Numbers, Empiricism and the A Priori

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The present paper deals with the ontological status of numbers and considers Frege´s proposal in *Grundlagen* upon the background of the Post-Kantian semantic turn in analytical philosophy. Through a more systematic study of his philosophical premises, it comes to unearth a first level paradox that would unset earlier still than it was exposed by Russell. It then studies an alternative path, that departin[1]g from Frege's initial premises, drives to a conception of numbers as synthetic a priori in a more Kantian sense. On this basis, it tentatively explores a possible derivation of basic logical rules on their behalf, suggesting a more rudimentary basis to inferential thinking, which supports reconsidering the difference between logical thinking and AI. Finally, it reflects upon the contributions of this approach to the problem of the *a priori.*

Keywords: philosophy of mathematics, Logical Empiricism, Frege, Russell, a priori.

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1 This paper is a contribution to the philosophy of mathematics from a non-technically trained philosophical perspective, as will become apparent for colleagues in the field. I do not pretend it to be otherwise. I, nevertheless, hope it can offer a valuable perspective on these problems. A first shorter version of this paper was presented at the conference *The Philosophy of Logical Atomism 1918-2018*, Complutense University Madrid, 28.01.2019.

1. Introduction. The Historical Backstage

Logical Empiricists critically rehabilitated the Kantian epistemic project meant to set apart genuine knowledge of the external world from our own contributions to it—a task that, though distinctive of philosophy from its very origins, has over and again become swallowed up by the outgrowths of different forms of undifferentiated idealisms and re-enchantments. Their project, though, adopted the specific form of dispelling those confusions brought about by misleading grammatical appearances, which often deceive us into believing a surplus of phantom realities and the pursuit of pseudo problems. But their Verificationist Criterion of Meaning (VCM) aimed nevertheless, as did Kant, to separate out experientially based knowledge that could serve scientific progress from speculative metaphysics and the possible projection onto the world of human emotions and values, characteristic, they thought, of morality and aesthetics—aspects that, significantly, Kant did not understand in any experientially based mode either.

Kant distinguished, however, two different ways that human beings might contribute to external world knowledge. These contributions could be due to extra content or they could be due to form, to our own form of cognition. The first characterised the excesses of transcendent metaphysics, illegitimately enhancing the world with further non-experientially based additions of our own. The second, though, constitutes his transcendental philosophy with the introduction of synthetic a priori judgments. These latter he found not only legitimate but absolutely essential if any knowledge of the world were to be possible at all. It is here where the Empiricists, getting rid of what they considered unnecessary, and misconceived, a priori conformations of experiential knowledge, most strongly departed from Kant. But, in doing away with the whole Kantian transcendental apparatus and his conception of synthetic a priori judgments in favour of just logic and language, they arguably arrived at much too restrictive criteria, which ended up making their own position untenable—since the removal left an explanatory lacuna when it came to giving an account of the constitution of the objects of experience from sensory data alone, the explanation of causality and other forms of necessity present in even in our most basic scientific laws. How successful their later attempts were to provide alternative accounts of these aspects by appeal to logic and language alone is still a troublesome issue. None of it obviates the important reasons that spoke against the Kantian position on this specific point—not just the revolutionary transformations brought about by Non-Euclidian Geometry, Einstein´s Relativity Theory and Quantum Mechanics into our scientific picture, but also the increased centrality gained by semantics owing to the writings of Bolzano and the later reception of Frege: the first appeared to directly contradict the Kantian theory; the second showed how well we could do without it. The perfect match between the difficulties of the theory and the incipient success of its abandonment, set the conditions for a paradigmatic overturn.

1.1. The Resulting Epistemic Setting

Once the Empiricists had renounced any other source of knowledge from a provenance external to our own, *prima facie* less mysterious, logico-linguistic equipment, experience became the only ground on whose basis to derive and validate our knowledge claims, the ultimate and sole criterion of existence. To this end, Russell´s analysis of definite descriptions opened up what can be considered the most consequent and properly empiricist line of existential analysis. If the grammatical surface can mislead us into believing in non-existent objects through deceitful singular terms, the way to expose it is precisely to lead them back to the ‘tribunal of experience’. There we could see whether or not there was an individuum satisfying the descriptions associated with the term. From this perspective, the claim that because our specific theoretical postulates require the existence of such objects they must be taken to exist in some other way, could not be taken to hold, simply because there is no other way to exist. Actually, as Coffa (1991, 109) points out, Russell saw himself as thereby 'neutralizing the tendency to produce false abstractions.' The kind of things that exist is, of course, a complicated issue, but at least we had to be able to find some basis in experience that allows us to confirm or disconfirm existential claims or else show how our terms are related to it. Otherwise, the whole fuss about transcendent metaphysics would have seemed superfluous were we to end up postulating entities as we see fit. The importance of Russell´s theory of descriptions was celebrated by Ramsey[2] who, following its lead, proposed his famous 'Ramsey Sentence' with the purpose of dealing with theoretical scientific terms in a similar way, a proposal that was later elaborately developed by Carnap (1966, Ch.26). As in Russell´s case, scientific sentences with singular terms seeming to refer to some abstract entities had at least to be seen as conditional to corresponding existential sentences from whose truth the truth of the theories would depend. Following this string of thought, the Fregean proposal to introduce numbers as abstract objects referred to by the corresponding singular terms in mathematical sentences, could scarcely be accommodated.

But the problem in this case was that neither of the options available appeared to provide the resources needed to deal with the status of mathematical knowledge—those options being either 1) to reduce numbers to experience or 2) to provide an account of them through mere logic and language. In the first case, neither a direct reduction of Mill's empiricist type, nor one analogous to Ramsey and Carnap's treatment of theoretical scientific terms, showed any means of success; but neither did the possibilities opened up

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[2] According to Coffa (1991), Ramsey would have seen in it as one of the greatest achievements of the century.

by the second—Conceptualism and Formalism—the preferred route of authors such as Schlick, Hahn or early Carnap[3]. Conceptualism, which was Russell´s option after the breakdown of Frege´s project, was problematic mainly for two reasons: concepts, even if understood as conventions, could not be *mere* conventions on pain of being absolutely hollow and useless; but if they weren't, showing them to be meaningful required remitting them to their verification conditions (as required by the VCM) or, at least, showing through an explicit conceptual analysis their ultimate possible connection to experience. This implied that there had to be something that these concepts were *about.* It had to be possible to prove whether what was said through them was the case or not, and this brought us back to the initial problem. Understanding them as some kind of properties, as Russell did, thus made things no better, since it equally required either showing how exactly they were to be derived from experience or accepting them as some new kind of abstract objects, giving rise to the consequent problems again. Formalism, on the other hand, attempted to find a solution by assimilation of them with logic, believing that, at least for some concepts, the question of their 'aboutness' could be dealt with differently. The corresponding concepts would actually concern rules, having more to do with relations among objects than referring to any objects or properties. But, far from being wholly unproblematic, implicit in this option was the assumption that the status of logical laws and our peculiar 'a priori grasp' of their necessity was absolutely no issue. Not even the conventionalist account, which according to Coffa (1991, Ch 14) would have provided the semantic tradition's solution to the problem of the a priori, can be considered to have given an appropriate response to this question. As Prior (1960) exemplified with the case of *Tonk,* the fact that we should set a concept with its corresponding rules of use to run,

[3] See, for example, Goldfarb (1998)

and then appeal back to those very inferential rules to justify it can be seen as circular[4]. The source of necessity of logical laws was through such explanations in no way exhausted. Actually, much of what is at issue here, as we will see later on[5], depends on this question. But, as an explanation of mathematical statements, Formalism could not give an account of their truth in any substantive manner. There are, of course, contemporary defences of Formalism of which I cannot pretend to give a proper account here, such as Field's (1980)[6], that ascribe to themselves the capacity to adopt talk of truth. But as long as there is nothing independent of those very forms capable of deeming mathematical statements true, I do not see how it could be defended that such truth is significant in any sense or that it could be nothing more than correct uses made of pre-given rules.

