A LIMITATION ON NEUTRO(NEUTROSOPHIC) SOFT SUBRINGS AND
SOFT SUBIDEALS

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Abstract. In this study, we introduce a type of neutrosophic subset concepts such as soft
subrings, soft subideals, the residual quotient of soft subideals, quotient rings of a ring by soft
subideals, soft subideals of soft subrings and investigate some of their properties and structured
characteristics. Also, we investigate them under homomorphisms and obtain some new results.
Indeed, the relation between uncertainty and semi-algebraic is presented and is analyzed to the
important results in neutrosophic subset theory.

1. Introduction

Dealing with uncertainties is a major problem in many areas such as economics, engineering,
environmental science, medical science, and social sciences. These kinds of problems can not be
dealt with by classical methods, because classical methods have inherent difficulties. Molodtsov’s
soft set theory [9] is a kind of new mathematical model for coping with uncertainty from a
parametrization point of view. In soft set theory, the problem of setting the membership function
does not arise, which makes the theory easily be applied to many different fields. Neutrosophy,
as a newly-born science, is a branch of philosophy that studies the origin, nature, and scope of
neutralities, as well as their interactions with different ideational spectra. It can be defined as the
incidence of the application of a law, an operation, an axiom, an idea, or a conceptual accredited
construction on an unclear, indeterminate phenomenon, contradictory to the purpose of making
it intelligible. Recently, Florentin Smarandache generalized the classical algebraic structures to
neutro algebraic structures neutro algebras) and anti algebraic structures (anti algebras) and he
proved that the neutro algebra is a generalization of partial algebra. Indeed the neutro algebras
are an extension of classical algebra or we can say that classical algebras are a limitation of
algebras. In this regard, some researchers have published the papaers on neutro algebras such as
neutro Bck algebra [5], valued-inverse dombi neutrosophic graph and application [6], extended
BCK-ideal based on single-valued neutrosophic hyper BCK-ideals [7], single-valued neutro hyper
BCK-subalgebras [8], new types of soft sets: hypersoft set, indetermsoft set, indetermhypoer
soft set, and tree soft set [14], cceev: modelling qos metrics in tree soft toward cloud services
evaluator based on uncertainty environment [15], extension of soft set to hypersoft set, and
then to plithogenic hypersoft set [16], applications of extended plithogenic sets in plithogenic
sociogram [17] and introduction to neutrosophic genetics [18]. At present, works on the soft set
theory as a limitation of neutrosophic subsets, with algebraic applications are progressing rapidly
[1, 2, 3, 4]. In this paper, we attempt to study the soft subrings and soft subideals and
their properties. The paper is organized as follows: Section 2 is devoted to introducing basic
definitions of soft subrings and soft subideals. In this section we investigate some conditions
such that $f_R^+$, $f_R^-$, and $f_R^*$ will be ideals of $R$. In Section 3, we define the soft subideals of $R$ which generated by $f_R$ and we prove that if $f_R \in S_I(U)$ and $g_R \in S_R(U)$, then $f_R g_R = \langle f_R \circ g_R \rangle$.

Section 4 contains the notions of the residual quotient of soft subideals and we discuss structural properties of them. In this section, we define the coset of $f_R$ and investigate the quotient ring of $R$ by $f_R$. Finally, in Section 5 we define soft subideals of soft subrings and characterize them. At the end of this section, we investigate the soft subrings, soft subideals, and soft subideals of soft subrings under the homomorphism of rings and prove some important results.

2. Soft Subrings and Soft Subideals

Throughout this work, $U$ refers to an initial universe set, $E$ is a set of parameters, $P(U)$ is the power set of $U$ and $R$ is commutative ring.

**Definition 2.1.** [9] For any subset $A$ of $E$, a soft set $f_A$ over $U$ is a set, defined by a function $f_A$, representing a mapping $f_A : E \rightarrow P(U)$, such that $f_A(x) = \emptyset$ if $x \notin A$. A soft set over $U$ can also be represented by the set of ordered pairs $f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\}$. Note that the set of all soft sets over $U$ will be denoted by $S(U)$. From here on, soft set will be used without over $U$.

**Definition 2.2.** [4] Let $f_A, f_B \in S(U)$. Then,

1. $f_A$ is called an empty soft set if $f_A(x) = \emptyset$ for all $x \in E$,
2. $f_A$ is a soft subset of $f_B$, denoted by $f_A \subseteq f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$,
3. $f_A$ and $f_B$ are soft equal, denoted by $f_A = f_B$, if and only if $f_A(x) = f_B(x)$ for all $x \in E$,
4. the set $(f_A \cup f_B)(x) = f_A(x) \cup f_B(x)$ for all $x \in E$ is called union of $f_A$ and $f_B$,
5. the set $(f_A \cap f_B)(x) = f_A(x) \cap f_B(x)$ for all $x \in E$ is called intersection of $f_A$ and $f_B$.

We consider the following example as an illustration.

**Example 2.3.** Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and $E = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of parameters. Let $A = \{x_1, x_2, x_3\}, B = \{x_3, x_4, x_5\}, C = \{x_3, x_4\}$.

Define $f_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\}), (x_3, \{u_4, u_5\})\}, f_B = \{(x_3, \{u_1, u_4\}), (x_4, U), (x_5, \{u_3\})\}, f_C = \{(x_3, \{\}, (x_4, \{\})\}$. Then for all $x \in E$ we have $(f_A \cup f_B)(x) = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\}), (x_3, \{u_4, u_5, u_1, u_2\}), (x_4, U), (x_5, \{u_3\})\}$ and $(f_A \cap f_B)(x) = \{(x_3, \{u_4\})\}$. Also $f_C$ is an empty soft set. Note that the definition of classical subset is not valid for the soft subset. For example $f_C \subseteq f_B$ as soft subset but $f_C \notin f_B$ as classical subset.

**Definition 2.4.** Let $f_R, g_R \in S(U)$. Define $f_R + g_R, -f_R, f_R - g_R, f_R \circ g_R \in S(U)$ as follows:

\[
(f_R + g_R)(x) = \cup\{(f_R(y) \cap g_R(z)) | y, z \in R, y + z = x\},
\]

\[
-f_R(x) = f_R(-x),
\]

\[
(f_R - g_R)(x) = \cup\{(f_R(y) \cap g_R(z)) | y, z \in R, y - z = x\},
\]

\[
(f_R \circ g_R)(x) = \cup\{(f_R(y) \cap g_R(z)) | y, z \in R, yz = x\},
\]

for all $x \in R$. $f_R + g_R, f_R - g_R$ and $f_R \circ g_R$ are called the sum, difference, and product of $f_R$ and $g_R$, respectively, and $-f_R$ is called the negative of $f_R$.

We can say that $f_R + g_R = g_R + f_R, f_R - g_R = f_R + (-g_R)$. Also since $R$ is commutative, then $f_R \circ g_R = g_R \circ f_R$.

**Proposition 2.5.** Let $f_R, g_R, h_R \in S(U)$. Then $f_R \circ (g_R + h_R) \subseteq f_R \circ g_R + f_R \circ h_R$. 

Proof. Let \( w \in R \) and let \( u, v \in R \) be such that \( uv = w \). Then

\[
f_R(u) \cap (g_R + h_R)(v) = f_R(u) \cap (\cup \{g_R(y) \cap h_R(z) \mid y, z \in R, y + z = v\})
= \cup \{(f_R(u) \cap g_R(y)) \cap (f_R(u) \cap h_R(z)) \mid y, z \in R, y + z = v\}
\subseteq \cup \{(f_R(u) \cap g_R(y)) \cap (f_R(u) \cap h_R(z)) \mid y, z \in R, uy + uz = uv\}
\subseteq \cup \{(f_R \circ g_R)(uy) \cap (f_R \circ h_R)(uz) \mid y, z \in R, uy + uz = uv = w\}
\subseteq (f_R \circ g_R + f_R \circ h_R)(w).
\]

Therefore,

\[
f_R \circ (g_R + h_R)(w) = \cup \{f_R(u) \cap (g_R + h_R)(v) \mid u, v \in R, uv = w\}
\subseteq (f_R \circ g_R + f_R \circ h_R)(w).
\]

\( \square \)

Definition 2.6. Let \( f_R, g_R, h_R \in S(U) \). Define \( f_R g_R \in S(U) \) as

\[
(f_R g_R)(x) = \cup \{\cap_{i=1}^{n} (f_R(y_i) \cap g_R(z_i)) \mid y_i, z_i \in R, 1 \leq n, n \in \mathbb{N}, \sum_{i=1}^{n} y_i z_i = x\}
\]

for all \( x \in R \). Since \( R \) is commutative, then \( f_R g_R = g_R f_R \).

