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Intuitionistic Logic and Its Philosophy

Formally, intuitionistic logic differs from classical logic in its denial of the universal validity of the law of the excluded middle, in short LEM (or the rule of double negation, which amounts to the same). This difference is based on the specific proof-interpretation, or BHK-interpretation (BHK stands for Brouwer-Heyting-Kolmogorov) of the meanings of logical connectives. It explains the meaning of the logical operators by describing the proofs of logically compound statements in terms of the proofs of their immediate subformulas.

The BHK-interpretation of the sentential connectives goes as follows:[1]

(1) There is a proof of $\neg A \Leftrightarrow$ there is a procedure for transforming any proof of A into a proof of $\perp$ ('absurdity' or 'the contradiction').

(2) There is a proof of $A \wedge B \Leftrightarrow$ there is a proof of *A* and there is a proof of *B*.

(3) There is a proof of $A \vee B \Leftrightarrow$ either there is a proof of *A* or there is a proof of *B*.

(4) There is a proof of $A \rightarrow B \Leftrightarrow$ there is a procedure for transforming any proof of *A* into a proof of *B*.

Now under such an interpretation of logical constants, LEM apparently fails to be valid. Namely, a proof of a disjunction $A \vee \neg A$ requires a proof of *A* or a proof of $\neg A$. But for some *A*, there may not exist a proof of either.

[1] I'll focus here only on propositional logic.

However, taken on their own, without any further motivation, these meaning explanations may appear only as arbitrary, conventional stipulations on using e.g. $\vee$ and $\neg$ differently (cf. Dummett 1977, p. 18). One may feel that there is no real conflict between intuitionistic and classical logic because the meanings of logical constants are defined differently. But certainly this is not what intuitionists have wanted to argue. Intuitionistic logic is rather put forward as a challenge and a genuine alternative to classical logic. However, nothing that has been said thus far gives a principled reason for rejecting the classical interpretation of logical constants in favor of the proof interpretation.

The proof interpretation can be given, and has been given, a principled motivation (cf. Dummett 1977, p. 17). This goes via the notion of truth and the Principle of Bivalence (in short, PB). This is because classical logic commits itself to PB, that is, to the claim that every statement is either true or false. The classical interpretation of logical connectives (given e.g. by truth tables) is indeed based on the assumption that every statement is either true or false, whether we can know this or not (in the mathematical context, whether we can prove or refute the statement, or not). But this assumption, intuitionists protest, at least tacitly assumes the existence of a mind-independent reality of mathematical objects. However, the latter is, according to intuitionists, a bold metaphysical assumption to which they do not want to commit themselves. Intuitionism, in contrast, prefers to explicate truth in terms of proof. From the intuitionistic point of view, to say that a mathematical statement is true amounts to the claim that there is a proof of the statement. In words of Troelstra and van Dalen: "A statement is *true* if we have a proof of it" (1988, p. 4). It is this understanding of the notion of truth that motivates the proof interpretation.

However, the phrase "there is a proof of the statement", or "we have a proof of it", can be understood in two radically different ways. First, it can be taken to mean that we have *actually proved* the statement. Second, it can be understood only as meaning that it would be *possible* for us to *prove* the statement. I have called these two interpretations the actualist view and the possibilist view, respectively (Raatikainen 2004). Although both views have had proponents in the intuitionistic camp, I shall argue that they have radically different implications concerning intuitionistic logic. But before going to that issue, let us clarify certain fundamental background issues.

The Law of the Excluded Middle and the Principle of Bivalence

Quite regularly in the literature LEM is not clearly distinguished from PB, but one easily slides from one to the other. Brouwer, for example, sometimes talks about LEM, but when he explains what it is, what he gives is in fact PB. Nevertheless, one should be aware of their difference. As the details of their exact relation seem to be often incompletely understood, they are worth a closer look.

