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Conceptions of truth in intuitionism

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Intuitionism's disagreement with classical logic is standardly based on its specific understanding of truth. But different intuitionists have actually explicated the notion of truth in fundamentally different ways. These are considered systematically and separately, and evaluated critically. It is argued that each account faces difficult problems. They all either have implausible consequences or are viciously circular.

1. Introduction

Intuitionism is famous for its denial of the validity of certain traditionally accepted laws of logic, in particular, the law of the excluded middle[1] (in short, LEM). Its revision of logic is, however, only a consequence of certain more fundamental theses of intuitionism, namely its specific way of understanding the notions of existence and truth.[2] Namely, a characteristic feature of intuitionism is the requirement that the notion of *truth* of a proposition should be explained in terms of the notion of proof, or verification,[3] rather than as correspondence with some sort of mind-independent realm of mathematical objects; from this one concludes that not every sentence is either true or false. Analogously, the *existence* of mathematical objects is analysed in terms of mental constructions rather than understood as some kind of mind-independent existence.

But how exactly are truth and existence explained in these constructivistic terms? It is my aim in this paper to show that the exact details are far less clear than has been usually thought. To begin with, let us distinguish the following two, fundamentally different cases:

(a) A has been proved;
(b) A is (in principle) provable;

1 Often what the intuitionists attack under the name of LEM is not actually LEM but the principle of bivalence; strictly speaking, these two should be distinguished (see e.g. *Dummett 1978*, p. xix). However, under certain default assumptions, accepted by intuitionists, these two are equivalent (see e.g. *Pagin 1998*). Consequently, in what follows I shall not be overly pedantic on this issue.

2 This is not the only possibility; some, most notably Heyting, rebut the whole notion of truth, and proceed directly to give a meaning-explanation of the logical constants, an explanation under which LEM etc. fail. But the basic question of this paper (see below), whether one should adopt the actualist or the possibilist approach, remains. I shall argue that Heyting is still deeply committed to the actualist picture, with all its problems; see below.

Also some recent constructivists such as Bishop and Bridges avoid talk of truth, but their explanation of constructivism closely leans on the possibilist picture (see below). Thus Bishop explains 'the basic constructivist goal' as being 'that mathematics concern itself with the precise description of finitely *performable* abstract operations ... Thus by "constructive" I shall mean a mathematics that describes or predicts the results of certain finitely *performable*, albeit hypothetical, computations within the set of integers' (*Bishop 1970*, p. 53 my emphasis). Bridges, in turn, writes: 'Constructive mathematics is distinguished from its traditional counterpart, classical mathematics, by the strict interpretation of the phrase "there exists" as "we *can* construct"' (*Bridges 1997*, my emphasis).

3 Some (e.g. Heyting; see below) rather want to eliminate the concept of truth and replace it e.g. by the possession of proof or by the possibility of proof. For my purposes here, however, this makes little difference.

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and, analogously,

(a′) *n* has been constructed;
(b′) *n* is (in principle) constructible.

Now it would be possible to equate truth and existence with either of these cases, respectively. In order to have convenient terms to speak of these alternative approaches, I shall refer to the first, temporal conception as the *actualist* notion, and to the second, non-temporal notion as the *possibilist* notion, of truth or existence, respectively. Although many recent expositions of intuitionism give the impression that it is definitely the possibilist notion that is intended by intuitionists, I shall argue that the real situation is much more complicated and problematic. Correspondingly, quite many critical discussions have actually focused solely on some particular variant of intuitionism. As too many treatments of intuitionism have given an oversimplified picture of the field, I shall first attempt to give a more accurate picture of the variety of intuitionistic views, and only after that proceed to more systematic issues and critical considerations.

2. Classical intuitionism: Brouwer and Heyting

When one considers intuitionism, it is natural to begin with L. E. J. Brouwer, the founding father of intuitionism. So what is Brouwer's view on truth? The answer is not fully clear. Many of Brouwer's statements on truth seem to commit him to straightforward actualism, according to which truth is significantly temporal, and assertions become true only when the relevant experience of thought-construction occurs, i.e. there are no truths which have not already been experienced:

> Truth is only in reality, i.e. in the present and past experiences of consciousness. ... expected experiences and experiences attributed to others are true only as anticipations and hypotheses; in their contents there is no truth.
>
> *Brouwer 1948*

> [The original Dutch version simply says: 'Expected experiences as such and the reputed experiences of others as such are not truths ... there are no non-experienced truths.'
>
> *van Stigt 1990* (p. 204)]

> ... in mathematics no truths could be recognized which had not been experienced.
>
> *Brouwer 1955*

> Correctness of an assertion then has no other meaning than that its content has in fact appeared in the consciousness of the subject.
>
> *Brouwer 1951*

In other words, here Brouwer thinks that a proposition is true only if it has been actually proved (and an object exists only if it has been actually constructed). Under this interpretation LEM is obviously false. Even a platonist must agree that not every proposition has yet been either proved or refuted, although he or she is likely to add that this is simply not what is normally meant by LEM.