The following proposition follows easily and we omit the proof.

Proposition 2.7. Let \( f_R, g_R, h_R \in S(U) \). Then the following assertions hold.
(1) \( f_R \circ g_R \subseteq f_R g_R \).
(2) If \( f_R \subseteq g_R \), then \( h_R f_R \subseteq h_R g_R \).
(3) \((f_R g_R) h_R = f_R (g_R h_R)\).
(4) \((f_R g_R)(x + y) \supseteq (f_R g_R)(x) \cap (f_R g_R)(y)\) for all \( x, y \in R \).

Definition 2.8. Let \( f_R \in S(U) \) and \( n \in \mathbb{N} \). Define \( f_R, f_R^{(n)} \) as follows

\[
f_R^{(1)} = f_R, f_R^{n} = f_R^{1} f_R^{n-1}
\]

and \( f_R^{(1)} = f_R, f_R^{(n)} = f_R^{(1)} f_R^{(n-1)} \).

Definition 2.9. Let \( f_R \in S(U) \). Then \( f_R \) is called soft subring of \( R \) if
(1) \( f_R(x - y) \supseteq f_R(x) \cap f_R(y) \),
(2) \( f_R(xy) \supseteq f_R(x) \cap f_R(y) \),
for all \( x, y \in R \). The set of all soft subrings of \( R \) is denoted by \( S_R(U) \).

Definition 2.10. Let \( f_R \in S(U) \). Then \( f_R \) is called soft subideal of \( R \) if
(1) \( f_R(x - y) \supseteq f_R(x) \cap f_R(y) \),
(2) \( f_R(xy) \supseteq f_R(x) \cup f_R(y) \),
for all \( x, y \in R \). The set of all soft subideals of \( R \) is denoted by \( S_I(U) \). As \( R \) is commutative so condition (2) is equivalent to \( f_R(xy) \supseteq f_R(x) \).

Definition 2.11. (3) Let \( f_R \in S(U) \) and \( \alpha \in P(U) \). Define
(1) \( f_R^{\alpha} = \{x \in R \mid f_R(x) \supseteq \alpha\} \) which is called \( \alpha \)-inclusion of the soft set \( f_A \).
(2) \( f_R^{-} = \{x \in R \mid f_R(x) \neq \emptyset\} \) which is called support of \( f_A \).
(3) \( f_R^{0} = \{x \in R \mid f_R(x) = f_R(0)\} \).

Proposition 2.12. Let \( f_R \in S(U) \). Then \( f_R \in S_I(U) \) if and only if \( f_R^{\alpha} \) is an ideal of \( R \), for all \( \alpha \in f_R(R) \cup \{\beta \in P(U) \mid \beta \subseteq f_R(0)\} \).
Proposition 2.13. Let $f_R, g_R \in S_1(U)$. Then the following assertions hold.

(1) $f_R(0) \supseteq f_R(x)$ for all $x \in R$.

(2) $f_R(x) = f_R(-x)$.

(3) If $R$ is with identity 1, then $f_R(1) \subseteq f_R(x)$ for all $x \in R$.

(4) If $f_R(x - y) = f_R(0)$, then $f_R(x) = f_R(y)$ for all $x, y \in R$.

(5) $f_R^*$ is an ideal of $R$.

(6) Let for all $\alpha, \beta \in P(U)$. If $\alpha \neq \emptyset \neq \beta$, then $\alpha \cap \beta \neq \emptyset$. Then $f_R^*$ is an ideal of $R$.

(7) $f_R \cap g_R \subseteq (f_R^* \cap g_R)^*$.

Proof. (1) $f_R(0) = f_R(x - x) \supseteq f_R(x) \cap f_R(x) = f_R(x)$.

(2) $f_R(x) = f_R(0 - (x - x)) \supseteq f_R(0) \cap f_R(-x) = f_R(-x) = f_R(0 - x) \supseteq f_R(0) \cap f_R(x) = f_R(x)$ and so $f_R(x) = f_R(-x)$.

(3) $f_R(x) = f_R(x + 1) \supseteq f_R(1)$ for all $x \in R$.

(4) $f_R(x) = f_R(x - y + y) = f_R(x - y - (y - y)) \supseteq f_R(x - y) \cap f_R(-y)$

and then $f_R(x) = f_R(y)$ for all $x, y \in R$.

(5) Let $x, y \in f_R^*$ and $r \in R$. Then $f_R(x - y) \supseteq f_R(x) \cap f_R(y) = f_R(0) \cap f_R(0) = f_R(0) \supseteq f_R(x - y)$ and so $f_R(x - y) = f_R(0)$. Also $f_R(x - x) \supseteq f_R(x) = f_R(0) \supseteq f_R(x)$.

Thus, $x - y, x \in f_R^*$, then $f_R^*$ is an ideal of $R$.

(6) Let $x, y \in f_R^*$ and $r \in R$. Since $f_R(x - y) \supseteq f_R(x) \cap f_R(y) \neq \emptyset$ and $f_R(x - x) \supseteq f_R(x) \neq \emptyset$ so $x - y, x \in f_R^*$ and then $f_R^*$ is an ideal of $R$.

(7) If $x \in f_R^* \cap g_R^*$, then $x \in f_R^*$ and $x \in g_R^*$. Hence, $f_R(x) = f_R(0)$ and $g_R(x) = g_R(0)$. Now $(f_R \cap g_R)(x) = f_R(x) \cap g_R(x) = f_R(0) \cap g_R(0) = (f_R \cap g_R)(0)$. Thus, $x \in (f_R \cap g_R)^*$ and hence, $f_R^* \cap g_R^* \subseteq (f_R \cap g_R)^*$.

Proposition 2.14. If $f_R, g_R \in S_1(U)$ and $f_R(0) = g_R(0)$, then $f_R^* \cap g_R^* = (f_R \cap g_R)^*$.

Proof. Let $x \in (f_R \cap g_R)^*$ and so $(f_R \cap g_R)(x) = (f_R \cap g_R)(0)$. Then $f_R(x) \cap g_R(x) = f_R(0) \cap g_R(0) = f_R(0) \subseteq f_R(x) \subseteq f_R(0)$ and hence, $f_R(x) = f_R(0)$. Similarly, $g_R(x) = g_R(0)$. Therefore $x \in f_R^* \cap g_R^*$ and so $(f_R \cap g_R)^* \subseteq f_R^* \cap g_R^*$. Also from Proposition 2.10 (7) we have that $f_R^* \cap g_R^* \subseteq (f_R \cap g_R)^*$. Thus, $f_R \cap g_R^* = (f_R \cap g_R)^*$.

Proposition 2.15. Let $f_R \in S_1(U)$. Then for all $i \in \mathbb{N}$ the following assertions hold.

(1) $f_R(0) = f_R(0)$.

(2) $f_R^{(i)}(0) = f_R(0)$.

Proof. (1) The result is true for $i = 1$. Assume the result is true for $i \geq 1$. Now $f_R^{i+1}(0) = \cup \{f_R(x) \cap f_R(y) \mid x, y \in R, 0 = xy \} = f_R(0)$. (Since the union is attained when $x = y = 0$.) The result now follows by induction.

(2) By (1) and from $f_R(0) = f_R^{(i)}(0) \subseteq f_R(0)$ we obtain $f_R^{(i)}(0) = f_R(0)$.
Proposition 2.16. Let \( f_R \in S_I(U) \). Then for all \( i \in \mathbb{N} \), we have the following statements.

(1) \((f_R^i)^+ \subseteq f_R^i\).
(2) \((f_R^i)^- \subseteq f_R^i\).

Proof. (1) Let \( x \in (f_R^i)^+ \) then \( f_R^i(x) = f_R^i(0) = f_R(0) \). Since \( f_R^i(x) \subseteq f_R(x) \) so \( f_R(0) \subseteq f_R(x) \) and hence, \( f_R(x) = f_R(0) \).