The Law of the Excluded Middle (LEM) is a schema in the object language: $A \vee \neg A$. Each instance of it may be taken, according to the classical logician, as a logical axiom (of course, not every formalization of classical logic in practice does exactly that; but they all have something logically equivalent to this). The Principle of Bivalence (PB), on the other hand, is a single universal statement in the metalanguage: every statement (of the object language) is either true or false. There is certainly a close connection between LEM and PB, but they are not trivially equivalent. There are logical systems (e.g. certain many-valued logics and supervaluational languages) in which LEM is valid but PB does not hold, and *vice versa* (see e.g. van Fraasen 1966, Day 1992, DeVidi & Solomon 1999). However, neither the intuitionist nor the classical logician can accept any such logics. Hence this possibility is not relevant for the dispute between them. They both make certain assumptions which rule out such alternatives. These are hardly ever stated explicitly, and there are some alternatives, depending also on how exactly one formulates LEM.

However, a plausible candidate is the set of the following basic principles governing the notion of truth:

(T1) A is true $\Rightarrow$ A

(T2) A $\Rightarrow$ A is true

(F1) A is false $\Rightarrow$ $\neg$A

(F2) $\neg$A $\Rightarrow$ A is false

(T1) and (T2) together form the familiar Tarskian T-equivalences. Principles (F1) and (F2) may be taken simply as the definition of falsity.

With the help of these principles, we can easily derive, for a given language L, PB and LEM from each other.

More exactly, we assume first order minimal logic as our background logic such that its rules of inference are applicable to any first order language. Let L be an arbitrary first order language, and let L′ be an extension of L which also contains two predicates $T(x)$ and $F(x)$ not in L, and individual constants $\ulcorner A \urcorner$, $\ulcorner B \urcorner$, … whose intended meaning is to denote sentences A, B, … of object language L.

We work in L′, and show that the assumption of the Principle of Bivalence for L entails the Law of the Excluded Middle for L, and *vice versa*, given (T1)-(F2) for $T(x)$ and $F(x)$, that is:

(T1) $T(\ulcorner A \urcorner) \rightarrow A$

(T2) $A \rightarrow T(\ulcorner A \urcorner)$

(F1) $F(\ulcorner A \urcorner) \rightarrow \neg A$

(F2) $\neg A \rightarrow F(\ulcorner A \urcorner)$

Nothing else is assumed about the interpretation of $T(x)$ and $F(x)$. Now let A be an arbitrary sentence of L. The derivations then go as follows:

$$\dfrac{T(\ulcorner A\urcorner)\vee F(\ulcorner A\urcorner) \qquad \dfrac{\dfrac{[\,T(\ulcorner A\urcorner)\,]^1 \qquad T(\ulcorner A\urcorner) \to A}{A}}{(A\vee\neg A)} \qquad \dfrac{\dfrac{[F(\ulcorner A\urcorner)\,]^1 \qquad F(\ulcorner A\urcorner) \to \neg A}{\neg A}}{(A\vee\neg A)}}{(A\vee\neg A)}\ {}_1\ \vee\mathrm{E}$$

$$\dfrac{(A\vee\neg A) \qquad \dfrac{\dfrac{[A\,]^1 \qquad A\to T(\ulcorner A\urcorner)}{T(\ulcorner A\urcorner)}}{T(\ulcorner A\urcorner)\vee F(\ulcorner A\urcorner)} \qquad \dfrac{\dfrac{[\neg A\,]^1 \qquad \neg A\to F(\ulcorner A\urcorner)}{F(\ulcorner A\urcorner)}}{T(\ulcorner A\urcorner)\vee F(\ulcorner A\urcorner)}}{T(\ulcorner A\urcorner)\vee F(\ulcorner A\urcorner)}\ {}_1\ \vee\mathrm{E}$$

These derivations do not use any essentially classical principles. Indeed, they go through even in minimal logic.[2]

The further question of whether “*A* is true ∨ *A* is false” *really* expresses bivalence remains. There are systems of logic where “*A* is true ∨ *A* is false” holds, but which nevertheless have non-two-valued semantics (see DeVidi & Solomon 1999). However, such systems are, again, acceptable neither for the intuitionist nor the classical logician. Hence I shall ignore this possibility here, and assume that “*A* is true ∨ *A* is false” indeed correctly expresses PB.

