A LIMITATION ON NEUTRO(NEUTROSOPHIC) SOFT SUBRINGS AND SOFT SUBIDEALS

MOHAMMAD HAMIDI AND RASUL RASULI

Abstract. In this study, we introduce a type of neutrosophic subset concepts such as soft subrings, soft subideals, the residual quotient of soft subideals, quotient rings of a ring by soft subideals, soft subideals of soft subrings and investigate some of their properties and structured characteristics. Also, we investigate them under homomorphisms and obtain some new results. Indeed, the relation between uncertainty and semi-algebraic is presented and is analyzed to the important results in neutrosophic subset theory.

1. Introduction

Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science, and social sciences. These kinds of problems can not be dealt with by classical methods, because classical methods have inherent difficulties. Molodtsov's soft set theory [9] is a kind of new mathematical model for coping with uncertainty from a parametrization point of view. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily be applied to many different fields. Neutrosophy, as a newly-born science, is a branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. It can be defined as the incidence of the application of a law, an operation, an axiom, an idea, or a conceptual accredited construction on an unclear, indeterminate phenomenon, contradictory to the purpose of making it intelligible. Recently, Florentin Smarandache generalized the classical algebraic structures to neutro algebraic structures neutro algebras) and anti algebraic structures (anti algebras) and he proved that the neutro algebra is a generalization of partial algebra. Indeed the neutro algebras are an extension of classical algebra or we can say that classical algebras are a limitation of algebras. In this regard, some researchers have published the papaers on neutro algebras such as neutro Bck algebra [5], valued-inverse dombi neutrosophic graph and application [6], extended BCK-ideal based on single-valued neutrosophic hyper BCK-ideals [7], single-valued neutro hyper BCK-subalgebras [8], new types of soft sets: hypersoft set, indetermsoft set, indetermhyper soft set, and tree soft set [14], ccsev: modelling qos metrics in tree soft toward cloud services evaluator based on uncertainty environment [15], extension of soft set to hypersoft set, and then to plithogenic hypersoft set [16], applications of extended plithogenic sets in plithogenic sociogram [17] and introduction to neutrosophic genetics [18]. At present, works on the soft set theory as a limitation of neutrosophic subsets, with algebraic applications are progressing rapidly [1, 2, 3, 4]. The author introduced and investigated some properties of soft algebraic structures of [10, 11, 12, 13]. In this paper, we attempt to study the soft subrings and soft subideals and their properties. The paper is organized as follows: Section 2 is devoted to introducing basic definitions of soft subrings and soft subideals. In this section we investigate some conditions

2020 *Mathematics Subject Classification.* 03E72, 13Axx, 16D25, 05C60.
Key words and phrases. Neutrosophic set theory, general commutative ring theory, soft ideals, homomorphisms.

such that $f_R^{\alpha}, f_R^{\star}$ and f_R^{*} will be ideals of R. In Section 3, we define the soft subideals of R which generated by f_R and we proof that if $f_R \in S_I(U)$ and $g_R \in S_R(U)$, then $f_R g_R =< f_R \circ g_R >$. Section 4 contains the notions of the residual quotient of soft subideals and we discuss structural properties of them. In this section, we define the coset of f_R and investigate the quotient ring of R by f_R. Finally, in Section 5 we define soft subideals of soft subrings and characterize them. At the end of this section, we investigate the soft subrings, soft subideals, and soft subideals of soft subrings under the homomorphism of rings and prove some important results.

2. Soft Subrings and Soft Subideals

Throughout this work, U refers to an initial universe set, E is a set of parameters, P(U) is the power set of U and R is commutative ring.

Definition 2.1. [9] For any subset A of E, a soft set f_A over U is a set, defined by a function f_A, representing a mapping $f_A : E \to P(U)$, such that $f_A(x) = \emptyset$ if $x \notin A$. A soft set over U can also be represented by the set of ordered pairs $f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\}$. Note that the set of all soft sets over U will be denoted by $S(U)$. From here on, soft set will be used without over U.

Definition 2.2. [4] Let $f_A, f_B \in S(U)$. Then,
(1) f_A is called an empty soft set if $f_A(x) = \emptyset$ for all $x \in E$,
(2) f_A is a soft subset of f_B, denoted by $f_A \subseteq f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$,
(3) f_A and f_B are soft equal, denoted by $f_A = f_B$, if and only if $f_A(x) = f_B(x)$ for all $x \in E$,
(4) the set $(f_A \cup f_B)(x) = f_A(x) \cup f_B(x)$ for all $x \in E$ is called union of f_A and f_B,
(5) the set $(f_A \tilde{\cap} f_B)(x) = f_A(x) \cap f_B(x)$ for all $x \in E$ is called intersection of f_A and f_B.

We consider the following example as an illustration.

Example 2.3. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and $E = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of parameters. Let $A = \{x_1, x_2, x_3\}, B = \{x_3, x_4, x_5\}, C = \{x_3, x_4\}$.
Define $f_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\}), (x_3, \{u_4, u_5\})\}, f_B = \{(x_3, \{u_1, u_4\}), (x_4, U), (x_5, \{u_3\})\}, f_C = \{(x_3, \{\}), (x_4, \{\})\}$. Then for all $x \in E$ we have $(f_A \cup f_B)(x) = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\}),$
$(x_3, \{u_4, u_5, u_1), (x_4, U), (x_5, \{u_3\})\}$ and $(f_A \cap f_B)(x) = \{(x_3, \{u_4)\}$. Also f_C is an empty soft set. Note that the difinition of classical subset is not valid for the soft subset. For example $f_C \subseteq f_B$ as soft subset but $f_C \nsubseteq f_B$ as classical subset.

Definition 2.4. Let $f_R, g_R \in S(U)$. Define $f_R + g_R, -f_R, f_R - g_R, f_R \circ g_R \in S(U)$ as follows:

$$(f_R + g_R)(x) = \cup\{f_R(y) \cap g_R(z) \mid y, z \in R, y + z = x\},$$

$$-f_R(x) = f_R(-x),$$

$$(f_R - g_R)(x) = \cup\{f_R(y) \cap g_R(z) \mid y, z \in R, y - z = x\},$$

$$(f_R \circ g_R)(x) = \cup\{f_R(y) \cap g_R(z) \mid y, z \in R, yz = x\},$$

for all $x \in R$. $f_R + g_R, f_R - g_R$ and $f_R \circ g_R$ are called the sum, difference, and product of f_R and g_R, respectively, and $-f_R$ is called the negative of f_R.
We can say that $f_R + g_R = g_R + f_R, f_R - g_R = f_R + (-g_R)$. Also since R is commutative, then $f_R \circ g_R = g_R \circ f_R$.

Proposition 2.5. *Let $f_R, g_R, h_R \in S(U)$. Then $f_R \circ (g_R + h_R) \subseteq f_R \circ g_R + f_R \circ h_R$.*

Proof. Let $w \in R$ and let $u, v \in R$ be such that $uv = w$. Then

$$\begin{aligned} f_R(u) \cap (g_R + h_R)(v) &= f_R(u) \cap (\cup\{g_R(y) \cap h_R(z) \mid y, z \in R, y + z = v\}) \\ &= \cup\{(f_R(u) \cap g_R(y)) \cap (f_R(u) \cap h_R(z)) \mid y, z \in R, y + z = v\} \\ &\subseteq \cup\{(f_R(u) \cap g_R(y)) \cap (f_R(u) \cap h_R(z)) \mid y, z \in R, uy + uz = uv\} \\ &\subseteq \cup\{(f_R \circ g_R)(uy) \cap (f_R \circ h_R)(uz) \mid y, z \in R, uy + uz = uv = w\} \\ &\subseteq (f_R \circ g_R + f_R \circ h_R)(w). \end{aligned}$$

Therefore,

$$\begin{aligned} f_R \circ (g_R + h_R)(w) &= \cup\{f_R(u) \cap (g_R + h_R)(v) \mid u, v \in R, uv = w\} \\ &\subseteq (f_R \circ g_R + f_R \circ h_R)(w). \end{aligned}$$

□

Definition 2.6. Let $f_R, g_R \in S(U)$. Define $f_R g_R \in S(U)$ as

$$(f_R g_R)(x) = \cup\{\cap_{i=1}^{n}(f_R(y_i) \cap g_R(z_i)) \mid y_i, z_i \in R, 1 \leq n, n \in \mathbb{N}, \sum_{i=1}^{n} y_i z_i = x\}$$

for all $x \in R$. Since R is commutative, then $f_R g_R = g_R f_R$.

The following proposition follows easily and we omit the proof.

Proposition 2.7. *Let $f_R, g_R, h_R \in S(U)$. Then the following assertions hold.*
(1) $f_R \circ g_R \subseteq f_R g_R$.
(2) If $f_R \subseteq g_R$, then $h_R f_R \subseteq h_R g_R$.
(3) $(f_R g_R) h_R = f_R(g_R h_R)$.
(4) $(f_R g_R)(x + y) \supseteq (f_R g_R)(x) \cap (f_R g_R)(y)$ for all $x, y \in R$.

