Leibniz's Calculus Proof of Snell's laws Violates Ptolemy's Theorem

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Abstract

Leibniz proposed the 'Most Determined Path Principle' in seventeenth century. According to it, 'ease' of travel is the end purpose of motion. Using this principle and his calculus method he demonstrated Snell's Laws of reflection and refraction. This method shows that light follows extremal (local minimum or maximum) time path in going from one point to another, either directly along a straight line path or along a broken line path when it undergoes reflection or refraction at plane or spherical (concave or convex) surfaces. The extremal time path avoided the criticism that Fermat's least time path was subjected to, by Cartesians who cited examples of reflections at spherical surfaces where light took the path of longest time. Thereby it became the standard method of demonstration of Snell's Laws. Ptolemy's theorem is a fundamental theorem in geometry. A special case of it offers a method of finding the minimum sum of the two distances of a point from two given fixed points. We show in this paper that Leibniz's calculus proof of Snell's Laws violates Ptolemy's theorem, whereby Leibniz's proof becomes invalid.

Key words

Optics, Reflection, Refraction, Descartes, Snell's laws, Extremal time path, Most Determined Path Principle, Ease of travel, Calculus Method, Fermat's Least Time Principle, Ptolemy's theorem, Special case of Ptolemy's theorem, Symmetry, Philosophy

Introduction

Gottfried Wilhelm Leibniz (1646-1716) was a mathematician, philosopher and cofounder of infinitesimal calculus. In the context of the behavior of light during reflection and refraction, he viewed it as a case of teleology[1-3]. Fermat developed a method for the study of maxima and minima[4]. He applied that method to derive Snell's law of refraction. He discovered that light follows the path that takes the least time in going from any one point A to any another point B. This came to be known as Fermat's least time principle (FLTP). The underlying philosophy was that there lies an '*end purpose*' in motion of light and exhibited here in its choosing the least time path in preference to the least distance path. This implied an ability of thinking and judging on the part of light – a '*thinking nature*'[1-3], in short. This was vehemently rejected by Descartes who insisted that natural phenomena including the behavior of light must be explained using the principles of mechanics. In his philosophy, things happen the way they happen because they had to happen that way as decided by God. Leibniz objected to Descartes' philosophy. He developed his own philosophy which was closer to Fermat's. In his philosophy a ray of light travels along that path which is unique with respect to *'ease'*, where ease is understood as the quantity obtained by multiplying the distance of the path by the resistance of the medium(s)[1-3]. His principle came to be known as 'Most Desired Path Principle' (MDPP)[3]. Using his infinitesimal calculus, and MDPP he was able to offer new derivations for reflection and refraction of light. According to MDPP light follows the extremal path of time. That is, it can take paths of local maximum or minimum time paths. With this result, he overcame the criticism that FLTP was subjected to by Cartesians, and others[5]. Cartesians cited the examples of light taking the path of maximum distance and maximum time in reflections at concave spherical surfaces[1-3]. By successfully overcoming that objection by the extremal path of time principle, Leibniz's solution became the standard and is taught nowadays in high school level physics courses. We show in this paper that Leibniz's solution of extremal time path violates Ptolemy's theorem[6-8] – a fundamental theorem in geometry. Therefore, it follows that Leibniz's derivation of Snell's laws using his calculus method is invalid.

Demonstration of the invalidity of Leibniz's calculus proof of extremal time path - reflection of light

Nature operates on the basis of symmetry. Laws of Nature are symmetry based. For example, if there was an 'end purpose', then there must be a 'start purpose'; one wouldn't exist without the other. Therefore, if one tries to base one's scientific demonstration on end purpose then the same must be demonstrated based on 'start purpose'! But we don't find any such. Therefore, there is a bias in Leibniz's philosophy towards the end purpose, in preference to the start purpose, that violates symmetry. Ptolemy's theorem has a built in symmetry. In our demonstration below we use symmetry in simplifying our method of demonstration of the invalidity of Leibniz's calculus proof of Snell's laws.

Figure 1 is taken from Leibniz's 'Tantamen Anagogicum'[9] and is slightly modified[1-3] (line CP in Tantamen Anagogicum does not appear to be perpendicular to AB., where as the text say that it is).