(2) The proof is similar to (1). \( \square \)

Proposition 2.17. Let \( f_R \in S_I(U) \) and \( k \in \mathbb{N} \). If \( x_1, x_2, \ldots, x_k \in R \), then

(1) \( f_R^k(x_1 x_2 \ldots x_k) \supseteq \cap_{i=1}^k f_R(x_i) \).
(2) \( f_R((x_1 x_2 \ldots x_k)^I) \supseteq \cap_{i=1}^k f_R(x_i)^I \).

Proof. (1) The result is true for \( k = 1 \). Suppose the result is true for \( k \geq 1 \).

Now \( f_R^{k+1}(x_1 x_2 \ldots x_{k+1}) = (f_R^k \circ f_R)(x_1 x_2 \ldots x_k) \supseteq \cap_{i=1}^k f_R^i(x_i) \cap f_R(x_{k+1}) = \cap_{i=1}^{k+1} f_R(x_i) \).

and so \( f_R^k(x_1 x_2 \ldots x_k) \supseteq \cap_{i=1}^k f_R(x_i)^I \).

(2) Use (1), and \( f_R^I \subseteq f_R^i \). \( \square \)

Proposition 2.18. Let \( f_R, g_R, h_R \in S_I(U) \). Then \( f_R \circ g_R \subseteq h_R \) if and only if \( f_R g_R \subseteq h_R \).

Proof. Let \( f_R g_R \subseteq h_R \) then \( f_R \circ g_R \subseteq f_R g_R \subseteq h_R \). Conversely, let \( f_R \circ g_R \subseteq h_R \). Let \( x \in R \) and \( x = \sum_{i=1}^n y_i z_i \) such that \( y_i, z_i \in R, i = 1, 2, \ldots, n \). We know that

\[ h_R(y_i z_i) \supseteq (f_R \circ g_R)(y_i z_i) \supseteq f_R(y_i) \cap g_R(z_i), \]

for all \( i = 1, 2, \ldots, n \). Now from Proposition 2.14, we obtain

\[ h_R(x) = h_R(\sum_{i=1}^n y_i z_i) \supseteq \cap_{i=1}^n h_R(y_i z_i) \supseteq \cap_{i=1}^n (f_R(y_i) \cap g_R(z_i)). \]

Thus, \( h_R(x) \supseteq \cup \{\cap_{i=1}^n f_R(y_i) \cap g_R(z_i) \mid x = \sum_{i=1}^n y_i z_i, n \in \mathbb{N} \} = f_R g_R(x) \). Therefore, \( f_R g_R \subseteq h_R \). \( \square \)

Proposition 2.19. If \( f_R, g_R \in S_R(U) \), then \( f_R^a + g_R^a \subseteq (f_R + g_R)^a \) for all \( \alpha \in P(U) \).

Proof. Let \( x \in f_R^a + g_R^a \). Then \( x = y + z \) such that \( y \in f_R^a \) and \( z \in g_R^a \). So \( f_R(y) \supseteq \alpha \) and \( g_R(z) \supseteq \alpha \). Now

\[ (f_R + g_R)(x) = \cup \{f_R(y) \cap g_R(z) \mid x = y + z \} \supseteq \alpha. \]

Hence, \( x \in (f_R + g_R)^a \) and then \( f_R^a + g_R^a \subseteq (f_R + g_R)^a \). \( \square \)

Proposition 2.20. Let \( \{f_R\}_{i \in I} \subseteq S_I(U) \). Then \( f_R = \cap_{i \in I} \{f_R\}_{i \in I} \subseteq S_I(U) \).

Proof. Let \( x, y \in R \). Then

\[ f_R(x - y) = \cap_{i \in I} \{f_R\}_{i \in I}(x - y) = \cap_{i \in I}(\{f_R\}_{i \in I})(x - y) \supseteq \cap_{i \in I}(\{f_R\}_{i \in I}(x) \cap \{f_R\}_{i \in I}(y)) = \cap_{i \in I} f_R(x) \cap f_R(y) = f_R(x) \cap f_R(y). \]

Also

\[ f_R(xy) = \cap_{i \in I} \{f_R\}_{i \in I}(xy) = \cap_{i \in I}(\{f_R\}_{i \in I}(xy) = \cap_{i \in I}(\{f_R\}_{i \in I}(xy)) \supseteq \cap_{i \in I}(\{f_R\}_{i \in I}(x)) = f_R(x). \]

Therefore, \( f_R \in S_I(U) \). \( \square \)
3. Generated Soft Subideals

In this section, we introduce the concepts of generated goft subideals and investigate their properties.

**Definition 3.1.** Let \( f_R \in S(U) \). Define \( f_R = \cap \{g_R \mid f_R \subseteq g_R, g_R \in S_I(U)\} \) and is called the soft subideal of \( R \) generated by \( f_R \). Note \( f_R \) is the smallest soft subideal of \( R \) containing \( f_R \).

**Corollary 3.2.** Let \( f_R, g_R \in S(U) \). Then
(1) \( f_R \in S_I(U) \) if and only if \( f_R = g_R \).
(2) If \( f_R \subseteq g_R \), then \( f_R < g_R \).

**Proposition 3.3.** Let \( A \) be a nonempty subset of \( R \) and \( < A > \) is the ideal of \( R \) generated by \( A \). Let \( f_R, g_R \in S(U) \) such that \( g_R(x) = \cup \{\cap_{y \in A} f_R(y) \mid A \subseteq R, 1 \leq |A| < \infty, x \in < A > \} \) for all \( x \in R \). Then \( < f_R > = g_R \).

**Proof.** It is obvious that \( f_R \subseteq g_R \) and \( g_R(-x) = g_R(x) \) for all \( x \in R \). Let \( x, y \in R \) and we prove \( g_R \in S_I(U) \).
(1) If \( x < A >, y < B >, A, B \subseteq R, 1 \leq |A| < \infty, 1 \leq |B| < \infty \), then \( x + y < A \cup B > \). Then
\[
g_R(x + y) \supseteq \cap_{z \in A \cup B} f_R(z) \supseteq (\cap_{u \in A} f_R(u)) \cap (\cap_{v \in B} f_R(v)),
\]
and so
\[
g_R(x + y) = \cup \{(\cap_{u \in A} f_R(u)) \cap (\cap_{v \in B} f_R(v)) \mid A \subseteq R, 1 \leq |A| < \infty, x \in < A >, y \in < B > \}
\]
and hence, \( g_R(x + y) \supseteq g_R(x) \cap g_R(y) \).
(2) Also if \( B \subseteq R, 1 \leq |B| < \infty, y \in < B > \), then \( xy < B > \). Therefore
\[
g_R(xy) = \cup \{(\cap_{z \in B} f_R(z)) \mid B \subseteq R, 1 \leq |B| < \infty, xy \in < B > \}
\]
and then \( g_R(xy) \supseteq g_R(y) \).

Now (1) and (2) show that \( g_R \in S_I(U) \). Let \( h_R \in S_I(U) \) such that \( f_R \subseteq h_R \). Suppose \( x \in R \) and \( A = \{y_1, y_2, ..., y_n\} \) with \( x \in < A > \). Then there exist \( x_1, x_2, ..., x_n \in R \) and integers \( m_1, m_2, ..., m_n \) such that \( x = \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} m_i y_i \). From \( h_R \in S_I(U) \) we obtain
\[
h_R(x) = h_R(\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} m_i y_i)
\]
and
\[
h_R(\sum_{i=1}^{n} x_i y_i) \cap h_R(\sum_{i=1}^{n} m_i y_i) \supseteq (\cap_{i=1}^{n} h_R(x_i y_i)) \cap (\cap_{i=1}^{n} h_R(m_i y_i))
\]
and
\[
\cap_{i=1}^{n} h_R(y_i).
\]
Now
\[ h_R(x) \supseteq \bigcup \{ R \supseteq A \mid A \subseteq R, 1 \leq |A| < \infty, x \in A \} \]
\[ \supseteq \bigcup \{ \cap_{y \in A} h_R(y) \mid A \subseteq R, 1 \leq |A| < \infty, x \in A \} \]
\[ = g_R(x) \]
and so \( h_R \supseteq g_R \). Hence, \( < f_R >= g_R \). \( \square \)

**Proposition 3.4.** Let \( f_R, g_R \in S_I(U) \) and \( f_R(0) = g_R(0) \). Then
(1) \( f_R + g_R \in S_I(U) \).
(2) \( f_R + g_R =< f_R \cup g_R > \).

