Ortega y Gasset on Georg Cantor's Theory of Transfinite Numbers

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Abstract Ortega y Gasset is known for his philosophy of life and his effort to propose an alternative to both realism and idealism. The goal of this article is to focus on an unfamiliar aspect of his thought. The focus will be given to Ortega's interpretation of the advancements in modern mathematics in general and Cantor's theory of transfinite numbers in particular. The main argument is that Ortega acknowledged the historical importance of the Cantor's Set Theory, analyzed it and articulated a response to it. In his writings he referred many times to the advancements in modern mathematics and argued that mathematics should be based on the intuition of counting. In response to Cantor's mathematics Ortega presented what he defined as an 'absolute positivism'. In this theory he did not mean to naturalize cognition or to follow the guidelines of the Comte's positivism, on the contrary. His aim was to present an alternative to Cantor's mathematics by claiming that mathematicians are allowed to deal only with objects that are immediately present and observable to intuition. Ortega argued that the infinite set cannot be present to the intuition and therefore there is no use to differentiate between cardinals of different infinite sets.

Keywords Ortega y Gasset; Georg Cantor; Galileo; infinite set; intuitionism.

DOI 10.1515/kjps-2016-0003

1. Introduction

The major difficulty of writing a paper about the manner in which Ortega y Gasset analyzed Cantor's new mathematics derives from the fact that Ortega never dedicated a paper to Cantor. Ortega wrote papers about physicists like Einstein and Galileo and also about mathematicians like Leibniz, but never did he complete a paper to a rigorous analysis of Cantor's set theory. My aim is to try to recover and reconstruct Ortega's analysis of the new developments in mathematics, especially Cantor's mathematics. I believe that this aim is accomplishable since in many passages of his writings Ortega did refer to advancements in mathematics, philosophy of mathematics and to Cantor.[1] By following and analyzing different passages I will try to rebuild Ortega's attitude towards Cantor's new concept regarding the infinity of infinite sets; my aim will be to convince the reader that Ortega was aware of the philosophical implications of Cantor's mathematics and that he had an articulated opinion regarding it. Cantor's new mathematics was conceived by him as part of a general approach towards science in general and mathematics in particular during the 19th century. This approach he defined as no less than "imperialism".[2]

According to Ortega's philosophy these are the circumstances which can help us understand human life, the person that stands in front of us. Cantor's circumstances were those of the 19th century. Ortega mentioned that during this century physicists aspired to become metaphysicians while philosophers wanted to become physicists[3]. It did not suffice for musicians to stay musicians; Wagner for example strived that his music will also perform as religion and philosophy.[4] Mathematicians felt ashamed that mathematics is based on intuition and not on logic.[5] These

1 Cantor's papers appear in Ortega's library in Madrid. The paper "Mathematische Annalen: Beitrage zur Begrundung der transfiniten Mengenlehre" from the 19th century appears at Ortega's library.

2 "Así, durante el siglo XIX, todas las ciencias ejercitaron el más desaforado imperialismo" (Ortega, 1995, 136).

3 Ortega defined this historical phenomena as "the terrorism of the laboratories" (terrorismo de los laboratorios), Ortega, 1995, 64.

4 "Cada cual aspiraba a ser ilimitado, a ser lo que eran los demás y él no era. Es el siglo en que una música – la de Wagner – no se contenta con ser música – sino sustituto de la filosofía y hasta de la religión –; es el siglo en que la física quiere ser metafísica, y la filosofía quiere ser física, y la poesia, pintura y melodía, y la política no se contenta con serlo, sino que aspiraba a ser credo religioso y, lo que es más desaforado, a hacer felices a los hombres" (Ortega, 1995, 71).

5 "La matemática se avergonzaba de no ser logia, de no poder constituirse en pura deducción conceptual, sino estar encadenada como un humilde can a la intuición" (Ortega, 2004, 136).

were the 19th century circumstances in which mathematicians and scientists worked. These were also Cantor's circumstances and therefore it would not be enough for us to focus only on the passages in which Ortega refers to Cantor's new mathematics, but it is also important to follow the relations he draws between Cantor's mathematics and the circumstances in which he lived. Hence, Ortega's approach towards Cantor's mathematics will also be examined according to his analysis of the relations between mathematics and physics and also between mathematics and metaphysics. We will see that Ortega criticized the Galilean's belief according to which the natural phenomena is behaving in a mathematical manner. Ortega believed that the natural world does not have geometrical preferences.[6] Mathematical objects are imaginary. Mathematics does not deal with metrical or visual space but rather with imaginary space. In the same manner mathematics does not deal with the trans-conceptual or meta-logical. Cantor wanted to reduce the infinite to concept or to logos.[7] This approach of Cantor was interpreted by Ortega as part of 19th century imperialism.[8] He argued that the intuition of the infinite cannot be reducible to concept or logos.[9] His basic attitude towards the new developments in mathematics derives from his basic idea that mathematics ought to be based on the irreducible intuition of counting.[10] Arithmetic or counting is described by him as an 'intuitive operation'.[11]

The main argument of the article is that Ortega presented a reading of Cantor's mathematics based on the idea that mathematical objects must be present to the intuition. Since the infinite cannot be present to the intuition, it cannot be part of mathematical theorems. We will analyze this idea of Ortega and our general effort will be to connect this understanding together with his conception of the differences between

6 "La material no tiene preferencias geométricas" (Ortega, 1930, 140).