[2] Minimal logic is the logic one obtains from intuitionistic logic by dropping the intuitionistic absurdity rule, which allows any statement to be derived from the absurdity.

No absolutely undecidable propositions

Let us next focus on an essential but often insufficiently understood characteristic of intuitionistic logic. We may capture it with the following slogan:

(NAUS) There are no absolutely undecidable statements.

More exactly, it is inconsistent for the intuitionistic logician to maintain, concerning any particular statement *S*, that there is neither a proof nor a refutation of *S*.

The main idea of the argument goes as follows: If an intuitionist logician is warranted to say that for a specific statement *S*, there is no proof of *S* and no proof of ¬*S*, the intuitionist logician must in particular possess a refutation of the assumption that there is a proof of *S*, that is, possess a proof of $S \rightarrow \perp$. But this constitutes proof of ¬S, contradicting the assumption that there is no proof of ¬S (see e.g. Dummett 1977, p. 17, van Atten 2004, p. 26; Heyting 1958). A somewhat different way to see essentially the same point is to note that one can derive, in intuitionistic logic (actually, even in minimal logic), a contradiction from the assumption $\neg (A \vee \neg A)$ – or, with (T1)-(F2), from ¬(A is true) ∨ ¬(A is false) (see the next section).

An interesting but bewildering historical detail is that already early on, Brouwer seems to have been aware of this argument, although he never published it (see van Dalen 2001, p. 174, van Atten 2004, p. 25-26). I find this puzzling for more than one reason. First, as I shall argue below, this idea does not harmonize very well with Brouwer's official, actualist view on truth. Second, Van Stigt submits that in the 1920s and 30s Brouwer attempted to provide a concrete example of an absolutely unsolvable problem which would convince the wider audience of the non-validity of LEM (see van Stigt 1990, p. 252–54; also van Atten (2004) seems to grant this, see endnote 18). Van Dalen (2001, p. 174; cf. van Atten 2004, 26), on the other hand, suggests that the fact that Brouwer had the above argument explains why he never searched for an absolutely undecidable proposition. But be that as it may, the fact remains that intuitionistic logic necessarily leads to NAUS.

NAUS is closely related to the issue of whether intuitionistic logic can be interpreted as a three-valued logic. Glivenko (1928) proved that it can not (see also Mancosu & van Stigt 1998). One may note in passing that Glivenko's proof too holds also for Minimal Logic. A few years later Gödel showed that intuitionistic logic is not *n*-valued logic for any natural number *n* (Gödel 1932).

Have we assumed bivalence?

Above, we have leaned, and will lean again below, on certain principles related to the notion of truth, namely the principles (T1)-(F2). However, it has been often argued that such principles entail the Principle of Bivalence (see e.g Haack 1975, p. xx). Let us see how the argument is supposed to go.

Assume that for some statement A, PB fails, that is, that $\neg T(\ulcorner A \urcorner) \wedge \neg F(\ulcorner A \urcorner)$. We can then reason, so the argument goes, as follows:

$$\dfrac{\dfrac{\dfrac{[A]^1 \quad A \to T(\ulcorner A \urcorner)}{T(\ulcorner A \urcorner)} \quad \dfrac{\neg T(\ulcorner A \urcorner) \wedge \neg F(\ulcorner A \urcorner)}{\neg T(\ulcorner A \urcorner)}}{\dfrac{T(\ulcorner A \urcorner) \wedge \neg T(\ulcorner A \urcorner)}{\neg A}\,\neg I\,1} \quad \dfrac{\dfrac{\dfrac{[\neg A]^2 \quad A \to F(\ulcorner A \urcorner)}{F(\ulcorner A \urcorner)} \quad \dfrac{\neg T(\ulcorner A \urcorner) \wedge \neg F(\ulcorner A \urcorner)}{\neg F(\ulcorner A \urcorner)}\,\wedge E}{F(\ulcorner A \urcorner) \wedge \neg F(\ulcorner A \urcorner)}\,\neg I\,2}{\neg\neg A}}{\neg A \wedge \neg\neg A}$$

We have thus derived a contradiction from the assumptions. The argument goes again through in intuitionistic and even in minimal logic. So it does not make any steps that essentially rely on classical logic.