The reasons why the attempt to reduce numbers to experience in accordance with the route of existential analysis opened up by Russell's Theory of Descriptions did not appear worthy of a try, might not be immediately obvious. So, I think it deserves at least a quick look, since it might bring out more clearly the starting point and motivations of Frege´s own account. Three possibilities can be distinguished here: 1) direct reduction; 2) existential conditionalized reduction; 3) functional conditionalized reduction.

The first can be taken to represent the position defended by Mill, for whom what we mean by natural numbers are compilations of objects. Natural number terms would be general terms obtained per induction from different sample groups. When we say there

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4 Questions of conservativeness and consistency might be appealed to here, but not even in this way is the matter clarified. There can be untruthful consistent systems, and non-conservative rules might be worth incorporating, forcing consistency to be rearranged.

5 See 4.2 in this paper.

6 In accordance with Shapiro (2000, 226)

are ‘Five apples on the table’, what we actually mean by ‘five’ is something to be found in the apples on the table, something they have in common with groups of five oranges or five peaches. The immediate problem, as it appears, is that the *fiveness* itself is nowhere to be experienced in the examples given.

The second corresponds to a parallel treatment of numbers to theoretical scientific terms. Here, we would discard claims containing numbers as singular terms, by way of conditionalizing such expressions upon some existential sentence no longer containing the term. The main difference with the previous option is that as with theoretical scientific terms, number statements would have conditions of application. If a track in a cloud chamber justifies claims about ‘neutrino’, the existence of specific compilations of objects does so with corresponding number statements. In both cases we assume that the application conditions do not exhaust the cognitive content of the terms. This would imply the existence of something else, ‘a surplus of content’ as Carnap puts it, going beyond what the application basis justifies (the presumed entity *neutrino*, and the *number* in question). The problem now is that while we can know what it would take to prove the existence of the assumed entity in the scientific case, and so make the truth of the initial statements dependent upon it, no similar hope is available in the case of numbers. We are not able to go beyond compilations of objects to a more adequate candidate of existential substitution. The problem of numbers reveals itself, therefore, as being clearly of a quite different sort. What makes this option interesting to consider is, though, that contemporary critiques of the Neo-Logicist programme (such as Field, 1984)[7] argue along similar lines to prove its implausibility. They assimilate Frege’s contextual introduction of number to a procedure aimed at introducing numbers as abstract objects on the basis of what might be seen as application conditions, but with no way to

[7] See Wright (2001, 160-164) discussion of this point

existentially legitimise the assumed further claim of the existence of numbers—no more than we could try to legitimise the existence of God through a conceptual introduction as in the Ontological Argument.

The third option, however, differs from this one and also from Russell´s own conceptual solution, coming from an empiricist perspective closer to Frege´s own proposal. This I call the 'Functional Conditionalization' option. The starting point would be the same, that is, the compilations of objects that would deliver the application conditions. It would provide the contact point with experience, but again would not exhaust the cognitive content of number claims. But now, instead of hoping for a hopeless existential candidate upon which to conditionalize the truth of such claims, we would make it dependent upon the existence of a recognisable and acceptable function (for pragmatic reasons acceptable, perhaps) that could justify the transition from application conditions (compilation of objects) to claims about numbers. This brings us into the vicinity of Frege´s own functional introduction of numbers, since we could imagine such a function in similar terms to Frege's '1-1 correlation' between the members of different compilations. But the point of the reconstruction from this reductionist perspective would, rather, be the opposite to Frege's: to deny the existence of numbers. Since the mediating function could just be a man-made one, not itself provided through experience, and since from its fulfilment the acceptability of number claims depends, the thereby legitimised claims can just be (however else understood) man-made products. The strategy could be seen as analogous to a similar treatment of thick moral concepts, which would justify the transition from behaviours to values through the fulfilment of a moral function[8]; the attribution of the one to the other being then implicitly registered in the concept. If the behaviour fulfils the function, we consider it good in the thereby defined moral sense. In

8 See, for example, Ramirez (2018)

our case, the transition from compilations of objects to 'numerical values', so to speak, would be made possible by a number-building function. That would be the idea. Could a response along these lines answer Field's (1984) type of complaints of having extracted an abstract object from an insufficient basis? Here, our result would be obtained through a specific mediating operator that takes application conditions as input and obtains numbers as output; by each added member to the compilation a successive number. The answer to our previous question would be that there is an 'intermediate reason' and that we have to do with a product, not a discovery. But what could be said of the number term so obtained? Does it refer to anything? Can it be considered to be justified in any empiricist-satisfying terms through our contact point with experience via application conditions? Even if we were to say that we have a constructed referential object, what would be its character? In the moral case, we can say that what we obtain is a moral *value* (in the sense of being good for the purpose of the fixed moral standard). But what is it that we obtain here? Would it make sense to talk of 'numerical values', as I did before (bringing, perhaps, the comparison to rely on the equal measurability of benefits, pains, lengths, weights or whatever, and arguing that actually the real 'value' is the number therein)? Would we not then again be required to give an account of their status? Or should we talk rather of 'a substitutive symbol' for such equivalences or maybe 'merely a term'? But even if we were to adopt a non-problematic position that reduces the obtained product to something like a 'shortage term' whenever the functional mediation is possible, the question is whether an interpretation along these lines is in fact available to our empiricist. As Frege´s approach makes clear, and for reasons we will see in a minute, the answer is that it is not.

From this perspective, we might be better able to see the very dimension of the solution that Frege proposes, since Frege, I believe, is the one who really makes an attempt to respond to the lacuna left by the Kantian synthetic a priori, not just in the philosophy of mathematics but as a whole.

2. Frege´s motivations

Although Frege was not as moved as others by the discovery of Non-Euclidean Geometry to abandon the notion of the synthetic a priori as an explanation of geometrical knowledge—nor might he necessarily have been by discoveries in astrophysics—he had his own reasons to abandon the realm of spatial and temporal Intuition[9], as he saw it, when it came to Arithmetic. It was the generality of arithmetical thinking, the certainty and necessity of its proofs, the fact that we could not, as he argued, question its basic principles without contradicting ourselves, that indicated an intimate connection with our own thinking processes. Arithmetical thinking was not simply a specific way of thinking but appeared to be our own way of thinking on itself. This would explain the fact that it would have such an overall embracing domain: 'to it belongs not only the actual, not only the intuitable but everything thinkable. Should not the laws of number, then, be connected very intimately to the laws of thought?' (Frege 1884, §.14, 21). Since it was logic that represented the laws of thought being equally general in character, it had to be possible to make this 'intimate connection' explicit and show how the concerns of arithmetic arose through pure logical thinking. It had to be possible to prove that the reason why arithmetical thinking applied with certainty and necessity (Dummett 1991) was because

9 Since Intuition here is meant in a sense akin to the Kantian notion of 'pure Intuition' I will use it with a capital I, to distinguish it from the idea of (non-sensible) 'intuition', understood as some special undetermined faculty capable of acquiring knowledge beyond the realm of experience, that Kant himself criticises.

of its derivability from logical laws and definitions alone.