7 "La intuition de lo continuo, de lo que llamamos y pensamos "infinito" es irreductible al concepto, al *logos o ratio.*" (Ortega, 1995, 142).

8 Ortega, 1995, 146.

9 Ortega, 1995, 142.

10 Ortega, 1995, 141. See also, Ortega, 1992b, 85-91.

11 "Aritmética es contar. Contar es una operación intuitiva, como son intuitivos sus resultados: los números" (Ortega, 1992a, 56).

physics and mathematics and the different objectives that lie between metaphysics and mathematics. Ortega's analysis of Cantor's new concept is part of a much bigger conception he had in relation to metaphysics and the objectives of philosophy. This fact which is evident as I will try to convince can allow us to conceive Ortega's attitude towards Cantor's mathematics as systematic.

2. How to Count an Infinite Set?

In his paper on 'mathematics and logic' the philosopher Bertrand Russell stated that in modern times mathematics has become more logical. We start with natural numbers and then move to cardinals. The more we progress in mathematics we realize that it hardly involves any essential reference to numbers: "the statement that mathematics is the science of number would be untrue in two different ways".[12] Russell states that modern mathematics is not about quantities. Modern mathematics focuses on new type of numbers like cardinal numbers or the cardinality of a set. Hence, the first two questions we should ask are the following:

1. What is a set?

2. What is a 'cardinal number' and 'how does it differ from a 'regular', natural or real number'?

A set is 'a collection of distinct objects, none of which is the set itself'.[13] It is important to emphasize that a set cannot be an element of itself. Therefore, a strange set like A = {1,2,3,A} cannot be a set since it contains the set itself, in this case it contains the letter A. On the other hand, the set B = {1,2,3} can be a set since none of its objects is the set itself. If we follow the old definition of a set: "A set is a collection of distinct objects', we will end up with paradox and therefore it is essential to stress that a set is not only a collection of different objects, but also a collection that none of its objects is the set itself.[14]

Each set has a cardinal number. The cardinal number of a set may be the same or different from other sets. The mathematician Leonard

12 Russell, 2010, 157.

13 Trudeau, 1993, 12.

14 Trudeau, 1993, 17–19.

Gillman defines a cardinal number in the following manner "the cardinal (number)" of a set is a generalization to all sets, nonfinite as well as finite, of the concept of "number of elements".[15] Cantor himself defined it in the following manner: "Every aggregate M has a definite 'power' which we will also call it 'cardinal number'".[16] Cantor relates cardinal number to the 'power' of a certain aggregate (Menge). For example, the cardinal number of a set with five elements is 5 ($|A| = 5$) and in the same manner the cardinal of an empty set is zero, $|\varnothing| = 0$. When we refer to finite set we can say that the cardinal of a finite set with n elements is denoted by $|n|$. However, when we refer to infinite set it is much more intricate to understand what its cardinal number is. We may even wonder how is it possible that infinite sets have different cardinal?! In his paper Cantor writes that "the cardinal numbers can be arranged according to their magnitude"[17]. Therefore, infinite sets may have different cardinals.

This statement might look odd to the layman who might immediately think that all the infinite set have the same cardinal number. However, we know from Cantor that this statement would be untrue. Two different infinite sets may have different cardinals and "they will have the same cardinality, $|X| = |Y|$" only "if there exists a one-to-one mapping of X onto Y".[18] So what is a cardinal number of an infinite set? How is it possible that infinite sets may have different cardinal number? Are not they all the same?!

Cantor's new idea is that infinite set may have different cardinal number and in order to expose this new idea he separates between countable set and uncountable set. At first before becoming familiar with the new definitions and Lemmas the philosopher or mathematician might wonder 'how can an infinite set be countable'? This philosophical question has some sense since the infinite cannot be directly present to our intuition and we have just mentioned that Ortega defined arithmetic as an intuitive operation. So how does Cantor deal with this problem? How did he man-

15 Gillman, 2012, 545.

16 Cantor, 2015, 86.

17 Cantor, 2015, 109.

18 Jech, 2006, 35.

age to prove his new contention according to which there are countable and uncountable sets?

Cantor emphasized that "mathematics, by virtue of its own independence from any constraints imposed by the external reality of spatial temporal world" is quite free.[19] One of Cantor's famous quotes was that "the essence of mathematics lies in its freedom".[20] He argued that controls and artificial philosophical presuppositions prevented any growth of mathematical knowledge.[21] Cantor succeeded to use this freedom in order to create a beautiful new theory of transfinite numbers. In his theory of transfinite numbers he managed to differentiate between countable and uncountable set by presenting the idea of 'one-to-one correspondence'. In modern mathematics we define a countable set as follows:

A set A is countable if and only if it is finite or there exists a bijection $f\colon \mathbb{N} \to A$.[22]

Therefore, a set B can be countable if and only if there can be an injection $f\colon B \to \mathbb{N}$.[23]

Two sets are equipotent when we can make a one-to-one correspondence between their elements. If we can draw a one-to-one correspondence between two different sets we can say that these two sets have the same power or the same cardinality.[24] For example, if we take the set of natural numbers and the set of even natural numbers we will notice that they have the same cardinality. This fact derives from our ability to

19 Dauben, 1989, 133.

20 Edwards, 1988, 140.

21 Cantor mentioned that "without this freedom Kummer for example never have been able to formulate his ideal numbers, and consequently the world would be in no position to appreciate the work of Kronecker and Dedekind". See: Dauben, 1989, 133.

