1 Informal Summary

1. We start with a propositional language $\mathcal{L}^-$ consisting of the following symbols:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Notation</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>propositional variables</td>
<td>$p_1, p_2, \ldots$</td>
<td>$\langle \rangle$</td>
</tr>
<tr>
<td>plural variables</td>
<td>$pp_1, pp_2, \ldots$</td>
<td>$\langle \langle \rangle$</td>
</tr>
<tr>
<td>operator variables</td>
<td>$O_1, O_2, \ldots$</td>
<td>$\langle \langle \rangle$</td>
</tr>
<tr>
<td>identity symbol</td>
<td>$=$</td>
<td>$\langle \langle \rangle, \langle \rangle \rangle$</td>
</tr>
<tr>
<td>inclusion symbol</td>
<td>$\prec$</td>
<td>$\langle \langle \rangle, \langle \langle \rangle \rangle \rangle$</td>
</tr>
<tr>
<td>existential quantifier</td>
<td>$\exists$</td>
<td>$\langle \langle \rangle \rangle$</td>
</tr>
<tr>
<td>negation symbol</td>
<td>$\neg$</td>
<td>$\langle \langle \rangle \rangle$</td>
</tr>
<tr>
<td>conjunction symbol</td>
<td>$\land$</td>
<td>$\langle \langle \rangle, \langle \rangle \rangle$</td>
</tr>
<tr>
<td>parentheses</td>
<td>$(),$</td>
<td>-</td>
</tr>
</tbody>
</table>

We also introduce some abbreviations:
2. We enrich $\mathcal{L}^-$ to a language $\mathcal{L}$, by adding the following symbols:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Notation</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>condition constants</td>
<td>$Q_1, \ldots, Q_r \langle\langle\rangle\rangle$</td>
<td></td>
</tr>
<tr>
<td>resolution increase</td>
<td>$\uparrow$</td>
<td>$\langle\langle\rangle\rangle$</td>
</tr>
<tr>
<td>resolution decrease</td>
<td>$\downarrow$</td>
<td>$\langle\langle\rangle\rangle$</td>
</tr>
</tbody>
</table>

The condition constants are used to express “procedures”. Intuitively, a procedure $Q$ might be used to characterize an operator $O_{\epsilon}$, relative to a space of propositions.

I’ll say more about the arrows below.

3. We work with a hierarchy of sets of “worlds”, of increasing levels of resolution:

- For $W$ a non-empty set,
  - $W^0 = W$
  - $P^n_W = \mathcal{O}(W^n)$
  - $W^{n+1} = \{ \langle w, e^n_1, \ldots, e^n_r \rangle : w \in W \land e^n_i \subseteq P^n_W \}^1$

4. This allows us to define “superworlds”:

- A **superworld** is a sequence $\langle w^0, w^1, w^2, \ldots \rangle$ such that:
  - $w^k \in W^k_W$
  - each $w^k$ is “refined” by $w^{k+1}$. $^2$

---

$^1$Intuitively, $e^n_i$ is the extension of $Q_i$ at world $\langle w, e^n_1, \ldots, e^n_r \rangle \in W^{n+1}$.

$^2$Intuitively, $w^k$ is refined by $w^{k+1}$ when it agrees about the extension of each $Q_i$ as far as propositions in $P^n_W$ are concerned.
• Superworlds are assessed at a given level of resolution.
  - A superworld \( \langle w^0, w^1, w^2, \ldots \rangle \) assessed at resolution level \( k \) behaves like \( w^k \).

5. The arrows, ↑ and ↓

• ↑ increases by 1 the level of resolution with respect to which superworlds are assessed.
• ↓ decreases by 1 the level of resolution with respect to which superworlds are assessed. \(^3\)

6. The result is a well-behaved system:

• One gets standard axioms, when attention is restricted to \( \mathcal{L} \)
• One gets sensible axioms for the general case, including a nice comprehension principle.

7. One gets a system that does not encourage lapsing into nonsense

• If logical space is genuinely open-ended, talking about “all possible refinements” is problematic. (For example, it can lead to revenge issues.) But having ↑ and ↓ instead of ♦ allows us to stay well within the range of sense.