**Proof.** Let \( x, y \in R \). It is clear that \( (f_R + g_R)(-x) = (f_R + g_R)(x) \). Now
(1)
\[ (f_R + g_R)(x + y) = \bigcup \{ f_R(u) \cap g_R(v) \mid u, v \in R, u + v = x + y \} \]
\[ \supseteq \bigcup \{ f_R(u_1 + v_1) \cap g_R(u_2 + v_2) \mid u_1, u_2, v_1, v_2 \in R, u_1 + u_2 = x, v_1 + v_2 = y \} \]
\[ \supseteq \bigcup \{ (f_R(u_1) \cap f_R(v_1)) \cap (g_R(u_2) \cap g_R(v_2)) \mid u_1, u_2, v_1, v_2 \in R, u_1 + u_2 = x, v_1 + v_2 = y \} \]
\[ = \bigcup \{ f_R(u_1) \cap g_R(u_2) \mid u_1, u_2 \in R, u_1 + u_2 = x \} \subseteq \bigcup \{ f_R(v_1) \cap g_R(v_2) \mid v_1, v_2 \in R, v_1 + v_2 = y \} \]
\[ = (f_R + g_R)(x) \cap (f_R + g_R)(y). \]

Also
\[ (f_R + g_R)(xy) \supseteq \bigcup \{ f_R(u) \cap g_R(v) \mid u, v \in R, u + v = y \} \]
\[ \supseteq \bigcup \{ f_R(u) \cap g_R(v) \mid u, v \in R, u + v = y \} \]
\[ = (f_R + g_R)(y). \]

Hence, \( f_R + g_R \in S_I(U) \).
(2)
\[ (f_R + g_R)(x) = \bigcup \{ f_R(u) \cap g_R(v) \mid u, v \in R, u + v = x \} \supseteq f_R(x) \cap g_R(0) = f_R(x) \]
and then \( f_R + g_R \supseteq f_R \). Similarly, \( f_R + g_R \supseteq g_R \). Thus, \( f_R + g_R \supseteq f_R \cup g_R \). Let \( h_R \in S_I(U) \) and \( f_R \cup g_R \subseteq h_R \). Then
\[ (f_R + g_R)(x) = \bigcup \{ f_R(u) \cap g_R(v) \mid u, v \in R, u + v = x \} \]
\[ \subseteq \bigcup \{ h_R(u) \cap h_R(v) \mid u, v \in R, u + v = x \} = h_R(x) \]
and so \( f_R + g_R \subseteq h_R \). Then \( f_R + g_R \) is the smallest soft subideal of \( R \) such that \( f_R \cup g_R \subseteq f_R + g_R \) and then \( f_R + g_R =< f_R \cup g_R > \). \( \square \)

The proof of the following proposition is straightforward.

**Proposition 3.5.** Let \( f_R, g_R \in S(U) \).
(1) If \( f_R \in S_I(U) \), then \( f_R \circ g_R \subseteq f_R \).
(2) If \( f_R, g_R \in S_I(U) \), then \( f_R \circ g_R \subseteq f_R \cap g_R \).

**Proposition 3.6.** If \( f_R, g_R \in S_R(U) \), then \( f_R g_R \in S_R(U) \).
Proposition 3.7. Let
\[
(f_{RGR})(-x) = (f_{RGR})(x)
\]
for all \( x \in R \). From Proposition 2.4 (4) we get that
\[
(f_{RGR})(x + y) \supseteq (f_{RGR})(x) \cap (f_{RGR})(y)
\]
for all \( x, y \in R \). Let \( x = \sum_{i=1}^{m} x_i s_i \) and \( y = \sum_{j=1}^{n} y_j t_j \) where \( x_i, s_i, y_j, t_j \in R \) for \( 1 \leq i \leq m, 1 \leq j \leq n \). Now
\[
xy = \left( \sum_{i=1}^{m} x_i s_i \right) \left( \sum_{j=1}^{n} y_j t_j \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i y_j)(s_i t_j)
\]
and by the definition of \( f_{RGR} \) we have
\[
(f_{RGR})(xy) \supseteq \bigcap_{i=1}^{m} \bigcap_{j=1}^{n} (f_{RGR}(x_i, y_j)) \cap (g_{R}(s_i, t_j))
\]
\[
\supseteq \bigcap_{i=1}^{m} \bigcap_{j=1}^{n} (f_{RGR}(x) \cap f_{RGR}(y)) \cap (g_{R}(s_i) \cap g_{R}(t_j))
\]
\[
= \bigcap_{i=1}^{m} \bigcap_{j=1}^{n} (f_{RGR}(x_i) \cap g_{R}(s_i)) \cap (f_{RGR}(y_j) \cap g_{R}(t_j))
\]
\[
= \bigcap_{i=1}^{m} (f_{RGR}(x_i) \cap g_{R}(s_i)) \cap \bigcap_{j=1}^{n} (f_{RGR}(y_j) \cap g_{R}(t_j))
\]
and hence,
\[
(f_{RGR})(xy) \supseteq \left( \bigcup \{ \bigcap_{i=1}^{m} (f_{RGR}(x_i) \cap g_{R}(s_i)) \mid x_i, s_i \in R, 1 \leq i \leq m, x = \sum_{i=1}^{m} x_i s_i \} \right) \cap \left( \bigcup \{ \bigcap_{j=1}^{n} (f_{RGR}(y_j) \cap g_{R}(t_j)) \mid y_j, t_j \in R, 1 \leq j \leq n, y = \sum_{j=1}^{n} y_j t_j \} \right)
\]
\[
= (f_{RGR})(x) \cap (f_{RGR})(y).
\]
Therefore \( f_{RGR} \in S_{R}(U) \).

\[ \square \]

**Proposition 3.7.** Let \( f_{R} \in S_{I}(U) \) and \( g_{R} \in S_{R}(U) \). Then
(1) \( f_{RGR} \in S_{I}(U) \).
(2) \( f_{RGR} = < f_{R} \circ g_{R} > \).

**Proof.** Let \( x, y \in R \). Then
(1) \( (f_{RGR})(-x) = (f_{RGR})(x) \) and from Proposition 2.4 (4) we have \( (f_{RGR})(x + y) \supseteq (f_{RGR})(x) \cap (f_{RGR})(y) \). Also
\[
(f_{RGR})(xy) \supseteq \bigcup \{ \bigcap_{i=1}^{n} (f_{RGR}(x) \cap g_{R}(v_i)) \mid u_i, v_i \in R, 1 \leq i \leq n, y = \sum_{i=1}^{n} u_i v_i \}
\]
\[
\supseteq \bigcup \{ \bigcap_{i=1}^{n} (f_{RGR}(u_i) \cap g_{R}(v_i)) \mid u_i, v_i \in R, 1 \leq i \leq n, y = \sum_{i=1}^{n} u_i v_i \} = (f_{RGR})(y).
\]
Thus, \( f_{RGR} \in S_{I}(U) \).
(2) By Proposition 2.4 (1) we obtain that \( f_{R} \circ g_{R} \subseteq f_{RGR} \). If \( h_{R} \in S_{R}(U) \) and \( f_{R} \circ g_{R} \subseteq h_{R} \), then
\[
(f_{RGR})(x) = \bigcup \{ \bigcap_{i=1}^{n} (f_{RGR}(y_i) \cap g_{R}(z_i)) \mid y_i, z_i \in R, 1 \leq i \leq n, x = \sum_{i=1}^{n} y_i z_i \}
\]
\[
\subseteq \bigcup \{ \bigcap_{i=1}^{n} (f_{R} \circ g_{R})(y_i z_i) \mid y_i, z_i \in R, 1 \leq i \leq n, x = \sum_{i=1}^{n} y_i z_i \}
\]
Now we prove

\[ f_R = \begin{cases} 
\end{cases} \]

\[
\subseteq \cup \{ h_R(y_i z_i) \mid y_i, z_i \in R, 1 \leq i \leq n, x = \sum_{i=1}^{n} y_i z_i \} \\
 \subseteq \cup \{ h_R(\sum_{i=1}^{n} y_i z_i) \mid y_i, z_i \in R, 1 \leq i \leq n, x = \sum_{i=1}^{n} y_i z_i \} \\
= h_R(x).
\]

Therefore \( f_R g_R \subseteq h_R \) and \( f_R g_R = \langle f_R \circ g_R \rangle \).

