Truth and Provability – A Comment on Redhead

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Michael Redhead puts forward, in his ambitious paper 'Mathematics and the Mind' (Redhead 2004), a simple argument which aims to show that humanly certifiable truth outruns provability. Redhead's arguments require a comment.

Redhead first discusses two possible answers to the question of how we know that the Gödel sentence *G* (for Peano Arithmetic PA) is true. He dismisses them both by stating that they presuppose that the axioms of PA are true. Strictly speaking, this is wrong – they only require that PA is consistent, which is a much weaker assumption. But Redhead is certainly on the right track in rebutting these strategies.[1]

Redhead's own argument focuses on the weaker Robinson Arithmetic Q (Redhead calls it, following Lucas, 'sorites arithmetic', but I prefer to use the standard name), which, unlike PA, does not have the induction scheme. His reason for this is that its axioms are 'arguably analytic': 'If any of these axioms were false we would not be talking about numbers.' Redhead contrasts them with the induction axiom (or scheme), which he calls 'notorious' and 'more mysterious'. With a reference to Poincaré, he concludes that the induction scheme is not analytically true.

Now the distinction between the analytic and the synthetic is famously elusive and problematic, but it is far from clear that the induction scheme is in any way less analytic than the other axioms. It is equivalent (assuming classical logic) to the least number principle, i.e. to the claim that if there exists a number with a property *P*, there exists the smallest number with the property P. But it is quite plausible to say that if this principle fails, one is not talking about natural numbers; in other words, that the principle, and hence, induction, is analytic in Redhead's sense.

But be that as it may, let us now consider Redhead's main argument. It begins with the well-known fact that while

For all pairs (m, n), it is provable in Q that $m \times n = n \times m$, (1)

holds, the following is *not* true:

It is provable in Q that for all pairs (m, n), $m \times n = n \times m$. (2)

The universal generalization

For all pairs (m, n), $m \times n = n \times m$ (3)

can be proved (e.g. in PA) with the help of the induction scheme, which Q does not have.

Redhead next submits that we can argue – presumably without using the induction scheme – that (3) is nevertheless true. Redhead introduces the notion of truth (or, 'is true'), and argues that since the axioms of Q are analytically true, we can replace (1) by

For all pairs (m, n), it is true that $m \times n = n \times m$, (4) which, according to Redhead, is strictly equivalent to

It is true that for all pairs (m, n), $m \times n = n \times m$. (5)

By eliminating the truth predicate, one gets (3). Redhead concludes that we have here a case in which certifiable truth outruns provability.

One problem with Redhead's discussion is that he doesn't make explicit which kind of notion of truth he is assuming in the above reasoning. His remarks at the end of the paper suggest that he has a Tarskian definition of truth in mind. However, such a definition can only be given in a sufficiently strong metatheory, a theory which must certainly contain the induction scheme. Hence there is a risk here that one smuggles in the very principle one is trying to avoid.

However, the most serious problem in Readhead's reasoning concerns (1) and the grounds of our knowledge of it. It is a statement about provability in Robinson Arithmetic Q, but this does not guarantee that it is itself provable in Q. And the fact is that it is provable only in a stronger metatheory, one which must have the induction scheme. Such a theory can directly prove (what Q cannot) also that for all pairs (m, n), $m \times n = n \times m$, that is, (3). The argument, and the appeal to the notion of truth, is redundant. Redhead's argument thus assumes, already in the beginning, something that goes beyond Q. Without induction, on the other hand, we just can't establish (1), the premise of the argument, and get the argument off the ground. I am afraid that one has no option but to conclude that the argument fails.

At the end of his paper, Redhead proposes that unless one is constructivist, his arguments confirm the anti-mechanist conclusion similar to that of Lucas and Penrose that minds are not machines, i.e. that the powers of the human mind outrun any formal system (these are equivalent, for an axiomatizable formal system is by definition a system whose theorems can be mechanically generated by a finite machine). But given the problems of Redhead's argument pointed out above, it seems fair to say that Redhead has not managed to give support to the anti-mechanist thesis either.

One of Redhead's conclusions was that human minds can know the truth of statements which can be expressed in a system but cannot be proved in the system. This is hardly controversial, if one focuses on relatively weak systems, as Redhead does. For example, a simple model-theoretic argument shows that $(\forall x)\ [x + 1 \neq x]$, which is an obvious truth about natural numbers, is not provable in Robinson Arithmetic Q. But this does not justify the conclusion that 'certifiable truth outruns provability' (neither would Redhead's main argument, were it successful, justify it). It only shows that Q is in many ways a too weak theory, and that the proof of the sentence requires a stronger axiom system. It is still possible, and indeed quite plausible, that all humanly certifiable mathematical truths are provable in some comprehensive formal system.

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References

Redhead, M. [2004]: ‘Mathematics and the Mind’, *British Journal for the Philosophy of Science* **55**, 731-737.

Notes

1 At one point (see p. 735), Redhead’s wording seems to suggest that Gödel’s theorem shows that there are true sentences of arithmetic which cannot be proved in any consistent, axiomatizable extension of Robinson Arithmetic. But of course it shows no such thing. It only shows that no such extension can be complete. Different extensions have different undecidable Gödel sentences. In all likelihood, Redhead did not really intended to claim the contrary, but given his misleading formulation, this point is perhaps worth making.