2 The language

Definition 1 \( \mathcal{L} \) is a language built from the following symbols:

• the propositional variables \( p_1, p_2, \ldots \), which are of type \( \langle \rangle \);
• the plural propositional variables \( pp_1, pp_2, \ldots \), which are of type \( \langle \rangle \langle \rangle \);
• the propositional identity symbol, =, which is of type \( \langle \rangle , \langle \rangle \rangle \);
• the propositional inclusion relation \( \prec \), which is of type \( \langle \rangle , \langle \rangle \langle \rangle \rangle \)
• the operator variables \( O_1, O_2, \ldots \), which are of type \( \langle \langle \rangle \rangle \);

\(^3\)Unless it is already 0, in which case ↓ does nothing.
• for $r > 0$, the indefinitely extensible constants $Q_1, \ldots, Q_r$, which are of type $\langle\langle\rangle\rangle$;

• the existential quantifier, $\exists$, which binds variables of any type;

• the negation symbol, $\neg$, the conjunction symbol, $\land$, and parentheses;

• the refinement operator, $\uparrow$, and unrefinement operator, $\downarrow$, which are of type $\langle\langle\rangle\rangle$.

Definition 2 The expressions “$\bot$”, “$\top$”, “$\forall$”, “$\lor$”, “$\rightarrow$”, and “$\leftrightarrow$” are defined, in the usual way. In addition:

- $\diamond \phi := \bot \neq \phi \quad \Box \phi := \phi = \top$

- $\phi \gg \psi := (\phi = (\phi \land \psi))$

Definition 3 The formulas of $\mathcal{L}$ are defined recursively, in the obvious way. A sentence is a formula in which every occurrence of a variable is bound by a quantifier.

3 Some Results

Here are some results, which presuppose that attention is restricted to “natural” models:

For

- $E^+ := \exists p(Op \land p)$
- $E^- := \exists p(Op \land \neg p)$

Prior $\models O(E^-) \rightarrow (E^+ \land E^-)$

Extensional Prior $\models \forall p[(p \leftrightarrow E^-) \rightarrow (Op \rightarrow (E^+ \land E^-))]$

An immediate consequence of Prior is:

Modal Prior $\models \neg \forall p \diamond \forall q(Oq \leftrightarrow (q = p))$

But we can also show:
Modal Prior Next: $\models \forall p \uparrow \Diamond q (Q_i q \leftrightarrow (q = p))$

or, equivalently:

Modal Prior Next: $\models \forall p \Diamond q (\uparrow Q_i q \leftrightarrow (q = p))$

There are obvious generalizations of Modal Prior and Modal Prior Next:

Kaplan $\models \neg \forall pp \Diamond \forall q (Oq \leftrightarrow (q \prec pp))$

Kaplan Next $\models \forall pp \uparrow \Diamond q (Q_i q \leftrightarrow q \prec pp)$

or, equivalently:

Kaplan Next: $\models \forall pp \Diamond q (\uparrow Q_i q \leftrightarrow q \prec pp)$

The intensional case yields different results. With no need to restrict to natural models, we have:

$\models \uparrow \exists O \Box \exists p (\uparrow Q_i p \not\leftrightarrow Op)$

and therefore

$\not\models \forall O \Diamond \forall p (\uparrow Q_i p \leftrightarrow Op)$

Regarding Russell-Myhill, we have:

Russell-Myhill $\models \exists O \exists P (Op = Pp \land \forall q (Oq \leftrightarrow Pq))$

But also:

Russell-Myhill Next Whenever $Q_i$ and $Q_j$ are independent, $\models \uparrow (Q_i p \neq Q_j p)$

Here is an outline of the behavior of $\uparrow$ and $\downarrow$:

- $\models (\neg \uparrow \phi) \leftrightarrow (\uparrow \neg \phi)$
- $\models (\Diamond \uparrow \phi) \leftrightarrow (\uparrow \Diamond \phi)$
- $\models (\uparrow \phi \land \uparrow \psi) \leftrightarrow \uparrow (\phi \land \psi)$
- $\models (\uparrow \phi = \uparrow \psi) \leftrightarrow \uparrow (\phi = \psi)$
• $\models \uparrow(p) \leftrightarrow p$
• $\models \uparrow(p \prec pp) \leftrightarrow p \prec pp$
• $\models \uparrow(Op) \leftrightarrow Op$
• $\models (\uparrow\downarrow\phi) \leftrightarrow (\uparrow\downarrow\phi)$
• $\models \phi \Rightarrow \models \uparrow\phi$