Sometimes, however, in some of his criticisms of classical logic, Brouwer rather seems to equate truth with the possibility of proof, that is, provability in principle.[4] In such contexts Brouwer does not at all speak about presently unsolved problems, as the above temporal conception of truth would suggest, but only about *absolutely unsolvable problems* (see *van Stigt 1990*, pp. 252–254), that is, he disputes Hilbert's thesis that every mathematical problem *can* be solved. As early as 1908 Brouwer wrote that the principle of the excluded middle 'claims that every supposition is either true or false' and that the question of its validity 'is equivalent to the question whether *unsolvable* mathematical questions can exist':

> Now consider the pricipium *tertii exclusi*: It claims that every supposition is either true or false; in mathematics this means that for every supposed imbedding of a system into another, satisfying certain given conditions, we *can* either accomplish such an imbedding by a construction, or we *can* arrive by a construction at the arrestment of the process which would lead to the imbedding. It follows that the question of the validity of the pricipium tertii exclusi is equivalent to the question whether *unsolvable* mathematical problems can exist. There is not a shred of a proof for the conviction ... that there exist no unsolvable mathematical problems.
>
> Insofar as only finite discrete systems are introduced, the investigation whether an imbedding is possible or not, *can* always be carried out and admits a definite results, so in this case the pricipium tertii exclusi is reliable as a principle of reasoning.
>
> *Brouwer 1908*, p. 109 (my emphasis)

In the 1920s and 1930s Brouwer attempted intensively to provide a concrete example of an absolutely unsolvable problem which would convince the wider audience of the non-validity of LEM (see *van Stigt 1990*, pp. 252–254). That was, at the time, the aim of Brouwer's well-known 'counterexamples'.

In his later years, Brouwer abandoned this search for absolutely unsolvable problems[5] and returned to the actualist formulations (see *van Stigt 1990*, pp. 254–255). In the 1950s, however, Brouwer endorsed actualism in a somewhat liberalized form. More exactly, he now insisted that the dichotomy between true and false must be replaced by the following four cases:[6]

(1) true = has been proved to be true;
(2) false/impossible/absurd = has been proved to be absurd;
(3) At present neither true or false, but we know an algorithm which decides the assertion;
(4) At present neither true or false, and we do not know an algorithm which would decide the assertion.

4 Possibly just because of the problem just mentioned, namely, that Brouwer realized that very few would accept his austere actualist notion of truth and falsity as being actually proved and being actually refuted, respectively (*van Stigt 1990*, p. 248).

5 One might think that he was destined to fail, because it is arguable that the idea of an absolutely unsolvable problem makes no sense intuitionistically; see e.g. *Heyting 1958b*, *Dummett* (*1977*, p. 17), *Pagin 1998*; see also below.

6 I have put this together from three different, very similar sources: *Brouwer 1951*, *1951–52* and *1955*.

According to Brouwer, case 3 is temporary. Moreover, Brouwer says that it 'obviously is reducible to the first and second cases' (*Brouwer 1951–52*, p. 454). I find it quite unclear what exactly he means here; if this is meant to say that a statement for which we know an algorithm which would decide it positively is true, it seems he already has compromised the actualist position. But in any case, Brouwer now claims that each mathematical statement which is in the fourth case yields a refutation of the LEM; he gives as an example then unproved (but, as is well known, now proved) Fermat's last theorem (*Brouwer 1955*)! Moreover, he also notes that an assertion in this fourth case may at some times pass into one of the other cases: 'one must take into account the possibility that one day a method may be found' (*Brouwer 1951*, p. 451). Thus also the line between cases 3 and 4 may change with time. Therefore, Brouwer's view here does not amount to the standard possibilist view either—the latter being non-temporal.[7]

In sum, Brouwer's formulations have oscillated—in a rather puzzling way—between the actualist and possibilist ones, that is, between the idea that truth equals actual possession of proof and the idea that truth consist of a mere possibility of constructing a proof. This may have been the source of related unclarity and confusion in the intuitionistic literature ever since.

Be that as it may, let us next consider the views of Arend Heyting, the most important student and follower of Brouwer, who among other things systematized and formalized intuitionistic logic (cf. *Frachella 1994*). Now at first sight, Heyting may seem rather unhelpful in the case of the notion of truth, for according to him, '[t]he notion of truth makes no sense [...] in intuitionistic mathematics' (*Heyting 1958a*). Heyting understands 'truth' here exclusively in the classical correspondence sense: 'One can only speak of genuine mathematical truth if there is a mathematical reality to which it is related. ... to me personally the assumption of an abstract reality of any sort seem meaningless' (*Heyting 1958a*).