But what Frege had set himself to do in his reconstruction of the logical form of our discursive thinking about the world, had a much wider reach. It amounted to including in the formal laws of logic, and thereby in the analytical realm, the epistemic possibility of our knowledge of objects[10] and those further necessary structures through which we would think about them, capturable themselves, in his view, through potential new, creative, conceptual synthesis[11], thereby reintroducing back into the field of logic, as I will come back to, essential features of the Kantian synthetic a priori.

Since his analysis of the logical form of linguistic discourse went beyond the mere reconstruction of its logical rules to include how such rules referred to objects, it was now possible too to reason about objects without the objects themselves; to do so in a universal and certain way about whatever objects we could possibly have to do with, and so to reason about the world without the world—precisely what would be needed in arithmetical thinking, whose objects had those very eternal and universal features too.

Frege´s *Begriffschrift* in this way brought logic much nearer to arithmetic. However, he thought that despite this communality, mathematics was not simply the same as logic but had a topic of its own, a topic it was *about*; something beyond the mere thinking procedures that made its statements true. Mathematical statements could be substantively true, and this was for him a non-negotiable idea. The task was, thus, to come to identify, through a similar logical procedure, the objects that made mathematical

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10 It is quite striking how far Kant's introduction of his idea of the recognition of transcendental objects of understanding is already in line with Frege´s proposal:

> What does one mean, then, if one speaks of an object corresponding to and therefore also distinct from the cognition? It is easy to see that this object must be thought of only as something in general = X, since outside of our cognition we have nothing that we could set over against this cognition as corresponding to it. (Kant 1781, KrV A104, 231)

11 See Dummett (1991, 305-36)

statements true, thereby giving our logical reasoning not just the capacity to think about objects but its own objects to think about. That this should be possible departing from mere logic and definitions, required somehow turning the forms of thinking, our very mechanism of objectual apprehension, upon themselves in such a way that we obtain a new form of second-order synthesis. Something along these lines is suggested by Dummett (1991, Ch.4): it would be synthetic in the sense of it being knowledge gained by encapsulating a content different from itself.

From Dummett's reading, what Frege attempted to do was a matter of dissecting some kind of second-order pattern 'within the expressed thoughts themselves'—the same procedure he would have taken himself to have used to come to his logical form in his *Begriffschrift* in recognising the hidden structure lying in our discursive thinking[12]. It would be possible not just to extract conceptual information about the objects we speak about, but to build new concepts in grasping the more complex patterns of inferential reasoning we were able to discern in our linguistic constructions. It is this very idea of creatively recognising new patterns whose justification would be independent of experience, that in my view very much resembles a form of synthetic a priori knowledge—the difference being, of course, the absence of reference to experience or Intuition. But I leave further discussion of this until later. However, in a parallel sense Frege would see it as possible to extract a pattern to arithmetical reasoning that would lay open what it is, we are referring to in talking about numbers. So, how is this to be understood?

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[12] The example he gives to illustrate this possibility is how it would be possible to 'dissect' a complex pattern from the proposition 'Either Jupiter is larger than Neptune and Neptune is larger than Mars, or Mars is larger than Neptune and Neptune is larger than Jupiter', into the pattern 'Either Jupiter is larger than x and x is larger than Mars, or Mars is larger than x and x is larger than Jupiter'. Which then can be captured with the concept 'Intermediate in size between Jupiter and Mars' It would require understanding the whole proposition, and not as a derived result from its components, to obtain the pattern. (Dummett (1991, 40-41)

Frege´s contextual introduction of the concept of number in *Grundlagen* attempts to explain the identity of what is referred to by the concept of number through an equivalence relation. The concept of being ‘equinumerous’ between two concepts is explained via an identity relation with a 1-1 correlation between the members of each concept.

> The number of Fs = the number of Gs if and only if there is a one – one correlation between the Fs and the Gs

The question is, therefore, what exactly is being done here? Dummett would say that Frege is attempting to explain the concept of number in terms of a new synthesis exercised upon the correlation 1-1 between the concepts on the right-hand side. That is, what this new synthesis records with the concept of ‘equinumerous’ is a pattern found in the established correlation on the right-hand side. There is i) the correlation 1-1- and there is ii) the recording of the pattern, being thereby created through a new concept: the concept of the specific number. This is supposedly the idea. But, first of all, what is the pattern supposed to be a pattern of? The fact that we establish a 1-1 correlation is, in principle, just the fact that we do so, even if we capture it with a new concept. What would be the difference between the concept ‘correlation 1-1 between the individuals of the two conceptual extensions’ and the concept of ‘equinumerosity’? [13] Unless we are ready to say that the first delivers the application conditions[14] and operational resources (via the

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[13] This is a possibility that even Wright (2001,164) considers as a possible counterargument, putting it in terms of whether the ontological commitments would be the same. He does give an answer to it, but I must say I am not sure what to make of it.

[14] Although I derive this from the proposed functional conditionalization offered before, Dummett comes to suggest too that Frege might be appealing to the truth conditions of the

correlation) on whose basis something else is to be proved (as in the case of conditionalizing upon functions in the third empiricist option before)[15], I fail to see how a difference in what each concept is supposed to refer to could be found. If we do follow such an explanation, we would say that the 1-1 correlation acts as a functional *operator* (similar to a multiplying one, for example) allowing us to derive a new product. But while a standard mathematical operation explains how in using it, we come to something new, here we would be doing something different. We are tracing a 1-1- correlation among the members of the extension of different concepts and are expected by virtue of it to grasp something new *there,* capturable through a new concept. But, in what sense is this extracting a higher order pattern *within thought*, as Dummett says, and not something more similar to the way a concept is extracted from a reality by finding something in common between two instances? It is usually explained that, if such a correlation holds,

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numerical terms, though not suggesting what I go on to say about a construction of number terms through the operational procedure.

15 It is the possibility of seeing such a procedure as opening a gap, through which the introduction per identity of all kind of imagined figures, that forms the core of Field´s arguments against Logicism, focusing specifically on the Ontological Argument of the existence of God. However, I do think that there is a difference here, to the extent that while in the Ontological Argument we require that through mere definition of existential possibility (by way of comparison with what there is) there must be such a Being, in the numerical case we are assumed to be able to grasp it in what is given to us. In this line also Wright (2001), according to which there would be in the numerical case, besides it, true instances of the application of the term. This brings us back, however, to the problem of what exactly are the application conditions here. If we adopt the functional-operational explanation, we would be assuming that there is a justifiable procedure that drives from one to the other. But defending this without the procedure delivering sufficient reasons, seems to me problematic on second thoughts.

the new numerical concept acts as a second-order conceptual function applied to the first concepts (F and G). The new synthesis thereby created in each case is said to be the same ‘number’. But I am not sure whether with it we really become aware of what is happening here and how the pattern is ultimately obtained. We are supposed to do this in view of the correlation on the right-hand side. So, let us try to be more specific. One could say, in accordance with Frege (1884, §46) that each of the members of the extension is turned into such through the concept that encloses it. It is through the concept of an apple that we sort out the unities of such. That is, the unities have been conceptually defined as such. So, it is upon two sets of such conceptually conformed unities, resulting, that is, out of a previous work of conceptualisation, that we are to find the correlation. The 1-1 correlation marks the conditions determining where attention should be directed. What he would be asking us to grasp is the common pattern in such groups of individuated conceptual apprehensions through a new conceptual synthesis. So just as a concept applied to a reality sorts out a unity, the concept of the group sorts out one too, a new unity upon already conceptualised ones, which would be the number.