22 Winskel, 2010, 55. A function $f\colon A \to B$ is bijective if it is both injective and surjective. In more detail: f is injective if for $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. It is equivalent to require here the contrapositive condition, namely, a function $f\colon A \to B$ is injective if for all $a_1, a_2 \in A$ $F(a_1) = F(a_2)$ implies $a_1 = a_2$. It is important to remember that "an injective function is also called an injection, an embedding or a one-to-one function". The function f is surjective if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$. We call f a surjection (or an onto function). See Beck and Geoghegan, 2011, 86.

23 Winskel, 2010, 56.

24 "M and N are uniquely and reciprocally referred to one another; and by it to the element *m of M* corresponds the element *n* of N" (Cantor, 2015, 88).

draw a one-to-one correspondence between the elements of these two sets. In other words, the set A of even natural numbers: A = {2,4,6…}, is countable since there is a 'one-to-one' correspondence (or in other words injection) between its elements and the elements of $\mathbb{N}$: f: $2n \rightarrow \mathbb{N}$.

$$\begin{array}{cccccc} 2 & 4 & 6 & 8 & 10 & 2n \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 & \mathbb{N} \end{array}$$

Since we know that a set A can be countable if and only if there can be an injection f: $A \rightarrow \mathbb{N}$, we can deduce that the set of even natural numbers has the same cardinality as the set of natural numbers, $|2n| = |\mathbb{N}|$. Their cardinal is aleph null, $|\mathbb{N}| = \aleph 0$

This simple example shows us that there may be two different infinite set which have the same cardinality. So when do we encounter two different infinite sets with different cardinals? The famous example is that of the natural numbers whose cardinal Cantor defines as 'Aleph zero' (or the smallest transfinite cardinal number) and the infinite set of real numbers.[25] While the first infinite set is countable, the second is uncountable and its cardinality is not aleph null, $|\mathbb{R}| = C$.

Any set that can be put into one-to-one correspondence with the natural is called a countable infinite. We have seen that the natural even numbers can be putted into one-to-one correspondence with the natural numbers and in the same manner the set of $\mathbb{N}$ can be putted into one-to-one correspondence with the set of integers, $\mathbb{Z}$. Therefore we can deduce two conclusions: one, the set of the even natural numbers is countable; second, it has the same cardinality as the set of natural number. In fact, a set is infinite if it can be put into one-to-one correspondence with a proper subset of itself. This fact is one of the main differences between finite and infinite sets. To better understand the importance of this innovation we will finish this section with Russell's interpretation in regard to the historical importance of Cantor's set theory.

Cantor's set theory was severely criticized during his life time. The famous mathematician Poincare regarded to Cantor's set theory as an

25 Cantor, 2015, 103-110.

illness.[26] From antiquity until the nineteen century the mathematicians rigorously distinguished between the 'potential infinite' and the 'actual infinite'. Aristotle, Descartes, Pascal, and Gauss had rejected the actual or complete infinite as unknowable and avoided its application. Thiele mentions that at the time of Cantor's life Kronecker was one of the most zealot defender of such views.[27] However, Cantor's ideas were taken up by Dedekind, Hilbert and Russell.[28] Hilbert praised Cantor as a scholar who is unrivaled by all mathematicians from Euler to Einstein and he is famous for his statement that "no one shall expel us from the paradise that Cantor is created for us".[29]

In his "Mathematics and Metaphysicians" Russell states that "Obviousness is always the enemy of correctness".[30] This statement of Russell is valid for many aspects in philosophy of motion, time and space. Russell refers for example to the maxim according to which "if one collection is part of another, one of which is a part has fewer terms than the one which it is a part"[31]. This maxim looked valid for many centuries and it might look valid today for many of us nowadays. Before a strict analysis we might think that "the whole has more terms than its part"[32]. However, these conceptions or the historical manner in which we defined our basic conceptions prevented us for seeing the entire picture. From Cantor we learned:

1. Every finite aggregate E is such that it is equipotent to none of its parts. For example, a set $B = \{1\}$ is a subset of $A = \{1,2\}$, $B \subseteq A$ [or $1 \in \{1,2\}$]. Different finite sets may have different cardinality since the bigger the set is the bigger its cardinality. These finite sets A and B have different cardinality ($|A| \neq |B|$) or $|A| > |B|$ because the set B has one more element

26 Thiele, 2005, 525.

27 Thiele, 2005, 528.

28 Edwards, 1988, 140.

29 Thiele believes that "It is rather remarkable that Hilbert uses here a biblical metaphor like the mathematicians of 18th century bygone used to do. The reason to take up biblical images such as this might root in the estimation of Cantor by Hilbert who regarded him as the profoundest mathematician of our age". See, Thiele, 2005, 525.

30 Russell, 1917, 59.

31 Russell, 1917, 66

32 Russell, 1917, 69.

in comparison with set A. It sounds very plausible whenever we speak about finite sets because we can see when and if one set has more elements than the other. The surprise comes when we examine Cantor's analysis of the transfinite sets and instead of an actual counting we use his own examination of one-to-one-correspondence.