However, Heyting is not altogether silent on the choice between the actualist and the possibilist approach. In 1930, Heyting first attacked the classical notion of '*p* is true' for implying the idea of transcendent existence. The continuation is very interesting: he next criticizes Levy for replacing '*p* is true' by '*p* is provable'. One does not, according to Heyting, thus escape the criticism, for '*p* is provable', being equivalent to 'there exists a proof of *p*', implies again the idea of transcendent existence. Instead, one must replace the classical notion with 'one knows how to prove *p*' (*Heyting 1930*). That is, although Heyting prefers to avoid the notion of truth, he nevertheless clearly rebuts here the possibilist notion in favour of the actualist approach. Again, he wrote in 1958: 'We simply cannot speak about the truth-value of a proposition which has neither been proved true nor proved false' (*Heyting 1958b*, p. 109). In the same spirit, he in 1959 explicated the notion of existence actualistically:

> ... object is only considered as existing after it *has been constructed* ... I am unable to give any intelligible sense to the assertion that a mathematical object which has not been constructed, exists. ... A formula of the form $(\exists x)A(x)$ can have no other meaning than: 'A mathematical object x satisfying the condition *has been constructed*'.
>
> *Heyting 1959*, p. 69 (my emphasis)

7 I shall return to views like this (which I shall call 'liberalized actualism') later in the paper.

Heyting, too, was not always faithful to his actualism. In some contexts he rather explains existence in possibilistic terms:

> ...the existence of [a natural number] *N* signifies nothing but the *possibility* of actually producing a number with the requisite property, and the non-existence of *N* signifies the *possibility* of deriving a contradiction from this property. Since we do not know whether or not one of these *possibilities* exists, we may not assert that *N* either exists or does not exist.
>
> *Heyting 1931*, p. 108 (my emphasis)

Nevertheless, it is safe to conclude that on most occasions, Heyting is inclined towards the actualist approach.

For Heyting (at least in the 1950s), however, intuitionistic and classical logic do not as such conflict, for they concern, according to him, wholly different issues; he called the former *the logic of knowledge* and the latter *the logic of being*, i.e. the former express, in his view, what is known as true, whereas the latter what is true (*Heyting 1956a*, *1958b*). There is as such no disagreement here between intuitionism and realism, for even a hard-core platonist agrees that not every statement is at present known to be true or known to be false. However we have also seen that Heyting condemns the logic of being (or truth, i.e. classical logic) as meaningless, leaving the (intuitionistic) logic of knowledge as the only acceptable logic.

Now given the sad history of failure for all attempts to provide an adequate criterion of meaningfulness, one should today be rather suspicious concerning any such complaints of meaninglessness. And the reasons Heyting gives are hardly conclusive; he simply notes that he personally cannot make sense of classical ideas. Further, if one accepts Brouwer's view on meaningfulness, the issue gets truly puzzling, for Brouwer apparently thought that a statement is meaningful only if it has been proved.[8] And certainly it is only with respect to meaningful statements that the question of truth, falsity, or the lack of truth-value can be at all raised. But restricted to them, Brouwer's view apparently entails the principle of bivalence: every *meaningful* statement is either true or false, because every meaningful (i.e. proved) statement is true (i.e. proved).

In sum, it is not altogether easy to form, from the basis of classical intuitionism, a coherent picture of the intuitionistic notion of truth (and existence). It simply seems as if classical intuitionism has not always been aware enough of the deep difference between the actualist and the possibilist view. Still, it has most often gravitated—contrary to the popular impression—towards actualism.

3. Contemporary intuitionism: Dummett and Prawitz

It is quite remarkable that apparently the first spokesman of intuitionism who has clearly distinguished what I have called the actualist and the possibilist conception, has been Michael Dummett, and as late as in the 1970s.[9] But even for him, the issue has not always been that clear, and his views on truth have altered much more than has been generally recognized.

8 See *van Stigt* (*1990*, p. 212); cf. also the quote from *Heyting 1959* above. Brouwer appeals especially often to the meaninglessness of classical mathematics in Brouwer 1912.

9 See, however, the comments in *Heyting 1930*, quoted above.

In his 1963 paper 'Realism',[10] and later in his 1973 book on Frege, Dummett in fact explicitly endorsed the actualist view (see *Dummett 1963*, pp. 163–164; *Dummett 1973*, p. 468). In *Elements of Intuitionism* (*Dummett 1977*) Dummett appeared to prefer the possibilist notion, for he wrote that it would be possible for a constructivist to agree with a platonist that a mathematical statement, if true, is timelessly true: when a statement is proved, then it is shown thereby to have been true all along. To say this, in effect, he continued, is to equate '*A* is true' with 'We can prove *A*' rather than with '*A* has been proved', and '*A* is false' with 'We cannot prove *A*' (*Dummett 1977*, p. 19). But the continuation is rather puzzling: Dummett concludes that 'We can prove *A*' must be understood as being rendered true only by our actually proving *A*, but as being rendered false only by our finding a purely mathematical obstacle of proving it (*Dummett 1977*, p. 19). This seems after all to reduce Dummett's view to some kind of actualism. I must say that I find it difficult to understand what exactly Dummett was trying to say here.