Connecting now to the reasons why the third functional empiricist option does not work for the empiricist, it becomes most clear what would be wrong. We have proceeded as though talking of a compilation of objects as a starting point would be no problem. We took a group of five apples or oranges as our point of contact with experience. But as Frege was well aware of, we are in no way appealing to the experientable apples or oranges themselves. They can be quite different—big, small, red, green or with different shades of orange—but what we expect someone to grasp in this context is the fact that they are a given set ‘of unities’ of a sort; what they have been turned into by virtue of conceptual differentiation. That is what is relevant in this case. Their ‘being unities’ is something that we can just recognise as a result of conceptual work but not as something experientable or abstractable in itself.[16]. This act of individuation is what makes it possible that no matter whether it is apples or half apples that we sort out, they can be equal when tracing a 1-1- correlation among the unities of corresponding extensions. Whatever we say about them in inferring from their being such a number of unities is necessary and certain independently of experience. About this, Frege was, of course, right.

The empiricist ambition, therefore, to get rid of abstract objects by reducing them to compilations of objects (when we actually mean their unities), starting from which we could then reconstruct functionally upwards what our numerical concepts refer to, is a fraud. But the importance this understanding of unities has in the whole Fregean enterprise is in my view greater than is commonly acknowledged, since it is upon this basis that the second-step proof for recognising the identity of numbers is built. While specific singular numbers would be based on this previous conceptual work, the notion of a unity allows two possible interpretations: a) the very idea of something being separated out through a concept, the content of it (the resulting unity); or, more in line

[16] It is important to understand this properly, since the idea is not that the concept makes up the reality, in some version of the idea that reality itself should be seen as conceptual. The distinctions in reality must previously have been there and recognisable for us first, in order to introduce the concept. The point would actually hold up if we were to adopt a version of non-conceptual contents, since it would be the act of distinguishing (the making of a synthesis upon) whatever aspect (even if we should not be talking yet of an intersubjective linguistic normative concept) it is that already separates out a unity. Actually, this possibility, that we should have this prior capacity, is what Frege would be appealing to when asking us to be able to grasp a pattern there, since, even if we should be grasping in a second-order synthesis the result of our own conceptualisations, its recognition requires exactly the same capacity as that in the first-order one.

with the procedure used with the singular numbers, b) the very act of synthesis done through the concept. If we follow Dummett in understanding how Frege's idea of analytic unities is to be understood, we should take it that he believes that here we must also do with a second-order synthesis upon concepts, since it would be not the conceptual synthesis itself but the second-order realisation of what is done in this process. Therefore, what Frege would be grasping through the concept of a unity would be what different acts of conceptual synthesis have in common, thus option b), the very conceptual unification. This makes sense, since this is what we would recognise when 'turning logical form upon itself' in a second order synthesis, while remaining in the realm of logic. The repercussions this will have for Frege´s project goes, in my view, to the very heart of his difficulties.

When asked, then, to recognise in a second-order synthesis the pattern in the 1-1 correlation at the right-hand side of the equivalence, we would be capturing such conceptual unifications (in b) in a new all-embracing one. Representing thereby the common pattern between both sides of the correlation.

Frege is known to have found this procedure unsatisfactory as is expressed in what has become known as the Julius Caesar Problem, because if you come to try to introduce a new entity per identity, how can you know what it is that you are finding in common with the other side of the equivalence if you cannot already presuppose that it is the unities that you mean? The fact of there being a correlation might be a pure casualty and what is meant is something else present there. Frege's solution to the difficulty was to opt to provide us directly in an already explicit form that it is the extensions that were meant. A concept G would be equinumeral to a concept F if its extensions were equivalent: $(x)F(x) \rightarrow G(x)$. This change of terms struck me like a sudden jump achieving its goal by departing from the careful epistemic derivation he had accustomed us to, to deliver a ready-made product without an explanation of how we came to it. As Wright (1983) argues, he seems to have thought that since classes were already part of logic, this was legitimate[17]. But wouldn´t there be an issue too regarding how we came to such classes of individuals with their own identity as unities in the first place? However that might be, this proposal, known as Basic Law V, delivered Frege's final understanding of numbers as class extensions. These class extensions would have, nevertheless, been constituted by equal numbers of unities understood in the sense of b) above.

3. Unities and how they interweave with Frege´s difficulties

The problem that arises through the understanding of unities is entrenched in the very issues Frege (1879) arrived at in his *Begriffschrift* with the discovery of variables as formal 'conditions' for objects. The idea was to be able to reconstruct the common formal structure of our thinking and talking about objects. Concepts were understood as incomplete functional expressions to which different (numbers of) objects could be assigned. This would allow, as he thought, a parallel treatment of numbers. Frege (1884) considered his most significant insight to be the idea that, as he said, it was only relative to a concept that we can count—just if you consider 'Books on the table' you can say there are (a, b, c) (if we are to represent each book) or if you consider the 'Moons of Jupiter' (a, b, c, d, n). Therefore, he concludes that in attributing numbers what we are doing is ascribing a given set of *unities* to a concept. These unities, thus, are not the apples or oranges we are experiencing but rather what makes them unities of the sort independently of what they exactly are. When he describes what is being done through this process Frege tells us

[17] Whether classes are part of logic is a disputed matter, see for example Wright (1983, 111)

> In the sentence: 'Jupiter has four Moons' the unity is 'Jupiter-Moon'. Under this concept fall the I as well as the II as well as the III, as well as the IV. That is why we can say: that the unity referred to by I is the same as the unity referred to by II, and so on. Here we have the Sameness. But when what we assert is the divergence of the unities, what we understand is that of the counted things. (Frege 1884, 42)[18]

In using the Roman numerals, he marks the distinction between the objects and the unities, thereby stressing that it is only through the concept that we can come to consider the different objects falling under it as equal in their being unities, that, as such, we can count. That is, when we say that Jupiter has four Moons, what we ascribe is the same as we ascribe when we say that there are four Russian armies in Stalingrad; however different the armies or the Moons are, what we are ascribing is a given amount to the respective concepts.

We can express this, following Frege, in representing the Moons of Jupiter through corresponding unities—not the objects, of course, but placeholders of them, such as in 'Moons of Jupiter' ($()_1$, $()_2$, $()_3$, $()_4$, $()_n$). Each would be individualised by the conceptual application and not independently of it. Frege dedicated some sections in *Grundlagen* to argue against others who claimed to obtain unities directly through an abstractionist process for getting rid of the particularities of an object. Through such a process, he argued, we would not end up with an abstract notion of unity, since being a unity is not something that we can somehow grasp in experience too (without the

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18 I use a more literal translation of the original German edition, even if it might sound a bit awkward since I find more clear the way Frege expresses this thought there; marking the distinction between the objects and the unities in starker form than might be apparent in the English version. Of course, the English translation attempts to say the same and you can read it that way too. I just think the original one makes this relevant contrast for the point I want to stress more apparent.

conceptual work) and then keep stripped of all other properties. It is in this sense that he rejected seeing numbers as sets of unities obtained per abstraction from reality. If we got rid of the experienced particularities of the reality, nothing would actually be left. Rather, in attributing unities to a concept we would be representing how many such conceptual individuations we can separate out. But, in this last sense, we do refer to what is common to them as such conceptual individuations, as explained in the previous section. If I can subsequently draw a correlation with some other concepts' unities, it will be precisely because as conceptually individuated ones they are the same.