2. Cantor argued that every transfinite aggregate T is such that it has parts T1 which are equipotent to it.[33] For example, the set of natural numbers $\mathbb{N}$ is equipotent with the set B = {7,14,21…}. Although it seems that B might have less numbers than the set of the natural numbers Cantor's one-to-one correspondence allows us to prove that these sets have the same cardinality, $|\mathbb{N}| = |B| = \aleph 0$. When we put the two sets in such a relation to one another that to every element of $\mathbb{N}$ corresponds one and only one element of B or in other words when we use the one-to-one-correspondence we manage to see that these two sets have the same cardinality. The conclusion is that the transfinite aggregate $\mathbb{N}$ is such that it has parts (the set B for example) which are equipotent to it. Furthermore, we can also see that B is a countable set because we can draw a function f: B→$\mathbb{N}$

$$\begin{array}{cccccc} 7 & 14 & 21 & 28 & 35 & 5n \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 & \mathbb{N} \end{array}$$

The idea of one-to-one-correspondence seems more basic than actual counting for infinite amount of time. I believe that this is the advantage of a mathematical proof by Cantor's new idea of counting and infinite set. It can allow us to predict the cardinality of a set without the need to count it for eternity. However, this proof is not based on a mental activity and contradicted Ortega's idea that 'counting is an intuitive operation'. Since Ortega believed that the arithmetical notions of equal/bigger/smaller are intuitive notions it could not have been easy for him to accept Cantor's new theory of counting.[34]

33 Cantor, 2015, 108.

34 Ortega, 1992, 47-60.

3. Mathematical Letters as the Language of Nature?

In his early writings Ortega followed the neo-Kantian philosophy of Hermann Cohen and Paul Natorp. He followed their definition of the relations between science and philosophy.[35] In one of his letters Ortega separated between the world of sensation and the world of truth. If we ask ourselves what is the sum of two plus two we will always arrive to the same result, we will answer that it is four. Therefore, this mathematical result belongs to "the world of truth". In contrast with the world of truth we can easily recognize the temporality of the world of the senses, a world in which things are constantly changing. The world of the senses might suggest that the sun revolves around us, but the world of science will reveal to us that it is vice versa.[36] In his youth the objectives of philosophy were interwoven together with these of science; it was the scientific idealism of the Marburg School. Furthermore, Ortega was the promoter of the Spanish science and he is also described in that manner in the historiography.[37] The focus on science always stood at the center of his philosophical writings.

In his adult life his description of the relations between the objectives of science and philosophy has changed. The youth approach towards Marburg's scientific idealism changed and from the year 1914 onwards (or even earlier in 1910) he started to formulate a new philosophy which criticized scientific idealism in general and idealism in particular.[38] In this section we will see how the new articulated approach of Ortega towards

35 It is important to note that Cohen and Natorp did not adopt metaphysics or dogmatism and that they did believe that experience plays a crucial role in the constant human effort to produce scientific knowledge. Their idealism can be also defined as "critical idealism". See: Natorp, 1912.

36 Ortega, Carta 175,1987, 551.

37 Zamora Bonilla, 2005, 83-99.

38 The researcher Ciriaco Morón Arroyo distinguishes between four periods in the thought of Ortega: The first period is defined by him as "rationalism" (1907-1914). The second is "perspectivisim" (perspectivismo) and it goes from 1914 to 1920. The third is "psychologisim" (psicologismo o antropocentrismo) and it is expressed in his writings from 1920-1927. The fourth is "rational vitalism", 1928-1955. In this article we will mainly focus on Ortega's thought from 1927 onwards. During this period Ortega focused on metaphysics and examined the potential of changes in technology and science. Our focus will focus on his analysis of the new advancements in mathematics. See: Morón, 1968, 77.

the relations between mathematics and physics can help us understand his understanding of the objectives of mathematics. His interpretation of Cantor's mathematics is derived from his new mature definition of the objectives of philosophy. The main claim of Ortega was that "reality is not composed of mathematical letters".[39] Galileo's error can be found in his belief that we should understand reality as written in mathematical letters. This belief led to the 19th century scientific 'imperialism' and caused many to forget that the mathematician captures his objects, like space and numbers, through intuition: "The mathematician captures his object- space or number- through concept according to some or through intuition according to others. However, both means are immediate for the cognizance mathematician".[40] The need was for the mathematicians will look back and base their research on intuition. According to Ortega mathematicians should focus only on the objects that are present for their human intuition: "This indicates that in each instance when we are thinking about the infinite, we compare our concept with the infinite object itself, therefore with its presence and by doing this comparison we find that our concept has been cut off".[41] This contention of Ortega is central for our understanding of his interpretation of Cantor's new attitude towards the infinite:

> "Pardon, but what we are asking is if when someone is thinking about the infinite as infinite points, is he really thinking about each and every point that makes this infinite. It is evident that the answer is no. We are thinking only on finite number of these and to this we are adding that we could have always thought another more point, another one and another one, without ever finishing. The result is that when we are thinking about an infinite number we are also thinking that we will never be able to finish thinking about it. We are thinking that the concept of the

39 Ortega, 1930, 140-141.

40 "El matemático captura su objecto- el espacio, el número -, o con el puro concepto, según unos, o con la intuición, según otros", Ortega, 2004, 137. When he refers to Cantor's infinity Ortega refers to the "intuición de lo continuo", Ortega, 1995, 142.