**Proposition 3.8.** Let \( f_R, g_R \in S_I(U) \). Then \( f_R g_R \subseteq f_R \cap g_R \).

**Proof.** The result follows from Proposition 2.15 and Proposition 3.5 (2).

**Proposition 3.9.** Let \( f_R, g_R, h_R \in S_I(U) \) and \( g_R(0) = h_R(0) \). Then \( f_R(g_R + h_R) = f_R g_R + f_R h_R \).

**Proof.** From \( g_R \subseteq g_R + h_R \) and \( h_R \subseteq g_R + h_R \) we obtain \( f_R g_R \subseteq f_R(g_R + h_R) \) and \( f_R h_R \subseteq f_R(g_R + h_R) \) and therefore \( f_R g_R + f_R h_R \subseteq f_R(g_R + h_R) \). Suppose hat \( x \in R \). Then

\[
(f_R(g_R + h_R))(x) = \cup \{ \sum_{i=1}^{n} (f_R(y_i) \cap (g_R + h_R)(z_i)) \mid y_i, z_i \in R, 1 \leq i \leq n, n \in N, x = \sum_{i=1}^{n} y_i z_i \} \\
= \cup \{ \sum_{i=1}^{n} (f_R(y_i) \cap h_R(v_i)) \mid u_i, v_i \in R, u_i + v_i = z_i \} \mid y_i, z_i \in R, 1 \leq i \leq n, n \in N, x = \sum_{i=1}^{n} (y_i u_i + y_i v_i) \} \\
\subseteq \cup \{ \sum_{i=1}^{n} (f_R(s_i) \cap g_R(t_i)) \cap (f_R(h_R(w_k))) \mid s_i, t_i, w_k \in R, 1 \leq i \leq p, 1 \leq k \leq q, p, q \in N, \\
x = \sum_{i=1}^{n} s_i t_i + \sum_{k=1}^{q} r_k w_k \} = \cup \{ (f_R g_R)(a) \cap (f_R h_R)(b) \mid a, b \in R, a + b = x \} = (f_R g_R + f_R h_R)(x).
\]

Thus, \( f_R(g_R + h_R) \subseteq f_R g_R + f_R h_R \). Therefore \( f_R(g_R + h_R) = f_R g_R + f_R h_R \).

4. **Residual Quotient of Soft Subideals**

In this section, we introduce the residual quotient of soft subideals.

**Definition 4.1.** Let \( f_R, g_R \in S_I(U) \). Define \( f_R : g_R \in S_I(U) \) as follows:

\[
f_R : g_R = \cup \{ h_R \mid h_R \in S_I(U), h_R \circ g_R \subseteq f_R \}.
\]

**Proposition 4.2.** If \( f_R, g_R \in S_I(U) \), then \( f_R \subseteq f_R : g_R \) and \( f_R : g_R \in S_I(U) \).

**Proof.** We know that \( f_R \circ g_R \subseteq f_R \) and then \( f_R : g_R = \cup \{ h_R \mid h_R \in S_I(U), h_R \circ g_R \subseteq f_R \} \supseteq f_R \).

Now we prove \( f_R : g_R \in S_I(U) \). Let \( x, y \in R \). It is clear \( (f_R : g_R)(-x) = (f_R : g_R)(x) \).

\[
(1)
( f_R : g_R)(x) \cap ( f_R : g_R)(y) = (\cup \{ h_R(x) \mid h_R \in S_I(U), h_R \circ g_R \subseteq f_R \}) \cap (\cup \{ k_R(y) \mid k_R \in S_I(U), k_R \circ g_R \subseteq f_R \}) \\
= \cup \{ h_R(x) \cap k_R(y) \mid h_R, k_R \in S_I(U), (h_R \circ g_R) \cup (k_R \circ g_R) \subseteq f_R \} \\
\subseteq \cup \{ (h_R + k_R)(x + y) \mid h_R, k_R \in S_I(U), (h_R + k_R) \circ g_R \subseteq f_R \} \\
\subseteq ( f_R : g_R)(x + y).
\]
Proof. (1)
\[
(f : g)(xy) = (\cup \{ h \mid h(x) \text{ or } h(y) \})
\]
\[
\supseteq \cup \{ h \mid h(x) \text{ or } h(y) \} \supseteq (f : g)(x).
\]
Then (1) and (2) show that \( f : g \in S(U) \).

Proposition 4.3. Let \( f, g, h \in S(U) \). If \( f \subseteq g \), then

(1) \( f : h \subseteq g : h \) and

(2) \( h : f \supseteq h : g \).

Proof. (1)
\[
(f : g) = \cup \{ k \mid k \in S(U), k \circ f \subseteq f \}
\]
\[
\subseteq \cup \{ k \mid k \in S(U), k \circ h \subseteq f \} \subseteq g : f.
\]
(2)
\[
(h : f) = \cup \{ k \mid k \in S(U), k \circ h \leq f \}
\]
\[
\supseteq \cup \{ k \mid k \in S(U), k \circ g = h : g \}.
\]

Proposition 4.4. Let \( f, g, h \in S(U) \). Then

(1) \( (f : g)g \subseteq f \).

(2) If \( f \subseteq h \) and \( h : g \subseteq h : f \), then \( h \subseteq f : g \).

Proof. Suppose that \( x \in R \) and \( x = \sum_{i=1}^{n} y_{i}z_{i} \) such that \( y_{i}, z_{i} \in R, 1 \leq i \leq n, n \in \mathbb{N} \). If \( f \supseteq h : g \), then
\[
f(x) = f(\sum_{i=1}^{n} y_{i}z_{i}) \supseteq \cap_{i=1}^{n} f(y_{i}z_{i}) \supseteq \cap_{i=1}^{n} (h : g)(y_{i}z_{i}) \supseteq \cap_{i=1}^{n} (h : g)(y_{i}) \cap g(z_{i})
\]
and so \( f(x) \supseteq \cap_{i=1}^{n} ((f : h)(y_{i}) \cap g(z_{i})) \). Then
\[
f(x) \supseteq \{ \cap_{i=1}^{n} ((f : h)(y_{i}) \cap g(z_{i})) \mid y_{i}, z_{i} \in R, 1 \leq i \leq n, n \in \mathbb{N}, x = \sum_{i=1}^{n} y_{i}z_{i} \},
\]
therefore, \( (f : g)g \subseteq f \).
(2) By \( h : g \subseteq h : g \), we get \( h \subseteq f : g \).

Proposition 4.5. If \( (f \text{ or } g) \in S(U) \), then \( (\cap_{i=1}^{n} (f \text{ or } g)) : g = (\cap_{i=1}^{n} (f \text{ or } g)) : g \) for all \( i = 1, 2, 3, \ldots, n \).

Proof. It is enough to prove for \( n = 2 \). Clearly, \( (f \text{ or } g) \supseteq ((f \text{ or } g) \cap (f \text{ or } g)) \).

