > 0$, for every $i \geq 0$, for every $P_i^{1,\dots,n} \in \mathsf{Pred}^n_{\mathcal{L}_{\mathsf{Q}}}$, a set $\mathsf{Reord}^n_{\mathcal{L}_{\mathsf{Q}}} = \{P_i^{\pi(1),\dots,\pi(n)} | \pi\colon \{1,\dots,n\} \to \{1,\dots,n\}$ a permutation$\}$ of *n-ary reorders*,

- the set of *logical symbols* including '¬', '∧', '∨', '→', '↔', '=', '∀', and '∃', and
- the set of *auxiliary symbols* including '(', ')', and ','.

For every $n \geq 1$, $\mathsf{Pred}^n_{\mathcal{L}_\mathsf{Q}} \subseteq \mathsf{Reord}^n_{\mathcal{L}_\mathsf{Q}}$; all other sets are assumed to be disjoint. Let $\mathsf{Pred}_{\mathcal{L}_\mathsf{Q}} := \bigcup_{n>0} \mathsf{Pred}^n_{\mathcal{L}_\mathsf{Q}}$ and $\mathsf{Reord}_{\mathcal{L}_\mathsf{Q}} := \bigcup_{n>0} \mathsf{Reord}^n_{\mathcal{L}_\mathsf{Q}}$.

Compared to Definition 17, Definition 27 is more complex. Firstly, what's analogous to the Fregean language, a QUARC-language contains *anaphors* which play a similar role to the *variables* of Fregean languages. However, Fregean languages need variables to achieve quantification, QUARC does not as witnessed by formulas of the form '$(\forall P)Q$' for $P, Q \in \mathsf{Pred}^1_{\mathcal{L}_\mathsf{Q}}$. Moreover, the QUARC-language contains *singular arguments* which correspond to Fregean (individual-)constants. The logical and auxiliary symbols of the languages are the same. However, QUARC specifies its predicates differently. In particular, I put the members of $\mathsf{Pred}^n_{\mathcal{L}_\mathsf{Q}}$ as '$P_i^{1,\ldots,n}$', not just indicating the arity n, but also the order of the slots. The reason for this is that this guarantees that they are identical to *reorders*. For any $P \in \mathsf{Pred}^n_{\mathcal{L}_\mathsf{Q}}$, there are $n!$-many reorders, generated by permutations on the predicate's argument-places. However, I just write 'P^π' instead of '$P^{\pi(1),\ldots,\pi(n)}$' ($P \in \mathsf{Pred}^n_{\mathcal{L}_\mathsf{Q}}$) to indicate the reorder if it is relevant.[107] Since $\mathsf{Pred}^n_{\mathcal{L}_\mathsf{Q}} \subseteq \mathsf{Reord}^n_{\mathcal{L}_\mathsf{Q}}$, we can often work with the latter in setting up the formalism; this helps reducing some complexity in specifying the QUARC-formulas.

Definition 28 ***($\mathcal{L}_\mathsf{Q}$-Formula)***
Let $\mathcal{L}_\mathsf{Q}$ be a QUARC-language. The set of $\mathcal{L}_\mathsf{Q}$*-formulas* ($\mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$) is recursively defined by:

(1) given $n > 0$ $s_1, \ldots, s_n \in \mathsf{SA}_{\mathcal{L}_\mathsf{Q}}$ and $P \in \mathsf{Reord}^n_{\mathcal{L}_\mathsf{Q}}$, then $(s_1, \ldots, s_n)P \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$;

(2) given $s_1, s_2 \in \mathsf{SA}_{\mathcal{L}_\mathsf{Q}}$, $(s_1, s_2) =\in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$ (usually written as '$(s_1 = s_2)$');

(3) given $n > 0$, $s_1, \ldots, s_n \in \mathsf{SA}_{\mathcal{L}_\mathsf{Q}}$, $P \in \mathsf{Reord}^n_{\mathcal{L}_\mathsf{Q}}$, and $*$ a string of negation-symbols ¬, $((s_1, \ldots, s_n) * P) \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$;

(4) if $\varphi \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$, then $\neg\varphi \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$;

(5) if $\varphi, \psi \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$ and $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $(\varphi \circ \psi) \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$;

(6) if $\varphi \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$ contains, from left to right, $s_1, \ldots, s_n$ ($n \geq 2$) occurrences of $s \in \mathsf{SA}_{\mathcal{L}_\mathsf{Q}}$, none of which is the source of $\beta \in \mathsf{Ana}_{\mathcal{L}_\mathsf{Q}}$ that occurs in φ, and φ does not contain $\alpha \in \mathsf{Ana}_{\mathcal{L}_\mathsf{Q}}$, then

[107] Hanoch (personal communication) prefers to think of π as an *operator* acting on predicates $P \in \mathsf{Pred}^n_{\mathcal{L}_\mathsf{Q}}$ so that the predicate stays the same, but gets reordered.

$\varphi[s_\alpha/s_1, \alpha/s_2, \ldots, \alpha/s_n] \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$ where $\varphi[s_\alpha/s_1, \alpha/s_2, \ldots, \alpha/s_n]$ is the result of substituting α for the occurrences $s_2, \ldots, s_n$ of s;

(7) if $\varphi[s] \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$, $q \in \{\forall, \exists\}$, $P \in \mathsf{Pred}^1_{\mathcal{L}_\mathsf{Q}}$, then $\varphi[qP/s] \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$ if qP *governs* φ (see Definition 30).

Let $\mathsf{QA}_{\mathcal{L}_\mathsf{Q}}$ be the set of *quantified arguments*, i.e., expressions of the form qP for $q \in \{\forall, \exists\}$ and $P \in \mathsf{Pred}^1_{\mathcal{L}_\mathsf{Q}}$.

Clauses (1)–(2) correspond to Definition 18's (1)–(2); the only difference is that QUARC takes predicates from $\mathsf{Reord}_{\mathcal{L}_\mathsf{Q}}$ and that we write the arguments *in front of* the predicate-symbol. Moreover, clauses (4)–(5) are standard too. Let me comment on the remaining clauses.

Clause (3) is QUARC-specific. It allows arbitrarily many negation-symbols inbetween a predicate-symbol's argument-slots and predicate-sign. Thus, we allow, e.g., '$(s)\neg\neg\neg P$' as an $\mathcal{L}_\mathsf{Q}$-formula.

Clause (6) allows for the introduction of *anaphors*. If a formula contains several occurrences of a singular argument s, we can replace all but the first by new anaphors. For example, we can move from '$(s, s)P$' to '$(s_\alpha, \alpha)P$'. As long as no quantified arguments are involved, these anaphors are not necessary, but they are once cross-reference is needed.

Clause (7), finally, allows the introduction of *quantified arguments*, i.e., expressions combining *quantifiers* with *unary predicates* so that quantification is understood as *plural*. These expressions can replace *singular* arguments given that they satisfy a certain condition, viz., that the quantified argument *governs* the formula—which we define below. As the quantified arguments can take the place of a singular argument that has anaphors referring to it, we also define the notion *source of anaphora*.

Definition 29 ***(Source of Anaphora)***
If an anaphor is introduced according to clause (6), then the term s *is the source of* α (indicated as 's_α') if it is the rightmost occurrence of s that is to the left of the anaphor α; if such a term is replaced by a $t \in \mathsf{QA}_{\mathcal{L}_\mathsf{Q}}$ due to an application of clause (7), then t *is the source of* α (indicated as 'qP_α' if $t = qP$).

Definition 30 ***(Governance)***
Let φ be a string of symbols and $t \in \mathsf{QA}_{\mathcal{L}_\mathsf{Q}}$. Then, t *governs* φ if it is the leftmost quantified argument and φ does not contain any other string of symbols ψ such that $\psi \in \mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$ contains t and all the anaphors of all arguments in ψ.

Given Definition 30, Definition 28 (7) is well-defined now. Roughly, the idea is that we can introduce *quantified arguments* if they are the *main symbol*, i.e., when breaking up the formula, one has to start with it.

As in Section 3.2, all formulas are *closed*. The mechanism to introduce anaphors and quantified expressions is via *substitution* and so we treat quantification *substitutionally.* This follows the treatment from Section 3.2. In particular, we use *models* to interpret QUARC-languages.

Definition 31 ***($\mathcal{L}_{\mathsf{Q}}$-Model)***
Let $\mathcal{L}_{\mathsf{Q}}$ be a QUARC-language. A *model for* $\mathcal{L}_{\mathsf{Q}}$ ($\mathcal{L}_{\mathsf{Q}}$-model) is an ordered pair $\mathfrak{M}_{\mathsf{Q}} = \langle D, \|\cdot\|_{\mathfrak{M}_{\mathsf{Q}}}\rangle$ such that

(1) D is a set (the *domain* of $\mathfrak{M}_{\mathsf{Q}}$);

(2) $\|\cdot\|_{\mathfrak{M}_{\mathsf{Q}}}$ is an *interpretation-function of* $\mathfrak{M}_{\mathsf{Q}}$ such that
   (a) if $s \in \mathsf{SA}_{\mathcal{L}_{\mathsf{Q}}}$, then $\|s\|_{\mathfrak{M}_{\mathsf{Q}}} \in D$;
   (b) if $P \in \mathsf{Pred}^1_{\mathcal{L}_{\mathsf{Q}}}$, then $\emptyset \neq \|P\|_{\mathfrak{M}_{\mathsf{Q}}} \subseteq D$;
   (c) if $n > 1$ and $P \in \mathsf{Pred}^n_{\mathcal{L}_{\mathsf{Q}}}$, then $\|P\|_{\mathfrak{M}_{\mathsf{Q}}} \subseteq D^n$;
   (d) if $n \geq 1$ and $P^\pi \in \mathsf{Reord}^n_{\mathcal{L}_{\mathsf{Q}}}$ for permutation $\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, $\|P^\pi\|_{\mathfrak{M}_{\mathsf{Q}}} =$
   $\{\langle \|s_{\pi(1)}\|_{\mathfrak{M}_{\mathsf{Q}}}, \ldots, \|s_{\pi(n)}\|_{\mathfrak{M}_{\mathsf{Q}}}\rangle | \langle \|s_1\|_{\mathfrak{M}_{\mathsf{Q}}}, \ldots, \|s_n\|_{\mathfrak{M}_{\mathsf{Q}}}\rangle \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}\}$.