The idea in *Begriffschrift* of representing the mere possibility of objects falling under a concept through conditions (again something like placeholders)[19] would allow us to make general claims. These conditions would be turned into realities when saturated by any real, corresponding individual. However, these placeholders (variables, in normal terms) were actually to count as 'numerical' (one place) unities differentiated through a concept too. The quantifiers, as their name implies, would then help to specify *how many* of such unities we are referring to, whether all of them or at least one, or whether we could talk about two of them falling under a concept without having to specify which determinate one it was. But here too we are talking about unities. We could say that three men crossed the road, without having to specify which particular ones they were. To the concept 'men crossing the road' three individuals could be assigned. Then we could try to see whether this was true, by finding as many corresponding particulars satisfying the

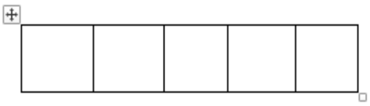
[19] We can see an allusion to this in the following quotation:

> This seen, we can also see the following possibility. Instead of linking our chain of deduction to any matter of fact, we can leave the fact where it is, while adopting its content in the form of a condition. By substituting in this way conditions for facts, throughout the whole of a train of reasoning we shall finally reduce it to a form in which a certain result is made dependent on a certain series of conditions…It is not impossible that the laws of number are of this sort. This would make them analytic judgments despite the fact that they would not be discovered by thought alone. (Frege 1884, 23)

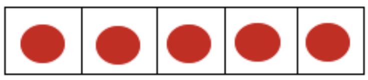
predicate (no matter those originally meant or others, since in either case the claim would be true). The same goes for ‘Jupiter has nine Moons’. I need not know which Moons these are to understand it, and if I happen to distinguish a corresponding number of them (even if completely different ones[20]) the claim would be true. Actually, this versatility is very important.

However, Frege´s final goal, as he himself says, was to set the conditions for referring to numbers[21], to quantify over numbers. So, precisely those numbers that were going to end up being understood in Frege’s work as classes of such (conceptually sorted out) unities, (capable of representing equally apples, oranges or whatever ‘equinumeral’ sets) were supposed to end up being seen as objects saturating those spaces. If we were to represent this to aid visualisation, we could picture it as follows:

1st We obtain possibilities of objects, variables, falling under a concept. Let us represent them as this



2nd These, we said, could be turned into realities when satisfied by experiential objects, such as here



3rd But, the idea was to come to see numbers (that are going to be understood as classes of unities, as explained) as objects saturating the spaces[22] of the variables.

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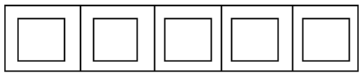
[20] Which is not unlikely, since they go up to as many as 79 now, according to NASA

[21] For example, here:

> As I remarked at the beginning, arithmetic was the point of departure for the train of thought that led me to my ideography. And that is why I intend to apply it first of all to that science. (Frege 1879, 6)

[22] This can be said of each numerical unity separately as well as for whole classes of them.

This is better seen if we first consider the separated *numerical* unities saturating the space of the variables, that amounts to something like this



What this does, therefore, is to allow us to take ‘possibilities of objects’ (since that is what numerical unities[23] are when considering the pattern obtained through the concept in the sense of b), sec. 2, turned into objects themselves, as saturating the very same possibilities of objects represented by the variables; that is, to take possibilities as realities—what we might call a Parmenidean monstrosity!—since possibilities define themselves by not-being. Therefore, if we treat possibilities as realities, as real objects, what we get is that Not-Being Is! A paradox thus arises, right at the moment that we are feeding, so to speak, the space of a possible object with itself.

That is, since variables are actually sets of one possible unity, if you saturate them with themselves (a unity of the same empty sort, with no determinate reality) you are already saying that a corresponding set belongs to itself and taking this path you go directly to Russell´s paradox. I will expound this point some more later.

If we try to follow Frege’s original line of thought, as developed in section 2., it required giving the obtained unities derived through conceptualization, and assigned to the corresponding concept, an identity as specific singular numbers. This, it seems to me, demands that we first establish an identity among unities in isolation[24], a process through

23 These numerical unities can already be considered equivalent to number one, since they represent the pattern obtained by sorting out what is common to two such conceptualized unifications. Figure 1. below refers to their obtention.

24 This would seem to cohere with Frege’s own thought process, which considered essential the process of identity to talk of numerical unities, as expressed for example here:

which we would obtain a synthesis of the conceptual unification in terms of b) above. This would be needed if we are to be able to identify what is it that is meant by the 1-1 correlation, since identifying the whole set of unities presupposes being able to identify the individual ones. Recognition of the whole would then be achieved through the equivalence relation between unities, through which one obtains corresponding empty entities (possibilities of classes of unities) that could be represented then shortened into the entity 5; since if the unities are conceptual, so is the unity of unities that the numerical concept introduces.

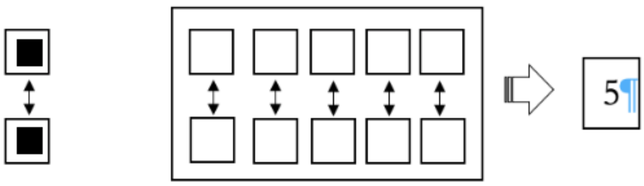


Figure 1.

On the left has been represented the identity of two conceptually sorted out unities, on the right the identity of the whole group. The patterns thereby obtained correspond in the case of the unity to the conceptual unification (in the sense of (b) sec.2.). The alternative possibility of having captured the content of the unity, as suggested in option (a) sec.2., is marked through the small black square. In the case of the identity of the group, the common pattern to both sides of the correlation, embracing all such previously identified conceptual unities, is represented in figure 2.

Is the dog conscious however dimly in that common element in two situations which we express by the word ‘one’? (Frege 1884, 42)

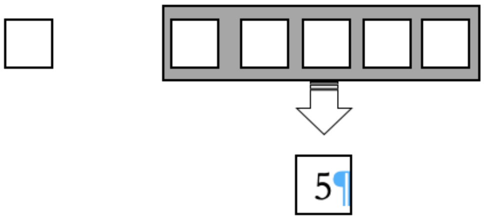


Figure 2.

Now, repeating the procedure of taking singular numbers as objects saturating our variables, we are allowing the new corresponding entity, the specific singular number (in darker grey), to take the place of this possibility of class as though it were a reality

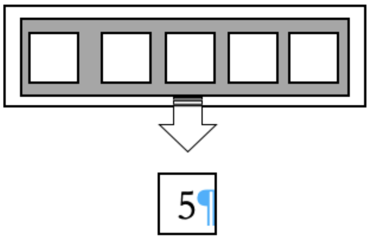


Figure 3.

If this is finally understood, as Frege does through his Basic Law V, as a class of equinumeral extensions, and we repeat the procedure of taking singular numbers as objects saturating our variables, we come to obtain Russell´s paradox.