Although Dummett has most often been rather careful not to commit himself too strongly to any particular doctrine, he has nevertheless pointed out, influentially, certain counter-intuitive consequences of the simple actualist temporal notion of truth, that is, the idea that the truth of a sentence consists of our actual possession of a proof (*Dummett 1975a*, *1975b*).[11] In order to present them, one must first make the distinction between a direct, or 'canonical' proof, and an indirect proof, that is, an effective means, in principle, for obtaining a (canonical) proof. Assume next that the actualist view of truth is correct. Must one then possess necessarily a direct proof of a statement, or does an indirect proof of it suffice, for the statement to be true?

Now the first of Dummett's arguments, based on the so-called paradox of inference, concerns the possibility that one equates truth of a statement with the actual possession of a direct (canonical) proof of it. If we assume that an indirect deductive inference must be of some use, that an epistemic advance is in its case possible, then it must be possible that the premise of the inference is recognized to be true without the conclusion having been. It is also plausible to require that the inference is truth preserving. But then, if truth is equated with possession of direct proof, no indirect valid (i.e. truth preserving) inference could be useful (*Dummett 1975b*, pp. 313–314; cf. *Prawitz 1987*, p. 151). Further, if truth is equated with possession of an actual direct proof, 'then we shall have to allow that a statement may be asserted even though it is not known to be true' (*Dummett 1975a*, p. 243).

If, on the other hand, one equates truth of a statement with actual possession of a mere indirect proof of it, then truth will not distribute over disjunction; i.e. it would then be possible that e.g. $A \vee B$ is true (that is, one actually possesses an indirect proof of it) although neither A nor B be is true (because one does not happen to possess proof of either) (see *Dummett 1975a*, p. 243; *Prawitz 1987*, pp. 139, 153). Moreover, '[on] either view, naturally, a valid rule of inference will not always lead from true premisses to a true conclusion, namely if we have not explicitly drawn the inference' (*Dummett 1975a*, p. 243).

10 *Dummett 1963* (not to be confused with *Dummett 1982* with the same title); in his 1978 introduction to the collection in which it was published Dummett regarded this paper 'as very crude and as mistaken in many respects' (*Dummett 1978*, p. xxx).

11 Very recently, Dummett has become less cautious; he now states that '[t]he canons of correct reasoning in intuitionistic mathematics debar us from restricting true statements to those we have actually proved' (*Dummett 2003*, p. 19).

The following example, which derives from Prawitz,[12] nicely illustrates the counter-intuitive character of the actualist notion of truth: Assume that '*A* is true' really meant the same as '*A* has been proved'. Consider then the following credible claim:

(1) If somebody possesses a proof that there exist infinitely many twin primes, then somebody knows a great deal about prime numbers.

But it is far from clear that one can on the basis of (1) conclude, as the actualist view would require, that:

(2) If it is true that there exist infinitely many twin primes, then somebody knows a great deal about prime numbers.

Apparently due to various counter-intuitive consequences such as these, the great majority of contemporary intuitionists have favoured unqualifiedly the possibilist view. However, although Dummett has certainly expressed clearly the problems that the simple actualist view faces, he has never adhered to the standard possibilist view either. Commenting on the above problems with actualism, Dummett once said that it is natural to relax slightly the requirement that a proof should have been explicitly given; the question is, he adds, how far we may consistently go along this path (*Dummett 1975a*, p. 243). And this is indeed a difficult problem. But before discussing in more detail how exactly Dummett has tried to answer to it, let us consider a different approach Dummett has also sympathised with on some occasions.

Namely, in some desperate moments,[13] Dummett has—in order to avoid the difficult problems we shall soon consider—preferred the view that actually intuitionism can dispense with any substantial notion of truth beyond the thinnest, disquotational one.[14] He then granted that if the intuitionist admits a more substantial of notion of truth, he will be driven to posit proofs as objectively existing, as Prawitz has done, but Dummett is certainly not ready to do:[15]

> But if the intuitionist were to admit a notion of truth more substantial than the thinnest, he would have to say that the proposition was true just in case a proof of it *existed*, where its existence was independent of whether the speaker or anyone else knew of it, just as, in fact, Prawitz says. I do not (at least at the moment) believe that the intuitionist needs to admit such a notion at all, involving himself in the labour of explaining that notion of existence.
>
> *Dummett 1994a* (p. 295)

> Instead of allowing ourselves in these difficulties, it seems better to represent a constructivist theory of meaning for mathematical statements as dispensing with the notion of truth altogether.
>
> *Dummett 1982* (p. 259)

12 See *Prawitz* (*1987*, p. 137). Prawitz originally presented it in a somewhat different form in a different context; I have slightly modified it.