This definition corresponds to Definition 19. The only QUARC-specific part is clause (2d) which interprets *reorders* in the obvious way. As a reorder $P^\pi \in \mathsf{Reord}^n_{\mathcal{L}_{\mathsf{Q}}}$ comes from reordering the argument-places of a predicate $P \in \mathsf{Pred}^n_{\mathcal{L}_{\mathsf{Q}}}$, the interpretation does the same.

As before, we do not want to be held hostage to the particular choice of what individuals the language can name, i.e., to the specific $\mathsf{SA}_{\mathcal{L}_{\mathsf{Q}}}$, so we expand the language (Definition 32), and specify the corresponding model expansions (Definition 33). With that, we can define the satisfaction-relation (Definition 34).

Definition 32 ***($\mathcal{L}_{\mathsf{Q}}$-A-Expansion)***
Let $\mathcal{L}_{\mathsf{Q}}$ be a QUARC-language and $\mathfrak{M}_{\mathsf{Q}} = \langle D, \|\cdot\|_{\mathsf{Q}}\rangle$ be an $\mathcal{L}_{\mathsf{Q}}$-model. Let $A \subseteq D$. The $\mathcal{L}_{\mathsf{Q}}$-*A-expansion of* $\mathcal{L}_{\mathsf{Q}}$ is the language $\mathcal{L}'_{\mathsf{Q}} \coloneqq \mathcal{L}_{\mathsf{Q}} \cup \{s_a | a \in A\}$ where the s_a are new singular arguments not contained in $\mathcal{L}_{\mathsf{Q}}$.

If $A = \{a\}$, we call $\mathcal{L}'_{\mathsf{Q}}$ an $\mathcal{L}_{\mathsf{Q}}$-*a-expansion.*

Definition 33 ***($\mathcal{L}_{\mathsf{Q}}$-Model Expansion)***
Let $\mathcal{L}_{\mathsf{Q}}$ be a QUARC-language and $\mathfrak{M}_{\mathsf{Q}} = \langle D, \|\cdot\|_{\mathfrak{M}_{\mathsf{Q}}}\rangle$ an $\mathcal{L}_{\mathsf{Q}}$-model. Let $A \subseteq D$, and $\mathcal{L}'_{\mathsf{Q}}$ be an $\mathcal{L}_{\mathsf{Q}}$-A-expansion. The *A-expansion of* $\mathfrak{M}_{\mathsf{Q}}$ *to* $\mathcal{L}'_{\mathsf{Q}}$ is the model $\mathfrak{M}'_{\mathsf{Q}} = \langle D', \|\cdot\|_{\mathfrak{M}'_{\mathsf{Q}}}\rangle$ such that

(1) $D' = D$;

(2) $\|\cdot\|_{\mathfrak{M}_{\mathsf{Q}}} \subseteq \|\cdot\|_{\mathfrak{M}'_{\mathsf{Q}}}$;

(3) $\|s_a\|_{\mathfrak{M}'_{\mathsf{Q}}} = a \in A$ for every new singular argument s_a.

Definition 34 ***(Satisfaction $\models_{\mathsf{Q}}$)***
Let the *QUARC satisfaction-relation* $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \varphi$ for $\varphi \in \mathsf{Form}_{\mathcal{L}_{\mathsf{Q}}}$ and $\mathcal{L}_{\mathsf{Q}}$-model $\mathfrak{M}_{\mathsf{Q}} = \langle D, \|\cdot\|_{\mathfrak{M}_{\mathsf{Q}}}\rangle$ be recursively defined as follows:

(1) $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} (s_1, \ldots, s_n)P$ iff $\langle \|s_1\|_{\mathfrak{M}_{\mathsf{Q}}}, \ldots, \|s_n\|_{\mathfrak{M}_{\mathsf{Q}}} \rangle \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$ ($P \in \mathsf{Reord}^n_{\mathcal{L}_{\mathsf{Q}}}$);

(2) $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} s_1 = s_2$ iff $\|s_1\|_{\mathfrak{M}_{\mathsf{Q}}} = \|s_2\|_{\mathfrak{M}_{\mathsf{Q}}}$;

(3) $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \neg\varphi$ iff it is not the case that $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \varphi$ ($\mathfrak{M}_{\mathsf{Q}} \not\models_{\mathsf{Q}} \varphi$);

(4) $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \varphi \wedge \psi$ iff $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \varphi$ and $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \psi$;

(5) $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} *((s_1, \ldots, s_n)\neg *' P)$ iff $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \neg * ((s_1, \ldots, s_n) *' P)$ (where $*, *'$ are possibly empty strings of negation-symbols $\neg$);

(6) $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \varphi[s_\alpha/s_1, \alpha/s_2, \ldots, \alpha/s_n]$ iff $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \varphi$;

(7) $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \varphi[\exists P_\alpha]$ iff for some a-expansion $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} \varphi[(s_a)_\alpha/\exists P_\alpha]$ ($\exists P$ governs φ and is the source of $\alpha \in \mathsf{Ana}_{\mathcal{L}_{\mathsf{Q}}}$ if there is one);

(8) $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \varphi[\forall P_\alpha]$ iff for all a-expansions $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} \varphi[(s_a)_\alpha/\forall P_\alpha]$ ($\forall P$ governs φ and is the source of $\alpha \in \mathsf{Ana}_{\mathcal{L}_{\mathsf{Q}}}$ if there is one).

Since the QUARC-models are pretty much the same as the Fregean-models, the satisfaction-relation is quite similar too. Clauses (1)–(4) correspond to Definition 22's (1)–(4). The remaining clauses are QUARC-specific.

Clause (5) concerns *predicate-negation*. As long as no quantified arguments occur in a formula, we just move the negation-symbols from the predicate-negation into sentence-negation; the resulting formulas are in the range of clause (3).

Clause (6) concerns anaphora. As the anaphors are just referring to whatever their source refers to, we interpret them accordingly. That is, as long as no quantified arguments occur, they refer to what their source singular argument refers. That is, a model satisfies it in exactly the same circumstances as when they are replaced by their source.

Clauses (7)–(8) concern the quantified arguments. The general idea is the same as it was in the case of Fregean languages, i.e., as specified in Definition 22 (5)–(6). However, as QUARC does not allow unrestricted quantification, we have to restrict the expansions in consonance with the *quantified argument*, consisting of a quantifier and unary predicate. Thus, instead of considering *some* or *all* a-expansions, we *only* consider those such that a is an element of the interpretation of the restricting unary predicate. If $qP \in \mathsf{QA}_{\mathcal{L}_{\mathsf{Q}}}$, we only consider those $a \in D$ such that $a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$, i.e., if for *some* (*all*) of these the expanded model $\mathfrak{M}'_{\mathsf{Q}}$ satisfies a formula φ, then the base model $\mathfrak{M}_{\mathsf{Q}}$ satisfies the formula involving the quantified argument $\exists P$ ($\forall P$), i.e., it satisfies that *some* P (*all* P) satisfy the formula.

Given the QUARC-language, its models, and the satisfaction-relation, we can define *logical consequence* etc. as in Definition 8, and obtain the QUARC-specific results below.

Theorem 35
$\models_{\mathsf{Q}} (\exists P)P$.

Proof. Let $\mathfrak{M}_{\mathsf{Q}} = \langle D, \|\cdot\|_{\mathfrak{M}_{\mathsf{Q}}}\rangle$ be an $\mathcal{L}_{\mathsf{Q}}$-model. By Definition 31 (2b), $\emptyset \neq \|P\|_{\mathfrak{M}_{\mathsf{Q}}} \subseteq D$ for $P \in \mathsf{Pred}^1_{\mathcal{L}_{\mathsf{Q}}}$. Let $a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$, and let $\mathcal{L}'_{\mathsf{Q}}$ be the $\mathcal{L}_{\mathsf{Q}}$-a-expansion of $\mathcal{L}_{\mathsf{Q}}$. Let $\mathfrak{M}'_{\mathsf{Q}}$ be the a-expansion of $\mathfrak{M}_{\mathsf{Q}}$ to $\mathcal{L}'_{\mathsf{Q}}$. Then, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} (s_a)P$ since $\|s_a\|_{\mathfrak{M}'_{\mathsf{Q}}} = a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}} \subseteq \|P\|_{\mathfrak{M}'_{\mathsf{Q}}}$ by Definition 33 (2)–(3). Therefore, by Definition 34 (7), $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} (\exists P)P$. □

Theorem 36
$(\forall P)Q \models_{\mathsf{Q}} (\exists P)Q$.