Definition 2.8. Let $f_R \in S(U)$ and $n \in \mathbb{N}$. Define $f_R^n, f_R^{(n)}$ as follows $f_R^1 = f_R, f_R^n = f_R^1 f_R^{n-1}$ and $f_R^{(1)} = f_R, f_R^{(n)} = f_R^{(1)} f_R^{(n-1)}$.

Definition 2.9. Let $f_R \in S(U)$. Then f_R is called soft subring of R if
(1) $f_R(x - y) \supseteq f_R(x) \cap f_R(y)$,
(2) $f_R(xy) \supseteq f_R(x) \cap f_R(y)$,
for all $x, y \in R$. The set of all soft subrings of R is denoted by $S_R(U)$.

Definition 2.10. Let $f_R \in S(U)$. Then f_R is called soft subideal of R if
(1) $f_R(x - y) \supseteq f_R(x) \cap f_R(y)$,
(2) $f_R(xy) \supseteq f_R(x) \cup f_R(y)$,
for all $x, y \in R$. The set of all soft subideals of R is denoted by $S_I(U)$. As R is commutative so condition (2) is equivalent to $f_R(xy) \supseteq f_R(x)$.

Definition 2.11. ([3]) Let $f_R \in S(U)$ and $\alpha \in P(U)$. Define
(1) $f_R^{\alpha} = \{x \in R \mid f_R(x) \supseteq \alpha\}$ which is called α-inclusion of the soft set f_A.
(2) $f_R^{\star} = \{x \in R \mid f_R(x) \neq \emptyset\}$ which is called support of f_A.
(3) $f_R^{*} = \{x \in R \mid f_R(x) = f_R(0)\}$.

Proposition 2.12. *Let $f_R \in S(U)$. Then $f_R \in S_I(U)$ if and only if f_R^{α} is an ideal of R, for all $\alpha \in f_R(R) \cup \{\beta \in P(U) \mid \beta \subseteq f_R(0)\}$.*

Proof. Let $f_R \in S_I(U), \alpha \in P(U)$. If $x, y \in f_R^\alpha$ and $r \in R$, then $f_R(x-y) \supseteq f_R(x) \cap f_R(y) \supseteq \alpha \cap \alpha = \alpha$ and $f_R(rx) \supseteq f_R(x) \supseteq \alpha$. Therefore $x-y, rx \in f_R^\alpha$ and so f_R^α is an ideal of R. Conversely, Suppose f_R^α be an idea of R for all $\alpha \in f_R(R) \cup \{\beta \in P(U) \mid \beta \subseteq f_R(0)\}$. Let $x, y, r \in R$ such that $\alpha = f_R(x) \cap f_R(y)$. Since $x - y \in f_R^\alpha$ so $f_R^\alpha(x-y) \supseteq \alpha = f_R(x) \cap f_R(y)$. If $\beta = f_R(x)$, then since $rx \in f_R^\alpha$ so $f_R^\alpha(rx) \supseteq \beta = f_R(x)$. Hence, $f_R \in S_I(U)$. □

Proposition 2.13. *Let $f_R, g_R \in S_I(U)$. Then the following assertions hold.*
(1) $f_R(0) \supseteq f_R(x)$ for all $x \in R$.
(2) $f_R(x) = f_R(-x)$.
(3) If R is with identity 1, then $f_R(1) \subseteq f_R(x)$ for all $x \in R$.
(4) If $f_R(x-y) = f_R(0)$, then $f_R(x) = f_R(y)$ for all $x, y \in R$.
(5) f_R^* is an ideal of R.
(6) Let for all $\alpha, \beta \in P(U)$, If $\alpha \neq \emptyset \neq \beta$, then $\alpha \cap \beta \neq \emptyset$. Then $f_R^\star$ is an ideal of R.
(7) $f_R^* \cap g_R^* \subseteq (f_R \cap g_R)^*$.

Proof. (1) $f_R(0) = f_R(x-x) \supseteq f_R(x) \cap f_R(x) = f_R(x)$.
(2) $f_R(x) = f_R(0-(-x)) \supseteq f_R(0) \cap f_R(-x) = f_R(-x) = f_R(0-x) \supseteq f_R(0) \cap f_R(x) = f_R(x)$ and so $f_R(x) = f_R(-x)$.
(3) $f_R(x) = f_R(x1) \supseteq f_R(1)$ for all $x \in R$.
(4)

$$f_R(x) = f_R(x-y+y) = f_R(x-y-(-y)) \supseteq f_R(x-y) \cap f_R(-y)$$

$$= f_R(0) \cap f_R(y) = f_R(y) = f_R(x-(x-y)) \supseteq f_R(x) \cap f_R(x-y) = f_R(x) \cap f_R(0) = f_R(x)$$

and then $f_R(x) = f_R(y)$ for all $x, y \in R$.
(5) Let $x, y \in f_R^*$ and $r \in R$. Then $f_R(x-y) \supseteq f_R(x) \cap f_R(y) = f_R(0) \cap f_R(0) = f_R(0) \supseteq f_R(x-y)$ and so $f_R(x-y) = f_R(0)$. Also $f_R(rx) \supseteq f_R(x) = f_R(0) \supseteq f_R(rx)$. Thus, $x-y, rx \in f_R^*$, then f_R^* is an ideal of R.
(6) Let $x, y \in f_R^\star$ and $r \in R$. Since $f_R(x-y) \supseteq f_R(x) \cap f_R(y) \neq \emptyset$ and $f_R(rx) \supseteq f_R(x) \neq \emptyset$ so $x-y, rx \in f_R^\star$ and then $f_R^\star$ is an ideal of R.
(7) If $x \in f_R^* \cap g_R^*$, then $x \in f_R^*$ and $x \in g_R^*$. Hence, $f_R(x) = f_R(0)$ and $g_R(x) = g_R(0)$. Now $(f_R \cap g_R)(x) = f_R(x) \cap g_R(x) = f_R(0) \cap g_R(0) = (f_R \cap g_R)(0)$. Thus, $x \in (f_R \cap g_R)^*$ and hence, $f_R^* \cap g_R^* \subseteq (f_R \cap g_R)^*$. □

Proposition 2.14. *If $f_R, g_R \in S_I(U)$ and $f_R(0) = g_R(0)$, then $f_R^* \cap g_R^* = (f_R \cap g_R)^*$.*

Proof. Let $x \in (f_R \cap g_R)^*$ and so $(f_R \cap g_R)(x) = (f_R \cap g_R)(0)$. Then $f_R(x) \cap g_R(x) = f_R(0) \cap g_R(0) = f_R(0) \subseteq f_R(x) \subseteq f_R(0)$ and hence, $f_R(x) = f_R(0)$. Similarly, $g_R(x) = g_R(0)$. Therefore $x \in f_R^* \cap g_R^*$ and so $(f_R \cap g_R)^* \subseteq f_R^* \cap g_R^*$. Also from Proposition 2.10 (7) we have that $f_R^* \cap g_R^* \subseteq (f_R \cap g_R)^*$. Thus, $f_R^* \cap g_R^* = (f_R \cap g_R)^*$. □

Proposition 2.15. *Let $f_R \in S_I(U)$. Then for all $i \in \mathbb{N}$ the following assertions hold.*
(1) $f_R^i(0) = f_R(0)$.
(2) $f_R^{(i)}(0) = f_R(0)$.

Proof. (1) The result is true for $i = 1$. Assume the result is true for $i \geq 1$. Now $f_R^{i+1}(0) = \cup\{f_R^i(x) \cap f_R(y) \mid x, y \in R, 0 = xy\} = f_R(0)$. (Since the union is attained when $x = y = 0$.) The result now follows by induction.
(2) By (1) and from $f_R(0) = f_R^i(0) \subseteq f_R^{(i)}(0) \subseteq f_R(0)$ we obtain $f_R^{(i)}(0) = f_R(0)$. □

Proposition 2.16. *Let $f_R \in S_I(U)$. Then for all $i \in \mathbb{N}$, we have the follwing statements.*
(1) $(f_R^i)^* \subseteq f_R^*$.
(2) $(f_R^{(i)})^* \subseteq f_R^*$.

Proof. (1) Let $x \in (f_R^i)^*$ then $f_R^i(x) = f_R^i(0) = f_R(0)$. Since $f_R^i(x) \subseteq f_R(x)$ so $f_R(0) \subseteq f_R(x) \subseteq f_R(0)$ and hence, $f_R(x) = f_R(0)$.
(2) The proof is similar to (1). □

Proposition 2.17. *Let $f_R \in S_I(U)$ and $k \in \mathbb{N}$. If $x_1, x_2, ..., x_k \in R$, then*
(1) $f_R^k(x_1x_2...x_k) \supseteq \cap_{i=1}^k f_R(x_i)$.
(2) $f_R^{(k)}(x_1x_2...x_k) \supseteq \cap_{i=1}^k f_R(x_i)$.