41 Ortega, 1995, 141.

> infinite implies recognition that it does not contain all that it intends, or that the object that we are thinking about- the infinite, exceeds our concept of it".[42]

Instead of focusing mainly on Cantor's mathematics I believe that Ortega thought it is important to focus on articulating a general argument whose aim is to remind the mathematic community that they should stop serving objectives which are foreign to their profession. In relation to the new developments in mathematics Ortega also argued that the logical 'law of excluded middle' cannot be applied to mathematical entities. The logical law can be expressed by the propositional formula $p \vee \neg p$. It means that if p is a preposition, then either p or its negation $\neg p$ is true. Ortega did not conceive classical mathematics as part of logic and therefore he argued that this law cannot be applied to mathematics: "The logical axiom of the excluded middle" does not apply "to mathematical entities".[43]

In his mature writings Ortega separated between the objectives or science and these of philosophy. In his book *En torno a Galileo* he described science as an interpretation of facts. He argued that reality is hidden behind the facts and data that we observe. The aim of science is to interpret the facts and allow us to reveal more and more aspects of reality.[44] By relating the objective of science to his own philosophy of life Ortega mentions that science helps us to better understand our circumstances. There are two aspects in science: the first is imaginary and creative (but not irrational or relative). The second aims to help us understand the facts and data that appear in our circumstances. Science gives us a better understanding of facts ad exposes some aspects of reality, however it can never replace or fulfill the objective of the mystic or philosopher. These fields of interest have different objective. Their goal is to deal with "everything that there is" and not only with certain aspects and facts.[45]

42 *Ibidem*.

43 "El axioma lógico llamado del <tercero excluso> no vale para las entitades matemáticas", Ortega, 1995, 74.

44 Ortega, 1983, 15.

45 Ortega, 1983, 92.

Science is a construction that is based on observation and imagination. Therefore, science is a construction. It cannot be strict reflection of the fact and it has an aspect of construction.[46] It is an act of interpretation that helps the human beings to navigate in their circumstances. Ortega argues that his age has a belief in science and this belief is just like any other belief and it is meant to allow us to navigate in our circumstances: "The only thing I am saying is that we are dealing with a belief, that science is a belief, a conviction we adhere to, in the same manner we adhere to a religious belief".[47] It is important to note that Ortega does not underestimate in science, On the contrary. Ortega's aim is to remind physicists and mathematicians that they should not try to replace metaphysics by giving us a picture of reality as a whole. This aspect cannot be achieved by science since its aims are not and cannot be metaphysical. Physicist looks at facts and data while the metaphysician looks at everything that there is. Furthermore, metaphysician will also look after what can or might exist. The objectives of science and philosophy are not the same.[48]

The mathematician or physicist starts his/her work by limiting their research object. They define the research object and by that they are focusing on certain attributes of the object they research. The scientific attitude of the mathematician reflects his belief that he will have the possibility to know his object or in other words to demonstrate and prove the theorem he is interested in proving. If we look at the history of mathematics and see how mathematicians tried to react to Fermat's theorem we can notice that they did hold a belief that it will be possible to prove or maybe even disprove it. The philosopher on the other hand looks after 'everything that there is' and tries to deal with metaphysical problems (like consciousness) while he knows it might be possible he/she would not be able to give a final solution to the problem. The philosophers deal with "absolute problem", a problem "without limits".[49] Why did mathematicians start to focus on the infinite or the infinite countable/

46 Ortega, 1983, 21.

47 Ortega, 1983, 82.

48 Ortega, 1995, 93.

49 Ortega, 1995, 92-94.

uncountable? This question will stand at the center of the next section. But before dealing with the most important question of this research it is important to understand how mathematics and physics crossed the lines into the realm of the infinite, the realm that is ascribed for philosophers.

According to Ortega, physics cannot be an exact science in the same manner mathematics is: "the exactness is a word that has a meaning, authenticity in mathematics"[50]. In physics we have only approximations. In physics it is exactitude within certain limits. Mathematician captures his objects through intuition and in this manner the objects are immediately present for him. Furthermore, the exactness in mathematics exists because it refers to quantitative objects.[51] In physics, on the other hand, the objects of research are not immediate and the physicist needs to measure them. The physicist's objects of research are not immediate for him in his intuition. The objects in physics have to be captured through measurements: "the measurement for the physicist is the intuition (or axiomatic procedure) for the mathematician".[52] For the mathematician the triangle is already integrated inside of the intuition; it can be conceived through the intuition.

The human being needs to measure the material objects because he does not possess them or "because he does not have them in his intelligence".[53] God does not need to measure. There is no entity that can do something that does not have any meaning for God.[54] Measuring is a human characteristic. However, the physicists of the 19th century believed that the aprioristic laws of geometry are physical laws. They believed that the natural objects are obeying them. The result was that the physicist whose profession is an empirical did not necessarily start with experiment, but rather with unconscious geometrical assumptions. Ortega mentions that one of the modern physic new assumptions is that there must be a geometrical docility of the natural phenomena.

50 Ortega, 2004, 139.

51 "La exactitude no puede existir sino cuando se habla de objectos cuantitativos, o como Descartes dice, de *quod recipit magis et minus*; por tanto, de lo que se cuenta y se mide" (Ortega, 1995, 88).