Let \( x \in R \) and \( \alpha, \beta \in P(U) \). Let
\[
\alpha = ((f_1 \text{ or } g_1) : g_2)(x) = \cup \{ h \mid h \in S(U), h \circ g_1 \subseteq (f_1 \text{ or } g_2) \}.
\]
Now
\[
((f_1 \text{ or } g_1) \cap (f_2 \text{ or } g_2))(x) = (f_1 \text{ or } g_1)(x) \cap (f_2 \text{ or } g_2)(x)
\]
\[
= \cup \{ k \mid k \in S(U), k \circ g_1 \subseteq (f_1 \text{ or } g_2) \} \cap (f_2 \text{ or } g_2)(x)
\]
\[
= \cup \{ k \cap l \mid k, l \in S(U), k \circ g_1 \subseteq (f_1 \text{ or } g_2) \}.
\]
Let \( k_R \odot g_R \subseteq (f_R)_1 \) and \( l_R \odot g_R \subseteq (f_R)_2 \). If \( \beta = k_R(x) \cap l_R(x) \), then \( A = k_R^2 \cap l_R^2 \) is an ideal of \( R \). Define \( m_R : R \to P(U) \) as

\[
m_R(x) = \begin{cases} \beta & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}
\]

then \( m_R \in S_I(U) \). Since \( m_R \subseteq k_R \) and \( m_R \subseteq l_R \) so \( m_R \odot g_R \subseteq k_R \odot g_R \subseteq (f_R)_1 \) and \( m_R \odot g_R \subseteq l_R \odot g_R \subseteq (f_R)_2 \) and then \( m_R \odot g_R \subseteq (f_R)_1 \cap (f_R)_2 \). Then \( \alpha = \cup \{ n_R \mid n_R \in S_I(U), n_R \odot g_R \subseteq (f_R)_1 \cap (f_R)_2 \} \supseteq m_R(x) = \beta = k_R(x) \cap l_R(x) \). Therefore

\[
((f_R)_1 \cap (f_R)_2) : g_R(x) = \alpha \supseteq \cup \{ k_R \cap l_R \mid k_R, l_R \in S_I(U), k_R \odot g_R \subseteq (f_R)_1, l_R \odot g_R \subseteq (f_R)_2 \}
\]

\[
= (((f_R)_1 : g_R) \cap ((f_R)_2 : g_R))(x).
\]

Therefore \( (\cap_{i=1}^2 (f_R)_i) : g_R \supseteq \cap_{i=1}^2 ((f_R)_i : g_R) \). Then \( (\cap_{i=1}^2 (f_R)_i) : g_R = \cap_{i=1}^2 ((f_R)_i : g_R). \)

**Definition 4.6.** Let \( f_R \in S(U) \) and \( x \in R \). Let \( 0_R \) denotes the zero element of \( R \). Define \( f_R(0) \{x\} : R \to P(U) \) as

\[
f_R(0) \{x\}(y) = \begin{cases} f_R(0) & \text{if } y = x \\ \emptyset & \text{if } y \neq x \end{cases}
\]

for all \( y \in R \).

**Definition 4.7.** Let \( f_R \in S(U) \) and \( x \in R \). Then \( f_R(0) \{x\} + f_R \) is called a coset of \( f_R \). Now for all \( y \in R \) we have that

\[
(f_R(0) \{x\} + f_R)(y) = \cup \{ f_R(0) \{x\}(z) \cap f_R(z_2) \mid y = z_1 + z_2 \}
\]

\[
= \cup \{ f_R(0) \cap f_R(z_2) \mid y = x + z_2 \} = f_R(z_2) = f_R(y - x).
\]

We write \( x + f_R \) for \( f_R(0) \{x\} + f_R \). Then \( (x + f_R)(y) = f_R(y - x) \).

**Proposition 4.8.** Let \( f_R \in S_I(U) \) and \( x, y \in R \). Then \( x + f_R = y + f_R \) if and only if \( f_R(x - y) = f_R(0) \).

**Proof.** If \( x + f_R = y + f_R \), then \( (x + f_R)(x) = (y + f_R)(x) \) and so \( f_R(0) = f_R(x - x) = f_R(x - y) \). Conversely, if \( z \in R \) and \( f_R(x - y) = f_R(0) \), then

\[
(x + f_R)(z) = f_R(z - x) = f_R(z - y + y - x) \supseteq f_R(z - y) \cap f_R(y - x)
\]

\[
= f_R(z - y) \cap f_R(0) = f_R(z - y) = (y + f_R)(z)
\]

and then \( x + f_R \supseteq y + f_R \). Similarly, \( y + f_R \subseteq x + f_R \). Therefore \( x + f_R = y + f_R \). \( \Box \)

**Proposition 4.9.** Let \( f_R \in S_I(U) \). Define \( R/f_R = \{ x + f_R \mid x \in R \} \). Then \( (R/f_R, +, \cdot) \) is a ring and is called the quotient ring of \( R \) by \( f_R \) such that \( + \) and \( \cdot \) on \( R/f_R \) are as \( (x + f_R) + (y + f_R) = (x + y) + f_R \) and \( (x + f_R) \cdot (y + f_R) = (xy) + f_R \) for all \( x, y \in R \).

**Proof.** We prove that \( + \) and \( \cdot \) are well defined. Let \( x, y, z, t \in R \) and \( x + f_R = z + f_R \) and \( y + f_R = t + f_R \). Then by Proposition 4.8 \( f_R(x - z) = f_R(y - t) = f_R(0) \). Therefore

\[
f_R(x + y - (z + t)) = f_R(x - z + y - t) \supseteq f_R(x - z) \cap f_R(y - t)
\]

\[
= f_R(0) \cap f_R(0) = f_R(0) \supseteq f_R(x + y - (z + t))
\]
so $f_R(x + y - (z + t)) = f_R(0)$. Now by Proposition 4.8 $(x + y) + f_R = (z + t) + f_R$ and so $+$ is well defined. Also

$$f_R(zt - xy) = f_R(zt - zy + zy - xy)$$
\[\geq f_R(zt - zy) \cap f_R(zy - xy)\]
\[= f_R(z(t - y)) \cap f_R(y(z - x))\]
\[\geq (f_R(z) \cup f_R(t - y)) \cap (f_R(y) \cup f_R(z - x))\]
\[(f_R(z) \cup f_R(0)) \cap (f_R(y) \cup f_R(0)) = f_R(0) \cap f_R(0) = f_R(0) \supseteq f_R(zt - xy)\]

and then $f_R(zt - xy) = f_R(0)$. Now by Proposition 4.8 $(xy) + f_R = (zt) + f_R$ and so is well defined.

Let $0_R$ denotes the zero element of $R$. Then $(0_R + f_R)(x) = f_R(x - 0) = f_R(x)$ and then $0_R + f_R = f_R$. Also $-x + f_R + x + f_R = f_R$ then $-(x + f_R) = (-x) + f_R$ for all $x \in R$. □

**Proposition 4.10.** If $f_R \in S_I(U)$, then $R/f_R^* \simeq R/f_R$.

**Proof.** Let $x, y \in R$ and define $\varphi : R \rightarrow R/f_R$ as $\varphi(x) = x + f_R$. Then $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(xy) = \varphi(x) \varphi(y)$ and so $\varphi$ is a homomorphism of $R$ onto $R/f_R$. Also

$$\ker \varphi = \{x \in R \mid \varphi(x) = 0\} = \{x \in R \mid x + f_R = f_R\} = \{x \in R \mid f_R(x) = f_R(0)\} = f_R^*.$$

Therefore $R/f_R^* \simeq R/f_R$. □

In the following propositions we introduce the quotient soft subrings and soft subideals $f_R$ relative to an ideal os $R$.

**Proposition 4.11.** Let $f_R \in S_I(U)$. Define $f_R^{(x)} : R/f_R \rightarrow P(U)$ as $f_R^{(x)}(x + f_R) = f_R(x)$ for all $x \in R$. Then $f_R^{(x)}$ is a soft subideal of $R/f_R$.

**Proof.** It is easy to prove that $f_R^{(x)}$ is well defined. Let $x, y \in R$, then

$$f_R(zt - xy) = f_R^{(x)}((xy) + f_R))$$
\[= f_R(xy) \supseteq f_R(x) \cup f_R(y) = f_R^{(x)}(x + f_R) \cup f_R^{(x)}(y + f_R).\]

Also

$$f_R^{(x)}((x + f_R)(y + f_R)) = f_R^{(x)}((x - y) + f_R)) = f_R(x - y)$$
\[\supseteq f_R(x) \cap f_R(y) = f_R^{(x)}(x + f_R) \cap f_R^{(x)}(y + f_R).\]

Thus, $f_R^{(x)}$ is a soft subideal of $R/f_R$. □

**Proposition 4.12.** Let $f_R \in S_R(U)$ and let $A$ be an ideal of $R$. Define $f_{R/A} : R/A \rightarrow P(U)$ as $f_{R/A}(x + A) = \cup \{f_R(z) \mid z \in [x]\}$ such that $[x] = x + A$ for all $x \in R$. Then $f_{R/A} \in S_{R/A}(U)$. 