Proof. Let $\mathfrak{M}_{\mathsf{Q}}$ be an $\mathcal{L}_{\mathsf{Q}}$-model such that $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} (\forall P)Q$. By Definition 34 (8), for all a-expansions $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} (s_a)Q$. Moreover, by Definition 31 (2b), $\|P\|_{\mathfrak{M}_{\mathsf{Q}}} \neq \emptyset$. Thus, there is an a-expansion $\mathfrak{M}^*_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}^*_{\mathsf{Q}} \models_{\mathsf{Q}} (s_a)Q$. Then, by Definition 34 (7), $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} (\exists P)Q$. □

The quantifiers still behave as one would expect them to:

Theorem 37
The following equivalences hold:

(1) $\models_{\mathsf{Q}} (\forall P)S \leftrightarrow \neg((\exists P)\neg S)$; (2) $\models_{\mathsf{Q}} (\exists P)S \leftrightarrow \neg((\forall P)\neg S)$;

(3) $\models_{\mathsf{Q}} \neg(\exists P)S \leftrightarrow (\forall P)\neg S$; (4) $\models_{\mathsf{Q}} \neg(\forall P)S \leftrightarrow (\exists P)\neg S$.

Proof. I only illustrate part of one case:

(3): Let $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \neg(\exists P)S$. Then, by Definition 34 (3), $\mathfrak{M}_{\mathsf{Q}} \not\models_{\mathsf{Q}} (\exists P)S$, i.e., by (7), it is not the case that for some a-expansion $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} (s_a)S$ iff for all a-expansions $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \not\models_{\mathsf{Q}} (s_a)S$, i.e., by (3), for all a-expansions $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} \neg(s_a)S$, and so, by (5), for all a-expansions $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|P\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} (s_a)\neg S$. Thus, by (8), $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} (\forall P)\neg S$.

□

QUARC also validates the conversions.

Theorem 38 ***(Conversion)***
The following conversions hold:

($\underline{\mathbf{a}}$-$\underline{\mathbf{i}}$-conv$^{\models_{\mathsf{Q}}}$) $(\forall A)B \models_{\mathsf{Q}} (\exists B)A$

($\underline{\mathsf{i}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{Q}}}$) $(\exists A)B \models_{\mathsf{Q}} (\exists B)A$

($\underline{\mathsf{e}}$-$\underline{\mathsf{e}}$-conv$^{\models_{\mathsf{Q}}}$) $(\forall A)\neg B \models_{\mathsf{Q}} (\forall B)\neg A$

Proof. Let $\mathfrak{M}_{\mathsf{Q}}$ be an $\mathcal{L}_{\mathsf{Q}}$-model.

($\underline{\mathsf{a}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{Q}}}$): Follows from Theorem 36 and ($\underline{\mathsf{i}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{Q}}}$).

($\underline{\mathsf{i}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{Q}}}$): Let $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} (\exists A)B$. By Definition 31 (2b), $\|A\|_{\mathfrak{M}_{\mathsf{Q}}} \neq \emptyset \neq \|B\|_{\mathfrak{M}_{\mathsf{Q}}}$. Then, by Definition 34 (7), for some a-expansion $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|A\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} (s_a)B$, i.e., by Definition 34 (1), $\|s_a\|_{\mathfrak{M}'_{\mathsf{Q}}} \in \|B\|_{\mathfrak{M}'_{\mathsf{Q}}}$. Since, by Definition 33 (3), $\|s_a\|_{\mathfrak{M}'_{\mathsf{Q}}} = a$, it follows that $a \in \|B\|_{\mathfrak{M}'_{\mathsf{Q}}}$. By Definition 33 (2), $\|B\|_{\mathfrak{M}_{\mathsf{Q}}} = \|B\|_{\mathfrak{M}'_{\mathsf{Q}}}$ and $\|A\|_{\mathfrak{M}_{\mathsf{Q}}} = \|A\|_{\mathfrak{M}'_{\mathsf{Q}}}$. Thus, $a \in \|A\|_{\mathfrak{M}_{\mathsf{Q}}} \cap \|B\|_{\mathfrak{M}_{\mathsf{Q}}} = \|A\|_{\mathfrak{M}'_{\mathsf{Q}}} \cap \|B\|_{\mathfrak{M}'_{\mathsf{Q}}}$, so also $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} (s_a)A$. Overall, for some a-expansion $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|B\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} (s_a)A$, i.e., by Definition 34 (7), $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} (\exists B)A$.

($\underline{\mathsf{e}}$-$\underline{\mathsf{e}}$-conv$^{\models_{\mathsf{Q}}}$): Let $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} (\forall A)\neg B$. Thus, by Definitions 34 (8), (5), and (3), it is not the case that for some a-expansion $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|A\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} (s_a)B$.

Suppose that $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} (\exists B)A$. Then, by ($\underline{\mathsf{i}}$-$\underline{\mathsf{i}}$-conv$^{\models_{\mathsf{Q}}}$), $\mathfrak{M}_{\mathsf{Q}} \models (\exists A)B$, i.e., for some a-expansion $\mathfrak{M}'_{\mathsf{Q}}$ of $\mathfrak{M}_{\mathsf{Q}}$ such that $a \in \|A\|_{\mathfrak{M}_{\mathsf{Q}}}$, $\mathfrak{M}'_{\mathsf{Q}} \models_{\mathsf{Q}} (s_a)B$, a contradiction. Therefore, $\mathfrak{M}_{\mathsf{Q}} \not\models_{\mathsf{Q}} (\exists B)A$, i.e., $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} \neg(\exists B)A$. Thus, by Theorem 37 (3), $\mathfrak{M}_{\mathsf{Q}} \models_{\mathsf{Q}} (\forall B)\neg A$.

□

Note also that the semantics distinguishes only between even and odd numbers of predicate-negations:

Theorem 39
$\models_{\mathsf{Q}} ((s_1, \ldots, s_n)\neg\neg * P) \leftrightarrow ((s_1, \ldots, s_n) * P)$.

Theorem 39 generalizes to cases including quantified arguments. Applied repeatedly, we get that if '$*$' contains an even number of negation-symbols, then '$((s_1, \ldots, s_n) * P)$' is equivalent to '$(s_1, \ldots, s_n)P$', and if it contains an odd number, it is equivalent to '$((s_1, \ldots, s_n)\neg P)$', and so, by Definition 34 (5), to '$\neg(s_1, \ldots, s_n)P$'.

This finishes the exposition of QUARC.

5 Sommers's Term Logic

5.1 Background

Fred Sommers is also not satisfied with the common approach to the semantics of natural language. He develops his *Term Functor Logic* (TFL) as an alternative approach. In this brief exposition, I focus on his book *The Logic of Natural Language* (1982), and only consider a few points that suggest themselves for comparison here (for a nice exposition, see Englebretsen 2016).

Sommers's conviction is that

> traditional formal logic is especially suited to the task of making perspicuous the logical form of sentences in the natural languages that are actually used in deductive reasoning and that, in virtue of this, traditional logic provides models for the study of what actually happens when we reckon the premisses and arrive at conclusion. (1982: 4)

Given that we generally reason in natural language, traditional formal logic is in a better position to make explicit how we do so; Fregean languages, with their machinery, rather distort this. In this context, Sommers emphasizes that the

> traditional logician emphasized syntactic simplicity, requiring of a canonical sentence that it have a straightforward noun-phrase verb-phrase structure (or be a compound of such 'categorical' sentences). (1982: 9)

The simple noun-phrase verb-phrase structure can be found in Aristotle's logic, though needs to be extended to overcome the syllogistic's shortcomings. Indeed, Sommers is concerned in constructing a language that is similarly powerful as Fregean languages while maintaining the basic analysis of sentences.

The basic analysis is into noun-phrase and verb-phrase; both are considered to be *terms*. Additionally, the noun-phrase as well as all other subject expressions are assigned a *quantity* (1982: 67). The general form of a sentence is then 'every/some S is (are)/is (are) not P' where 'every/some' is the quantity of the subject S (e.g., 1982: 95).

In order to increase the expressive power, Sommers introduces *proterms* and allows *complex* terms. As in Aristotle's logic, terms can play the role of *both* subject and predicate in sentences (1982: 116). Moreover, Sommers also allows *n-ary* terms (1982: 139), construed in a way so that the subject-predicate structure remains via *nesting* them (1982: 148). As terms can play several roles, Sommers (1982: 116f.) argues that there is no need to include *identity* in the way Fregean languages do. This also means that TFL is more parsimonious than Fregean languages are with respect to their primitive symbols.

Overall, Sommers claims that his TFL, already in a more basic form which he calls 'Primitive Term Logic' (PTL), is

> roughly equivalent to that of a standard first-order logic whose logical particles consist of the existential quantifier and the signs for conjunction, negation and identity. (1982: 174)

He goes on to *amplify* PTL to full TFL. However, for the purposes of comparing the systems, I stick to the more basic system, though even depart from Sommers's presentation and particular claims regarding it. Moreover, I continue the model-theoretic approach which is significantly different from Sommers's algebraic treatment of term logic.

5.2 A Formalism

Definition 40 ***(TFL-Language)***
A *TFL-language* ($\mathcal{L}_\mathsf{T}$) consists of the following:

- a countably infinite set $\mathsf{PTerm}_{\mathcal{L}_\mathsf{T}} = \{\alpha_0, \alpha_1, \alpha_2, \dots\}$ of *proterms*,
- a countable set $\mathsf{ITerm}_{\mathcal{L}_\mathsf{T}} = \{t_0, t_1, t_2, \dots\}$ of *individual-terms*,
- for every $n > 0$, a countable set $\mathsf{STerm}^n_{\mathcal{L}_\mathsf{T}} = \{T^n_0, T^n_1, T^n_2, \dots\}$ of (*simple*) *n-ary term-symbols*,
- the set of *logical symbols* including '¬', '–', '∧', '∨', '→', '↔', '∀', and '∃', and
- the set of *auxiliary symbols* including '(', ')', and ','.