52 Ortega, 2004, 137.

53 Ortega, 2004, 138.

54 Ortega, 2004; "Dios es desmesurado *(exuperantissimus*), *Idem*, 138.

This new historical attitude started with the work of Galileo who defined the new science as consisted from measuring everything that is measureable and in succeeding to measure also the things that it is impossible to measure. Ortega defined Galileo's new approach as 'cosmometry'. Galileo believed that mathematics is physics or that the natural phenomena behave mathematically: "Galileo believed that the space and time of the things are mathematical time and space, not metrical time and space".[55] However, Galileo never conducted any experiment that was supposed to show that the natural phenomena follow Euclidean theorems. For Galileo the objective of physics was to discover the special laws that rule the phonemes aside from "the general geometrical laws".[56] According to this interpretation of Ortega, it was Einstein who 'freed' humanity from this prejudice. When Einstein realized that the natural phenomenon does not behave according to the Euclidean geometry he did not hesitate to declare physics' sovereignty in relation to mathematics. Ortega mentions that Einstein demanded that geometry will adapt itself to physics and not vice-versa. On the other hand, Lorentz expected that physics will adapt itself to mathematics since he was influenced by Galileo's approach and the dependence physics had on mathematics.[57]

In one of his late writings Ortega refers to Einstein's short paper from 1921 "On Geometry and experience" and presents his agreement with Einstein's separation between "practical geometry" and "purely axiomatic geometry". In his paper from 1921 entitled "Geometry and Experience" Einstein argues that "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality".[58] Ortega argued that this separation between practical and axiomatic geometry allowed Einstein to overcome Galileo's assumption regarding what he defined as the 'the general laws of geometry" (Euclidean geometry) that govern reality.[59] Ortega's argument was that reality is

55 Ortega, 2004, 138.

56 Ortega, 1995, 73.

57 Ortega, 1995, 73.

58 Einstein, 2007, 247.

59 "Para Galileo, la misión de la física consistía en descubrir las leyes especiales que rigen sobre los cuerpos, ademas de las leyes generales geometricas " (Ortega, 1995, 73).

not written with mathematical letters. In the next section we will closely center on how Ortega approached the new changes in mathematics. First, He made an effort to overcome Galileo's mathematical approach to reality. Then, he articulated an "absolute positivism" which meant to pose a different approach towards reality and the mathematical infinite.

4. The Mathematical Infinite: Is it Accessible to the Intuition?

One of the main passages where Ortega refers to Cantor's Theory of Infinite Numbers is to be found in chapter VI of his book *What is Philosophy.* This is not accidental. The interpretation Ortega gave to Cantor's mathematics derives from his effort in his book to overcome scientific idealism and to separate between the objectives of mathematics and philosophy. In this book from 1929 Ortega interprets Cantor's mathematics as the highlight of the movement of rationalism:

"The rationalism of the last times wanted to make illusions; rationalism is by its essence living proudly with illusions of reducing to concept, to *logos* the mathematical infinite. With Cantor it managed to extend the mathematical science by the use of the so called pure logic. The mathematical science extended its fields with excessive imperialism of the 19th century… This movement that has incalculable importance is being fulfilled in these years, these months. The new mathematics acknowledges the irrational part that lies in its object, in other words it accepts its proper and non-transferable destiny, leaving to logic its own destiny".[60]

In his book *What is Philosophy* Ortega distinguished between two types of positivism: the first was criticized by him while he was the follower of the other. The one that was represented by Comte was criticized by him while he tried to convince his readers in the advantages of his new revised positivism. Ortega's separation between two types of positivism can allow us to understand his analysis of Cantor's new mathematics. When referring to positivism Ortega did not support the naturalization of the

60 Ortega, 1995, 142. In this paragraph Ortega refers indirectly also to Russell's intention to base mathematic on logic. In his late writing Ortega mentioned that Russell's goal to base mathematics on logic failed: "Russell quiso reformar radicalmente la vieja lógica elaborando una lógica de clases, pero fracasó, como no podia menos y tuvo que fundarla en una lógica de relaciones" (Ortega,1992, 76).

consciousness. His "absolute positivism" as he defined it was articulated in order to explain what should stand at the center of the mathematicians' attention. A mathematician should not focus on concepts which are not immediately present to his intuition, concepts like the infinite. Mathematics should focus on the positively observable to the intuition and since the infinite is not immediately present to the intuition, it cannot stay at the center of the mathematical research.[61]

Ortega argued that Comte's philosophy represent bourgeoisie's point of view, a practical point of view. Its main aim is practical: "science, hence foresight; foresight therefore action".[62] Comte allowed technique to control science. Ortega mentions that in the same manner the pragmatists saw truth as something that is being examined according to its practical use. If something leads to practical results the pragmatist will consider it as true.[63] Ortega rejects this very general aspect in both positivism and pragmatism. Instead of these approaches he refers to the traditional metaphysics and makes an effort to articulate a new version of positivism that will suffice to represent his attitude towards the mathematics of the end of the 19th century.

When Ortega refers to mathematics and especially to Cantor's Set Theory he uses some definitions that might resonate as positivistic. Ortega himself defines these ideas as "absolute positivism": "It deals with a radical extension of positivism and like I have said few years ago in a paper, actual philosophy could have been characterized by saying that "in front of the partial and limited positivism, an absolute positivism"[64]. In other words, when he refers to Cantor's new approach to mathematical infinite Ortega uses a positivistic terminology, but gives it completely different meaning. It is important to note that Ortega's main aim in his book *What is Philosophy* was to overcome scientific idealism and to propose a new theory of knowledge that overcomes both idealism and realism. In his book and others he presents his philosophy of life as a

61 Ortega, 1995, 127-146.

62 Ortega, 1995, 62.

63 Ortega, 1995, 63.

64 Ortega, 1995, 138.

radical and indubitable reality in the universe.[65] In the historiography there is an agreement regarding this topic.[66] My aim is not to contradict this analysis with which I agree. My aim is different. I believe that in 1929 Ortega's goals stood a little far ahead from the mere need to express how his own philosophy of life overcomes both realism and idealism. I believe that this can be shown in his articulation of what he himself defined as "absolute positivism", a view that is designed to propose an alternative to the mathematics of the end of the 19th century.