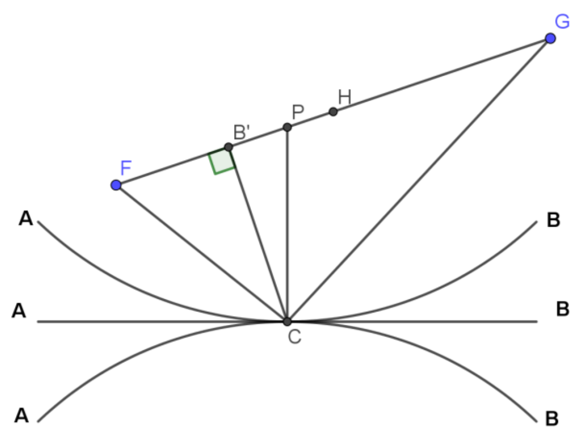

Fig.1 Figure shows Leibniz's diagram for reflection. AB is a reflecting surface - plane, spherical concave or convex. F and G are the end points of a ray of light incident and reflected at a point C on AB. CB' is perpendicular to FG from C. CP is perpendicular to AB at C. H is the midpoint of FG.

AB is a reflecting surface - plane, spherical concave or convex. F and G are the end points of a ray of light incident and reflected at a point C on AB. CB is perpendicular to FG from C. CP is perpendicular to AB from C. H is the midpoint of FG. Tacitly assuming that the medium through which light travels is homogeneous and isotropic, Leibniz reduced the problem of finding the unique path with respect to the product of distance and resistance of the medium to the problem of finding the point C such that the path FCG is unique with respect to its length. He then constructs an equation for the length of the path from F to G via some point C on ACB and uses calculus to find the value of the equation of the path such that the sum of the distances FP and PG is unique or stationary with respect to location of P on FG. He proves that FP : PG = FC : CG. From elementary geometry it follows that this demands angle FCP = angle GCP, which is Snell's law of reflection. He uses essentially the same strategy in proving the law of refraction.

His method for reflection runs thus[1-3]:

Let a ray of light from F be incident at a point C on AB and be reflected to reach the end point of the path at G. Draw a perpendicular CB' from C to FG. Applying Pythogorus theorem to the right triangle FCB', calculate FB' from FC and CB'. Similarly, calculate B'G from CG and CB' of the right triangle GCB'. Minimize the sum of these two distances, FB' and B'G by letting the derivative of the sum with respect to the location of C (or P on FG) be zero. This yielded him the result that the ratio of FP to GP equals the FC to CG. It led to the equality of the angle FCB' - the angle of incidence and the angle GCB'- the angle of reflection. That is the Snell's law of reflection.

We demonstrate below that it is impossible for sum of the two distances, FB' and B'G to be a minimum without violating Ptolemy's theorem (PT). Since Lebniz's derivation of Snell's law of reflection using the method of calculus violates PT which is a fundamental theorem of geometry, Lebniz's derivation of Snell's law of reflection is invalid.

Statement of Ptolemy's theorem[8]

Ptolemy's theorem states that: The sum of the products of the opposite pairs of sides of a cyclic quadrilateral is equal to the product of the diagonals. Let ABCD be a cyclic quadrilateral (Fig. 2). Then according to Ptolemy's theorem, we get Eq. (1).

$$AB \times CD + BC \times AD = AC \times BD \qquad (1)$$

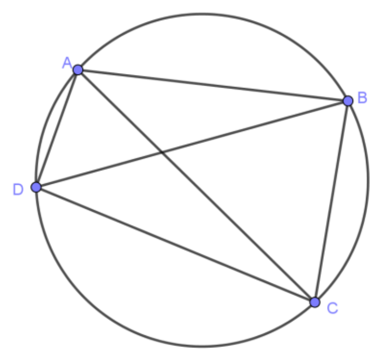

Fig. 2 The figure shows a cyclic quadrilateral ABCD used to describe Ptolemy's theorem.

A special case of Ptolemy's theorem arises when three sides of the quadrilateral are equal[6-8], that is, when there is an equilateral triangle in the quadrilateral (Fig. 3). The symmetry of the triangle leads to a simplification of Eq. (1) and gives Eq. (2) below.

