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LEON HORSTEN and PHILIP WELCH (eds.), *Gödel's Disjunction: The Scope and Limits of Mathematical Knowledge*. Oxford: Oxford University Press, 2016. 288 pp. $110.00. ISBN 9780198759591

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Austrian-born Kurt Gödel is widely considered the greatest logician of modern times. It is above all his celebrated incompleteness theorems – rigorous mathematical results about the necessary limits of any formalised theory – that have earned him this fame.

There have been many ambitious attempts to draw out the philosophical consequences of these results (see *Raatikainen 2005*; *Franzén 2005*). The view that the human mind is in some sense equivalent to a finite computing machine is commonly called 'mechanism'. Lucas (*1962*), for example, has argued that Gödel's results prove conclusively that the human mind can surpass any computing machine, and that mechanism is false. However, his argument is controversial; many experts think it is simply flawed.

Gödel himself was quite cautious about outlining the strong philosophical implications of his results. However, he did suggest a more careful philosophical conclusion of a disjunctive form. There are different formulations, but the common idea is the following:

(GD) Either the human mind (even within the realm of pure mathematics) can surpass the power of any finite computing machine, or there are absolutely undecidable mathematical problems.

Gödel characterised this as a 'mathematically established fact'. The epithet 'absolutely' here means that 'they would be undecidable, not just within some particular axiomatic system, but by any mathematical proof the human mind can conceive' (*Gödel 1951*, p. 310). Furthermore, Gödel suggested that the philosophical implications are, by either alternative, 'very decidedly opposed to materialistic philosophy' (*ibid*.).

The former conclusion is now widely known as 'Gödel's disjunction' (in short: GD), hence the title of the book at hand. The volume aims to bring together the best up-to-date knowledge related to GD, and to illuminate it from various perspectives. The book includes ten original articles, and a substantial and very helpful introduction by the editors.

Walter Dean seeks to analyse the metaphysical status of algorithms, in particular the view he calls 'algorithmic realism'; this is the view according to which algorithms can be considered mathematical objects. Though popular, Dean finds this position to be ultimately problematic. The opposite of algorithmic computability is randomness, and Joan Rand Moschovakis' chapter provides a survey of the notions of randomness and lawless sequences.

Gödel's second incompleteness theorem concerns, in essence, the unprovability of a theory's consistency in the theory itself. It is quite well known that there are certain technical obstacles to proving a fully general result. The chapter by Albert Visser is a valuable survey of certain more recent research, by e.g. Harvey Friedman, Pavel Pudlák, Craig Smorynski, and Visser himself, which revolves around the second incompleteness theorem and aims at more general results. This work is not yet widely known, and consequently, this review fills a gap in the literature.

Graham Leach-Krouse compares Gödel's views to those of another pioneer in the field, the American logician Emil Post, who independently achieved results similar to those of Gödel. These two figures both held interesting philosophical views about mechanism and absolute unprovability, although they sometimes pulled in opposite directions. Leach-Krouse's discussion is not, however, purely historical; it has a systematic aspect too.

The notion of an absolutely undecidable problem in GD is grounded on the concept of provability-in-principle, or absolute provability. *Epistemic Arithmetic* is a formal framework initiated by Stewart Shapiro (*1985*) and William Reinhardt (*1986*) in which a formal arithmetical theory is enriched with a modal operator whose intended interpretation is *absolute provability*. There has been some debate about the usefulness of this approach (see *Horsten 1998*), but it has proved to be quite a fruitful formal tool for the rigorous study of issues such as GD. The chapters by Timothy Carlson and by Marianna Antonutti Marfori and Leon Horsten are contributions to this research programme; they examine in particular issues around the so-called Epistemic Church's Thesis (ECT). Carlson goes on to study what he calls 'knowing machines'. Antonutti and Horsten demonstrate an analogue of GD in this context formulated in terms of ECT. The chapter by Theodora Achourioti is also related to this general programme; it considers an alternative semantic interpretation of the modal operator of absolute provability.

More traditionally, it has been common to take it for granted that the notion of absolute provability is sufficiently well understood. However, in different but complementary ways, Shapiro, Timothy Williamson and Peter Koellner all raise doubts about the very clarity of the concept. Shapiro's contribution is a compact and very clear systematic discussion of GD. He points out that any systematic discussion of mechanism and Gödel's theorems must presuppose a certain amount of idealisation. However, Shapiro argues that it is doubtful there would be any sufficiently sharp and stable notion of idealised human knowability which would support a Gödelian anti-mechanist argument. Williamson argues that we are on a slippery slope with the notion of absolute provability: there is no principled stopping point, and on closer inspection, any mathematical truth is provable-in-principle.

However, it is above all Koellner's chapter – at least to the present reviewer's mind – that is the true gem of the collection, as it really takes the discussion concerning GD to a wholly new level. Gödel himself believed that if we had an adequate paradox-free theory of truth, it would be possible to demonstrate the first anti-mechanist disjunct – he did not consider the existence of absolutely unsolvable problems plausible. Accordingly, Koellner tries different formal theories of truth: both the Tarskian approach involving a hierarchy of languages and a hierarchy-free Kripke-Feferman-style approach. Building on the earlier work of Reinhardt, and assuming for the sake of argument that the notion of absolute provability is well defined, Koellner shows that GD can be rigorously demonstrated in the setting of Epistemic Arithmetic. However, he also shows that the prospects of demonstrating either disjunct are dim. Koellner also puts forward the possibility that, under some interpretations, both disjuncts of GD may be separately 'absolutely undecidable'.

It is perhaps inevitable that any collection of this kind is at least a bit uneven. All in all, however, this book is a major contribution to this interesting and important topic, and obligatory reading for anyone interested in issues related to GD.

Errata

p. 5: ‘Liar sentence which says of itself that it is true’ – should read: ‘…says of itself that it is false’ (or ‘… is not true’).

References

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