Proof. (1) The result is true for $k = 1$. Suppose the result is true for $k \geq 1$. Now

$$\begin{aligned} f_R^{k+1}(x_1x_2...x_{k+1}) &= (f_R^k \circ f_R)(x_1x_2...x_k) \\ &\supseteq f_R^k(x_1x_2...x_k) \cap f_R(x_{k+1}) \supseteq \cap_{i=1}^k f_R(x_i) \cap f_R(x_{k+1}) = \cap_{i=1}^{k+1} f_R(x_i). \end{aligned}$$

and so $f_R^k(x_1x_2...x_k) \supseteq \cap_{i=1}^k f_R(x_i)$.
(2) Use (1), and $f_R^k \subseteq f_R^{(k)} \subseteq f_R$. □

Proposition 2.18. *Let $f_R, g_R, h_R \in S_I(U)$. Then $f_R \circ g_R \subseteq h_R$ if and only if $f_R g_R \subseteq h_R$.*

Proof. Let $f_R g_R \subseteq h_R$ then $f_R \circ g_R \subseteq f_R g_R \subseteq h_R$. Conversely, let $f_R \circ g_R \subseteq h_R$. Let $x \in R$ and $x = \sum_{i=1}^n y_i z_i$ such that $y_i, z_i \in R, i = 1, 2, .., n$. We know that

$$h_R(y_i z_i) \supseteq (f_R \circ g_R)(y_i z_i) \supseteq f_R(y_i) \cap g_R(z_i),$$

for all $i = 1, 2, .., n$. Now from Proposisition 2.14, we obtain

$$h_R(x) = h_R(\sum_{i=1}^n y_i z_i) \supseteq \cap_{i=1}^n h_R(y_i z_i) \supseteq \cap_{i=1}^n (f_R(y_i) \cap g_R(z_i)).$$

Thus, $h_R(x) \supseteq \cup\{\cap_{i=1}^n (f_R(y_i) \cap g_R(z_i)) \mid x = \sum_{i=1}^n y_i z_i, n \in \mathbb{N}\} = f_R g_R(x)$. Therefore, $f_R g_R \subseteq h_R$. □

Proposition 2.19. *If $f_R, g_R \in S_R(U)$, then $f_R^\alpha + g_R^\alpha \subseteq (f_R + g_R)^\alpha$ for all $\alpha \in P(U)$.*

Proof. Let $x \in f_R^\alpha + g_R^\alpha$. Then $x = y + z$ such that $y \in f_R^\alpha$ and $z \in g_R^\alpha$. So $f_R(y) \supseteq \alpha$ and $g_R(z) \supseteq \alpha$. Now

$$(f_R + g_R)(x) = \cup\{f_R(y) \cap g_R(z) \mid x = y + z\} \supseteq \alpha.$$

Hence, $x \in (f_R + g_R)^\alpha$ and then $f_R^\alpha + g_R^\alpha \subseteq (f_R + g_R)^\alpha$. □

Proposition 2.20. *Let $\{f_R\}_{i\in I} \in S_I(U)$. Then $f_R = \cap_{i\in I}\{f_R\}_{i\in I} \in S_I(U)$.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned} f_R(x - y) &= \cap_{i\in I}\{f_R\}_{i\in I}(x - y) = \cap_{i\in I}(\{f_R\}_{i\in I})(x - y) \\ &\supseteq \cap_{i\in I}(\{f_R\}_{i\in I}(x) \cap \{f_R\}_{i\in I}(y)) = \cap_{i\in I}\{f_R\}_{i\in I}(x) \cap \cap_{i\in I}\{f_R\}_{i\in I}(y) \\ &= f_R(x) \cap f_R(y). \end{aligned}$$

Also

$$\begin{aligned} f_R(xy) &= \cap_{i\in I}\{f_R\}_{i\in I}(xy) = \cap_{i\in I}(\{f_R\}_{i\in I})(xy) = \cap_{i\in I}(\{f_R\}_{i\in I}(xy)) \\ &\supseteq \cap_{i\in I}(\{f_R\}_{i\in I}(x)) = f_R(x). \end{aligned}$$

Therefore, $f_R \in S_I(U)$. □

3. **Generated Soft Subideals**

In this section, we introduce the concepts of generated goft subideals and investigte their properties.

Definition 3.1. Let $f_R \in S(U)$. Define $< f_R >= \cap\{g_R \mid f_R \subseteq g_R, g_R \in S_I(U)\}$ and is called the soft subideal of R generated by f_R. Note $< f_R >$ is the smallest soft subideal of R containing f_R.

Corollary 3.2. *Let $f_R, g_R \in S(U)$. Then*
(1) $f_R \in S_I(U)$ if and only if $< f_R >= f_R$.
(2) If $f_R \subseteq g_R$, then $< f_R > \subseteq < g_R >$.

Proposition 3.3. *Let A be a nonempty subset of R and $< A >$ is the ideal of R generated by A. Let $f_R, g_R \in S(U)$ such that $g_R(x) = \cup\{\cap_{y\in A} f_R(y) \mid A \subseteq R, 1 \leq |A| < \infty, x \in < A >\}$ for all $x \in R$. Then $< f_R >= g_R$.*

Proof. It is obvious that $f_R \subseteq g_R$ and $g_R(-x) = g_R(x)$ for all $x \in R$. Let $x, y \in R$ and we prove $g_R \in S_I(U)$.
(1) If $x \in < A >, y \in < B >, A, B \subseteq R, 1 \leq |A| < \infty, 1 \leq |B| < \infty$, then $x + y \in < A \cup B >$. Then

$$g_R(x + y) \supseteq \cap_{z\in A\cup B} f_R(z) \supseteq (\cap_{u\in A} f_R(u)) \cap (\cap_{v\in B} f_R(v)),$$

and so

$$\begin{aligned}
g_R(x + y) &\supseteq \cup\{(\cap_{u\in A} f_R(u)) \cap (\cap_{v\in B} f_R(v)) \mid A, B \subseteq R, 1 \leq |A|, |B| < \infty, x \in < A >, y \in < B >\} \\
&= (\cup\{(\cap_{u\in A} f_R(u)) \mid A \subseteq R, 1 \leq |A| < \infty, x \in < A >\}) \\
&\cap (\cup\{(\cap_{v\in B} f_R(v)) \mid B \subseteq R, 1 \leq |B| < \infty, y \in < B >\}) \\
&= g_R(x) \cap g_R(y)
\end{aligned}$$

and hence, $g_R(x + y) \supseteq g_R(x) \cap g_R(y)$.
(2) Also if $B \subseteq R, 1 \leq |B| < \infty, y \in < B >$, then $xy \in < B >$. Therefore

$$\begin{aligned}
g_R(xy) &= \cup\{(\cap_{z\in B} f_R(z)) \mid B \subseteq R, 1 \leq |B| < \infty, xy \in < B >\} \\
&\supseteq \cup\{(\cap_{z\in B} f_R(z)) \mid B \subseteq R, 1 \leq |B| < \infty, y \in < B >\} \\
&= g_R(y)
\end{aligned}$$

and then $g_R(xy) \supseteq g_R(y)$.
Now (1) and (2) show that $g_R \in S_I(U)$. Let $h_R \in S_I(U)$ such that $f_R \subseteq h_R$. Suppose $x \in R$ and $A = \{y_1, y_2, ..., y_n\}$ with $x \in < A >$. Then there exist $x_1, x_2, ..., x_n \in R$ and integers $m_1, m_2, ..., m_n$ such that $x = \sum_{i=1}^n x_i y_i + \sum_{i=1}^n m_i y_i$. From $h_R \in S_I(U)$ we obtain

$$\begin{aligned}
h_R(x) &= h_R(\sum_{i=1}^n x_i y_i + \sum_{i=1}^n m_i y_i) \\
&\supseteq h_R(\sum_{i=1}^n x_i y_i) \cap h_R(\sum_{i=1}^n m_i y_i) \supseteq (\cap_{i=1}^n h_R(x_i y_i)) \cap (\cap_{i=1}^n h_R(m_i y_i)) \\
&\supseteq \cap_{i=1}^n h_R(y_i).
\end{aligned}$$

Now

$$\begin{aligned}h_R(x) &\supseteq \cup\{\cap_{y\in A} h_R(y) \mid A \subseteq R, 1 \leq |A| < \infty, x \in < A >\}\\ &\supseteq \cup\{\cap_{y\in A} f_R(y) \mid A \subseteq R, 1 \leq |A| < \infty, x \in < A >\}\\ &= g_R(x)\end{aligned}$$

and so $h_R \supseteq g_R$. Hence, $< f_R >= g_R$. □

Proposition 3.4. *Let $f_R, g_R \in S_I(U)$ and $f_R(0) = g_R(0)$. Then*
(1) $f_R + g_R \in S_I(U)$.
(2) $f_R + g_R =< f_R \cup g_R >$.