There are several things to comment here. To start with, it should be noticed that in this process the paradox actually presents itself twice: first at the level of unities, as explained before—what we might call a 'First-Level Paradox'; and then, at the level of class extensions—a 'Second-Level Paradox'. Another aspect I wish to comment on is the step that I have included in Frege´s process requiring an identity for a unity first. It seems to me that the lack of one is related to the difficulties Frege found in Hume´s principle expressed in his Julius Caesar Problem. Maybe Frege initially believed it sufficient to have determined what ascriptions of unities to concepts amount to. But, identifying their 'unicity', what is meant by such, as expressed in a second-order pattern dissected within the very conceptual synthesis issued by the specific concepts, seems to me not to be done with it yet.[25] Once we have already sorted out the pattern of the unity, it could have been argued that it is not the object or something else we mean, since in obtaining the pattern of the whole we are already working upon the extracted unity patterns.

In general terms, this reconstruction seems to me to allow one to see more clearly a deeper source of difficulties in Frege´s project than that usually considered to have doomed the project to failure from early on. But we have so far given no consideration to what a proper reconstruction would have to look like, and that is what I want to try next.

4. Flipping things around

Understanding the difficulties that Frege arrived at requires understanding the whole scale of what he had attempted to do, his whole understanding of logical form and how he thought the grasping of it took place. As we saw, Frege included in his understanding of logical form the conceptual conformation of objects and the necessary relations through which we reason about them in our linguistic discourse, equally capturable through a second-order conceptual synthesis. So, if at a first level we were to conceptualise the world linguistically and establish such connections, at a second one we would capture the structure of this very work of conceptualisation, the formal unification introduced by our concepts and the implications traced among them, by way of new concepts. Remaining all the way through within the limits of the formal domain. This last step is to be differentiated from the idea of capturing, through a higher-order concept, an already conceptually understood reality. It is more like capturing the very act of capturing, the unifying synthesis thereby taking place.

[25] Consider the previous footnote.

The way Frege understands what is achieved through this second-order conceptualisation is essential to his project. Therefore, I wish to consider first, how the route of interpretation he took affects the understanding of the objects he was after, and where the alternative would have led. And secondly, I will focus on the understanding of the logical connections at the first-order level and the idea of a second-order conceptualisation of their patterns.

4.1. The objects we were after

Frege's central idea—that it is relative to concepts that we can count—together with his realisation that we could represent the role of objects through corresponding 'conditions', were the two important moves that connected his 'objectual' reconstruction of logical form with the arithmetical understanding of number. But, although what he finally represented as the numerical unities was the very form of the unification done through the concept, and thus the concept of the unity, his intention was to differentiate between the logical unities (the variables) and the rightly so-called numerical ones. The second were supposed to be the objects of the former (as also, later, the specific numbers). Therefore, I think that while, among the options drawn at the end of section 2, he was actually aiming at a), the resulting content of the unification, what he ended up representing was b) the concept of the unity. This was not without reasons, of course; it was all that he could allow himself, if he was to remain within the boundaries of the formal framework, he had set himself. He thought he could do well enough with the formality of the unity, since the alternative in a) went beyond the logico-linguistic realm representing the result of the synthesis done by the concept into something *other*. Probably a key aspect of it was that he was paying less attention to the notion of the unity he was arriving at and more to the possibility of obtaining the pattern of the whole conceptual extension through a second-order synthesis, which he hoped could achieve his aim. But since this second synthesis was done on behalf of what were already conceptual ones, what he obtained was again of a conceptual, formal nature.

What would the alternative have looked like? Where would route a) have taken us? What are the numbers we would have arrived at and how are they to be understood? I think that the metaphor of 'figuring out' what it is that we are doing when separating out a unity through a concept into some background extension, into *something other* to itself, into a virtual representation of reality itself, is more than a metaphor. I think the only way to make sense of this is through a representation that requires both sides: what the concept does, and its counterpart, what it does it upon. In this way we can gain not just the concept of a unity but what it is a concept of. This way of putting things brings us back to something akin to Kantian transcendental Intuition.

Actually, if we are to properly comprehend what it is that we are doing through conceptual application we have to represent not just the concept but the whole background extension of conceptual activity, since differentiating something from something else always requires some extension where the severed out remains in a different position at some level other than that from which it detaches itself. An illustration of the process might be helpful to see better,

Figure 1.

Although conceptual demarcations normally take place in reality, pre-existent differences being necessary for us to discriminate conceptually upon, to understand what is it that we do in conceptual use we must represent it to ourselves, such as in Figure 1. Since we cannot really be said to *perceive* the result of our own conceptual labour, such representation enables us to get a proper grasp of it. This way we can separate out what is taking the place of reality, the extension in the horizontal line, and what is done through conceptualisation in separating out a unity by way of using a concept. On the basis of such a representation, we can then better understand what we mean by talk of 'being' (being real), 'being something', being 'an object' from the perspective of a concept; we can differentiate what becomes a real unity (symbolised by the 1 in Figure 1.), a Moon of Jupiter or whatever, and what is not (symbolised by the 0).

The importance of representing the conceptualisation in its context, is that through it we can also realise that what remains outside the conceptualized, 0, has its own part in the process, since on the one hand it makes 1 actually possible as a unity (otherwise both would be the same undifferentiated whole). However, at the same time it is cut off as 'not being', that is, not being there to be counted in the Fregean conceptual sense, and is, therefore, literally 0. Now, through this process 0 becomes 'some kind of unity' too, through being separated out, but one that 'is-not' from a countable perspective. Thus, it exists, but not in any linguistically accountable sense: it is nothing. Mystifying as it might sound, this represents I believe, the *noumenons*, *non-being*s, *limits of language* and *silences* of our literature.

If this interpretation is right, then maybe the insight to be gained is that 0 is not to be defined as the class of all objects that are not identical to themselves, as Frege said, but rather as 'what remains outside of any class'[26], since if counting, as well as in the

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[26] Someone might wonder how we then explain the existence of empty classes, if we cannot say that there are classes with 0 members. This doubt was expressed by Peter Simons at a Conference

representation we can make of it through Imagination[27], is always counting from a perspective (virtually *ad infinitum*) 0 is always what can never be included. Also, if we were to count all there is, *being*, since this requires a perspective too, it must be conceptually detached, leaving something outside—the consequence thereof being, which can hardly be news, that no account of what there is can ever be complete.

A further consequence of this perspective is that it would allow to answer problems such as those relative to the truth of negative sentences that interested Russell. Since in any complete account of the individuals existent in a world, given by the conjunctive set of conceptually individuated ones, 0 would always have to be taken to exist. Therefore, negative sentences would be true, because now we must say that there is something, 0, that is somehow there too. But how do we explain a sentence such as "there is no *rhinoceros* in the room"? Since that doesn´t just say that there is nothing in the room, but what kind of individual there-is-not. In a sense, it is absurd to say that "that nothing" that exists in the room is of one kind rather than another. But we do want to speak that way, that there is nothing relative to a specific concept and in that sense to speak of *possibilities* of this and that, that do not exist in this world. So maybe we could also represent such non-existence, but we have to differentiate such representations from

on *The Philosophy of Logical Atomism 1918-2018* at the Complutense University in Madrid on the 28.01.2019. My answer would be the following: saying that an empty set has one peculiar member not identical with itself which happens not to exist is not accurate, since it would be like saying its content is a Meinongian figure. By this account, on the contrary, we are saying precisely that what corresponds to such a class is non-existence (not as a mere modal issue). It captures no reality. We can write 0, but we need not say that it is a peculiar impossible entity.

27 Since 'Imagination' is meant in a sense akin to the Kantian notion of Pure Transcendental Imagination I will use it with a capital 'I' to make this more explicit.

those of existence. So, on the one hand they would be forms of 0, but relative to a concept, giving rise to corresponding unities of such sort, they would 'exist' in the negative way of 0, we could represent them as the result of conceptualising in that realm. That is, they would correspond to the negative numbers. This are, of course, tentative approaches, but I think, they are worth considering.