All the sets are assumed to be disjoint. Let $\mathsf{STerm}_{\mathcal{L}_\mathsf{T}} := \bigcup_{n>0} \mathsf{STerm}^n_{\mathcal{L}_\mathsf{T}} \cup \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$.

The TFL-language $\mathcal{L}_\mathsf{T}$ is different from the one Sommers actually uses, and changes certain aspects. What's left are *proterms* $\mathsf{PTerm}_{\mathcal{L}_\mathsf{T}}$ which play a similar role to Fregean variables and QUARC-anaphora. The language does not contain anything like individual-constants or singular arguments, but only terms. One kind of term are the *individual-terms* $\mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$—playing a similar role as individual-constants—another *n-ary terms* $\mathsf{STerm}^n_{\mathcal{L}_\mathsf{T}}$. Similar to the language $\mathcal{L}_\mathsf{A}$ and in contrast to $\mathcal{L}_\mathsf{F}$ and $\mathcal{L}_\mathsf{Q}$, $\mathcal{L}_\mathsf{T}$ does not include an identity-symbol '=' among its logical symbols, but includes a second negation-symbol '–' which figures in the introduction of complex terms.

One important difference to Definition 1 of $\mathcal{L}_\mathsf{A}$ is that $\mathcal{L}_\mathsf{T}$ includes *n-ary term*. These are necessary to capture relational predications that Aristotle's syllogistic misses.

Given this basic vocabulary, we can introduce the complex terms.

Definition 41 ***(Complex $\mathcal{L}_\mathsf{T}$-Terms)***
For each $n > 0$, the *set of complex n-ary $\mathcal{L}_\mathsf{T}$-terms* ($\mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$) is recursively defined as follows:

(1) if $t \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$, then $t \in \mathsf{CTerm}^1_{\mathcal{L}_\mathsf{T}}$;

(2) if $A \in \mathsf{STerm}^n_{\mathcal{L}_\mathsf{T}}$, then $A \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$;

(3) if $A \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$, then $\overline{A} \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$;

(4) if $A, B \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$, then $(A \circ B) \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$ ($\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$);

(5) if $1 \leq i \leq n-1$, $t_1, \ldots, t_i \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$, $q_1, \ldots, q_i \in \{\forall, \exists\}$, and $A \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$, then $(q_1 t_1, \ldots, q_i t_i, \_{i+1}, \ldots, \_n)A \in \mathsf{CTerm}^{n-i}_{\mathcal{L}_\mathsf{T}}$ and so are all ways of putting the i terms into the n slots of A (where '$\_k$' indicates the kth argument-slot of A, $1 \leq k \leq n$);

(6) if $A \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$ and π: $\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ is a permutation, then $A^\pi \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$ where 'A^π' is the result of permuting A's slots according to π.

Terms generated by clause (5) are called *n-ary reduced terms* ($\mathsf{RTerm}^n_{\mathcal{L}_\mathsf{T}}$). Let $\mathsf{RTerm}_{\mathcal{L}_\mathsf{T}} := \bigcup_{n>1} \mathsf{RTerm}^n_{\mathcal{L}_\mathsf{T}}$.

Clauses (2)–(4) are analogous to the clauses (1)–(3) of Definition 2 of $\mathsf{Term}_{\mathcal{L}_\mathsf{A}}$, just generalized from only *unary* terms to *n-ary* terms. These allow to capture relational predications in more complex settings. Moreover, clause (1) includes the individual terms among the *unary* complex terms.

Clause (5) additionally allows to form further terms, reducing an n-ary term A to an m-ary term B by filling up slots with elements from $\mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$. In the spirit of TFL, each term is assigned a *quantity*. However, as the particular quantity does not make a difference for the individual-terms, both '$\forall$' and '$\exists$' are allowed as quantities.

Note, too, that clause (5) also sticks to the QUARC convention to place the argument-places to the left of the term symbol.

Clause (6), finally, allows for *reordered* terms analogous to QUARC's reorders in Definition 27. The clause allows to reorder reorders, but it is clear that there are only $n!$-many different ones. For example, '$(\_1, \_2)A$' only leads to '$(\_2, \_1)A^\pi$' as $A^{\pi^\pi} = A$.

Given the vocabulary and the set of terms, we can define what counts as formula in a way mirroring Definition 28 of $\mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$.

Definition 42 ***($\mathcal{L}_\mathsf{T}$-Formula)***
Let $\mathcal{L}_\mathsf{T}$ be a TFL-language. The *set of $\mathcal{L}_\mathsf{T}$-formulas* ($\mathsf{Form}_{\mathcal{L}_\mathsf{T}}$) is recursively defined by:

(1) if $n \geq 1$, $A \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$, $t_1, \ldots, t_n \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$, $q_1, \ldots, q_n \in \{\forall, \exists\}$, and $*$ a possibly empty string of negation-symbols $\neg$, then $((q_1 t_1, \ldots, q_n t_n) * A) \in \mathsf{Form}_{\mathcal{L}_\mathsf{T}}$;

(2) if $\varphi \in \mathsf{Form}_{\mathcal{L}_\mathsf{T}}$, then $\neg\varphi \in \mathsf{Form}_{\mathcal{L}_\mathsf{T}}$;

(3) if $\varphi, \psi \in \mathsf{Form}_{\mathcal{L}_\mathsf{T}}$, then $(\varphi \circ \psi) \in \mathsf{Form}_{\mathcal{L}_\mathsf{T}}$ ($\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$);

(4) if $\varphi \in \mathsf{Form}_{\mathcal{L}_\mathsf{T}}$ contains, from left to right, $t_1, \ldots, t_m$ ($m \geq 2$) occurrences of $t \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$, none of which is the source of $\beta \in \mathsf{PTerm}_{\mathcal{L}_\mathsf{T}}$ that occurs in φ, and φ does not contain $\alpha \in \mathsf{PTerm}_{\mathcal{L}_\mathsf{T}}$, then $\varphi[t_\alpha/t_1, \alpha/t_2, \ldots, \alpha/t_m] \in \mathsf{Form}_{\mathcal{L}_\mathsf{T}}$ (which is the result of substituting α for the occurrences $t_2, \ldots, t_m$ of t);

(5) if $q \in \{\forall, \exists\}$, $t \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$, $\varphi[qt] \in \mathsf{Form}_{\mathcal{L}_\mathsf{T}}$, $A \in \mathsf{CTerm}^1_{\mathcal{L}_\mathsf{T}}$, then $\varphi[qA/qt] \in \mathsf{Form}_{\mathcal{L}_\mathsf{T}}$ if qA *governs* φ.

Definition 42 resembles Definition 28 which defines $\mathsf{Form}_{\mathcal{L}_\mathsf{Q}}$. Indeed, *governance* in clause (5) is to be understood analogous to Definition 30. Moreover, an analogue of Definition 29 applies to the proterms in clause (4) and once the $t \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$ gets substituted by $A \in \mathsf{CTerm}^1_{\mathcal{L}_\mathsf{T}}$. Also, we collapsed Definition 28 (1) and (3) into one clause (1).

What's TFL-specific in Definition 42 is the assignment of *quantities* to *all* terms. Thus, the basic formulas are n-ary terms applying to n *individual-terms* $t_i \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$ while assigning them a quantity, i.e., one of the quantifiers. As these terms are such that the particular quantifier does not make a difference, both are allowed. Definition 42 does not introduce *wild* quantities, but just assigns *both* quantities and the rest will be taken care by the interpretation.

Moreover, we allow for the usual combination of sentences via clauses (2)–(3). Terms for which the quantity makes a difference are only introduced in the last clause (5), and they always replace individual terms for which they are substituted—and this includes individual terms used in the *reduced terms* $R \in \mathsf{RTerm}_{\mathcal{L}_\mathsf{T}}$.

As before, all formulas are *closed*, i.e., sentences. The complexity introduced by Definition 41 is mirrored in the interpretation of terms.