Proof. Let \( x, y \in R \). Then
\[
\begin{align*}
\frac{R}{A}(x - y) &= \frac{R}{A}(xy) \\
&= \square
\end{align*}
\]

Also
\[
\begin{align*}
\frac{R}{A}(x)[y] &= \frac{R}{A}(xy) = \cup \{ R(xy + z) \mid z \in A \} \\
&\subseteq \cup \{ R(xy + (xb + ay + ab)) \mid a, b \in A \} \\
&= \cup \{ R((x + a)(y + b)) \mid a, b \in A \} \\
&= \cup \{ R(x + a) \cap R(y + b) \mid a, b \in A \} \\
&= \cup \{ R(x + a) \cap R(y + b) \mid a, b \in A \} \\
&= \frac{R}{A}(x)[y] \cap \frac{R}{A}(y).
\end{align*}
\]
This completes the proof. \( \square \)

The soft subideal \( R/fR \) defined in the above Proposition is called the quotient soft subring of \( fR \) relative to \( A \) and denoted by \( fR/A \).

5. Soft Subideals of Soft Subrings

In this section, we introduce the concepts of soft subideals of soft subrings.

**Definition 5.1.** Let \( fR \in S(U) \) and \( gR \in S_R(U) \) such that \( fR \subseteq gR \). Then \( fR \) is called soft subideal of \( gR \) if

1. \( fR(x - y) \supseteq fR(x) \cap fR(y) \),
2. \( fR(xy) \supseteq fR(y) \cap gR(x) \),

for all \( x, y \in R \).

By using similar method as in the proof of of Proposition 2.9 we obtain the following Proposition.

**Proposition 5.2.** Let \( fR \in S(U) \) and \( gR \in S_R(U) \). Then \( fR \) is a soft subideal of \( gR \) if and only if \( fR \) is an ideal of \( gR \) for all \( \alpha \in fR(R) \cup \{ \beta \in P(U) \mid \beta \subseteq fR(0) \} \).

**Proposition 5.3.** Let \( gR \in S_R(U) \) and \( fR \) is a soft subideal of \( gR \). Then

1. \( fR^* \) is an ideal of \( gR^* \),
2. Let for all \( \alpha, \beta \in P(U) \), if \( \alpha \neq 0 \neq \beta \), then \( \alpha \cap \beta \neq 0 \). Then \( fR \) is an ideal of \( gR^* \).

**Proof.** (1) Let \( x, y \in fR^* \) and then \( fR(x) = fR(0) \) and \( fR(y) = fR(0) \). Now
\[
\begin{align*}
fR(x - y) &\supseteq fR(x) \cap fR(y) = fR(0) \cap fR(0) = fR(0) \supseteq fR(x - y),
\end{align*}
\]
and consequently \( fR(x - y) = fR(0) \). Thus, \( x - y \in fR^* \). Also if \( y \in fR^*, x \in gR^* \), then
\[
\begin{align*}
fR(xy) &\supseteq fR(y) \cap gR(x) = fR(0) \cap gR(0) \supseteq fR(0) \cap fR(0) = fR(0) \supseteq fR(xy),
\end{align*}
\]
and so \( f_R(xy) = f_R(0) \). Hence, \( xy \in f_R^n \). Therefore, \( f_R^n \) is an ideal of \( g_R^* \).

(2) Suppose that \( x, y \in f_R^n \). Hence, \( f_R(x - y) \supseteq f_R(x) \cap f_R(y) \neq \emptyset \) and so \( f_R(x - y) \neq \emptyset \). On the other hand, \( x - y \in f_R^n \). Also if \( x \in g_R^* \) and \( y \in f_R^n \), then \( f_R(xy) \supseteq f_R(y) \cap g_R(x) \neq \emptyset \) and then \( f_R(xy) \neq \emptyset \). Hence, \( xy \in f_R^n \). Therefore, \( f_R^n \) will be an ideal of \( g_R^* \).

Proposition 5.4. If \( f_R \in S_I(U) \) and \( g_R \in S_R(U) \), then \( f_R \cap g_R \) is a soft subideal of \( g_R \).

Proof. Clearly, \( f_R \cap g_R \subseteq g_R \). Let \( x, y \in R \) then

\[
(f_R \cap g_R)(x - y) = f_R(x - y) \cap g_R(x - y) \\
\supseteq f_R(x) \cap f_R(y) \cap g_R(x) \cap g_R(y) \\
= (f_R \cap g_R)(x) \cap (f_R \cap g_R)(y).
\]

Also

\[
(f_R \cap g_R)(xy) = f_R(xy) \cap g_R(xy) \supseteq f_R(x) \cap f_R(y) \cap g_R(x) \cap g_R(y) \\
= (f_R \cap g_R)(x) \cap (f_R \cap g_R)(y).
\]

Hence, \( f_R \cap g_R \) is a soft subideal of \( g_R \).

Proposition 5.5. Let \( h_R \in S_R(U) \). If \( f_R \) and \( g_R \) be two soft subideals of \( h_R \), then \( f_R \cap g_R \) is also a soft subideal of \( h_R \).

Proof. Obviously, \( f_R \cap g_R \subseteq h_R \). Let \( x, y \in R \). Then

\[
(f_R \cap g_R)(x - y) = f_R(x - y) \cap g_R(x - y) \\
\supseteq f_R(x) \cap f_R(y) \cap g_R(x) \cap g_R(y) = (f_R \cap g_R)(x) \cap (f_R \cap g_R)(y).
\]

Also

\[
(f_R \cap g_R)(xy) = f_R(xy) \cap g_R(xy) \supseteq f_R(y) \cap h_R(x) \cap g_R(y) \cap h_R(x) \\
= (f_R \cap g_R)(y) \cap h_R(x).
\]

Therefore, \( f_R \cap g_R \) is a soft subideal of \( h_R \).

Definition 5.6. ([3]) Let \( \varphi \) be a function from \( A \) into \( B \) and \( f_A, f_B \in S(U) \). Then soft image \( \varphi(f_A) \) of \( f_A \) under \( \varphi \) is defined by

\[
\varphi(f_A)(y) = \begin{cases} 
\cup \{ f_A(x) \mid x \in A, \varphi(x) = y \} & \text{if } \varphi^{-1}(y) \neq \emptyset \\
\emptyset & \text{if } \varphi^{-1}(y) = \emptyset
\end{cases}
\]

and soft pre-image (or soft inverse image) of \( f_B \) under \( \varphi \) is \( \varphi^{-1}(f_B)(x) = f_B(\varphi(x)) \) for all \( x \in A \).

Proposition 5.7. Let \( R \) and \( S \) be rings and \( f_R \in S_I(U), f_S \in S_I(U) \). Let \( \varphi : R \to S \) be a ring homomorphism and \( 0_R, 0_S \) denote the zero elements of \( R \) and \( S \) respectively. Then

(1) \( \varphi(f_R)(0_S) = f_R(0_R) \).

(2) \( \varphi(f_R^0) \subseteq (\varphi(f_R))^* \).

(3) if \( f_R \) be constant on \( \ker \varphi \), then \( \varphi(f_R)(\varphi(x)) = f_R(x) \) for all \( x \in R \).

(4) if \( \varphi \) is onto, then \( \varphi(f_R) \in S_I(U) \). Moreover if \( f_R \) is constant on \( \ker \varphi \), then \( \varphi(f_R^0) = (\varphi(f_R))^* \).

(5) \( \varphi^{-1}(f_S) \in S_I(U) \) and also \( \varphi^{-1}(f_S) \) is constant on \( \ker \varphi \).

(6) \( \varphi^{-1}(f_S^0) \subseteq (\varphi^{-1}(f_S))^* \).

(7) if \( \varphi \) is onto, then \( \varphi(\varphi^{-1}(f_S)) = f_S \).