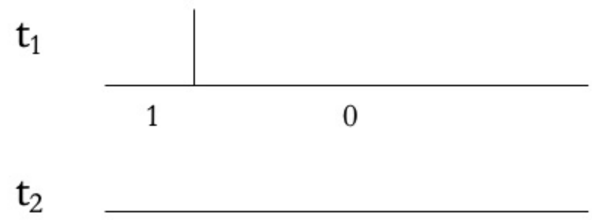
4.2. The logical rules

Going now back to the understanding of logical rules in Frege's picture, and actually in Logical Empiricism more generally, we distinguished at the first level conceptualisations and inferential relations whose patterns were likely to be reconstructed conceptually again at a higher, second-order level. A first question, then, is how the inferential relations themselves, at the first level, are justified in this picture. Frege explains how we might recognise and conceptualise new patterns in our linguistic discourse, but how these are introduced in the first place, why we put them forward with inferential necessity, is not explained. It is also insufficiently explained in the conventionalist picture, as we briefly saw in discussing Formalism. As Coffa (1991, Ch.14) argues, the solution given by Wittgenstein and Carnap to the problem of the a priori was to turn things around: instead of saying that we grasp the meaning of logical constants or geometrical undefinable terms through some form of intuition (or Intuition in the geometrical case), deriving then from them further axioms and a priori truths, it is the methods of measuring and those axioms themselves, or the logical rules in the case of the constants, that determine meaning, this being the reason why their truth struck us as necessary. But this just delays the question, since our problem now is how we come to those rules that determine the constitution of meaning, how we derive their necessity if it is not to be seen as conventional. We have singled out two models: a) meaning determines rules and requires intuition (or Intuition);

b) rules determine meaning and the necessity of rules is therefore seen as unquestionably presumed, unless we reject meaning as a whole. This, Coffa argues, was initially shown for the field of Geometry by Poincaré and Gilbert, for whom the measurements and axioms of Geometry defined primitive notions such as ‘distance’, and not the other way around. But then we are driven back to the question of the origin of those rules that define the semantic primitives.[28]

My proposal now is to see how this problem turns out from the perspective we have adopted, if we were to accept that understanding the very idea of unities, and therefore the very notion of measurement units, requires representation of what it is to obtain a unity (in general) through conceptualisation in some virtual representation of the extended context upon which it takes place. Then, whatever conclusions we might derive in thinking about them would apply to whichever units and conceptual applications we were talking about. Again, visualising might be helpful,

Figure 2.



If we depart from Figure 1 above (t_1 in Figure 2), once we have this representation of what severing a unity from its background amounts to, we can come to a few further

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28 According to Coffa, Wittgenstein was aware of this problem and thought that ‘grammar’ itself could not be regarded as conventional; there was a way grammar should be, but no justification could be given not requiring a justification itself in an infinite regress.

conclusions. We can, for example, come to realise that if the conceptual detachment did not take place, the unity, 1, gained thereby would not be a unity at all. This necessary conclusion is not gained through some mysterious capacity of intuition, but simply because in modifying things through Imagination in some extension akin to Kantian Intuition, we can come to *see* it, as pictured in the transition from t_1 to t_2 in Figure 2 where the vertical line is taken away. We literally see that the very existence of the generated unity is only the case through its being differentiated from what remains outside of it; were it not so, it doesn´t exist as a unity. Since this represents any unity whatsoever, as we said, it applies generally and therefore necessarily in all cases. In other words, this allows the *introduction* of a necessary connection $Not\ 0\ \rightarrow\ Not\ 1$. Starting from here, we can also come to conclude that for this unity, 1, to be itself, the unity we might identify with the singular term '1', it cannot be whatever other (conceptualised or not) it leaves outside, call it 'Not 1', in this case 0. Otherwise, as Frege in his own context puts it, we go back to an undifferentiated whole. We see, then, how through this very rudimentary process we can come to a first law of identity, expressing exactly that $'1' \rightarrow Not\ ('Not\ 1')$, equivalent in more standard terms to this other $\alpha \rightarrow\ \neg(\neg\alpha\ )$.[29] These simple relations hold from the very fact of something being differentiated as a unity and, since this is what characterises any object conformation whatsoever, it will necessarily apply to any relation among objects we happen to consider.

However simple this reconstruction might seem, I think it stands, and coheres, for example, with some counterfactual reconstructions of necessity. The idea is that the capacity to Imaginatively represent general epistemic contexts in 'Intuition' in order to

[29] Notice that while with the numbers 1 and 0 we move at the ontological level of unities, by using names for them we are identifying them linguistically. The same goes for the representation in terms of $\alpha\ and\ \neg\alpha$.

keep them fixed and allow them to be modified easily (while keeping the different moments of the transition present, moving back and forth between them)[30] allows us to understand how the most basic logical rules themselves come to stand. Showing that they are not primitive and, in that sense, not a priori given in our cognitive equipment but developed — although we should be taking them, once acquired, as a priori justified for further uses—. We can study what is or is not possible by virtue of what actually happens when introducing such modifications and advance on that basis what is *necessary*. Since here what we are reconstructing, as explained, is the *constitution* of inferential relations affecting any unity on the grounds of being such, we have to do with a general idea. That is, it is not because other unities should be similar to this one that we are allowed to make a generalising inference, requiring an explanation of a supposedly pre-existent capacity of so inferring. It is rather that the represented unity is instantiated in any occurrent one; and so, the question—how do you know that you can infer from this case to all others? —does not pose itself. It would be tantamount to asking: how do you know that what happens to this, happens to this? Notice too, that advancing the first inference, its necessity, is simply a matter of acknowledging, as a matter of fact, that this 'unity' ceases to be what it is if some conditions are removed. Since, again, we have to do with a general claim, this will be so in all cases, and thus we can advance an inferential claim.

Returning now to the previous conventionalist idea, by picturing the basics of arithmetic in these terms we would be explaining the constitution of those very

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[30] This is an essential feature in Kant (1781 KrV A101-102, 230) and I think an essential one in any reconstruction we are to give of such a background extension. As he points out in the paragraph 'On the Synthesis of Reproduction in Imagination':

> Now it is obvious that if I draw a line in thought or think of the time from one noon to the next, or even want to represent a certain number to myself, I must necessarily first grasp one of these manifold representations after another in my thoughts. But if I were always to lose the preceding representations (…) then no whole representation (…) could ever arise

measurement units and the further necessary conclusions we advance on their basis whereupon meanings can then be said to be built. We would thus be delivering a deeper *constitutive* account of the a priori. This would not deliver a justification of the laws then requiring, in an infinite regress, a further justification for it (as Wittgenstein argued), but rather simply depicts their very constitution. The difference with the Kantian picture is that here the constitution of logic too comes out as synthetic a priori. That is, if it is right, as argued, that representation in 'Intuition' is required in order to arrive at the notion of necessity and the necessity of identity, then the conclusion to be drawn is that not just mathematics but also logic is synthetic a priori. Furthermore, since inferential connections emerge as *derived* from more basic distinctions (Figure 1), we appear to come closer to a computational picture of human's most rudimentary cognitive capacities.