Definition 43 ***($\mathcal{L}_\mathsf{T}$-Model)***
Let $\mathcal{L}_\mathsf{T}$ be a TFL-language. An $\mathcal{L}_\mathsf{T}$-*model* is a tuple $\mathfrak{M}_\mathsf{T} = \langle D, \|\cdot\|_{\mathfrak{M}_\mathsf{T}}\rangle$ such that

(1) D is a set (the *universe*)

(2) $\|\cdot\|_{\mathfrak{M}_\mathsf{T}}$ is an *interpretation-function of* $\mathfrak{M}_\mathsf{T}$ such that

  (a) if $t \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$, then $\|t\|_{\mathfrak{M}_\mathsf{T}} = \{a\}$ for an $a \in D$;

  (b) if $n = 1$ and $A \in \mathsf{STerm}^n_{\mathcal{L}_\mathsf{T}}$, then $\emptyset \neq \|A\|_{\mathfrak{M}_\mathsf{T}} \subseteq D$;

  (c) if $n > 1$ and $A \in \mathsf{STerm}^n_{\mathcal{L}_\mathsf{T}}$, then $\|A\|_{\mathfrak{M}_\mathsf{T}} \subseteq D^n$;

  (d) if $n > 0$ and $A \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$ is of the form '$\overline{B}$' for a $B \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$, then $\|A\|_{\mathfrak{M}_\mathsf{T}} = \{\langle a_1, \ldots, a_n\rangle \in D^n | \langle a_1, \ldots, a_n\rangle \notin \|B\|_{\mathfrak{M}_\mathsf{T}}\}$;

  (e) if $n > 0$ and $A \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$ is of the form '$(B \circ C)$' for $B, C \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$, then $\|A\|_{\mathfrak{M}_\mathsf{T}} = \{\langle a_1, \ldots, a_n\rangle \in D^n | \langle a_1, \ldots, a_n\rangle \in \|B\|_{\mathfrak{M}_\mathsf{T}} \circ \langle a_1, \ldots, a_n\rangle \in \|C\|_{\mathfrak{M}_\mathsf{T}}\}$ ($\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$);

  (f) if $n > 0$ and $A \in \mathsf{RTerm}^n_{\mathcal{L}_\mathsf{T}}$ stemming from $B \in \mathsf{CTerm}^m_{\mathcal{L}_\mathsf{T}}$ ($m > n$), $i = m - n$ individual terms $t_1, \ldots, t_i \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$ and $q_1, \ldots, q_i \in \{\forall, \exists\}$ such that A is of the form '$(q_1 t_1, \ldots, q_i t_i, \_{i+1}, \ldots, \_m)B$', then $\|A\|_{\mathfrak{M}_\mathsf{T}} = \{\langle a_1, \ldots, a_n\rangle \in D^n | \langle \bigcup \|t_1\|_{\mathfrak{M}_\mathsf{T}}, \ldots, \bigcup \|t_i\|_{\mathfrak{M}_\mathsf{T}}, a_1, \ldots, a_n\rangle \in \|B\|_{\mathfrak{M}_\mathsf{T}}\}$; similarly for all other ways of generating an $A \in \mathsf{RTerm}^n_{\mathcal{L}_\mathsf{T}}$;

(g) if $n > 0$ and $A^\pi \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$ for permutation π, then $\|A^\pi\|_{\mathfrak{M}_\mathsf{T}} = \{\langle \bigcup \|t_{\pi(1)}\|_{\mathfrak{M}_\mathsf{T}}, \ldots, \bigcup \|t_{\pi(n)}\|_{\mathfrak{M}_\mathsf{T}}\rangle \in D^n | \langle \bigcup \|t_1\|_{\mathfrak{M}_\mathsf{T}}, \ldots, \bigcup \|t_n\|_{\mathfrak{M}_\mathsf{T}}\rangle \in \|A\|_{\mathfrak{M}_\mathsf{T}}\}$.

The $\mathcal{L}_\mathsf{T}$-models $\mathfrak{M}_\mathsf{T}$ are similar to the models seen so far. However, similar to the $\mathcal{L}_\mathsf{A}$-models $\mathfrak{M}_\mathsf{A}$, they have to take care of the interpretation of the complex terms.

Clause (2a) interprets individual terms *as terms*, i.e., as a set; they are *individual* as the sets are singletons.

In line with how I introduced it before, *unary* terms are interpreted by *non-empty* sets. The reason is again to facilitate comparison with QUARC.

Complex n-ary terms are interpreted analogous to how $\mathcal{L}_\mathsf{A}$-models $\mathfrak{M}_\mathsf{A}$ interpreted complex *unary* terms; clause (2c) just generalizes from unary to n-ary terms, i.e., from subsets of the domain to n-ary relations on the domain.

Clause (2f) interprets the reduced terms. These are n-ary terms generated out of m-ary terms ($m > n$) by filling up slots with individual terms. These individual terms have quantities assigned, though as they are *individual*, the quantity does not make a difference. Thus, they are simply interpreted as $\bigcup \|t\|_{\mathfrak{M}_\mathsf{T}}$ ($t \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$). If $\|t\|_{\mathfrak{M}_\mathsf{T}} = \{a\}$, $\bigcup \|t\|_{\mathfrak{M}_\mathsf{T}} = a$.

The last clause (2g) is analogous to Definition 31 (2d), i.e., it interprets *reorders* by considering what they reorder; simply apply the permutation π to the n-tuples in the interpretation of term A in order to get the interpretation of A^π.

Since we treat quantification substitutionally and $\mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$ plays the role of individual-constants, we need to make sure that the particular choice of $\mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$ does not lead to problematic results; we do that as before by expanding the language.

Definition 44 ***($\mathcal{L}_\mathsf{T}$-A-Expansion)***
Let $\mathcal{L}_\mathsf{T}$ be a TFL-language and $\mathfrak{M}_\mathsf{T} = \langle D, \|\cdot\|_{\mathfrak{M}_\mathsf{T}}\rangle$ be an $\mathcal{L}_\mathsf{T}$-model. Let $A \subseteq D$. The *$\mathcal{L}_\mathsf{T}$-A-expansion of* $\mathcal{L}_\mathsf{T}$ is the language $\mathcal{L}'_\mathsf{T} := \mathcal{L}_\mathsf{T} \cup \{t_a | a \in A\}$ where the t_a are new individual-terms not contained in $\mathcal{L}_\mathsf{T}$.

If $A = \{a\}$, we call $\mathcal{L}'_\mathsf{T}$ an *$\mathcal{L}_\mathsf{T}$-a-expansion.*

Once the language is expanded, we need to make sure that the interpretation keeps up.

Definition 45 ***($\mathcal{L}_\mathsf{T}$-Model Expansion)***
Let $\mathcal{L}_\mathsf{T}$ be a TFL-language and $\mathfrak{M}_\mathsf{T} = \langle D, \|\cdot\|_{\mathfrak{M}_\mathsf{T}}\rangle$ be an $\mathcal{L}_\mathsf{T}$-model. Let $A \subseteq D$ and $\mathcal{L}'_\mathsf{T}$ be an $\mathcal{L}_\mathsf{T}$-A-expansion. The *A-expansion of* $\mathfrak{M}_\mathsf{T}$ *to* $\mathcal{L}'_\mathsf{T}$ is the model $\mathfrak{M}'_\mathsf{T} = \langle D', \|\cdot\|_{\mathfrak{M}'_\mathsf{T}}\rangle$ such that

(1) $D' = D$;

(2) $\|\cdot\|_{\mathfrak{M}_\mathsf{T}} \subseteq \|\cdot\|_{\mathfrak{M}'_\mathsf{T}}$;

(3) $\|t_a\|_{\mathfrak{M}'_\mathsf{T}} = \{a\} \subseteq A$ for every new individual term t_a.

As in the cases before, Definition 45 keeps the domain the same, and extends the interpretation-function to $\|\cdot\|_{\mathfrak{M}'_{\mathsf{T}}}$ so that the new individual-terms t_a are interpreted in alignment as they have been introduced. In accordance with Definition 43 (2a), these are not elements of the domain, but singleton-subsets.

Definition 46 ***(Satisfaction*** $\models_{\mathsf{T}}$***)***
Let the *TFL satisfaction-relation* $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \varphi$ for $\varphi \in \mathsf{Form}_{\mathcal{L}_{\mathsf{T}}}$ and $\mathcal{L}_{\mathsf{T}}$-model $\mathfrak{M}_{\mathsf{T}} = \langle D, \|\cdot\|_{\mathfrak{M}_{\mathsf{T}}}\rangle$ be recursively defined as follows:

(1) $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} (q_1 t_1, \ldots, q_n t_n)A$ iff $\langle \bigcup \|t_1\|_{\mathfrak{M}_{\mathsf{T}}}, \ldots, \bigcup \|t_n\|_{\mathfrak{M}_{\mathsf{T}}}\rangle \in \|A\|_{\mathfrak{M}_{\mathsf{T}}}$;

(2) $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} (q_1 t_1, \ldots, q_n t_n)\neg * A$ iff $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \neg(q_1 t_1, \ldots, q_n t_n) * A$;

(3) $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \neg\varphi$ iff it is not the case that $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \varphi$ ($\mathfrak{M}_{\mathsf{T}} \not\models_{\mathsf{T}} \varphi$);

(4) $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \varphi \wedge \psi$ iff $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \varphi$ and $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \psi$;

(5) $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \varphi[t_\alpha/t_1, \alpha/t_2, \ldots, \alpha/t_n]$ iff $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \varphi$;

(6) $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \varphi[\exists A]$ iff for some a-expansions $\mathfrak{M}'_{\mathsf{T}}$ of $\mathfrak{M}_{\mathsf{T}}$ such that $a \in \|A\|_{\mathfrak{M}_{\mathsf{T}}}$, $\mathfrak{M}'_{\mathsf{T}} \models_{\mathsf{T}} \varphi[\exists t_a]$;

(7) $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} \varphi[\forall A]$ iff for all a-expansion $\mathfrak{M}'_{\mathsf{T}}$ of $\mathfrak{M}_{\mathsf{T}}$ such that $a \in \|A\|_{\mathfrak{M}_{\mathsf{T}}}$, $\mathfrak{M}'_{\mathsf{T}} \models_{\mathsf{T}} \varphi[\forall t_a]$.

As already done in Definition 43 (2f), individual-terms are interpreted regardless of their specific quantity as done in clause (1); individual-terms are pretty much treated as individual-constants in Definition 22 (1), as is predication.

As QUARC, TFL allows for negative predication; clause (2) is analogous to clause (5) of Definition 34. The negation-symbols $\neg$ are moved in front of formulas and then interpreted via clause (2) as long as only individual-terms are involved.