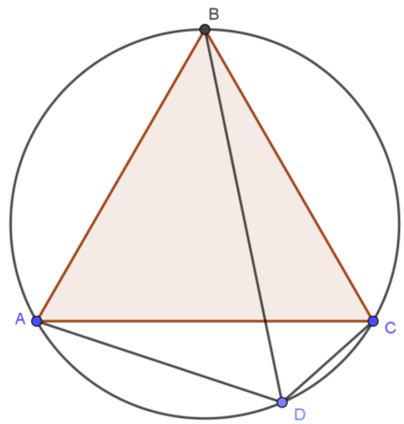

Fig. 3 The figure shows a cyclic quadrilateral ABCD inscribing the equilateral triangle ABC (shaded) used to describe the special case of Ptolemy's theorem.

Let ABC be an equilateral triangle, and D be any point on the circumcircle of the triangle. Then, the largest of the distances DA, DB, DC is equal to the sum of the other two distances.

$$DB = DA + DC \qquad since\ AC = BC = AB \qquad (2)$$

Construction of the equilateral triangle and its circum circle using the reflected ray path[6-8]

Let us construct an equilateral triangle FEG on the line segment FG (Fig. 4) with end points F, G of the path of the reflected ray couple, FC, CG. Let us also construct the circumcircle of the equilateral triangle FEG. Now let us consider the quadrilateral FEGC.

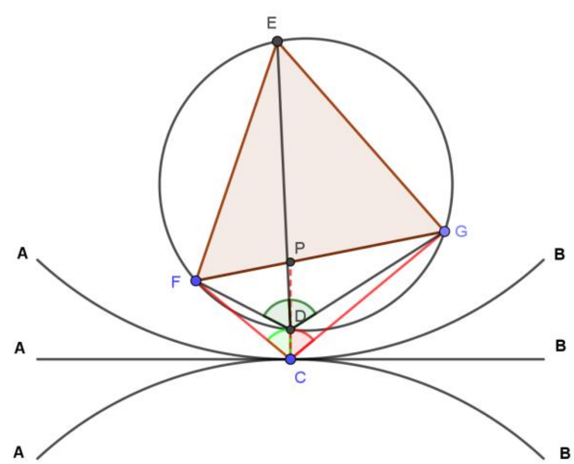

Fig. 4 The figure shows the equilateral triangle FGE (shaded) on the side FG and the circumcircle of the equilateral triangle FGE. Draw a perpendicular CP to AB at C. It intersects the circle at D.

The circumcircle does not pass through C. The quadrilateral FEGC is not cyclic. This is to be expected since the angle FCG is an arbitrary angle (since the incident ray FC is arbitrary) and the sum of the opposite angles at E and C of the quadrilateral is arbitrary.

Let the circumcircle intersect CP at D. Join D to F, E and G. Since all the four points F, E, G and D are concyclic by construction, the quadrilateral FEGD is cyclic.

According to the above special case of the Ptolemy's theorem we get,

$$(FD + GD) = ED \text{ (is a minimum)} \qquad (3)$$

However, according to Leibniz

$$(FC + GC) \text{ (is a minimum)} \qquad (4)$$

$$But, \quad (FD + GD) \neq (FC + GC) \text{ since } C \text{ is not on the circle} \qquad (5)$$

From Eq. (5) we see that Leibniz's result Eq. (4) violates Ptolemy's theorem, Eq. (3). Since Leibniz's result violates a fundamental theorem of geometry, viz., the Ptolemy's theorem, Leibniz's proof of Snell's law of reflection is invalid.

Least distance path is the least time path

In the case of reflection, the speed of travel is constant throughout the path. Therefore, we get the result that if the distance of travel is a minimum then the time of travel is necessarily a minimum. Let the constant speed of travel be *v*. If s_1 and s_2 are the distances of travel before and after reflection, then, we get.

$$\left(\frac{s_1}{v} + \frac{s_2}{v}\right) = (t_1 + t_2) = t = \left[\frac{(s_1 + s_2)}{v}\right] \qquad (6)$$

If $(s_1 + s_2)$ is a minimum then a constant multiple (or fraction) of $(s_1 + s_2)$ is also minimum. We demonstrate this geometrically below.