Proof. Let $x, y \in R$. It is clear that $(f_R + g_R)(-x) = (f_R + g_R)(x)$. Now
(1)

$$(f_R + g_R)(x + y) = \cup\{f_R(u) \cap g_R(v) \mid u, v \in R, u + v = x + y\}$$

$$\supseteq \cup\{f_R(u_1 + v_1) \cap g_R(u_2 + v_2) \mid u_1, u_2, v_1, v_2 \in R, u_1 + u_2 = x, v_1 + v_2 = y\}$$

$$\supseteq \cup\{(f_R(u_1) \cap f_R(v_1)) \cap (g_R(u_2) \cap g_R(v_2)) \mid u_1, u_2, v_1, v_2 \in R, u_1 + u_2 = x, v_1 + v_2 = y\}$$

$$= \cup\{(f_R(u_1) \cap g_R(u_2)) \cap (f_R(v_1) \cap f_R(v_2)) \mid u_1, u_2, v_1, v_2 \in R, u_1 + u_2 = x, v_1 + v_2 = y\}$$

$$= (\cup\{f_R(u_1) \cap g_R(u_2) \mid u_1, u_2 \in R, u_1 + u_2 = x\}) \cap (\cup\{f_R(v_1) \cap g_R(v_2) \mid v_1, v_2 \in R, v_1 + v_2 = y\})$$

$$= (f_R + g_R)(x) \cap (f_R + g_R)(y).$$

Also

$$\begin{aligned}(f_R + g_R)(xy) &\supseteq \cup\{f_R(xu) \cap g_R(xv) \mid u, v \in R, u + v = y\}\\ &\supseteq \cup\{f_R(u) \cap g_R(v) \mid u, v \in R, u + v = y\}\\ &= (f_R + g_R)(y).\end{aligned}$$

Hence, $f_R + g_R \in S_I(U)$.
(2)

$$(f_R + g_R)(x) = \cup\{f_R(u) \cap g_R(v) \mid u, v \in R, u + v = x\} \supseteq f_R(x) \cap g_R(0) = f_R(x)$$

and then $f_R + g_R \supseteq f_R$. Similarly, $f_R + g_R \supseteq g_R$. Thus, $f_R + g_R \supseteq f_R \cup g_R$. Let $h_R \in S_I(U)$ and $f_R \cup g_R \subseteq h_R$. Then

$$\begin{aligned}(f_R + g_R)(x) &= \cup\{f_R(u) \cap g_R(v) \mid u, v \in R, u + v = x\}\\ &\subseteq \cup\{h_R(u) \cap h_R(v) \mid u, v \in R, u + v = x\} = h_R(x)\end{aligned}$$

and so $f_R + g_R \subseteq h_R$. Then $f_R + g_R$ is the smallest soft subideal of R such that $f_R \cup g_R \subseteq f_R + g_R$ and then $f_R + g_R =< f_R \cup g_R >$. □

The proof of the following proposition is straightforward.

Proposition 3.5. *Let $f_R, g_R \in S(U)$.*
(1) If $f_R \in S_I(U)$, then $f_R \circ g_R \subseteq f_R$.
(2) If $f_R, g_R \in S_I(U)$, then $f_R \circ g_R \subseteq f_R \cap g_R$.

Proposition 3.6. *If $f_R, g_R \in S_R(U)$, then $f_R g_R \in S_R(U)$.*

Proof. It is easy to prove that $(f_Rg_R)(-x) = (f_Rg_R)(x)$ for all $x \in R$. From Prpposition 2.4 (4) we get that

$$(f_Rg_R)(x+y) \supseteq (f_Rg_R)(x) \cap (f_Rg_R)(y)$$

for all $x, y \in R$. Let $x = \sum_{i=1}^{m} x_i s_i$ and $y = \sum_{j=1}^{n} y_j t_i$ where $x_i, s_i, y_j, t_j \in R$ for $1 \leq i \leq m, 1 \leq j \leq n$. Now

$$xy = (\sum_{i=1}^{m} x_i s_i)(\sum_{j=1}^{n} y_j t_i) = \sum_{i=1}^{m}\sum_{j=1}^{n}(x_i y_j)(s_i t_j)$$

and by the definition of f_Rg_R we have

$$\begin{aligned}(f_Rg_R)(xy) &\supseteq \cap_{i=1}^{m} \cap_{j=1}^{n} (f_R(x_i y_j)) \cap (g_R(s_i t_j)) \\ &\supseteq \cap_{i=1}^{m} \cap_{j=1}^{n} (f_R(x_i) \cap f_R(y_j)) \cap (g_R(s_i) \cap g_R(t_j)) \\ &= \cap_{i=1}^{m} \cap_{j=1}^{n} (f_R(x_i) \cap g_R(s_i)) \cap (f_R(y_j) \cap g_R(t_j)) \\ &= (\cap_{i=1}^{m}(f_R(x_i) \cap g_R(s_i))) \cap (\cap_{j=1}^{n}(f_R(y_j) \cap g_R(t_j)))\end{aligned}$$

and hence,

$$\begin{aligned}(f_Rg_R)(xy) \supseteq\ & (\cup\{\cap_{i=1}^{m}(f_R(x_i) \cap g_R(s_i)) \mid x_i, s_i \in R, 1 \leq i \leq m, x = \sum_{i=1}^{m} x_i s_i\}) \\ & \cap(\cup\{\cap_{j=1}^{n}(f_R(y_j) \cap g_R(t_j)) \mid y_j, t_j \in R, 1 \leq j \leq n, y = \sum_{j=1}^{n} y_j t_i\}) \\ & = (f_Rg_R)(x) \cap (f_Rg_R)(y).\end{aligned}$$

Therefore $f_Rg_R \in S_R(U)$. □

Proposition 3.7. *Let $f_R \in S_I(U)$ and $g_R \in S_R(U)$. Then*
(1) $f_Rg_R \in S_I(U)$.
(2) $f_Rg_R =< f_R \circ g_R >$.

Proof. Let $x, y \in R$. Then
(1) $(f_Rg_R)(-x) = (f_Rg_R)(x)$ and from Proposition 2.4 (4) we have $(f_Rg_R)(x+y) \supseteq (f_Rg_R)(x) \cap (f_Rg_R)(y)$. Also

$$\begin{aligned}(f_Rg_R)(xy) &\supseteq \cup\{\cap_{i=1}^{n}(f_R(xu_i) \cap g_R(v_i)) \mid u_i, v_i \in R, 1 \leq i \leq n, y = \sum_{i=1}^{n} u_i v_i\} \\ &\supseteq \cup\{\cap_{i=1}^{n}(f_R(u_i) \cap g_R(v_i)) \mid u_i, v_i \in R, 1 \leq i \leq n, y = \sum_{i=1}^{n} u_i v_i\} = (f_Rg_R)(y).\end{aligned}$$

Thus, $f_Rg_R \in S_I(U)$.
(2) By Proposision 2.4 (1) we obtain that $f_R \circ g_R \subseteq f_Rg_R$. If $h_R \in S_R(U)$ and $f_R \circ g_R \subseteq h_R$, then

$$\begin{aligned}(f_Rg_R)(x) &= \cup\{\cap_{i=1}^{n}(f_R(y_i) \cap g_R(z_i)) \mid y_i, z_i \in R, 1 \leq i \leq n, x = \sum_{i=1}^{n} y_i z_i\} \\ &\subseteq \cup\{\cap_{i=1}^{n}(f_R \circ g_R)(y_i z_i) \mid y_i, z_i \in R, 1 \leq i \leq n, x = \sum_{i=1}^{n} y_i z_i\}\end{aligned}$$

$$\subseteq \cup\{\cap_{i=1}^{n} h_R(y_i z_i) \mid y_i, z_i \in R, 1 \le i \le n, x = \sum_{i=1}^{n} y_i z_i\}$$
$$\subseteq \cup\{h_R(\sum_{i=1}^{n} y_i z_i) \mid y_i, z_i \in R, 1 \le i \le n, x = \sum_{i=1}^{n} y_i z_i\}$$
$$= h_R(x).$$

Therefore $f_R g_R \subseteq h_R$ and $f_R g_R =< f_R \circ g_R >$. □

Proposition 3.8. *Let $f_R, g_R \in S_I(U)$. Then $f_R g_R \subseteq f_R \cap g_R$.*

Proof. The result follows from Proposition 2.15 and Proposition 3.5 (2). □

Proposition 3.9. *Let $f_R, g_R, h_R \in S_I(U)$ and $g_R(0) = h_R(0)$. Then $f_R(g_R+h_R) = f_R g_R + f_R h_R$.*

Proof. From $g_R \subseteq g_R + h_R$ and $h_R \subseteq g_R + h_R$ we obtain $f_R g_R \subseteq f_R(g_R + h_R)$ and $f_R h_R \subseteq f_R(g_R + h_R)$ and therefore $f_R g_R + f_R h_R \subseteq f_R(g_R + h_R)$. Suppose hat $x \in R$. Then

$$(f_R(g_R + h_R))(x) = \cup\{\cap_{i=1}^{n}(f_R(y_i) \cap (g_R + h_R)(z_i)) \mid y_i, z_i \in R, 1 \le i \le n, n \in \mathbb{N}, x = \sum_{i=1}^{n} y_i z_i\}$$

$$= \cup\{\cap_{i=1}^{n}(f_R(y_i) \cap (\cup\{g_R(u_i) \cap h_R(v_i) \mid u_i, v_i \in R, u_i + v_i = z_i\}) \mid y_i, z_i \in R, 1 \le i \le n, n \in \mathbb{N},$$
$$x = \sum_{i=1}^{n} y_i z_i\} = \cup\{\cap_{i=1}^{n}(f_R(y_i) \cap g_R(u_i) \cap h_R(v_i)) \mid u_i, v_i, y_i \in R, 1 \le i \le n, n \in \mathbb{N}, x = \sum_{i=1}^{n}(y_i u_i + y_i v_i)\}$$
$$\subseteq \cup\{(\cap_{i=1}^{p}(f_R(s_i) \cap g_R(t_i))) \cap (\cap_{k=1}^{q}(f_R(r_k) \cap h_R(w_k))) \mid s_i, t_i, r_k, w_k \in R, 1 \le i \le p, 1 \le k \le q, p, q \in \mathbb{N},$$
$$x = \sum_{i=1}^{p} s_i t_i + \sum_{k=1}^{q} r_k w_k\} = \cup\{(f_R g_R)(a) \cap (f_R g_R)(b) \mid a, b \in R, a + b = x\} = (f_R g_R + f_R h_R)(x).$$

Thus, $f_R(g_R + h_R) \subseteq f_R g_R + f_R h_R$. Therefore $f_R(g_R + h_R) = f_R g_R + f_R h_R$. □

4. Residual Quotient of Soft Subideals

In this section, we introduce the residual quotient of soft subideals.