Now there are ways to resist this argument (see e.g. Holton 2000, Beall 2002). However, they involve many-valued logics, deviant connectives and several distinct negations. Therefore, they are not relevant for the debate between the classical and the intuitionistic logicians, for this way is not open for either. So in this context, apparently we must accept the above line of reasoning.

Does this mean that the principles (T1)-(F2) are not available to the intuitionist? Or that appeal to them is illegitimate in the present context? I don't think so; the argument is in fact based on the assumption that we can exhibit a particular statement *A* which is neither true nor false. But we have seen that it is exactly this case that intuitionistic logic does not allow (NAUS). And without that assumption the principles (T1)-(F2) seem to be harmless and acceptable even from the intuitionist point of view.

Actualism

Intuitionism as a distinct approach in the philosophy of mathematics was founded by L.E.J. Brouwer in the beginning of the 20th century. It was Brouwer who first started to criticize the use of classical logic, and LEM in particular, in mathematics. This was often based on his view on truth. Most of Brouwer's statements on truth seem to commit him to straightforward actualism, according to which a statement is true only when it has been proved; truth for him is thus significantly temporal (see Raatikainen 2004; cf. van Atten 2004, p. 18-20). Under this interpretation PB, and consequently LEM, is obviously false. Indeed, Brouwer often stated that each mathematical statement which is at the present neither proved nor refuted provides a counterexample for LEM (cf. Raatikainen 2004). Arend Heyting, the most important student and follower of Brouwer, who systematized and formalized intuitionistic logic, did not say much about truth directly. Nevertheless, it can be argued that on most occasions, Heyting too is inclined towards the actualist approach (see Raatikainen 2004). But be that as it may, my real interest here is not to do an exegesis of Brouwer or Heyting, but consider systematically of the relations between actualism and intuitionistic logic.

As noted, the actualist interpretation of the existence of proofs and truth, seemingly favored by the founding fathers of intuitionism, certainly entails that PB and LEM are not valid. However, it does not follow that intuitionistic logic can be founded on the actualist view. In fact, although this has rarely been clearly noted, the actualist view and intuitionistic logic are in conflict.

Consider for example Goldbach's Conjecture, which is at present neither proved nor refuted. So according to the actualist understanding of truth and falsity, we know that it is neither true nor false. But introduce now intuitionistic logic and NAUS, and you must conclude that we know that Goldbach's Conjecture is false. But it was assumed that it is not false. This is not only counterintuitive; it is contradiction. We can simply use the argument of the previous section, which derived in intuitionistic logic (with the help of (T1)-(F2)) a contradiction from the assumption $\neg(A \text{ is true}) \wedge \neg(A \text{ is false})$.

Van Atten suggests that there are reasons to doubt that Brouwer agreed with all of the rules of intuitionistic logic. In particular, he has doubts about 'ex falso sequitur quodlibet', that is, about the rule which allows one to derive anything from a contradiction (van Atten 2004, p. 24).[3] Not only am I inclined to agree, but I think that even more is true. The derivation of contradiction above (from the assumption that Goldbach's Conjecture is neither true nor false) did not use 'ex falso sequitur quodlibet'. Hence, inasmuch as Brouwer really committed himself to the actualist position, his view of truth leads to even more radical divergence from intuitionistic logic: even minimal logic is too much. In fact, if one strictly sticks to the actualist interpretation of truth, one cannot really have any reasonable logic, however weak. That is, a valid rule of inference will not always lead from true premises to a true conclusion, simply because we have not explicitly drawn the inference.[4]

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[3] It is partly this issue that motivated me to check whether the various derivations discussed in this paper require the full power of intuitionistic logic, or whether minimal logic suffices. In all of them, the latter is the case.

[4] For some other puzzling consequences of the actualist view, see Raatikainen 2004.

In sum, intuitionistic logic necessarily requires a more liberal and idealized notion of the existence of proof. Instead of the actual possession of proof of a statement *A*, one has to focus on the provability in principle of *A*.