The remaining clauses are analogous to those of QUARC in Definition 34. In particular, we interpret quantifiers via the expansions, where, as in the QUARC-case given in Definition 34 (7)–(8), we consider appropriate expansions, i.e., expansions which expand with elements in the interpretation of the subject-term A and consider as many as the quantity q of A specifies.

As before, we can define *logical consequence* as done in Definition 8. Given these notions, we can formulate the TFL-specific treatment of individual-terms.

Theorem 47
For $t \in \mathsf{ITerm}_{\mathcal{L}_{\mathsf{T}}}$, $\models_{\mathsf{T}} (\exists t)A \leftrightarrow (\forall t)A$.

Proof. Let $\mathfrak{M}_{\mathsf{T}}$ be an $\mathcal{L}_{\mathsf{T}}$-model and $t \in \mathsf{ITerm}_{\mathcal{L}_{\mathsf{T}}}$. Let $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} (\exists t)A$. By Definition 46 (1), $\bigcup \|t\|_{\mathfrak{M}_{\mathsf{T}}} \in \|A\|_{\mathfrak{M}_{\mathsf{T}}}$ and so $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} (\forall t)A$. □

Moreover, we get a similar result regarding non-emptiness as Theorem 35, though extended to include individual-terms.

Theorem 48
For $A \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}} \cup \mathsf{STerm}^1_{\mathcal{L}_\mathsf{T}}$, $\models_\mathsf{T} (\exists A)A$.

Proof. Let $\mathfrak{M}_\mathsf{T}$ be an $\mathcal{L}_\mathsf{T}$-model.

- If $A \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$, by Definition 43 (2a) $\|A\|_{\mathfrak{M}_\mathsf{T}} = \{a\}$ for an $a \in D$. Thus, $\bigcup \|A\|_{\mathfrak{M}_\mathsf{T}} = a \in \|A\|_{\mathfrak{M}_\mathsf{T}}$. Therefore, by Definition 46 (1), $\mathfrak{M}_\mathsf{T} \models_\mathsf{T} (\exists A)A$.
- If $A \in \mathsf{STerm}^1_{\mathcal{L}_\mathsf{T}}$, by Definition 43 (2b), $\|A\|_{\mathfrak{M}_\mathsf{T}} \neq \emptyset$. Let $a \in \|A\|_{\mathfrak{M}_\mathsf{T}}$. Then, for some a-expansion $\mathfrak{M}'_\mathsf{T}$ of $\mathfrak{M}_\mathsf{T}$ such that $a \in \|A\|_{\mathfrak{M}_\mathsf{T}}$, $\mathfrak{M}'_\mathsf{T} \models_\mathsf{T} (\exists t_a)A$. By Definition 46 (6), $\mathfrak{M}_\mathsf{T} \models_\mathsf{T} (\exists A)A$.

□

However, as we allow for complex terms, this does not hold in general.

Theorem 49
$\not\models_\mathsf{T} (\exists A)A$.

Proof. Let $\mathfrak{M}_\mathsf{T}$ be an $\mathcal{L}_\mathsf{T}$-model. Consider $A \in \mathsf{STerm}^1_{\mathcal{L}_\mathsf{T}}$ such that $\|A\|_{\mathfrak{M}_\mathsf{T}} = D$. Then, by Definition 43 (2d), $\|\overline{A}\|_{\mathfrak{M}_\mathsf{T}} = \emptyset$. Thus, there is no a-expansion $\mathfrak{M}'_\mathsf{T}$ of $\mathfrak{M}_\mathsf{T}$ such that $a \in \|\overline{A}\|_{\mathfrak{M}_\mathsf{T}}$, so $\mathfrak{M}_\mathsf{T} \not\models_\mathsf{T} (\exists \overline{A})\overline{A}$. □

For similar reasons, we get that that the universal doesn't imply the particular.

Corollary 50
$(\forall A)B \not\models_\mathsf{T} (\exists A)B$.

Proof. Consider the model in the proof of Theorem 49. Since there is no a-expansion $\mathfrak{M}'_\mathsf{T}$ of $\mathfrak{M}_\mathsf{T}$ such that $a \in \|\overline{A}\|_{\mathfrak{M}_\mathsf{T}}$ it follows that for all a-expansions $\mathfrak{M}'_\mathsf{T}$ of $\mathfrak{M}_\mathsf{T}$ such that $a \in \|\overline{A}\|_{\mathfrak{M}_\mathsf{T}}$, $\mathfrak{M}'_\mathsf{T} \models_\mathsf{T} (\forall t_a)B$, i.e., by Definition 46 (7), $\mathfrak{M}_\mathsf{T} \models_\mathsf{T} (\forall \overline{A})B$. However, as there are no a-expansions $\mathfrak{M}'_\mathsf{T}$ of $\mathfrak{M}_\mathsf{T}$ such that $a \in \|\overline{A}\|_{\mathfrak{M}_\mathsf{T}}$, $\mathfrak{M}_\mathsf{T} \not\models_\mathsf{T} (\exists \overline{A})B$. □

Of course, as in the case of Theorem 15, we obtain a restricted version.

Theorem 51
$(\exists A)A, (\forall A)B \models_\mathsf{T} (\exists A)B$.

Overall, as was to be expected, the $\mathcal{L}_\mathsf{T}$-models $\mathfrak{M}_\mathsf{T}$ behave similar to the rejected non-empty $\mathcal{L}_\mathsf{A}$-models $\mathfrak{M}_\mathsf{ne}$.

Moreover, the quantifiers still behave as expected.

Theorem 52
The following equivalences hold:

(1) $\models_{\mathsf{T}} (\forall A)B \leftrightarrow \neg((\exists A)\neg B)$; (2) $\models_{\mathsf{T}} (\exists A)B \leftrightarrow \neg((\forall A)\neg B)$;

(3) $\models_{\mathsf{T}} \neg(\exists A)B \leftrightarrow (\forall A)\neg B$; (4) $\models_{\mathsf{T}} \neg(\forall A)B \leftrightarrow (\exists A)\neg B$.

Given the way term-negation '$\overline{}$' is interpreted, it is equivalent to a negative predication.

Theorem 53
$\models_{\mathsf{T}} (qA)\neg B \leftrightarrow (qA)\overline{B}$ $(q \in \{\forall, \exists\})$.

Proof. Let $\mathfrak{M}_{\mathsf{T}}$ be an $\mathcal{L}_{\mathsf{T}}$-model such that $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} (qA)\neg B$. Then, by Definition 46 (6)/(7), for some/all a-expansions $\mathfrak{M}'_{\mathsf{T}}$ of $\mathfrak{M}_{\mathsf{T}}$ such that $a \in \|A\|_{\mathfrak{M}_{\mathsf{T}}}$, $\mathfrak{M}'_{\mathsf{T}} \models_{\mathsf{T}} (q't_a)\neg B$. Thus, by Definition 46 (2), for some/all a-expansions $\mathfrak{M}'_{\mathsf{T}}$ of $\mathfrak{M}_{\mathsf{T}}$ such that $a \in \|A\|_{\mathfrak{M}_{\mathsf{T}}}$, $\mathfrak{M}'_{\mathsf{T}} \models_{\mathsf{T}} \neg((q't_a)B)$, i.e., by Definition 46 (3) and (1), $\bigcup \|t_a\|_{\mathfrak{M}'_{\mathsf{T}}} \notin \|B\|_{\mathfrak{M}'_{\mathsf{T}}}$. Then, by Definition 43 (2d), $\bigcup \|t_a\|_{\mathfrak{M}'_{\mathsf{T}}} \in \|\overline{B}\|_{\mathfrak{M}'_{\mathsf{T}}}$, i.e., for some/all a-expansions $\mathfrak{M}'_{\mathsf{T}}$ of $\mathfrak{M}_{\mathsf{T}}$ such that $a \in \|A\|_{\mathfrak{M}_{\mathsf{T}}}$, $\mathfrak{M}'_{\mathsf{T}} \models_{\mathsf{T}} (q't_a)\overline{B}$. Thus, by Definition 46 (6)/(7), $\mathfrak{M}_{\mathsf{T}} \models_{\mathsf{T}} (qA)\overline{B}$. □

Similar again to the non-empty models of Aristotelian syllogistic, only two conversions hold generally, and the third one with a restriction in place.

Theorem 54 ***(Conversion)***
The following conversions hold:

($\underline{\mathbf{a}}$-$\underline{\mathbf{i}}$-conv$^{\models_{\mathsf{T}}} \restriction \exists A$) $(\exists A)A, (\forall A)B \models_{\mathsf{T}} (\exists B)A$

($\underline{\mathbf{i}}$-$\underline{\mathbf{i}}$-conv$^{\models_{\mathsf{T}}}$) $(\exists A)B \models_{\mathsf{T}} (\exists B)A$

($\underline{\mathbf{e}}$-$\underline{\mathbf{e}}$-conv$^{\models_{\mathsf{T}}}$) $(\forall A)\neg B \models_{\mathsf{T}} (\forall B)\neg A$

Lastly, negation works as expected as well.

Theorem 55
The following hold ('$*$' being a possibly empty string of negation-symbols '$\neg$'):

(1) $\models_{\mathsf{T}} (q_1t_1, \ldots, q_nt_n)\neg\neg * A \leftrightarrow (q_1t_1, \ldots, q_nt_n) * A$;

(2) $\models_{\mathsf{T}} (q_1t_1, \ldots, q_nt_n) * \overline{\overline{A}} \leftrightarrow (q_1t_1, \ldots, q_nt_n) * A$;

(3) $\models_{\mathsf{T}} (q_1t_1, \ldots, q_nt_n) * \neg\overline{A} \leftrightarrow (q_1t_1, \ldots, q_nt_n) * A$.