Definition 4.1. Let $f_R, g_R \in S(U)$. Define $f_R : g_R \in S(U)$ as follows:

$$f_R : g_R = \cup\{h_R \mid h_R \in S_I(U), h_R \circ g_R \subseteq f_R\}.$$

Proposition 4.2. *If $f_R, g_R \in S_I(U)$, then $f_R \subseteq f_R : g_R$ and $f_R : g_R \in S_I(U)$.*

Proof. We know that $f_R \circ g_R \subseteq f_R$ and then $f_R : g_R = \cup\{h_R \mid h_R \in S_I(U), h_R \circ g_R \subseteq f_R\} \supseteq f_R$. Now we prove $f_R : g_R \in S_I(U)$. Let $x, y \in R$. It is clear $(f_R : g_R)(-x) = (f_R : g_R)(x)$.
(1)

$$\begin{aligned}(f_R : g_R)(x) \cap (f_R : g_R)(y) &= (\cup\{h_R(x) \mid h_R \in S_I(U), h_R \circ g_R \subseteq f_R\}) \\ &\cap (\cup\{k_R(y) \mid k_R \in S_I(U), k_R \circ g_R \subseteq f_R\}) \\ &= \cup\{h_R(x) \cap k_R(y) \mid h_R, k_R \in S_I(U), (h_R \circ g_R) \cup (k_R \circ g_R) \subseteq f_R\} \\ &\subseteq \cup\{(h_R + k_R)(x + y) \mid h_R, k_R \in S_I(U), (h_R + k_R) o g_R \\ &\subseteq (f_R : g_R)(x + y).\end{aligned}$$

(2)

$$(f_R : g_R)(xy) = (\cup\{h_R(x) \mid h_R \in S_I(U), h_R \circ g_R \subseteq f_R\})$$
$$\supseteq \cup\{h_R(x) \mid h_R \in S_I(U), h_R \circ g_R \subseteq f_R\} \supseteq (f_R : g_R)(x).$$

Then (1) and (2) show that $f_R : g_R \in S_I(U)$. □

Proposition 4.3. *Let $f_R, g_R, h_R \in S_I(U)$. If $f_R \subseteq g_R$, then*
(1) $f_R : h_R \subseteq g_R : h_R$ and
(2) $h_R : f_R \supseteq h_R : g_R$.

Proof. (1)

$$(f_R : g_R) = \cup\{k_R \mid k_R \in S_I(U), k_R \circ h_R \subseteq f_R\}$$
$$\subseteq \cup\{k_R \mid k_R \in S_I(U), k_R \circ h_R \subseteq g_R\} \subseteq g_R : h_R.$$

(2)

$$h_R : f_R = \cup\{k_R \mid k_R \in S_I(U), k_R \circ f_R \subseteq h_R\}$$
$$\supseteq \cup\{k_R \mid k_R \in S_I(U), k_R \circ g_R = h_R : g_R.$$

□

Proposition 4.4. *Let $f_R, g_R, h_R \in S_I(U)$. Then*
(1) $(f_R : g_R)g_R \subseteq f_R$.
(2) If $f_R \subseteq h_R$ and $h_R g_R \subseteq f_R$, then $h_R \subseteq f_R : g_R$.

Proof. Suppose that $x \in R$ and $x = \sum_{i=1}^{n} y_i z_i$ such that $y_i, z_i \in R, 1 \leq i \leq n, n \in \mathbb{N}$. If $f_R \supseteq h_R \circ g_R$, then

$$f_R(x) = f_R(\sum_{i=1}^{n} y_i z_i) \supseteq \cap_{i=1}^{n} f_R(y_i z_i) \supseteq \cap_{i=1}^{n} (h_R \circ g_R)(y_i z_i) \supseteq \cap_{i=1}^{n} (h_R(y_i) \cap g_R(z_i))$$

and so $f_R(x) \supseteq \cap_{i=1}^{n}((f_R : h_R)(y_i) \cap g_R(z_i))$. Then

$$f_R(x) \supseteq \cup\{\cap_{i=1}^{n}((f_R : h_R)(y_i) \cap g_R(z_i)) \mid y_i, z_i \in R, 1 \leq i \leq n, n \in \mathbb{N}, x = \sum_{i=1}^{n} y_i z_i\},$$

therefore, $(f_R : g_R)g_R \subseteq f_R$.
(2) By $h_R \circ g_R \subseteq h_R g_R \subseteq f_R$, we get $h_R \subseteq f_R : g_R$. □

Proposition 4.5. *If $(f_R)_i, g_R \in S_I(U)$, then $(\cap_{i=1}^{n}(f_R)_i) : g_R = \cap_{i=1}^{n}((f_R)_i : g_R)$ for all $i = 1, 2, 3, ..., n$.*

Proof. It is enough to prove for $n = 2$. Clearly, $((f_R)_1 \cap (f_R)_2) : g_R \subseteq ((f_R)_1 : g_R) \cap ((f_R)_2 : g_R)$. Let $x \in R$ and $\alpha, \beta \in P(U)$. Let

$$\alpha = (((f_R)_1 \cap (f_R)_2) : g_R)(x) = \cup\{h_R \mid h_R \in S_I(U), h_R \circ g_R \subseteq (f_R)_1 \cap (f_R)_2\}.$$

Now

$$(((f_R)_1 : g_R) \cap ((f_R)_2 : g_R))(x) = ((f_R)_1 : g_R)(x) \cap ((f_R)_2 : g_R)(x)$$
$$= (\cup\{k_R \mid k_R \in S_I(U), k_R \circ g_R \subseteq (f_R)_1\}) \cap (\cup\{l_R \mid l_R \in S_I(U), l_R \circ g_R \subseteq (f_R)_2\})$$
$$= \cup\{k_R \cap l_R \mid k_R, l_R \in S_I(U), k_R \circ g_R \subseteq (f_R)_1, l_R \circ g_R \subseteq (f_R)_2\}.$$

Let $k_R \circ g_R \subseteq (f_R)_1$ and $l_R \circ g_R \subseteq (f_R)_2$. If $\beta = k_R(x) \cap l_R(x)$, then $A = k_R^\beta \cap l_R^\beta$ is an ideal of R. Define $m_R : R \to P(U)$ as

$$m_R(x) = \begin{cases} \beta & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

then $m_R \in S_I(U)$. Since $m_R \subseteq k_R$ and $m_R \subseteq l_R$ so $m_R \circ g_R \subseteq k_R \circ g_R \subseteq (f_R)_1$ and $m_R \circ g_R \subseteq l_R \circ g_R \subseteq (f_R)_2$ and then $m_R \circ g_R \subseteq (f_R)_1 \cap (f_R)_2$. Then $\alpha = \cup\{n_R \mid n_R \in S_I(U), n_R \circ g_R \subseteq (f_R)_1 \cap (f_R)_2\} \supseteq m_R(x) = \beta = k_R(x) \cap l_R(x)$. Therefore

$$(((f_R)_1 \cap (f_R)_2) : g_R)(x) = \alpha \supseteq \cup\{k_R \cap l_R \mid k_R, l_R \in S_I(U), k_R \circ g_R \subseteq (f_R)_1, l_R \circ g_R \subseteq (f_R)_2\}$$

$$= (((f_R)_1 : g_R) \cap ((f_R)_2 : g_R))(x).$$

Therefore $(\cap_{i=1}^2 (f_R)_i) : g_R \supseteq \cap_{i=1}^2((f_R)_i : g_R)$. Then $(\cap_{i=1}^2 (f_R)_i) : g_R = \cap_{i=1}^2((f_R)_i : g_R)$. □

Definition 4.6. Let $f_R \in S(U)$ and $x \in R$. Let 0_R denotes the zero element of R. Define $f_R(0)_{\{x\}} : R \to P(U)$ as

$$f_R(0)_{\{x\}}(y) = \begin{cases} f_R(0) & \text{if } y = x \\ \emptyset & \text{if } y \neq x \end{cases}$$

for all $y \in R$.

Definition 4.7. Let $f_R \in S(U)$ and $x \in R$. Then $f_R(0)_{\{x\}} + f_R$ is called a coset of f_R. Now for all $y \in R$ we have that

$$\begin{aligned}(f_R(0)_{\{x\}} + f_R)(y) &= \cup\{f_R(0)_{\{x\}}(z_1) \cap f_R(z_2)\} \mid y = z_1 + z_2\} \\ &= \cup\{f_R(0) \cap f_R(z_2)\} \mid y = x + z_2\} = f_R(z_2) = f_R(y - x).\end{aligned}$$

We write $x + f_R$ for $f_R(0)_{\{x\}} + f_R$. Then $(x + f_R)(y) = f_R(y - x)$.