Construction of the equilateral triangle and its circum circle for travel time

Let (s_1/v) = A'C. We draw a parallel line A'B' to AB (Fig. 5). A'B' represents the time of travel along AB; A'C the time of travel along AC and CB' the time of travel along CB. We draw an equilateral triangle A'E'B' with A'B' as the side length. Draw the circumcircle of the triangle A'E'B'. Let P be any point on this circle. Join P to A', E' and B'.

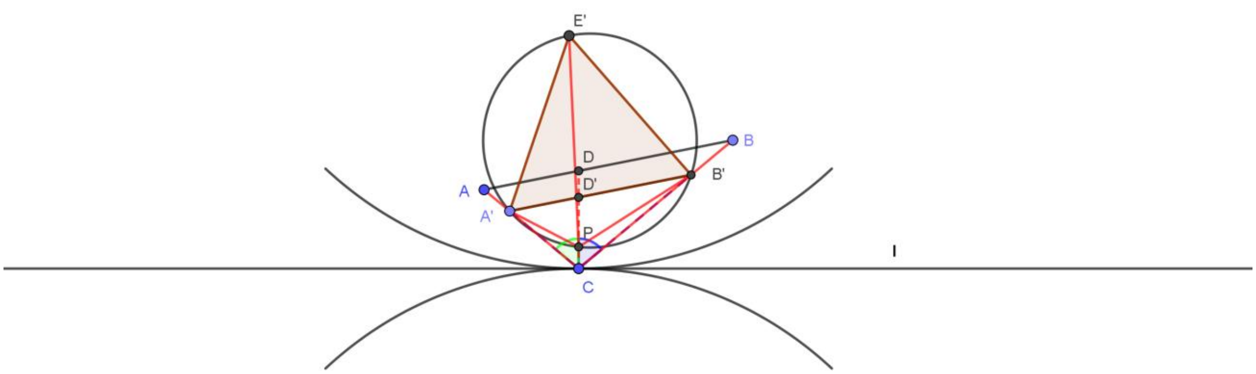

Fig. 5. The figure shows the equilateral triangle A'E'B' (shaded) on the side A'B' and the circum circle of the equilateral triangle A'E'B'. A'C and B'C represent times of travel along AC and BC.

A'C (=AC/*v*) and B'C (=B'C/*v*) represent times of travel along AC and BC respectively. According to Ptolemy's theorem we get,

$$A'P + B'P = E'P \quad \textit{is a minimum.} \qquad (7)$$

However, according to Leibniz

$$A'C + B'C = EC \qquad (\textit{is a minimum}) \qquad (8)$$

$$\textit{But,} \qquad (A'P + B'P) \neq (A'C + B'C) \quad \textit{since C is not on the circle} \qquad (9)$$

From Eqs (7), (8) and (9) we see that Leibniz 's result Eq, (8) violates Ptolemy's theorem Eq, (7). Thus, since Leibniz's result of extremal travel time violates a fundamental theorem of geometry, viz., the Ptolemy's theorem, Leibniz's proof of Snell's law of reflection is invalid.

If light were to take the least time path between any two points in a given medium, then, when it undergoes reflection on its paths, the point of incidence must lie on the circumcircle of the equilateral triangle drawn with the line segment joining the end points of the path, as the side.

This completes the demonstration of the invalidity of Leibniz's proof of Snell's law of reflection.

We now proceed to the demonstration of the invalidity of Leibniz's proof of Snell's law of refraction phenomena.

Demonstration of the invalidity of Leibniz's proof of Snell's law of refraction phenomena

It is very easy now for us to demonstrate the invalidity of Leibniz's proof of Snell's law of refraction phenomena, because the argument follows similar lines as in the case of reflection.

Let m_1, m_2 be two media through which light passes from a point A in m_1 to a point B in m_2 by refraction at a point C on the interface l, of the two media (see Fig. 6). Let us assume m_1 be the rarer medium and m_2 be the denser medium. Let n be the normal to l at C. It is well known that light bends towards the normal when it goes from a lighter to a denser medium.