The contrast between this picture and a traditional reconstruction of the undefinable concepts of geometry, for example, as synthetic a priori, on the other hand, is that in putting arithmetic and thereupon derived logical laws as more basic first, they are no longer primitives, but could, nevertheless, be explained further in the conventionalist way. All these results are put down with care for the weight I know they carry, but I want to put them down for further reflection. What is clear is that whatever this might otherwise imply, this way of looking at things definitively turns things around for Frege´s project.

One last point I must return to is the idea I have been putting forward that Frege actually reintroduces the idea of the synthetic a priori even if he claims to do away with any recourse to anything like experience or Intuition. This relates to the role of the second-order synthesis of deductive patterns 'within thought', as Dummett puts it. From Frege´s perspective, since it all takes place in the logic-linguistic realm they can safely be regarded as analytic. The idea of there being such second-order patterns of reasoning procedures seems to me perfectly fine. But the question is, whether understanding the necessity of such reasoning procedures, grasping their pattern, is at all possible without going all the way down to their application in the first-order realm? Whether we could make any sense of them without figuring out, as we have argued, what their application in some counterpart extension amounts to? Think of it this way. Take the three models of *a priori* necessity considered: 1) meaning determines rules; that is why, in knowing the meaning of a term, we can immediately (a priori) see that the predicate belongs to the understanding of the subject; 2) conventionalist, model, we said that it is actually because the rules themselves determine meaning that this is the case. Frege would actually say something along these second lines, since it is in grasping new patterns of such rules that we come to a new synthesis. But if we now, 3) constitutive model, ask ourselves how such rules themselves come to stand, we conclude that it is through such a representation of what being an object[31] amounts to that we can derive some further necessary relations with other objects. So, we end up building the different models upon each other (in inverse order), with the ultimate one giving sense to the necessity of the others. And this, in a way that cannot be dismissed as being merely the triggering origin that says nothing to an a priori knowledge we would grasp as having been always necessarily there,[32] since it is the very grounds for why such rules are introduced as certain at all. Would we not then have rather to say that the (supposedly analytic) synthesis built thereupon, is ultimately understandable on this basis? Is Frege then, inadvertedly, ultimately appealing to that which he had wanted to ban, that is, the synthetic a priori in a more traditional sense? On the other hand, there is also an issue with the idea itself of a second synthesis

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[31] Of course, the idea of an object is meant here in much more general sense than the usual one.

[32] Frege argues that the fact that experience (or here Intuition) might be the source is irrelevant for the justification of the claims. This is true once the claims are already obtained, of course.

obtaining *a pattern* required for conceptualisation, since the very act of doing so and not deriving it from experience is precisely what Kant characterises as a synthesis a priori. From the Kantian perspective it would not be that we grasp it, but that we can get it because we ourselves *do* it.

5. But then, what are numbers?

Well, before we make ourselves a picture of the singular numbers, if doing so requires prior identification of the unities that conform their extensions, we must first ask what unities are. The answer would not be the concepts of unities, as it ended up being for Frege, but rather what we can represent as the result of applying concepts; something like shadowy representations of real unities through Imagination. This is what Frege himself realised at the end of his life and what Kant said. But these are no realities, for reality requires still something further: corresponding unifications in experience, which result from the application of specific concepts to specific experience.

Frege´s difficulties arise if we understand his proposal in terms of b), as the concept of the unity, and then understand them as objects saturating variables. Doing so gave us the First-Level Paradox, a one-possible unity inside another, representable through a unitary set with another unitary subset as its object, such as this: $U = ((x))$. The initial unity now contains the numerical (conceptual) unity, *and* its occupiable space as its subset. The new empty unity in the subset could be saturated again and again by another and another one-place unity, giving rise to unities with increasing members, and the question is whether this recursive process is not paradoxical in itself. However, I believe the most important problem is that it raises a version of Russell´s paradox, which we might call the Paradox of the Unity (or unitary set) since the question would be whether the set of all (unitary) should or should not include itself as a member. If it includes itself as a member, can it still be considered the set of all? Since as a member it is not the set of all anymore unless it includes itself again (and then again recursively), it cannot really include itself without ceasing to be what it is; however, if it does not include itself, it is no longer the set of all. If we have a set with one possible individual as a member, and we allow number one (understood as a set of one possible member) to be a possible substitution instance, we are doing exactly what we have described: allowing our set to include itself, with the consequence of sabotaging its very possibility of being the set of all possible one individuals anymore, since as an instance it is not. But if we do not allow number one as an instance, it could not be the set of all possible one individuum either (since number one is to be considered one such).

By considering numerical unities now as Imaginative representations of the real individuals obtained by conceptual application, we are simply representing the results of conceptualisation. So, the situation is a different one: numerical unities are then being used as mere (non-saturable) *representatives* of individuated realities in order to figure out things about relations among them. Being representatives not of *factive* reality but of what individual reality *is like*, they therefore need not constrain themselves to a real number of individuals but can exceed this with ease (ad infinitum). Since such representatives are not sets, no paradox applies.

One last question I wish to address is the status of such Imaginative representations. I have not made it clear whether I am referring to a mental representation or whether we are talking of possible intersubjective representations. In Kant´s own transcendental philosophy, the point was that such synthetic a priori knowledge is not obtained per exceptional capacities of direct intuition of something going beyond the realm of experience. This view amounts to reopening the door to transcendent metaphysics with all its potentially intuitable creatures. Rather, it is when we try to reconstruct how it is possible that we arrive at something not derived from experience, if not by such means that we try to represent to ourselves how this must take place for things to be how they are for us. I think this is right in the sense that, as previously said, we cannot claim to epistemically perceive our own conceptual conformation of unities, for example, but rather come to grasp what is it that *we do* when representing this very performance to ourselves. But the question is how this exactly takes place. In the Kantian model, it is actually in abstracting the experiential (in the sense of stripping it away) from our recognition of already singled-out objects that we come to realise that there is something that we are taking for granted about them, but which cannot be said to have been experienced or grasped by special faculties. In this way too, in Kant's view, we would come to the consciousness of pure spatiotemporal Intuition. It too, is understood ultimately as a form of a priori synthesis (remitting to the a priori synthesis of apperception). We would realise that in our very ordering of impressions in our keeping of their transition present, there is also something, not in itself experientable but which is necessary for our knowledge of experience to be possible at all. The contemporary critic would see a problem in the mentalistic aspect of such a reconstruction, but things could be put the other way around. We can come to represent in intersubjective sharing ways, virtual computer representations, architectonic schemes or pictures, formal universal aspects of our reality, because we must be capable of schematising them in some such form. We can all agree in view of such shared depicted representations, but it must be because we can recognise something there, the same structure that enables us to come personally to depict it in such a way in the first place. I am afraid there is much too discuss on this point, so I will have to leave it here.

There are many other issues open for further thought. One, for example, is to what extent the proposal put forward is coincident with positions such as those of Brouwer or Dummett, who defend understanding numbers as synthetic a priori too. This is something

I still have to think about. I have intentionally remained relatively neutral about how the background extension is to be understood, in order to reserve myself the right to think further on the extent to which I share the Kantian picture as a whole and make up my mind on such matters. Another issue is whether the position outlined implies that numbers are constructed or not, or whether the possibility of such representations reproducing themselves ad infinitum is enough to consider them given—which actually seems a plausible option, since Wittgenstein´s 'Rules as Rails' picture, which in the standard conceptual case fails (for the need of human assessment to determine further conceptual application), appears perfectly unproblematic in this one. Finally, the initial question of how we are to come to the singular number from here, although I think it enables a reconstruction in some such representative terms, is left for a further occasion too.

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