Proposition 4.8. *Let $f_R \in S_I(U)$ and $x, y \in R$. Then $x + f_R = y + f_R$ if and only if $f_R(x - y) = f_R(0)$.*

Proof. If $x + f_R = y + f_R$, then $(x + f_R)(x) = (y + f_R)(x)$ and so $f_R(0) = f_R(x - x) = f_R(x - y)$. Conversely, if $z \in R$ and $f_R(x - y) = f_R(0)$, then

$$\begin{aligned}(x + f_R)(z) &= f_R(z - x) = f_R(z - y + y - x) \supseteq f_R(z - y) \cap f_R(y - x) \\ &= f_R(z - y) \cap f_R(0) = f_R(z - y) = (y + f_R)(z)\end{aligned}$$

and then $x + f_R \supseteq y + f_R$. Similarly, $y + f_R \subseteq x + f_R$. Therefore $x + f_R = y + f_R$. □

Proposition 4.9. *Let $f_R \in S_I(U)$. Define $R/f_R = \{x + f_R \mid x \in R\}$. Then $(R/f_R, +, .)$ is a ring and is called the quotient ring of R by f_R such that $+$ and $.$ on R/f_R are as $(x + f_R) + (y + f_R) = (x + y) + f_R$ and $(x + f_R).(y + f_R) = (xy) + f_R$ for all $x, y \in R$.*

Proof. we prove that $+$ and $.$ are well defined. Let $x, y, z, t \in R$ and $x + f_R = z + f_R$ and $y + f_R = t + f_R$. Then by Proposition 4.8 $f_R(x - z) = f_R(y - t) = f_R(0)$. Therefore

$$\begin{aligned}f_R(x + y - (z + t)) &= f_R(x - z + y - t) \supseteq f_R(x - z) \cap f_R(y - t) \\ &= f_R(0) \cap f_R(0) = f_R(0) \supseteq f_R(x + y - (z + t))\end{aligned}$$

so $f_R(x+y-(z+t)) = f_R(0)$. Now by Proposition 4.8 $(x+y)+f_R = (z+t)+f_R$ and so $+$ is well defined. Also

$$\begin{aligned} f_R(zt-xy) &= f_R(zt-zy+zy-xy) \\ &\supseteq f_R(zt-zy)\cap f_R(zy-xy) \\ &= f_R(z(t-y))\cap f_R(y(z-x)) \\ &\supseteq (f_R(z)\cup f_R(t-y))\cap (f_R(y)\cup f_R(z-x)) \\ &= (f_R(z)\cup f_R(0))\cap (f_R(y)\cup f_R(0)) = f_R(0)\cap f_R(0) = f_R(0)\supseteq f_R(zt-xy) \end{aligned}$$

and then $f_R(zt-xy) = f_R(0)$. Now by Proposition 4.8 $(xy)+f_R = (zt)+f_R$ and so . is well defined.
Let 0_R denotes the zero element of R. Then $(0_R+f_R)(x) = f_R(x-0) = f_R(x)$ and then $0_R+f_R = f_R$. Also $-x+f_R+x+f_R = f_R$ then $-(x+f_R) = (-x)+f_R$ for all $x\in R$. □

Proposition 4.10. *If $f_R\in S_I(U)$, then $R/f_R^* \simeq R/f_R$.*

Proof. Let $x,y\in R$ and define $\varphi : R\to R/f_R$ as $\varphi(x) = x+f_R$. Then $\varphi(x+y) = \varphi(x)+\varphi(y)$ and $\varphi(xy) = \varphi(x)\varphi(y)$ and so φ is a homomorphism of R onto R/f_R. Also

$$\ker\varphi = \{x\in R \mid \varphi(x) = 0\} = \{x\in R \mid x+f_R = f_R\} = \{x\in R \mid f_R(x) = f_R(0)\} = f_R^*.$$

Therefore $R/f_R^*\simeq R/f_R$. □

In the following propositions we introduce the quotient soft subrings and soft subideals f_R relative to an ideal os R.

Proposition 4.11. *Let $f_R\in S_I(U)$. Define $f_R^{(*)} : R/f_R\to P(U)$ as $f_R^{(*)}(x+f_R) = f_R(x)$ for all $x\in R$. Then $f_R^{(*)}$ is a soft subideal of R/f_R.*

Proof. It is easy to prove that $f_R^{(*)}$ is well defined. Let $x,y\in R$, then

$$\begin{aligned} f_R(zt-xy) &= f_R^{(*)}((xy)+f_R)) \\ &= f_R(xy)\supseteq f_R(x)\cup f_R(y) = f_R^{(*)}(x+f_R)\cup f_R^{(*)}(y+f_R). \end{aligned}$$

Also

$$\begin{aligned} f_R^{(*)}((x+f_R)(y+f_R)) &= f_R^{(*)}((x-y)+f_R)) = f_R(x-y) \\ &\supseteq f_R(x)\cap f_R(y) = f_R^{(*)}(x+f_R)\cap f_R^{(*)}(y+f_R). \end{aligned}$$

Thus, $f_R^{(*)}$ is a soft subideal of R/f_R. □

Proposition 4.12. *Let $f_R\in S_R(U)$ and let A be an ideal of R. Define $f_{R/A} : R/A\to P(U)$ as $f_{R/A}(x+A) = \cup\{f_R(z)\mid z\in [x]\}$ such that $[x] = x+A$ for all $x\in R$. Then $f_{R/A}\in S_{R/A}(U)$.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned} f_{R/A}([x]-[y]) &= f_{R/A}([x-y]) \\ &= \cup\{f_R(x-y+z) \mid z \in A\} \\ &\supseteq \cup\{f_R(x-y+a-b) \mid a,b \in A\} \\ &\supseteq \cup\{f_R(x+a) \cap f_R(y+b) \mid a,b \in A\} \\ &= (\cup\{f_R(x+a) \mid a \in A\}) \cap (\cup\{f_R(y+b) \mid b \in A\}) \\ &= f_{R/A}([x]) \cap f_{R/A}([y]). \end{aligned}$$

Also

$$\begin{aligned} f_{R/A}([x][y]) &= f_{R/A}([xy]) = \cup\{f_R(xy+z) \mid z \in A\} \\ &\supseteq \cup\{f_R(xy+(xb+ay+ab)) \mid a,b \in A\} \\ &= \cup\{f_R((x+a)(y+b)) \mid a,b \in A\} \\ &= (\cup\{f_R(x+a) \mid a \in A\}) \cap (\cup\{f_R(y+b) \mid b \in A\}) \\ &\supseteq \cup\{f_R(x+a) \cap f_R(y+b) \mid a,b \in A\} \\ &= (\cup\{f_R(x+a) \mid a \in A\}) \cap (\cup\{f_R(y+b) \mid b \in A\}) \\ &= f_{R/A}([x]) \cap f_{R/A}([y]). \end{aligned}$$

This completes the proof. □

The soft subideal R/f_R defined in the above Proposition is called the quotient soft subring of f_R relative to A and denoted by f_R/A.

5. Soft Subideals of Soft Subrings

In this section, we introduce the concepts of soft subideals of soft subrings.

Definition 5.1. Let $f_R \in S(U)$ and $g_R \in S_R(U)$ such that $f_R \subseteq g_R$. Then f_R is called soft subideal of g_R if
(1) $f_R(x-y) \supseteq f_R(x) \cap f_R(y)$,
(2) $f_R(xy) \supseteq f_R(y) \cap g_R(x)$,
for all $x, y \in R$.

By using similar method as in the proof of of Proposition 2.9 we obtain the following Proposition.

Proposition 5.2. *Let $f_R \in S(U)$ and $g_R \in S_R(U)$. Then f_R is a soft subideal of g_R if and only if f_R^α is an ideal of g_R^α for all $\alpha \in f_R(R) \cup \{\beta \in P(U) \mid \beta \subseteq f_R(0)\}$.*

Proposition 5.3. *Let $g_R \in S_R(U)$ and f_R is a soft subideal of g_R. Then*
(1) f_R^* is an ideal of g_R^*.
(2) Let for all $\alpha, \beta \in P(U)$, If $\alpha \neq \emptyset \neq \beta$, then $\alpha \cap \beta \neq \emptyset$. Then $f_R^\star$ is an ideal of $g_R^\star$.