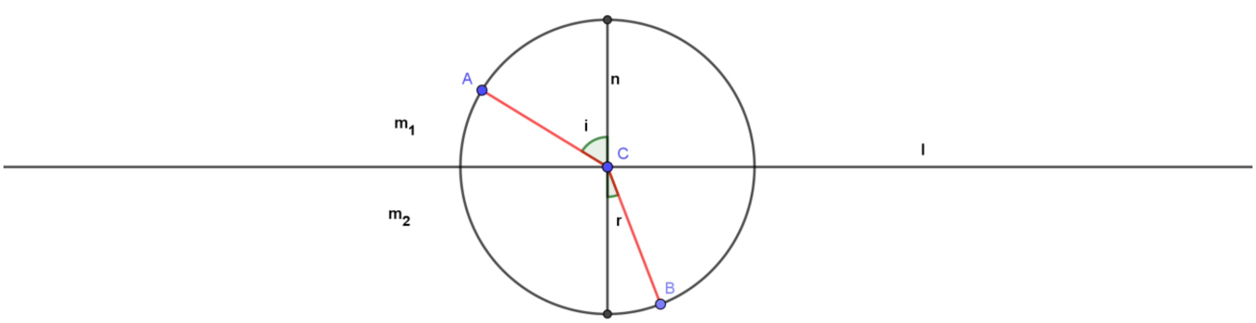

Fig. 6 The figure shows the path ACB of a ray of light refracted at the point C on l, the surface of separation of two media m_1, m_2. n is the normal to l at C.

Let AC = CB = s. Let v_1 and v_2 be the speeds of travel of light in the two media m_1, m_2 respectively. Let i, r be the angles of incidence and refraction respectively. According to the Snell' law of refraction, we get,

$$\frac{\sin(i)}{\sin(r)} = \frac{v_1}{v_2} = \text{constant} \qquad (10)$$

Let t_1, t_2 be the times if travel along AC and CB respectively. Then we get,

$$\frac{s}{v_1} = t_1, \qquad \frac{s}{v_2} = t_2 \qquad (11)$$

According to Leibniz $(t_1 + t_2)$ is a minimum.

To see if it is a minimum we proceed as follows.

Without loss of generality we take $v_2 = 1$, then we get, $s = t_2 =$ CB, Let $(s/\,v_1) = t_1 =$ A'C (see Fig. 7).

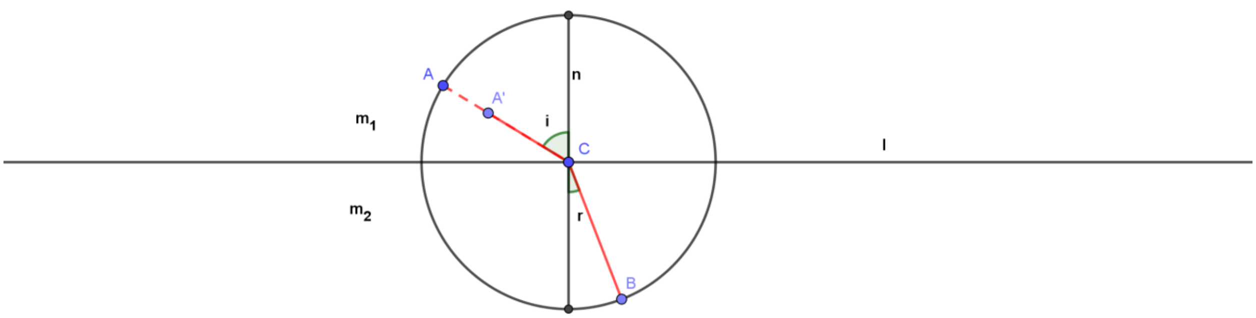

Fig. 7 The figure shows the times of travel (red) along the segments AC and CB.

Thus we get A'C and CB as the times of travel along the distance segments AC and CB.

We construct an equilateral triangle A'BF with A'B as its side (see Fig. 8) and the circum circle of the triangle. We find the circumcircle does not pass through C. The quadrilateral FA'CB is not cyclic. This is to be expected since the angle A'CG is an arbitrary angle (since the incident ray A'C is arbitrary) and the sum of the opposite angles at F and C of the quadrilateral is arbitrary.