13 *Dummett 1982*, *1994a*.

14 In his early classic, 1959 paper 'Truth', Dummett had already recommended accepting the redundancy theory of truth.

15 See below. Note how similar Dummett's view here is to that of Heyting's.

However, the price of this line of Dummett is also high: Usually intuitionists admit that every natural number is either prime or composite, since we have a method that in principle decides the question. But this, Dummett admits, already commits one to a substantial notion of truth:

> The intuitionist sanctions the assertion, for any natural number, however large, that it is either prime or composite, since we have a method that will, at least in principle, decide the question. But suppose that we do not, and perhaps in practice cannot, apply the method: is there nevertheless a fact of the matter concerning whether the number is prime or not? There is a strong impulse to say that there must be: for surely there must be a definite answer to the question what we should get, were we to apply our decision method. If we say this ... [w]e shall nevertheless adopt a notion of mathematical truth more robust than the pure disquotational notion. If the intuitionist follows the strategy I proposed for him, he will have nothing to do with this or any other notion of truth stronger than the disquotational one. In doing so, however, he will require to be resolute: for he is resisting a line of thought that is overpoweringly natural for us.
>
> *Dummett 1994a* (pp. 296–297)

That is, an intuitionist following this strategy is obliged to deny this, in Dummett's own words, 'overpoweringly natural' idea. This austere view, although it aims to avoid the difficult problems related to the more substantial intuitionistic notion of truth, seems to lead to counter-intuitive consequences similar to those of the simple actualist view. Heyting's view seems to face the same problems.

More recently, Dummett has admitted that a semantics for intuitionistic mathematics needs a notion of truth (*Dummett 1998*; see also *Dummett 2003*). Dummett has on several occasions favoured a view that bears certain affinity to the view endorsed by the later Brouwer—he considered it already in 1975 as a solution to the above-mentioned problems with strict actualism (*Dummett 1975a*), and it is his last published word on the issue (*Dummett 1998*; see also *Dummett 1987*). According to this view, a statement is true if we are in fact in possession of a means of obtaining a proof of it, whether or not we are aware of the fact. Let us call this middle position 'liberalized actualism'.

But even though this liberalized notion avoids some of the problems of the simple actualism mentioned above, it still makes truth significantly temporal (as Brouwer explicitly pointed out, and as Dummett himself has at least in one occassion explicitly admitted (*Dummett 1994b*), and for many, this may be a reason enough to rebut it.[16]

Prawitz has remarked that the analysis of such activities as conjecturing and wondering seems to demand another notion of truth than the one suggested by Dummett here (*Prawitz 1998b*). I think that the criticism in these terms could be actually developed much further. Consider for example the following case:[17]

(3) Bob wonders whether the hypothesis H is true.

16 Indeed, Dummett himself has often expressed the desirability of a non-temporal notion of truth.

17 To make the examples more concrete: take H to be e.g. the Paris-Harrington sentence (finitary Ramsey's theorem), and assume that the methods of proof available in Bob's time are limited to (a conservative extension of) Heyting Arithmetic **HA**, which does not prove H. It is, however, provable with the help of (intuitionistically acceptable) transfinite induction, which we may imagine to become accepted only much later.

If we follow Dummett's explication of the notion of truth, this should be equivalent to:

(4) Bob wonders whether the hypothesis *H* can be proved by the methods we presently possess.

But it should be abundantly clear that (3) and (4) express quite different thoughts, and are in no way equivalent.

Or let us imagine that Bob conjectures that *H* is true. Assume then that it was shown, in the distant future, that:

(i) *H* is actually undecidable by means of the mathematical methods of Bob's time;

and:

(ii) *H* is provable by some new methods unknown at Bob's time.

Should one then conclude that Bob's conjecture that *H* is true was incorrect? Certainly not! But this should be the case if one accepted Dummett's explication of the notion of truth.

To put the point somewhat differently, we are apparently capable of acquiring new methods of proofs, but Dummett's position cannot account for the idea that these must be sound or correct—or, in terms of axioms, the idea that when we introduce new axioms, these must be true. The liberalized actualist notion of truth favoured by Dummett attributes no truth-value to a statement undecidable by our present axioms and methods, and seems to allow us to extend our theories arbitrarily. Thus this view seems to lead to radical conventionalist with respect to all such presently undecidable statements—a view Dummett seemingly cannot accept (see e.g. *Dummett 1993*).

Finally, as the totality of the methods of proof we possess at an any given time is presumably in some sense finite, it is natural to assume that they can be captured by a formalized system;[18] but if this were the case, one could conclude, by Gödel's theorem, that there is a true sentence which is not provable by these methods, contrary to the claim that provability (by these methods) and truth coincide.