Possibilist interpretation

For most people, even for most enthusiasts of the intuitionist criticism of classical mathematics and logic, the actualist view of truth as significantly temporal is simply too radical and counter-intuitive an idea to be swallowed (even independently of its problematic relation to intuitionistic logic). Accordingly, the large majority of later-day intuitionists and constructivists have rather favored the possibilist view, which identifies truth with provability in principle. This interpretation is intended to make truth non-temporal and stable, and more suitable as a foundation for intuitionistic logic.

Now the possibilist interpretation of intuitionist truth must lean on some notion of possibility. For *A*'s provability in principle — in contradistinction to the actual possession of a proof of *A* — means that it is (in some sense) *possible* to prove *A*. But one may then ask exactly what kind of possibility is meant here by the intuitionists. And this question, when pressed, is much harder than is usually recognized.[5]

On the one hand, logical possibility (in addition to the problem that it already assumes that the correct logic has been fixed) is too liberal for this purpose. On the other hand, physical or psychological possibility is too restrictive. Between these, the kind of possibility that most naturally suggests itself here is certainly *mathematical possibility*. It appears, however, that the notion of mathematical possibility amounts to the consistency with mathematical truth, and therefore already assumes the notion of mathematical truth. But that would make the possibilist explication of truth viciously circular. The more demanding interpretation, which requires a positive guarantee of the possibility, namely

[5] The following is based on (Raatikainen 2004).

its provability (that is, the *possibility* of a proof), is equally question-begging (see also Raatikainen 2004).

At this point, someone might ask – and I have indeed sometimes met this question – why not equate the intuitionistic notion of provability with derivability in some comprehensive, axiomatizable formal system; this suggestion may not be faithful to some more philosophical views of traditional intuitionists, but isn't this a coherent view, and one which still refutes LEM? Couldn't one base intuitionistic logic on this interpretation?

The correct answer is, however, negative. Most directly, it is undermined by Gödel's first incompleteness theorem, which entails that for any sufficiently rich formal system, there is a true sentence not provable in the system. But even if one had doubts about this way of stating the moral of Gödel's result, the above suggestion would be at odds with logical facts. Gödel (1933) has pointed out that e.g. the schema $Provable(Provable(\mathrm{S}) \rightarrow \mathrm{S})$ – or, "(S is true $\rightarrow$ S) is true" (that is, that (T1) is true) – holds for the notion of provability assumed by intuitionistic logic, but cannot hold for provability in a formal system, for this would contradict the second incompleteness theorem. Today, we can easily see that it is also at odds with Löb's theorem.

Actually, more is true. Not only must the notion of provability (even if we focus on truth and the provability of arithmetical statements) assumed in intuitionistic logic be not recursively enumerable (which is the same as being provable in some formal system). It cannot even be arithmetically definable. In other words, it is nowhere in the arithmetical hierarchy.

As to this, assume that intuitionistic provability is definable in the language of arithmetic. Let us denote the defining formula by $Prov(x)$. Then, by the diagonalization lemma (which is intuitionistically provable, in, e.g., Heyting Arithmetic HA), there is a formula S such that:

(D) $\quad \mathrm{HA} \vdash \; S \leftrightarrow \neg Prov(\ulcorner S \urcorner)$.

Consequently, we take $S \leftrightarrow \neg\, Prov(\ulcorner S \urcorner)$, and also (T1), that is, $\mathrm{Prov}(\ulcorner S \urcorner) \rightarrow S$, for granted:

$$[Prov(\ulcorner S \urcorner)]^{2} \quad [Prov(\ulcorner S \urcorner) \rightarrow \neg\, Prov(\ulcorner S \urcorner)]^{3}$$

$$Prov(\ulcorner S \urcorner) \rightarrow S \quad [Prov(\ulcorner S \urcorner)]^{1} \qquad [Prov(\ulcorner S \urcorner)]^{2} \qquad \neg\, Prov(\ulcorner S \urcorner)$$

$$S \quad S \leftrightarrow \neg\, Prov(\ulcorner S \urcorner) \qquad Prov(\ulcorner S \urcorner) \wedge \neg\, Prov(\ulcorner S \urcorner)\ {}_{(2)}$$