6 Comparison

Having sketched the different systems, let's compare them. Aristotle's syllogistic and Fregean logic function as base; we consider how QUARC and TFL compare to them and differ from each other. The comparison, however, does not account for all the subtleties and differences between QUARC and TFL, but is restricted to more general points. It also remains open to see whether QUARC can be developed along TFL-lines and vice versa. For this reason, among others, I do not argue for the superiority of either of these systems when it comes to the question of which one better captures the semantics of natural language—the underlying motivation of both QUARC and TFL. The comparison is rather meant to consider potential differences which might lead to further development of either of these approaches along the lines of the other.

6.1 Aristotelian Roots

As we have seen in Sections 4.1 and 5.1, both Sommers and Ben-Yami claim a strong connection to Aristotelian logic. Ben-Yami (2004: 62) sees his understanding of predication as fundamentally in agreement with that of Aristotle, and Sommers considers several of Aristotle's points throughout the development of TFL.

In the version of TFL developed in Section 5.2, I excluded many of Sommers's more specific points that show a strong similarity to Aristotle's logical discussions. For example, I did not include *categories* and, as a consequence, excluded Sommers's discussion of *contrariety* (see, e.g., Sommers 1982: 80).

TFL, in contrast to QUARC, takes the subject-predicate structure of (basic) sentences to be fundamental. The formalism from Section 5.2 does not fully reflect that, though takes some steps towards it with the introduction of *reduced terms* collected in $\mathsf{RTerm}_{\mathcal{L}_\mathsf{T}}$ in Definition 41 (5). This allows to reduce n-ary terms to *unary* terms which can be the predicate in the subject-predicate structure. For example, a binary predicate like 'loves' can be reduced to a unary predicate 'loves t' (t a term) serving as predicate to a subject. Similarly, we can iterate this and use reduced terms to reduce further terms. This can account for the intended nesting of terms to keep the subject-predicate structure intact (cf., e.g., Sommers 1982: 113ff.).

Moreover, TFL does not include *individual-constants*, but does include *individual-terms* in form of $\mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$. As all the descriptive signs are *terms*, each term can play the role of subject *and* predicate. This is reflected in the conversions (Theorem 54) which only hold in QUARC for the unary predicates.

Each term in subject position is assigned a *quantity*—indicated by a *quantifier*. In the case of individual terms, the quantity does not make a difference (Theorem 47). In the formalism of Section 5.2, a quantity is assigned, but not in the form of a *wild* quantity (as in Sommers 1982: 18). Given $t \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$, $\mathcal{L}_\mathsf{T}$-models interpret them accordingly as *singletons* which puts them on a par with other *unary* terms. Indeed, Definition 43 (2b) allows for unary terms to be interpreted as singletons, too. The difference between an $A \in \mathsf{STerm}^1_{\mathcal{L}_\mathsf{T}}$ and a $t \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$ would only show up once the system is modalized; t would still be interpreted as singleton, A might not.

QUARC follows the Fregean line of dividing the language into individual-constants ($\mathsf{Const}_{\mathcal{L}_\mathsf{F}}$)/singular arguments ($\mathsf{SA}_{\mathcal{L}_\mathsf{Q}}$) and n-ary predicates ($\mathsf{Pred}^n_{\mathcal{L}_\mathsf{F}}/\mathsf{Pred}^n_{\mathcal{L}_\mathsf{Q}}$). The Aristotelian root that Ben-Yami sees for QUARC is when it comes to predication. The sentences of the syllogistic ($\mathsf{Form}_{\mathcal{L}_\mathsf{A}}$) follow the subject-predicate pattern, where the predication can be *universally* or *particularly* and so assign the subject a *quantity*. This general structure is not kept for all the QUARC-sentences though, but only for those with *unary* predicates. In particular, only quantified sentences come with the assignment of quantities, not all sentences. TFL, in contrast, takes every sentence to come with a quantity.

Relational predications, on the other hand, are treated by QUARC as they are in Fregean languages. This contrasts with TFL-sentences which keep the subject-predicate structure also for those. However, $\mathsf{Form}_{\mathcal{L}_\mathsf{T}}$ also allows for sentences involving *connectives* so that complex sentences without this subject-predicate structure are included, too, but such complex sentences bottom out in sentences with subject-predicate structure in TFL; in QUARC, they do not.

6.2 Identity

Sommers (1982: ch. 6) argues that there is no need to include an identity-symbol '=' into TFL. Rather, we can understand Aristotle's basic notion of *predicated of all/none* (Section 2.7) as providing us with a *substitution principle* so that identity becomes superfluous. This substitution principle can be taken to be a formal rendering of (B<u>a</u>rb<u>a</u>r<u>a</u>) which allows to conclude AaC from AaB and BaC. In the languages of TFL and QUARC, this can be captured as (where '$\models$' is either '$\models_\mathsf{T}$' or '$\models_\mathsf{Q}$')

$$(\forall C)B, (\forall B)A \models (\forall C)A.$$

However, TFL comprises more notions here as we are allowed to use individual-terms. QUARC, on the other hand, only allows unary predicates, and so the formal rending of (B<u>a</u>rb<u>a</u>r<u>a</u>) does *not* apply to individuals as such. Nevertheless, as there is nothing ruling out unary predicates which are interpreted as *singletons*—i.e., as the $t \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$—it can be taken to apply indirectly, via establishing a connection between the singular arguments and specific predicates.

Moreover, TFL allows this substitution also in cases where the predicate is n-ary. For example, if $B \in \mathsf{CTerm}^n_{\mathcal{L}_\mathsf{T}}$, we get from

$$(q_1A_1, \ldots, q_{i-1}A_{i-1}, \forall A_i, q_{i+1}A_{i+1}, \ldots, q_nA_n)B$$

and

$$(\forall C)A_i$$

that

$$(q_1A_1, \ldots, q_{i-1}A_{i-1}, \forall C, q_{i+1}A_{i+1}, \ldots, q_nA_n)B.$$

Even though QUARC can validate such consequences too, $\mathcal{L}_\mathsf{Q}$ contains an identity symbol '=' among its logical constants. Given the different understanding of *predication*, though, it behaves slightly different compared to the Fregean case. As Fregean languages quantify *unrestrictedly* over individuals, it can capture that everything is self-identical ('$\forall x(x = x)$'). QUARC, on the other hand, cannot (see Section 6.4), though identity works similar. For example, given two singular arguments $s_1, s_2 \in \mathsf{SA}_{\mathcal{L}_\mathsf{Q}}$, '$s_1 = s_2$' is a QUARC-sentence. However, for $\alpha, \beta \in \mathsf{Ana}_{\mathcal{L}_\mathsf{Q}}$, '$\alpha = \beta$' would not be well-formed (and neither would be '$\forall_\alpha \alpha = \beta$' or something similar). Anaphora can only be introduced by replacing singular arguments; see Definition 28 (6). Thus, '$s = s$' can lead to '$s_\alpha = \alpha$' which, in turn, can lead to '$\forall P_\alpha = \alpha$'.

As $\mathcal{L}_\mathsf{T}$ does not contain any individual-constants or variables, identity cannot be introduced as in $\mathcal{L}_\mathsf{F}$ or $\mathcal{L}_\mathsf{Q}$. Nevertheless, in principle, it could be introduced as restricted to individual-terms. For example, if $t_1, t_2 \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$, '$t_1 = t_2$' could be interpreted via Definition 46 (1), i.e., an $\mathcal{L}_\mathsf{T}$-model $\mathfrak{M}_\mathsf{T}$ satisfies it iff $\langle \bigcup \|t_1\|_{\mathfrak{M}_\mathsf{T}}, \bigcup \|t_2\|_{\mathfrak{M}_\mathsf{T}} \rangle \in \| = \|_{\mathfrak{M}_\mathsf{T}}$ (or, equivalently, $\bigcup \|t_1\|_{\mathfrak{M}_\mathsf{T}} = \bigcup \|t_2\|_{\mathfrak{M}_\mathsf{T}}$ or simply $\|t_1\|_{\mathfrak{M}_\mathsf{T}} = \|t_2\|_{\mathfrak{M}_\mathsf{T}}$). One could then also show that $(\forall t_1)t_2 \models_\mathsf{T} t_1 = t_2$ (and so use '$(\forall t_1)t_2$' as definition of '$t_1 = t_2$'). In principle, this could also be achieved in QUARC.

6.3 Negation

As in the case of $\mathcal{L}_\mathsf{A}$, several ways to negate have been introduced into the systems. In the syllogistic, *terms* can be negated ('$\overline{A}$') and sentences can be *negative* ('$(qA)\neg B$'). Fregean languages, on the other hand, only contain sentence-negations ('$\neg\varphi$').[108]

The version of QUARC presented in Section 4.2 incorporates *sentence*- and *predication*-negation. The former works as it does in Fregean languages, the latter negates predication and so compares to the negative sentences of the syllogistic ('ti apo tinos'). What is captured by predicate-negation is that a predicate such as 'friendly' can be *affirmed* or *denied.* However, as long as there is no quantification involved, these are treated as equivalent to sentence-negations as specified in Definition 34 (5).

Similarly, TFL, as presented in Section 5.2, contains both sentence- and predicate-negation. Additionally, it contains negated terms as the syllo-

[108] Or, given a different set-up, formula-negation.

gistic does. As I did not incorporate categories and contrariety into the formalism, these are also treated as equivalent as shown in Theorems 53 and 55.