(8) if \( f_R \) is constant on \( \ker \varphi \), then \( \varphi^{-1} \circ \varphi(f_R) = f_R \).
Proof. (1) \( \varphi(f_R)(0_S) = \cup\{f_R(0_R) \mid \varphi(0_R) = 0_S\} = f_R(0_R) \).
(2) Let \( y \in \varphi(f_R) \), then \( y = \varphi(x) \) and \( f_R(x) = f_R(0_R) \). Now
\[
\varphi(f_R)(y) = \cup\{f_R(x) \mid y = \varphi(x)\} = \cup\{f_R(0_R) \mid y = \varphi(x)\} = f_R(0_R) = \varphi(f_R)(0_S).
\]
Then \( y \in (\varphi(f_R))^* \).
(3)
\[
(\varphi(f_R))(\varphi(x_1)) = \cup\{f_R(x_2) \mid x_2 \in R, \varphi(x_1) = \varphi(x_2)\}
= \cup\{f_R(x_2) \mid x_2 \in R, \varphi(x_1 - x_2) = 0_S\}
= \cup\{f_R(x_2) \mid x_2 \in R, x_1 - x_2 \in \ker \varphi\}
= \cup\{f_R(x_2) \mid x_2 \in R, f_R(x_1 - x_2) = f_R(0_R)\}
= \cup\{f_R(x_2) \mid x_2 \in R, f_R(x_1) = f_R(x_2)\}
= f_R(x_1)
\]
for all \( x_1 \in R \).
(4) Let \( y_1, y_2 \in S \) such that \( y_1 = \varphi(x_1), y_2 = \varphi(x_2) \) for some \( x_1, x_2 \in R \). Then
\[
\varphi(f_R)(y_1 - y_2) = \cup\{f_R(z) \mid z \in R, f_R(z) = y_1 - y_2\}
\supseteq \cup\{f_R(x_1 - x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\}
\supseteq \cup\{f_R(x_1) \cap f_R(x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\}
= (\cup\{f_R(x_1) \mid x_1 \in R, y_1 = \varphi(x_1)\}) \cap (\cup\{f_R(x_2) \mid x_2 \in R, y_2 = \varphi(x_2)\})
= \varphi(f_R)(y_1) \cap \varphi(f_R)(y_2).
\]
Also
\[
\varphi(f_R)(y_1 y_2) = \cup\{f_R(z) \mid z \in R, f_R(z) = y_1 y_2\}
\supseteq \cup\{f_R(x_1 x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\}
\supseteq \cup\{f_R(x_1) \cup f_R(x_2) \mid x_1, x_2 \in R, y_1 = \varphi(x_1), y_2 = \varphi(x_2)\}
= (\cup\{f_R(x_1) \mid x_1 \in R, y_1 = \varphi(x_1)\}) \cup (\cup\{f_R(x_2) \mid x_2 \in R, y_2 = \varphi(x_2)\})
= \varphi(f_R)(y_1) \cup \varphi(f_R)(y_2).
\]
Thus, \( \varphi(f_R) \in S_I(U) \). From (2) we have that \( \varphi(f_R^*) \subseteq (\varphi(f_R))^* \) and we must prove \( (\varphi(f_R))^* \supseteq \varphi(f_R^*) \). Let \( y \in S \) such that \( y = \varphi(x) \) for some \( x \in R \). If \( y \in (\varphi(f_R))^* \), then \( \varphi(f_R)(y) = \varphi(f_R^*)(0_S) = f_R(0_R) \) (from (1)). Now from (3) \( (\varphi(f_R))(\varphi(x)) = f_R(x) = f_R(0_R) \) and so \( x \in f_R^* \) and then \( y = \varphi(x) \in \varphi(f_R)^* \).
(5) Let \( x_1, x_2 \in R \). Then
\[
\varphi^{-1}(f_S)(x_1 - x_2) = f_S(\varphi(x_1) - \varphi(x_2)) = f_S(\varphi(x_1) - \varphi(x_2))
\supseteq f_S(\varphi(x_1)) \cap f_S(\varphi(x_2)) = \varphi^{-1}(f_S)(x_1) \cap \varphi^{-1}(f_S)(x_2).
\]
Also
\[
\varphi^{-1}(f_S)(x_1 x_2) = f_S(\varphi(x_1 x_2)) = f_S(\varphi(x_1)\varphi(x_2))
\supseteq f_S(\varphi(x_1)) \cup f_S(\varphi(x_2)) = \varphi^{-1}(f_S)(x_1) \cup \varphi^{-1}(f_S)(x_2).
\]
Hence, \( \varphi^{-1}(f_S) \in S_I(U) \). If \( x \in \ker \varphi \), then
\[
\varphi^{-1}(f_S)(x) = f_S(\varphi(x)) = f_S(\varphi(0_R)) = f_S(0_S),
\]
Proof. (1) Let \( \varphi \) be onto. If \( f_R \in S_R(U) \), then \( \varphi(f_R) \in S_S(U) \). Moreover if \( g_R \in S_R(U) \) and \( f_R \) is a subideal of \( g_R \), then \( \varphi(f_R) \) is a soft subideal of \( \varphi(g_R) \).

(2) If \( f_S \in S_S(U) \), then \( \varphi^{-1}(f_S) \in S_R(U) \). Moreover if \( g_S \in S_S(U) \) and \( f_S \) is a soft subideal of \( g_S \), then \( \varphi^{-1}(f_S) \) is a soft subideal of \( \varphi^{-1}(g_S) \).

Proposition 5.8. Let \( R \) and \( S \) be two rings and \( \varphi : R \to S \) be a ring homomorphism.

(1) Let \( \varphi \) be onto. If \( f_R \in S_R(U) \), then \( \varphi(f_R) \in S_S(U) \). Moreover if \( g_R \in S_R(U) \) and \( f_R \) is a soft subideal of \( g_R \), then \( \varphi(f_R) \) is a soft subideal of \( \varphi(g_R) \).

(2) If \( f_S \in S_S(U) \), then \( \varphi^{-1}(f_S) \in S_R(U) \). Moreover if \( g_S \in S_S(U) \) and \( f_S \) is a soft subideal of \( g_S \), then \( \varphi^{-1}(f_S) \) is a soft subideal of \( \varphi^{-1}(g_S) \).

Proof. (1) Let \( y_1, y_2 \in S \) such that \( y_1 = \varphi(x_1), y_2 = \varphi(x_2) \) for some \( x_1, x_2 \in R \). Then

\[
\varphi(f_R)(y_1 - y_2) = \varphi(f_R)(y_1, y_2) = \varphi(f_R)(x_1 - x_2) = \varphi(f_R)(x_1, x_2) = \varphi(f_R)(x_1) \cap \varphi(f_R)(x_2) \cap (\varphi(f_R)(x_1) \cap \varphi(f_R)(x_2))
\]

Also

\[
\varphi(f_R)(y_1y_2) = \varphi(f_R)(y_1, y_2) = \varphi(f_R)(x_1, x_2) = \varphi(f_R)(x_1) \cap \varphi(f_R)(x_2) \cap (\varphi(f_R)(x_1) \cap \varphi(f_R)(x_2))
\]

Thus, \( \varphi(f_R) \in S_S(U) \). Now from

\[
\varphi(f_R)(y_1y_2) = \varphi(f_R)(y_1, y_2) = \varphi(f_R)(x_1, x_2) = \varphi(f_R)(x_1) \cap \varphi(f_R)(x_2) \cap (\varphi(f_R)(x_1) \cap \varphi(f_R)(x_2))
\]

we obtain that \( \varphi(f_R) \) is a soft subideal of \( \varphi(g_R) \).

(2) Let \( x_1, x_2 \in R \). Then

\[
\varphi^{-1}(f_S)(x_1 - x_2) = f_S(\varphi(x_1 - x_2)) = f_S(\varphi(x_1) - \varphi(x_2)) \supseteq f_S(\varphi(x_1)) \cap f_S(\varphi(x_2)) = \varphi^{-1}(f_S)(x_1) \cap \varphi^{-1}(f_S)(x_2).
\]
Also
\[ \varphi^{-1}(f_S(x_1x_2)) = f_S(\varphi(x_1x_2)) = f_S(\varphi(x_1)\varphi(x_2)) \]
\[ \supseteq f_S(\varphi(x_1)) \cap f_S(\varphi(x_2)) = \varphi^{-1}(f_S(x_1)) \cap \varphi^{-1}(f_S(x_2)). \]

Therefore, \( \varphi^{-1}(f_S) \in S_R(U) \). Moreover
\[ \varphi^{-1}(f_S(x_1x_2)) = f_S(\varphi(x_1x_2)) = f_S(\varphi(x_1)\varphi(x_2)) \]
\[ \supseteq f_S(\varphi(x_1)) \cap g_S(\varphi(x_2)) = \varphi^{-1}(f_S(x_1)) \cap \varphi^{-1}(g_S(x_2)). \]

and then \( \varphi^{-1}(f_S) \) is a soft subideal of \( \varphi^{-1}(g_S) \).

References


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