Prawitz, on the other hand, has always seen the possibilist notion of truth as the only plausible option, convinced by the considerations such as those above. However, the long dispute between him and Dummett on the notion of truth (*Prawitz 1980*, *1987*, *1994*, *1998a*, *1998b*; *Dummett 1980*, *1987*, *1994a*, *1998*) has revealed how extremely difficult it is to formulate more exactly the possibilist intuitionistic conception of truth in a coherent way.

That is, the attempts to explicate the possibilist notion have led Prawitz, as was noted above, to postulate an objective realm of proofs: 'We may say that a mathematical sentence is true if there exists a proof of it, in a tenseless or abstract sense of exists' (*Prawitz 1987*, p. 153). Prawitz admits that the identification of truth with the mere possibility of a verification not necessarily realized by us 'gives some kind of objective reality to these verifications' (*Prawitz 1994*, pp. 88–89).

18 The situation is even worse; the Gödelian argument goes through if one merely assumes that these methods are, even if not captured by a formalized system (i.e. not recursively enumerable, or Σ^0_1), at least somewhere in the arithmetical hierarchy (i.e. Σ^0_n for some *n*); cf. Appendix.

Dummett, however, has found this unacceptable, for it leads, according to him, either back to realism (i.e. to bivalence, which for Dummett is the essence of realism) or to the conclusion that there are statements that absolutely have no truth-value, which, according to Dummett, makes no sense in intuitionism (see below):

> There is a well-known difficulty about thinking of mathematical proofs ... as existing independently of our hitting them, which insisting that they are proofs we are capable of grasping of giving fails to resolve. Namely, it is hard to see how the equation of the falsity of a statement (the truth of its negation) with the non-existence of a proof or a verification could be resisted: but, then, it is equally hard to see how, on this conception of the existence of proofs, we can resist supposing that that a proof of a given statement determinately either exists or fails to exist. We shall then have driven ourselves into a realist position, with a justification of bivalence.
>
> If we refuse to identify falsity with the non-existence of a proof, we shall be little better of, because we shall find it hard to resist concluding that there are statements which are determinately neither true of false ...
>
> *Dummett 1987* (p. 285)

And as has been mentioned, one can argue[19] that the latter alternative is inconsistent with intuitionism. The argument goes as follows: Assume that A is a sentence such that neither A nor $\neg A$ can be proved; A cannot be proved only if it can be proved that A cannot be proved (since truth amounts to provability), but such a proof constitutes proof of $\neg A$,[20] contradicting the assumption that $\neg A$ cannot be proved. Prawitz also agrees with this, see e.g. *Prawitz 1994*, *1998b*.

Prawitz has nevertheless defended his view of proofs as existing independently of our knowledge of them against Dummett's criticism, see *Prawitz 1994*, *1998a,b*. He says that although his view contains a flavour of realism, it is consistent with intuitionism because one need not take the disjunction 'either there exists a proof of A or there does not exist a proof of A' classically, but that one can interpret such a disjunction intuitionistically (*Prawitz 1998a*).

Although this reply enables one to avoid the fatal consequences Dummett mentions, it seems to me, nevertheless, to be highly problematic. It is a basic idea of intuitionism that provability is a fundamental notion in terms of which the meanings of logical constants are explained. If the explication of the notion of provability in turn presupposes intuitionistic interpretation of logical constants, the whole account appears to be viciously circular, or to lead to an infinite regress. Interestingly, Dummett has in one occasion[21] made essentially the same point; viz. in his seminal 'The philosophical basis for intuitionistic logic', Dummett stressed repeatedly that it is 'wholly legitimate, and, indeed, essential' to require that the intuitionistic notion of truth must be explainable in terms which do not already presuppose the intuitionistic understanding of the logical constants. For otherwise, intuitionists have no way of conveying what it is that they were about to anyone who does not already accept their ideas, and communication between intuitionists and non-converts would be

19 Cf. note 5.

20 This is because 'A is not provable' is true (according to intuitionism) if and only if it is provable; a proof that A is not provable entails that $A \to B$ holds for any B, and thus in particular that $A \to \bot$, that is, that $\neg A$, cf. *Dummett* (*1977*, p. 17).

21 Not as a reaction to Prawitz but already before their debate begun.

impossible (*Dummett 1975a*, pp. 237–239). Further, if one admits that proofs have an independent existence, it is quite unclear what motivation there is left for not allowing the use of classical logic here.

Moreover, if one is prepared to be realist with respect to proofs, it is unclear why one could not be realist with respect to, say, natural numbers,[22] as Dummett again has remarked in another context: 'If we admit such a conception of proofs [as existing independently of our knowledge], we can have no objection to a parallel conception of mathematical objects such as natural numbers, real numbers, metric spaces, etc.; and then shall have no motivation for abandoning a realistic, that is, platonist, interpretation of mathematical statements in the first place' (*Dummett 1982*, pp. 258–259). Here too I can only agree with Dummett.