In the same manner the positivist demands that we refer to observable objects, Ortega demands from the mathematicians to refer only to objects that can be "immediately present" to the human intuition: "the word intuition means immediate present".[67] Ortega's goal is to replace the "partial positivism" of Comte by a new articulated version of "absolute positivism".[68] At this point we must stop to reflect on Ortega's contentions and to ask ourselves how is it possible for Ortega to speak about positivism together with intuition, how can these contradictory approaches in theory of knowledge can be related? The answer is that these new definitions are designed especially to reflect the meaning of Cantor's new mathematics. Ortega's "absolute positivism" is comprehensible only when it is related to his approach towards the new mathematical concept of the infinite. If we read chapter number six of Ortega's book *What is Philosophy* we notice it is a chapter designed to present an alternative to Cantor's attitude towards the infinite based on what Ortega defined as "immediately present" or "intuition". Cantor's mathematics tries to focus on objects that lie beyond what appears in the intuition. In other words, Ortega's argument is that instead of focusing on infinite set of infinites (that are obviously not present to our intuition), mathematics

65 Ortega, 1995, 213.

66 In the introduction to the book Sánchez Cámara rightly mentions that Ortega believed that the theme and goal of his time was to overcome idealism. Another important researcher Sán Martin claims that in 1929/30 Ortega connected between phenomenology and modern idealism and subjectivism. In these years Ortega argued that our first relation with the things that surround us is not a relation of consciousness (conciencia-de) and in that manner he overcame the idealistic aspect of phenomenology. See: Sánchez 1995, 9-25; and also, San Martín, 1994, 50.

67 Ortega, 1995, 138.

68 *Idem.*

should strictly on the objects that are present to the intuition. In the same manner the positivist calls to focus on what is present to the human mind through the senses Ortega calls to "look" (ver) at the objects that appearing to the intuition, no need to go beyond them. Ortega's new definition of partial/absolute positivism is his own answer to what he conceived as a mathematical error, a focus on objects that are not present to the intuition: "The concept or the idea is always an idea of something and this "something" has to be present for us in some manner in order for us to be able to think of it. Even if we had had the power to create ex nihilo, first we would have needed to create the object, then to have it present and only then to think about it"[69].

The separation Ortega makes between partial and absolute positivism is aimed to allow him to articulate a new philosophical concept whose aim is to demonstrate an alternative to mathematicians' new approach towards the infinite: "Intuition is the least mystical and magical of the things in the world: it strictly signifies a mental state in which the object becomes present for us. There is an intuition of the sensible and also an intuition of the insensible"[70]. Ortega's "absolute positivism" is designed to allow putting at the center of the discussion the intuition of the insensible. The partial positivism was based on sensualist theory of knowledge and did not refer to the existence of the intuitionally sensible. The aim of Ortega was therefore to go beyond the identification of positivism with the sensualist philosophy: "This absolute positivism – as we will see – corrects and overcomes for the first time the fault that philosophy suffered more or less all the years: sensualist philosophy"[71].

The sensualist philosophy focuses on the things that we learn from our sensations and perceptions. The sensualist philosopher does not admit that there can be things in the mind that were not before in the sensations. In order to articulate an alternative to the new approach in mathematics Ortega demanded that we should overcome this contention of the sensualist philosopher. The theory of knowledge of the sensualists influenced the articulation of partial positivism and therefore instead of

69 Ortega, 1995, 141.

70 Ortega, 1995, 138.

71 *Idem.*

this sensualist approach Ortega wanted to offer a new approach which will help him define his main argument in relation to the new mathematics of the 19^{th} century.

What is Ortega's "absolute positivism"? First, Ortega argues that truth must be based on evidence. This implies that we need to see the objects; we need them to be immediately present for us. However, if it is positivism are we doomed to say that we cannot speak on the objects that we can "see" directly by using the power of our intuition? At this point, Ortega's response is that his "absolute positivism" does not imply that we should focus only on the objects we conceive through our sensations or experience in general. In Ortega's "absolute positivism" 'seeing' objects means focusing also on the mathematical figures and numbers that appear in our intuition. Ortega calls us to watch these objects without assuming that these were derived and learned from experience. If we adopt Ortega's absolute positivism we will conclude that mathematicians should focus only on objects that have 'immediately present' for the human intuition. Mathematical objects are observable, observable for the intuition. At this point we can see how big is the difference between Ortega's definition of positivism and the traditional definition. This new definition, 'the absolute positivism', is designed to preserve only the part where the positivists claim that the objects must be observable for us, evident to us. Ortega's addition is that the mathematical objects must be present to us in our intuition: "the word intuition means immediate presence".[72]

What happens when the mathematician thinks about the infinite? Can the infinite be observable or immediately present to us? Ortega's answer is negative and in this manner he aims to show how his absolute positivism exposes the philosophical weakness in the method of Cantor's new mathematics. In Ortega's 'absolute positivism' the focus is on seeing and observing, on focusing on the immediate presence of the real objects on the one hand and the mathematical objects on the other hand. We are not allowed to speak of objects that are not present to our sensations

72 Ortega, 1995, 138.

in general or to our eyes in particular or (and this is important) for the intuition.