$$\neg\, Prov(\ulcorner S \urcorner)\ {}_{(1)} \qquad \neg\, Prov(\ulcorner S \urcorner)$$

$$Prov(\ulcorner S \urcorner) \rightarrow \neg\, Prov(\ulcorner S \urcorner) \qquad (Prov(\ulcorner S \urcorner) \rightarrow \neg\, Prov(\ulcorner S \urcorner)) \rightarrow \neg\, Prov(\ulcorner S \urcorner)\ {}_{(3)}$$

$$\neg\, Prov(\ulcorner S \urcorner)$$

Using again (D), we can conclude *S*. *S* has been proved, i.e., $Prov(\ulcorner S \urcorner)$ has been established. But we have also proved $\neg\, Prov(\ulcorner S \urcorner)$. Thus we have ended in a contradiction. Hence the intuitionistic provability (of arithmetical truths) cannot be anywhere in the arithmetical hierarchy.

This means, though, that in terms of the the arithmetical hierarchy, or degrees of unsolvability, the intuitionistic notion of provability is just as abstract, undecidable and inaccessible as the classical notion of truth (for arithmetical statements: in terms of recursion-theoretic hierarchies, it must be at least Δ^1_1 – but this is the level of classical arithmetical truth). And this, in turn, has important implications in my opinion.

To begin with, the intuition that there possibly isn't, for every proposition, either proof of it or proof of its negation, begins to waver. The idea seems plausible as long as we focus on a more concrete idea of provability, but is much less clear with this highly abstract notion of proof.

Further, it seems to be in various ways vital for intuitionism that one should be able to recognize a proof when one sees one – that is, to require that the proof relation must be decidable. The intuitive idea here is that proofs begin with immediate truths (axioms), which themselves are not justified further by proof, and continue with steps of immediate inference, each of which cannot be further justified by proof (see e.g. Sundholm 1983).

However, the above logical facts raise a serious doubt about how well the notion of provability assumed by intuitionistic logic really fits into this intuitive picture. Unless the human mind can see the truth of infinitely many independent mathematical facts, an alternative which seems highly implausible and in various ways contrary to the general spirit of intuitionism, that notion of provability has very little to do with the notion of provability in any intuitive sense, and in particular with the provability by us humans (even in a somewhat idealized sense).

At this point, some intuitionists might protest against the use of concepts from the recursive function theory, for certain brands of intuitionists may deny the very meaningfulness of them, or at least not accept the Church-Turing thesis about the equivalence of recursivity and intuitive computability. Would this strategy resolve the above problem? Not really, I think. First, it is well known that the elementary part of recursive function theory can be developed inside the intuitionistically acceptable Heyting Arithmetic HA. And second, even if one did not accept the concept of an arbitrary formal system (that is, a theory whose set of theorems is recursively enumerable), the following should make one think twice: by *Kleene's finite axiomatizability theorem* (Kleene 1952) (strengthened by Craig and Vaught 1958), any recursively axiomatizable theory can be *finitely* axiomatized (conservatively) by the addition of new predicate symbols. Consequently, we can restate our basic worry as follows: the notion of provability presupposed by intuitionistic logic cannot coincide with provability in any theory with only finitely many axioms. And surely the notion of finiteness is meaningful for any intuitionist.[6]

In sum, it turns out, on closer scrutiny, to be quite unclear what exactly the notion of provability that intuitionistic logic tacitly postulates really is. I, for one, find it difficult to

[6] Actually, there are various distinct notions of finiteness in intuitionistic mathematics, all equivalent classically but not intuitionistically. However, the sets whose finiteness is at stake here are (intuitively) decidable, and hence there should not be any problems even from the intuitionistic perspective.

have any coherent picture of it. It remains to be seen whether one can make more sense of it.

Conclusions

The actualist view on the existence of proofs is a very radical position, and arguably incompatible with intuitionistic logic, which necessarily requires a much more abstract and idealized notion of provability. However, it has turned out to be extremely difficult to explain more clearly, without moving in circles, what exactly this notion is.

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