Proof. (1) Let $x, y \in f_R^*$ and then $f_R(x) = f_R(0)$ and $f_R(y) = f_R(0)$. Now

$$f_R(x-y) \supseteq f_R(x) \cap f_R(y) = f_R(0) \cap f_R(0) = f_R(0) \supseteq f_R(x-y),$$

and consequently $f_R(x-y) = f_R(0)$. Thus, $x - y \in f_R^*$. Also if $y \in f_R^*, x \in g_R^*$, then

$$f_R(xy) \supseteq f_R(y) \cap g_R(x) = f_R(0) \cap g_R(0) \supseteq f_R(0) \cap f_R(0) = f_R(0) \supseteq f_R(xy),$$

and so $f_R(xy) = f_R(0)$. Hence, $xy \in f_R^*$. Therefore, f_R^* is an ideal of g_R^*.
(2) Suppose that $x, y \in f_R^\star$. Hence, $f_R(x-y) \supseteq f_R(x) \cap f_R(y) \neq \emptyset$ and so $f_R(x-y) \neq \emptyset$. On the other hand, $x - y \in f_R^\star$. Also if $x \in g_R^\star$ and $y \in f_R^\star$, then $f_R(xy) \supseteq f_R(y) \cap g_R(x) \neq \emptyset$ and then $f_R(xy) \neq \emptyset$. Hence, $xy \in f_R^\star$. Therefore, $f_R^\star$ will be an ideal of $g_R^\star$. □

Proposition 5.4. *If $f_R \in S_I(U)$ and $g_R \in S_R(U)$, then $f_R \cap g_R$ is a soft subideal of g_R.*

Proof. Clearly, $f_R \cap g_R \subseteq g_R$. Let $x, y \in R$ then

$$\begin{aligned}(f_R \cap g_R)(x-y) &= f_R(x-y) \cap g_R(x-y)\\ &\supseteq f_R(x) \cap f_R(y) \cap g_R(x) \cap g_R(y)\\ &= (f_R \cap g_R)(x) \cap (f_R \cap g_R)(y).\end{aligned}$$

Also

$$\begin{aligned}(f_R \cap g_R)(xy) &= f_R(xy) \cap g_R(xy) \supseteq f_R(x) \cap f_R(y) \cap g_R(x) \cap g_R(y)\\ &= (f_R \cap g_R)(x) \cap (f_R \cap g_R)(y).\end{aligned}$$

Hence, $f_R \cap g_R$ is a soft subideal of g_R. □

Proposition 5.5. *Let $h_R \in S_R(U)$. If f_R and g_R be two soft subideals of h_R, then $f_R \cap g_R$ is also a soft subideal of h_R.*

Proof. Obviously, $f_R \cap g_R \subseteq h_R$. Let $x, y \in R$. Then

$$\begin{aligned}(f_R \cap g_R)(x-y) &= f_R(x-y) \cap g_R(x-y)\\ &\supseteq f_R(x) \cap f_R(y) \cap g_R(x) \cap g_R(y) = (f_R \cap g_R)(x) \cap (f_R \cap g_R)(y).\end{aligned}$$

Also

$$\begin{aligned}(f_R \cap g_R)(xy) &= f_R(xy) \cap g_R(xy) \supseteq f_R(y) \cap h_R(x) \cap g_R(y) \cap h_R(x)\\ &= (f_R \cap g_R)(y) \cap h_R(x).\end{aligned}$$

Therefore, $f_R \cap g_R$ is a soft subideal of h_R. □

Definition 5.6. ([3]) Let φ be a function from A into B and $f_A, f_B \in S(U)$. Then soft image $\varphi(f_A)$ of f_A under φ is defined by

$$\varphi(f_A)(y) = \begin{cases} \cup\{f_A(x) \mid x \in A, \varphi(x) = y\} & \text{if } \varphi^{-1}(y) \neq \emptyset \\ \emptyset & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

and soft pre-image (or soft inverse image) of f_B under φ is $\varphi^{-1}(f_B)(x) = f_B(\varphi(x))$ for all $x \in A$.

Proposition 5.7. *Let R and S be rings and $f_R \in S_I(U), f_S \in S_I(U)$. Let $\varphi : R \to S$ be a ring homomorphism and $0_R, 0_S$ denote the zero elements of R and S respectively. Then*
(1) $\varphi(f_R)(0_S) = f_R(0_R)$.
(2) $\varphi(f_R^*) \subseteq (\varphi(f_R))^*$.
(3) if f_R be constant on $\ker\varphi$, then $(\varphi(f_R))(\varphi(x)) = f_R(x)$ for all $x \in R$.
(4) if φ is onto, then $\varphi(f_R) \in S_I(U)$. Moreover if f_R is constant on $\ker\varphi$, then $\varphi(f_R^*) = (\varphi(f_R))^*$.
(5) $\varphi^{-1}(f_S) \in S_I(U)$ and also $\varphi^{-1}(f_S)$ is constant on $\ker\varphi$.
(6) $\varphi^{-1}(f_S^*) = (\varphi^{-1}(f_S))^*$.
(7) if φ is onto, then $(\varphi \circ \varphi^{-1})(f_S) = f_S$.
(8) if f_R is constant on $\ker\varphi$, then $(\varphi^{-1} \circ \varphi)(f_R) = f_R$.

Proof. (1) $\varphi(f_R)(0_S) = \cup\{f_R(0_R) \mid \varphi(0_R) = 0_S\} = f_R(0_R)$.
(2) Let $y \in \varphi(f_R^*)$, then $y = \varphi(x)$ and $f_R(x) = f_R(0_R)$. Now

$$\varphi(f_R)(y) = \cup\{f_R(x) \mid y = \varphi(x)\} = \cup\{f_R(0_R) \mid y = \varphi(x)\} = f_R(0_R) = \varphi(f_R)(0_S).$$

Then $y \in (\varphi(f_R))^*$.
(3)

$$\begin{aligned}
(\varphi(f_R))(\varphi(x_1)) &= \cup\{f_R(x_2) \mid x_2 \in R, \varphi(x_1) = \varphi(x_2)\} \\
&= \cup\{f_R(x_2) \mid x_2 \in R, \varphi(x_1 - x_2) = 0_S\} \\
&= \cup\{f_R(x_2) \mid x_2 \in R, x_1 - x_2 \in \ker\varphi\} \\
&= \cup\{f_R(x_2) \mid x_2 \in R, f_R(x_1 - x_2) = f_R(0_R)\} \\
&= \cup\{f_R(x_2) \mid x_2 \in R, f_R(x_1) = f_R(x_2)\} \\
&= f_R(x_1)
\end{aligned}$$

for all $x_1 \in R$.
(4) Let $y_1, y_2 \in S$ such that $y_1 = \varphi(x_1), y_2 = \varphi(x_2)$ for some $x_1, x_2 \in R$. Then

$$\begin{aligned}
\varphi(f_R)(y_1 - y_2) &= \cup\{f_R(z) \mid z \in R, f_R(z) = y_1 - y_2\} \\
&\supseteq \cup\{f_R(x_1 - x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\} \\
&\supseteq \cup\{f_R(x_1) \cap f_R(x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\} \\
&= (\cup\{f_R(x_1) \mid x_1 \in R, y_1 = \varphi(x_1)\}) \cap (\cup\{f_R(x_2) \mid x_2 \in R, y_2 = \varphi(x_2)\}) \\
&= \varphi(f_R)(y_1) \cap \varphi(f_R)(y_2).
\end{aligned}$$

Also

$$\begin{aligned}
\varphi(f_R)(y_1 y_2) &= \cup\{f_R(z) \mid z \in R, f_R(z) = y_1 y_2\} \\
&\supseteq \cup\{f_R(x_1 x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\} \\
&\supseteq \cup\{f_R(x_1) \cup f_R(x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\} \\
&= (\cup\{f_R(x_1) \mid x_1 \in R, y_1 = \varphi(x_1)\}) \cup (\cup\{f_R(x_2) \mid x_2 \in R, y_2 = \varphi(x_2)\}) \\
&= \varphi(f_R)(y_1) \cup \varphi(f_R)(y_2).
\end{aligned}$$

Thus, $\varphi(f_R) \in S_I(U)$. From (2) we have that $\varphi(f_R^*) \subseteq (\varphi(f_R))^*$ and we must prove $(\varphi(f_R))^* \supseteq \varphi(f_R^*)$. Let $y \in S$ such that $y = \varphi(x)$ for some $x \in R$. If $y \in (\varphi(f_R))^*$, then $\varphi(f_R))(y) = \varphi(f_R))(0_S) = f_R(0_R)$ (from (1)). Now from (3) $(\varphi(f_R))(\varphi(x)) = f_R(x) = f_R(0_R)$ and so $x \in f_R^*$ and then $y = \varphi(x) \in \varphi(f_R^*)$.
(5) Let $x_1, x_2 \in R$. Then

$$\begin{aligned}
\varphi^{-1}(f_S)(x_1 - x_2) &= f_S(\varphi(x_1 - x_2)) = f_S(\varphi(x_1) - \varphi(x_2)) \\
&\supseteq f_S(\varphi(x_1)) \cap f_S(\varphi(x_2)) = \varphi^{-1}(f_S)(x_1) \cap \varphi^{-1}(f_S)(x_2).
\end{aligned}$$

Also

$$\begin{aligned}
\varphi^{-1}(f_S)(x_1 x_2) &= f_S(\varphi(x_1 x_2)) = f_S(\varphi(x_1)\varphi(x_2)) \\
&\supseteq f_S(\varphi(x_1)) \cup f_S(\varphi(x_2)) = \varphi^{-1}(f_S)(x_1) \cup \varphi^{-1}(f_S)(x_2).
\end{aligned}$$