In his article "Kronecker's Place in History" the mathematician Harold Edwards states that Kronecker had a grand conception that "all of mathematics would be based on the intuition of natural numbers".[73] He believes that Kronecker had a "unified view of all the branches of mathematics and had, in many instances, fully thought-out ideas on how to base them on intuitionist principles"[74]. Kronecker opposed to Cantor's new mathematics based on intuitionist view and the idea in general was that mathematics should be based on the "irreducible intuition of counting"[75]. This point of view of Kronecker should not surprise us. The idea of the function of the intuition stood also at the center of Ortega's interpretation to Cantor's Set Theory. Ortega did not refer specifically to the methods applied in Cantor's mathematics. He did not give enough attention to Cantor's new method of one-to-one correspondence, the diagonal argument or the new concept of 'cardinal number' of finite or infinite series. This should not surprise us since Ortega was not a trained mathematician. However, as a philosopher he had a tremendous interest in the mathematics of the end of the 19th century and his response to Cantor's mathematics was articulated by what he defined as "absolute positivism". This point of view might also shed some light on the difficulty Cantor had to endure in his life before his mathematics had been accepted by Hilbert and Russell. Cantor went beyond intuition and for philosophers like Ortega or mathematicians like Kronecker this step was too extreme.

5. Ortega as a Positivist?

One of the surprising results of this research is to be found in Ortega's new definition of his own philosophy– the philosophy of "absolute positivism". This is a surprise since Ortega was not positivist, so why does he use this term? As we have seen this term is used by him in response to

73 Edwards,1988, 142.

74 Edward, 1988, 144.

75 Edward, 1988, 140.

Cantor's mathematics. However, there might be another reason for using such a term. The separation between practical and axiomatic geometry, a separation that was presented in Einstein's famous article from 1921 was developed by the Logical Positivists. One of the important philosophers Hans Reichenbach also followed Einstein's separation between practical and axiomatic and argued that there is a need to separate between physical space and mathematical space. It was after the discoveries of non-Euclidean geometries that the duality between physical space and mathematical space was recognized. Reichenbach's argument was that "mathematics reveals the possible spaces; physics decides which among them corresponds to physical space".[76] In the third section we analyzed Ortega's conception of mathematics and we have seen that he also introduced the separation between physical and axiomatic geometry. His argument was that in physics space and geometry should adapt to the natural phenomena: "the most energetic geniality in the work of Einstein for me is to be found in his decision to liberate this traditional prejudice".[77] When he saw that the natural phenomena do not comport according to the law of Euclides he decided in favor of the sovereignty of physics. In this regard we can see that Ortega follows the logical positivists' interpretation of Einstein's physics.

Ortega's presentation of the new concept of absolute positivism and his support for the use of this concept in order to analyze Cantor's new mathematics should not surprise us. There is no need to mention that Ortega's philosophy is not positivistic, on the contrary. However, his reading of Cantor's set theory and his analysis of the mathematics of the 19th century was inspired by philosophers like Comte and Reichenbach. Positivism was not his real philosophy, but a reaction towards advancements in science in general and mathematics in particular. His own use of the term 'absolute positivism' is designed to highlight his reading of Cantor's mathematics from intuitionist point of view. The demand that mathematics can deal only with objects that are immediately present to intuition.

76 Reichenbach, 2014, 6.

77 Ortega, 1995, 73.

Ortega's main argument is that the infinite cannot be immediately present for the human intuition. In this respect one might argue that a very large finite number also cannot be immediately present in intuition. So does this imply that mathematics cannot deal with large finite number?! In other words, in the realm of human intuition when does the separation between finite and infinite begin?[78] It is plausible that we cannot know for sure Ortega's answer to these dilemmas since he did not write in details on these issues. Unfortunately, his analysis was more philosophical and less mathematical. His argument was "we have to conserve the positivistic demand for immediate presence and to save it from its positivistic narrowness".[79] The question we need to ask is why a philosopher whose philosophy of life is metaphysical redefines his philosophy as "absolute positivism" when he refers to modern tendencies in mathematics and physics? Comte's positivism is a philosophy that is designed to go beyond metaphysics and in this respect Ortega's 'absolute positivism' is used to refer to what he considered as metaphysical tendencies in the science of physics and mathematics: the focus on objects that cannot be immediately observable in intuition and/or empirical reality. His argument is that metaphysics should be left to philosophers; it should be used only in philosophy. Physics cannot replace philosophy since it cannot solve metaphysical dilemmas and mathematics should not deal with potential infinite but should focus mainly on the objects that are immediately observable for the intuition.

6. Conclusion

Ortega's philosophy of absolute positivism was a response to what he conceived as the entrance or even 'invasion' of metaphysics to physics

78 This question is important since it might help us to historically contextualize Ortega's philosophy of mathematics. The mathematician Brouwer criticized Cantor's mathematics and articulated a philosophy of mathematics, which he defined as "Intuitionism". For Brouwer, a mathematical symbol is meaningful only when it refers to a mental construction. Since transfinite cardinals cannot be regarded as referring to mental constructions Brouwer considered Cantor's mathematics as meaningless. However, when Brouwer referred to Cantor's smallest transfinite number ω he argued that it exists since it can be mentally constructed. See: Gillies, 1980, 105–126.

79 Ortega, 1995, 137–138.

and mathematics. Cantor's mathematics, together with its separation between countable and uncountable infinite sets, was conceived by Ortega as a contradiction to his demand for absolute positivism. He demanded to speak only about observable objects: observable for intuition of the mathematician and also observable for the physicist in his daily work at the laboratory.

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