In sum, if one accepts the objective existence of proof, the only way to avoid being totally unfaithful to the whole spirit of intuitionism and to be lead to full-blown realism, is evidently to appeal to already understood intuitionistic interpretation of logical constants, which is seriously question-begging. Poincaré once remarked that people do not understand each other, because they do not speak the same language, and because there are languages that cannot be learned.[23] In Prawitz's hands, intuitionism threatens to turn into such a language.

4. A dilemma for the possibilist intuitionism

Let us now focus on the possibilist account of truth, which seems to be dominant in the contemporary intuitionism.[24] Whether or not it must necessarily commit itself to an objective existence of proofs, there are certain considerations that in my mind point out a decisive problem for it.

To begin with, it should be clear that the possibilist interpretation of intuitionist truth must lean on some notion of possibility. For *A*'s provability in principle—in contradistinction to the actual possession of a proof of *A*—means that it is (in some sense) *possible* to prove *A* (and similarly for the possibilist notion of existence). But one may then ask exactly what kind of possibility is meant here by the intuitionists.[25]

Certainly one cannot mean here *physical possibility*, nor psychological or in general empirical possibility; intuitionists of different variants all agree with this: an intuitionist must accept a certain amount of idealization and abstraction from such empirical, non-mathematical contingencies. Otherwise one may be forced to conclude a mathematical statement is false for some purely non-mathematical reasons.[26] For it is obvious that there are many mathematical statements for which it is psychologically or physically impossible for us to construct a proof, but which would be provable in

22 The set of natural numbers is effectively generated, and the standard model of arithmetic is recursive (it is the only recursive model of PA), whereas the totality of proofs (if the idea is assumed to make sense) cannot be even arithmetical; see Appendix. Hence the former notion is much more simple and accessible than the latter. Therefore, it would be quite preposterous to be realist only with respect to the latter but not with respect to the former.

23 Quoted in *Brouwer 1912*!

24 Although, as I have argued, it is a widespread mistake to count Dummett as an adherent of this view.

25 While later surveying the literature, I have found out that the same question ('What kind of possibility?') has been raised, and sometimes a related argument hinted, by others: *Parsons 1983* (passim), *1986*, *1997*, *George 1993*, and *Moore 1998*. For me, however, the argument occurred independently of these; it was rather inspired by Sellars' criticism of phenomenalism (see *Sellars 1963*). Interestingly, also Parsons (*1986*) notes the affinity between intuitionism and phenomenalism.

26 Cf. e.g. *Brouwer 1933*, *Heyting 1956b*, *Troelstra* (*1969*, pp. 3–4), *Dummett* (*1977*, p. 19), *Prawitz 1987*, *1994*, *Martin-Löf 1995*.

principle—even by the methods we now possess. The possibility of proof intended by the intuitionists must therefore be wider and more abstract than just the physical possibility.

On the other extreme, there is the notion of *logical possibility*. And this is, in fact, the choice of Per Martin-Löf. He says that the notion of possibility, implicitly built into the intuitionistic notion of truth, 'is the notion of logical possibility, or possibility in principle, as opposed to real, or practical, possibility, which takes resources and so on into account' (*Martin-Löf 1995*, p. 193).

Now what is logically possible is obviously relative to a logic one uses. But if intuitionistic logic is already assumed, a vicious circle threatens again. Also a strange phenomenon occurs: The more metaphysically cautious and hence restricted logic one chooses, the more becomes logically possibly provable and hence true. If we, on the other hand, try to formulate the notion of logical possibility, or of freedom from contradiction, without assuming any particular logic, we can only speak about immediate contradictions—and freedom from them. Then any judgement that is not immediately contradicting the set of previously somehow chosen set of truths (for example, the standard axioms of elementary arithmetic) is free from contradiction and hence possibly provable and true. For example, Goldbach's conjecture and its negation are both possibly provable and hence true; but then a contradiction would be true.

The same problem occurs even if one would be allowed to use, say, intuitionistic logic. Apparently for a very few sentences, the possibility of proof of the sentence is ruled out by pure logic—intuitionistic or whatever. Certainly for most sentences, both the provability of the sentence and the provability of its negation are logically possible. However if this makes them true, then a contradiction is also true.

It thus appears that the idea that the notion of possibility used in the intuitionistic notion of truth is logical possibility leads to various intolerable problems. Some further constraints than just logical consistency are needed to delimit the possibility of proof. Hence the possibility in question must be something narrower than logical possibility.

Between physical and logical possibility, the kind of possibility that most naturally suggests itself here is certainly *mathematical possibility*—it is even hard to imagine what other kind of possibility it could be.[27] And indeed, some passages of e.g. Dummett, where he speaks about 'a purely mathematical obstacle of proving' a given sentence A, seem to suggest that it may be mathematical possibility that intuitionists have had in mind when they have spoken about provability in principle. It appears, however, that the notion of mathematical possibility amounts to the consistency with mathematical truth, and therefore already assumes the notion of mathematical truth. But that would make the possibilist explication of truth absolutely circular. The more restricted interpretation, which requires a positive guarantee for the possibility, i.e. its provability, is equally question-begging.