There is no reason to treat predicate-negation as equivalent to sentence-negation in quantifier-free cases. Following Sommers's discussion, we can understand predicate-negation as connected to categories and category mistakes. For example, the number 2 is neither friendly nor not friendly; the sentences '*it is not the case that* the number 2 is friendly' (which is true) and 'the number 2 *is not* friendly' (which is false) come apart. This, too, could be incorporated into QUARC.

Given TFL's additional *negative terms*, TFL can also treat predicate-negation as introduced, and construe the negated terms as connected to categories directly. It could also understand the predicate-negation so and the term-negation as introduced. The different ways to negate open different possibilities to introduce where negation can "go wrong".

In the empty semantics for the syllogistic, on the other hand, negative predication and negated terms are not equivalent; this is shown in Theorem 16. The reason is that the $\mathcal{L}_\mathsf{A}$-models $\mathfrak{M}_\mathsf{A}$ allow terms with empty extensions which rule out the validation of **a**-type sentences by ($\mathbf{a}_+$). In the alternative semantics $\mathfrak{M}_\mathsf{ne}$, simple terms are taken to be non-empty—as are the simple terms in TFL according to Definition 43 (2b) as well as the (Fregean/QUARC) unary predicates according to Definition 19 (2b)/Definition 31 (2b). If incorporated into TFL or QUARC, this opens different ways of interpreting the different ways to negate.

6.4 Quantification

As the 'QUAR' in 'QUARC' suggests, Ben-Yami considers QUARC's treatment of *quantification* as one of its major divergences from Fregean languages. Firstly, Ben-Yami (2004: §9.8) argues that quantification comes with what he calls 'referential import' in his book ('instantiation' in his 2014).[109] However, in my presentation of the Fregean language, I incorporated this already; see Definition 19 (2b), Theorem 23, and Corollary 24.

Secondly, Ben-Yami (2004: §6.1) argues that Fregean languages *presuppose* a domain of quantification whereas QUARC does not. Rather, quantification in natural language is always combined with a specification as to what is quantified over, i.e., a plurality is identified and the quantifier specifies *how much* of that plurality is relevant. For example, in 'all human beings are mortal', 'human beings' refers to a plurality of

[109] By now he prefers 'instantial import'. He also insists (personal communication) that there are two issues that are mixed together, viz., unary predicates are not empty as to keep QUARC *bivalent*, and instantial import is about quantification, viz., a sentence of the form '$\varphi[\forall P]$' can only be true or false if there are Ps. Hanoch points out that he has been clear about the distinction since after the publication of his (2014).

human beings (i.e., reference is to be construed *plurally*) and 'all' suggests how much of that plurality is relevant. The Fregean analysis, on the other hand, quantifies over the whole domain which, therefore, has to be presupposed.[110] Since TFL considers sentences to have subject-predicate structure where the subject is assigned a quantity, the treatment aligns with that of QUARC, viz., quantification is always *restricted* by a term. Thus, insofar as QUARC's treatment differs from that of Fregean languages, TFL's does too.

However, Sommers treats 'human beings' as the *subject* of the sentence, whereas Ben-Yami takes it to be 'all human beings'. The 'all' only indicates the quantity of the subject, but does not figure as part of it in TFL. Formally, this does not make a difference, though, as can be seen by Definitions 34 (7)–(8) and 46 (6)–(7). Nevertheless, the underlying understanding of *reference* is a different one, though one that I do not discuss here.

Another difference is that every sentence comes with quantities according to TFL but not to QUARC. The reason is that TFL takes all the descriptive signs to be *terms* for which one can specify quantities. QUARC, on the other hand, follows the Fregean approach. However, as treated in Definition 46 (1), the quantity does not make a difference for individual terms; we might as well reformulate Definition 42 so as to allow $t \in \mathsf{ITerm}_{\mathcal{L}_\mathsf{T}}$ to occur *without* quantifier in $\mathcal{L}_\mathsf{T}$-formulas. Similarly, we could reformulate Definition 28 (1) to *include* quantifiers which don't affect the interpretation.

6.5 Expressive Power

Both Sommers and Ben-Yami are concerned with the *expressive power* of their systems. Indeed, both consider expressive power as an adequacy criterion when it comes to alternatives to the Fregean approach. As Aristotelian syllogistic is clearly inferior in this respect, it fails to meet

[110] I have to admit that I—still; see Raab (2018: n. 29, p. 315)—don't fully grasp Ben-Yami's claim that domains are not needed. As interpreted here via Definitions 31 and 34 (7)–(8), it is true that all quantification is *restricted* by the interpretation of the quantified argument: $\mathfrak{M}_\mathsf{Q} \models_\mathsf{Q} \varphi[\forall/\exists P]$ iff for all/some a-expansions $\mathfrak{M}'_\mathsf{Q}$ of $\mathfrak{M}_\mathsf{Q}$ such that $a \in \|P\|_{\mathfrak{M}_\mathsf{Q}}$, $\mathfrak{M}'_\mathsf{Q} \models_\mathsf{Q} \varphi[s_a]$. However, that still *presupposes* a domain in which $\|P\|_{\mathfrak{M}_\mathsf{Q}}$ lives.

Lanzet likewise claims to develop a "domain-free semantics" (2017: 550) and goes on to suggest that when "reference is made to the domain of an interpretation $\mathcal{M}$, what will be meant is the domain of $\mathcal{M}$ *as a function*" (2017: 565, his emphasis). However, unless the *function* maps *into* somewhere—its range or our domain—it is not a function, and so the model would not be well-defined.

One might suggest that the problem is the *model-theoretic* approach, but I don't see how the problem disappears by going for a *valuational semantics* (seemingly, Ben-Yami's preferred approach). Whether I presuppose for each predicate P what exactly is referred to or whether I presuppose a domain and then restrict it to predicates seems to me to amount to the same (with the latter option to be in many cases more convenient and expressively richer; see Section 6.5).

the criterion.

Both QUARC and TFL have a legitimate claim as to satisfy the criterion. QUARC achieves the expressive power by including *anaphora* and *reorders* of any arity; TFL by including *proterms* and *complex terms* of any arity. Both systems also have formal results to show their expressive power in comparison to Fregean languages. Sommers claims that "the expressive power of [PTL] is that of a standard language of modern predicate logic" (1982: 176), i.e., of a Fregean language; see also (1982: Appendix A). Given that TFL extends PTL, it is clear that TFL does not fall behind with respect to its expressive power.

QUARC, too, has been investigated with respect to its expressive power compared to a Fregean language. Once we expand $\mathcal{L}_{\mathsf{Q}}$ by a unary predicate T such that $\|T\|_{\mathfrak{M}_{\mathsf{Q}}} = D$, all $\varphi \in \mathsf{Form}_{\mathcal{L}_{\mathsf{F}}}$ can be translated into QUARC and vice versa. One way to introduce such a predicate is to allow complex predicates into QUARC; see my (2016: ch. 5) and, for a fuller treatment, my (ms). For, we can then define $(\cdot)T$ as $(\cdot)(P \vee \neg P)$. This, then, allows to capture quantified sentences which don't have restricting predicates such as '$\forall x(x = x)$'; QUARC captures it as '$\forall T_\alpha = \alpha$'. A similar approach works when showing that TFL can capture all $\varphi \in \mathcal{L}_{\mathsf{F}}$.

What has not been investigated is how exactly TFL and QUARC compare. Once translations between the systems and a Fregean language have been introduced, they can be used to establish the relation between them. However, this has not been done yet. Nevertheless, if the formal systems that have been introduced here are adequate representations of the intended systems, translations between them suggest themselves. Since the presentation of TFL has been quite diminished compared to Sommers's developments, I would think that TFL is the most expressive systems among those considered here. However, there does not seem to be a principled reason to suggest that QUARC couldn't similarly developed further to match this expressive richness.

7 Conclusion

I have developed four formalisms here, one for each of Aristotelian syllogistic, Fregean languages, QUARC, and TFL. Both QUARC and TFL are meant to favourably compare to Aristotle's logic. QUARC's understanding of predication and quantification and TFL's understanding of terms and the subject-predicate structure of basic sentences is claimed to be close to Aristotle's understanding of these. Moreover, both systems have been developed as a better way to the semantics of natural language compared to what Fregean languages are capable. Again, it's the Aristotelian root that does much of the heavy lifting.

The expressive power of Fregean languages remains one of the main arguments to adopt the Fregean approach. However, QUARC and TFL have a claim to match this power, and so undermine at least the argument from expressive power. On the other hand, the availability of translations of both TFL and QUARC into Fregean languages also shows that the expressive power alone cannot decide here. One major way in which the case is made for either QUARC or TFL is by the *syntactic* similarity of their formal grammars compared to that of natural language. Given that those formal grammars differ from one another while claiming to fit that of natural language well, it needs to be seen in which ways these formalisms can be extended to capture more and more of natural language. But even once that is done, if we can establish the precise relationship between these systems, it might well be that both can be developed to incorporate parts of the other so that nothing might decide between the two. As it stands, it's focus on the terms and the subject-predicate structure of basic sentences means that TFL is a more radical alternative to Fregean languages; whether it is a better one than QUARC, I leave the readers to decide for themselves.

Acknowledgements

I gratefully acknowledge the Irish Research Council's funding (project-number GOIPD/2022/635). I would also like to thank Vasilis Politis, Norbert Gratzl, and Hanoch Ben-Yami.

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