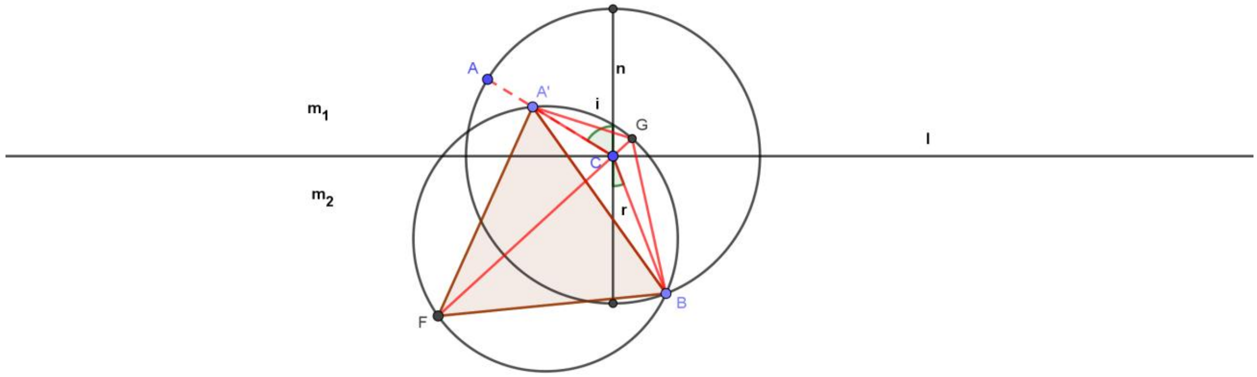

Fig. 8 The figure shows the equilateral triangle A'BF with A'B as its side and the circum circle of the triangle.

Join G to A', B and F. Since all the four points F, A', G and B are concyclic by construction, the quadrilateral FA'GB is cyclic.

According to the above special case of the Ptolemy's theorem we get,

$$A'G + GB = GF \quad is\ a\ minimum. \qquad (12)$$

However, according to Leibniz

$$A'C + CB \qquad (is\ a\ minimum) \qquad (13)$$

$$But, \quad (A'G + GB) \neq (A'C + CB) \quad since\ C\ is\ not\ on\ the\ circle \qquad (14)$$

From Eqs (12), (13) and (14) we see that Leibniz 's result Eq. (13) violates Ptolemy's theorem Eq. (12). Thus, since Leibniz's result violates a fundamental theorem of geometry, viz., the Ptolemy's theorem, Leibniz's proof of Snell's law of refraction is invalid.

If light were to take the least time path between any two points in a given medium, then, when it undergoes refraction on its paths, the point of incidence must lie on the circumcircle of the equilateral triangle drawn with the line segment joining the end points of the path, as the side.

This completes the demonstration of the invalidity of Leibniz's proof of Snell's law of refraction.

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References

1. McDonogh J. K., Leibniz on Natural Teleology and the Laws of Optics, http://nrs.harvard.edu/urn-3:HUL.InstRepos:5130440 (2009). Doi :10.1111/j.1933-1592.2009.00254.x

2. McDonogh J. K., Leibniz and Optics,The Oxford Handbook of Leibniz, Ch. 23, Ed. Maria Rosa Antognazza, Oxford Univ. Press. (2012),

3. Richard Lamborn, “Thinking Nature, Pierre Maupertuis and the Charge of Error Against Fermat and Leibniz”. Ph. D. thesis, Univ. South Florida (2015).

4. Pierre de Fermat, Study of Maxima and Minima, Original in Latin, This English translation was made by Jason Ross from the French translation in the CEuvres de Fermat vol. 3, (1638) pp. 121-156.

5. Per Stromholm, Fermat’s Methods of Maxima and Minima and of Tangents, A reconstruction, Arch. Hist. Exact Sci. Vol. 5 (1968) pp. 47-69

6. Radhakrishnamurty Padyala, ‘A geometrical proof of Ptolemy's theorem’ (03/05/2020). http://viXra.org/abs/2005.0026, RG DOI: 10.13140/RG.2.2.20787.17446.

7. Radhakrishnamurty Padyala, ‘A geometrical proof of an Application of Ptolemy's theorem’, Azim Premji University At Right Angles, March (2020).

8. Shailesh Shirali, ‘How to Prove It’, At Right Angles, **6** (3), (2016).

9. W. Leibniz, Tentamen Anagogicum: An Anagogical Essay in The Investigation of Causes, Acta Eruditorum (1682), Philosophical Papers and Letters a Selection Translated and Edited by Leroy E Loemkar, Holland. Boston, D Reidel Pub. Co. (1976).