There does not seem to exist any clear way to explicate the intuitionistic notion of truth in the possibilist terms which does not either have unbearably implausible consequences or is not hopelessly circular.

27 Timothy Williamson once suggested in conversation that there is one more option, namely *metaphysical* possibility. I am personally inclined to agree, but the resulting view would certainly be quite alien for the anti-metaphysical spirit of intuitionism, and I doubt that no intuitionist is prepared to base intuitionism on this notion.

5. Conclusion

I will end by recalling what Heyting once said is the aim of intuitionism: 'We look for a basis of mathematics which is directly given and which we can immediately understand without philosophical subtleties' (*Heyting 1974*, p. 79). It is arguable that after almost a hundred years of intensive attempts, intuitionism has not yet succeeded in this. Above, we have examined the three basic choices there are for the intuitionistic theory of truth, the strict actualism, the liberalized actualism and possibilism, and found all them wanting.

6. Appendix: On the knowability of intuitionistic proofs

In this Appendix, I shall deal solely with the later intuitionism which has a more positive view of logic than the orthodox intuitionism of Brouwer, according to which mathematics is absolutely independent of logic. Indeed, much of contemporary intuitionism, or constructivism, views mathematics simply as deriving theorems with the help of intuitionistic logic from intuitionistically acceptable axioms (and whatever principles used in proofs not covered by intuitionistic logic). Thus e.g. Bridges says that in practice what the contemporary constructive mathematicians are doing amounts to 'doing mathematics with intuitionistic logic' (*Bridges 1997*).

Another popular trend in the present-day intuitionism is to emphasize that one should recognize a proof when one sees one.[28] This idea derives from Kreisel, and has been pressed repeatedly especially by Dummett. More formally, it is expressed by the requirement that the proof relation must be decidable. It is indeed arguable that such a requirement is necessary for the intuitionistic epistemology. It also harmonizes well with Heyting's view that '[a] mathematical construction ought to be so immediate to the mind and its result so clear that it needs no foundation whatsoever' (*Heyting 1956b*, p. 6).

The whole picture I want to consider here is beautifully expressed by Sundholm: 'Proofs begin with immediate truths (axioms), which themselves are not justified further by proof, and continue with steps of immediate inference, each of which cannot further justified by proof' (*Sundholm 1983*, p. 162). I shall next argue that the two above ideas are incompatible. (Interestingly, also Beeson (*1985*) denies the decidability of proof relation. He ends up with this conclusion somewhat differently than the way I do.)

For simplicity, let us focus on the provability in the language of arithmetic L(HA). Now given a finite sequence of formulas, it is certainly possible to check effectively whether every step in it is an application of intuitionistically acceptable rule of inference. But how about the premises? Only if one can in addition see that all the premises of a derivation are intuitionistically true one can say that one has a proof of the conclusion at hand. This is at least in principle possible if axiomhood is a decidable property. However, in the intuitionistic setting, it cannot be! For if it was, the intuitionistic provability could be captured by a formalized system. And then, by Gödel's theorem, there would be truths that are unprovable, contrary to the basic principle of intuitionism, which equates truth with provability.

The situation is actually even much worse—I doubt that it is generally realized how bad it really is. Not only must the set of admissible axioms be undecidable. It cannot be semi-decidable, i.e. recursively enumerable (Σ^0_1), it cannot be Trial-and-Er-

28 See e.g. *Sundholm 1983* and the many references therein.

ror decidable (Δ^0_2); it cannot be anywhere in the arithmetical hierarchy (not Σ^0_n for any *n*). (Here I assume that the notions of arithmetical hierarchy, or at least the idea of being definable in the language of arithmetic, make sense; in practice quite many contemporary intuitionists seem to accept them.) For assume that the property of being an admissible axiom were definable by an arithmetical formula (however complex). This implies that also provability is definable in the language of arithmetic. Then one can apply Gödel's technique and construct a statement of the language which is unprovable but true.

Thus the totality of intuitionistically provable sentences (already, restricted to L(HA) i.e. the arithmetical sentences) necessarily is non-arithmetical, i.e. at least hyperarithmetical (Δ^1_1). But this means that they are just as abstract and inaccessible as truth in classical arithmetic. The same holds already for the alleged axioms, that is, 'the immediate truths'. But certainly non-arithmeticality makes the sphere of 'the immediate truths' implausibly complex and inaccessible. If one cannot tell whether the premises used in a derivation are acceptable, that is, true, or not, one cannot tell whether one has a genuine proof before one's eyes or not, contrary to the standard assumption of contemporary intuitionism.

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