Hence, $\varphi^{-1}(f_S) \in S_I(U)$. If $x \in \ker\varphi$, then

$$\varphi^{-1}(f_S)(x) = f_S(\varphi(x)) = f_S(\varphi(0_R)) = f_S(0_S),$$

and then $\varphi^{-1}(f_S)$ is constant on $\ker\varphi$.
(6) Let $x \in R$. Then $x \in \varphi^{-1}(f_S^*)$ if and only if $f_S(\varphi(x)) = f_S(0_S) = f_S(\varphi(0_R))$ if and only if $\varphi^{-1}(f_S)(x) = \varphi^{-1}(f_S)(0_R)$ if and only if $x \in (\varphi^{-1}(f_S))^*$.
(7) Let $y \in S$ such that $y = \varphi(x)$ for some $x \in R$. Then

$$\begin{aligned}(\varphi \circ \varphi^{-1})(f_S)(y) &= \varphi(\varphi^{-1}(f_S))(y) = \varphi(\varphi^{-1}(f_S))(\varphi(x)) = \cup\{\varphi^{-1}(f_S)(x) \mid \varphi(x) = \varphi(x)\}\\ &= \varphi^{-1}(f_S)(x) = f_S(\varphi(x)) = f_S(y).\end{aligned}$$

Therefore, $(\varphi \circ \varphi^{-1})(f_S) = f_S$.
(8) Let $x \in R$. Then $(\varphi^{-1} \circ \varphi)(f_R)(x) = \varphi^{-1}(\varphi(f_R))(x) = \varphi(f_R)(\varphi(x)) = f_R(x)$ (from (3)). $\square$

Proposition 5.8. *Let R and S be two rings and $\varphi : R \to S$ be a ring homomorphism.*
(1) Let φ be onto. If $f_R \in S_R(U)$, then $\varphi(f_R) \in S_S(U)$. Moreover if $g_R \in S_R(U)$ and f_R is a soft subideal of g_R, then $\varphi(f_R)$ is a soft subideal of $\varphi(g_R)$.
(2) If $f_S \in S_S(U)$, then $\varphi^{-1}(f_S) \in S_R(U)$. Moreover if $g_S \in S_S(U)$ and f_S is a soft subideal of g_S, then $\varphi^{-1}(f_S)$ is a soft subideal of $\varphi^{-1}(g_S)$.

Proof. (1) Let $y_1, y_2 \in S$ such that $y_1 = \varphi(x_1), y_2 = \varphi(x_2)$ for some $x_1, x_2 \in R$. Then

$$\begin{aligned}\varphi(f_R)(y_1 - y_2) &= \cup\{f_R(z) \mid z \in R, f_R(z) = y_1 - y_2\}\\ &\supseteq \cup\{f_R(x_1 - x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\}\\ &\supseteq \cup\{f_R(x_1) \cap f_R(x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\}\\ &= (\cup\{f_R(x_1) \mid x_1 \in R, y_1 = \varphi(x_1)\}) \cap (\cup\{f_R(x_2) \mid x_2 \in R, y_2 = \varphi(x_2)\})\\ &= \varphi(f_R)(y_1) \cap \varphi(f_R)(y_2).\end{aligned}$$

Also

$$\begin{aligned}\varphi(f_R)(y_1 y_2) &= \cup\{f_R(z) \mid z \in R, f_R(z) = y_1 y_2\}\\ &\supseteq \cup\{f_R(x_1 x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\}\\ &\supseteq \cup\{f_R(x_1) \cap f_R(x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\}\\ &= (\cup\{f_R(x_1) \mid x_1 \in R, y_1 = \varphi(x_1)\}) \cap (\cup\{f_R(x_2) \mid x_2 \in R, y_2 = \varphi(x_2)\})\\ &= \varphi(f_R)(y_1) \cap \varphi(f_R)(y_2).\end{aligned}$$

Thus, $\varphi(f_R) \in S_S(U)$. Now from

$$\begin{aligned}\varphi(f_R)(y_1 y_2) &= \cup\{f_R(z) \mid z \in R, f_R(z) = y_1 y_2\}\\ &\supseteq \cup\{f_R(x_1 x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\}\\ &\supseteq \cup\{f_R(x_1) \cap g_R(x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\}\\ &= (\cup\{f_R(x_1) \mid x_1 \in R, y_1 = \varphi(x_1)\}) \cap (\cup\{g_R(x_2) \mid x_2 \in R, y_2 = \varphi(x_2)\})\\ &= \varphi(f_R)(y_1) \cap \varphi(g_R)(y_2).\end{aligned}$$

we obtain that $\varphi(f_R)$ is a soft subideal of $\varphi(g_R)$.
(2) Let $x_1, x_2 \in R$. Then

$$\begin{aligned}\varphi^{-1}(f_S)(x_1 - x_2) &= f_S(\varphi(x_1 - x_2)) = f_S(\varphi(x_1) - \varphi(x_2))\\ &\supseteq f_S(\varphi(x_1)) \cap f_S(\varphi(x_2)) = \varphi^{-1}(f_S)(x_1) \cap \varphi^{-1}(f_S)(x_2).\end{aligned}$$

Also

$$\begin{aligned}\varphi^{-1}(f_S)(x_1x_2) &= f_S(\varphi(x_1x_2)) = f_S(\varphi(x_1)\varphi(x_2))\\ &\supseteq f_S(\varphi(x_1)) \cap f_S(\varphi(x_2)) = \varphi^{-1}(f_S)(x_1) \cap \varphi^{-1}(f_S)(x_2).\end{aligned}$$

Therefore, $\varphi^{-1}(f_S) \in S_R(U)$. Moreover

$$\begin{aligned}\varphi^{-1}(f_S)(x_1x_2) &= f_S(\varphi(x_1x_2)) = f_S(\varphi(x_1)\varphi(x_2))\\ &\supseteq f_S(\varphi(x_1)) \cap g_S(\varphi(x_2)) = \varphi^{-1}(f_S)(x_1) \cap \varphi^{-1}(g_S)(x_2).\end{aligned}$$

and then $\varphi^{-1}(f_S)$ is a soft subideal of $\varphi^{-1}(g_S)$. □

References

[1] U. Acar, F. Koyuncu and B.Tanay, *Soft sets and soft rings*, Comput. Math. Appl., **59** (2010), 3458-3463.

[2] H. Aktas and N. Cagman, *Soft sets and soft groups*, Inform. Sci., **177** (2007), 2726-2735.

[3] A. O. Atagun and A. Sezgin, *Soft substructures of rings, fields and modules*, Comput. Math. Appl., **61**(3)(2011), 592-601.

[4] N. Cagman and S. Enginoglu, *Soft set theory and uni-int decision making*, European J. Oper. Res. **207** (2010), 848-855.

[5] M. Hamidi, F. Smarandache, Neutro-BCK-Algebra, International Journal of Neutrosophic Science, **8(2)** (2020), 110–117.

[6] M. Hamidi, and F. Smarandache, Valued-inverse Dombi neutrosophic graph and application, AIMS Mathematics, **8(11)** (2023), 26614–26631.

[7] M. Hamidi, Extended BCK-ideal Based On Single-valued Neutrosophic Hyper BCK-Ideals, Bulletin of the Section of Logic, **52/4** (2023), 411-440.

[8] M. Hamidi, F. Smarandache, Single-Valued Neutro Hyper BCK-Subalgebras, Journal of Mathematics, **2021** (2021), 1–11.

[9] D. Molodsov, Soft set theory - First results, Computers & Mathematics with Applications, **37**(4/5) (1999), 19-31.

[10] R. Rasuli, Soft Lie Ideals and Anti Soft Lie Ideals, The Journal of Fuzzy Mathematics Los Angeles **26(1)** (2018), 193-202.

[11] R. Rasuli, Extension of Q-soft ideals in semigroups, Int. J. Open Problems Compt. Math., **10(2)** (2017), 6-13.

[12] R. Rasuli and F. Hassani, Q-soft Subgroups and Anti-Q-soft Subgroups in Universal Algebra, The Journal of Fuzzy Mathematics Los Angeles, **26(1)** (2018), 139-152.

[13] R. Rasuli, Anti Q-soft Normal Subgroups, The Journal of Fuzzy Mathematics, **21(1)** (2020), 237-248.

[14] F. Smarandache, New Types of Soft Sets: HyperSoft Set, IndetermSoft Set, IndetermHyper Soft Set, and Tree Soft Set, International Journal of Neutrosophic Science, **20(04)** (2023), 58–64.

[15] M. Gharib, F. Smarandache & M. Mohamed, CSsEv: Modelling QoS Metrics in Tree Soft Toward Cloud Services Evaluator based on Uncertainty Environment, International Journal of Neutrosophic Science, **23 (2)** (2024), 32–41.

[16] F. Smarandache, Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set, Neutrosophic Sets and Systems, **22 (1)** (2018), 168-170.

[17] F. Smarandache, Applications of Extended Plithogenic Sets in Plithogenic Sociogram, International Journal of Neutrosophic Science, **20 (4)** (2023), 8-35.

[18] F. Smarandache, Introduction to Neutrosophic Genetics, International Journal of Neutrosophic Science, **1 (1)** (2021), 1-5.

Department of Mathematics, Payame Noor University(PNU), P. O. Box 19395-4697, Tehran, Iran.
E-mail address: `m.hamidi@pnu.ac.ir, rasuli@pnu.ac.ir`