**Existential Generalization** Let $\psi$ be free for $p$ in $\phi$. For $k = v_0(\psi),$\(^4\)

$$\models \phi[\psi/p] \rightarrow \uparrow^k \exists p \downarrow^k \phi$$

**Comprehension** Let $k = v_0(\phi)$ and let $p$ be a variable not occurring free in $\phi$. Then:

$$\models \uparrow^k \exists p(p = \downarrow^k \phi)$$

### 4 Frames

We use a non-empty set of “worlds” $W$ to characterize a hierarchy with one level for each natural number. At level $n$, we introduce a set of $n$-level worlds ($W^n$), a set of $n$-level propositions ($P^n$), a set of $n$-level “extensions” ($E^n$), and a set of $n$-level intensions ($I^n$). (An $n$-level proposition is a set of $n$-level worlds; an $n$-level extension is a set of $n$-level propositions; an $n$-level intension is a function from $n$-level propositions to $n$-level propositions.) The $0$-level worlds are just the members of $W$. An $(n + 1)$-level world $w^{n+1}$ is a sequence consisting of a $0$-level world and an $n$-level extension for each indefinitely extensible constant $Q_1, \ldots, Q_r$. Formally,

**Definition 4 (Worlds, propositions, extensions, intensions)**

For $W$ a non-empty set,

\[
\begin{align*}
\uparrow^k & := \uparrow \ldots \uparrow \\
\downarrow^k & := \downarrow \ldots \downarrow
\end{align*}
\]

The **valence** of $\psi$, $v^0(\psi)$, is a syntactically characterized upper bound on the resolution that is needed to describe the proposition expressed by $\psi$, when evaluated externally at resolution 0 (assuming a variable assignment of level 0).
In some applications we may not want to count some worlds in $W^\infty$ as “inadmissible”, on metaphysical grounds. We therefore introduce the following additional definitions:

**Definition 5 (Frames)** A frame is a pair $(W, \mathcal{A})$, where $W$ is a non-empty set and $\mathcal{A} \subseteq W^\infty$.

**Definition 6 (Admissible Worlds)** For $(W, \mathcal{A})$ a frame, we let:

- $W^0 = W$
- $P^n_W = \wp(W^n)$
- $E^n_W = \wp(P^n_W)$
- $I^n_W = \{ f : P^n_W \to P^n_W \}$
- $W^{n+1} = \{ \langle w, e^n_1, \ldots, e^n_r \rangle : w \in W \land e^n_i \in E^n_W \}$
- $W^\infty = \bigcup_{n \in \mathbb{N}} W^n$

**Definition 7 (Refinements)** Fix a frame $(W, \mathcal{A})$. Intuitively speaking,

- For $w^n \in W^n_{\mathcal{A}}$ and $w^{n+1} \in W^{n+1}_{\mathcal{A}}$, $w^n \triangleright_{\mathcal{A}} w^{n+1}$ states that world $w^n$ is “refined” by world $w^{n+1}$, relative to $(W, \mathcal{A})$.

- For $p^n \in P^n_{W_{\mathcal{A}}}$ and $p^{n+1} \in P^n_{W_{\mathcal{A}}}$, the $(n+1)$-level proposition $[p^n]^{n+1}_{W_{\mathcal{A}}}$ is the set of worlds in $W^{n+1}_{\mathcal{A}}$ that are “refinements” of some world in $p^n$.

Formally:

- $w^0 \triangleright_{\mathcal{A}} w^1 := \exists e^0_1 \ldots e^0_r \in E^0_{W_{\mathcal{A}}}(w^1 = \langle w^0, e^0_1, \ldots, e^0_r \rangle)$
\[ [p^n]_{W_A}^{n+1} := \{ w^{n+1} \in W_A^{n+1} : \exists w^n \in P^n (w^n \rhd_{W_A} w^{n+1}) \} \]

\[ w^{n+1} \rhd_{W_A} w^{n+2} := \exists w \in W \ \exists e_1 \ldots e_r \in E_{W_A}^{n} \ \exists e_1^{n+1} \ldots e_r^{n+1} \in E_{W_A}^{n+1} \]
\[ (w^{n+1} = \langle w, e^n_1, \ldots, e^n_r \rangle) \land w^{n+1} = \langle w, e^{n+1}_1, \ldots, e^{n+1}_r \rangle \land \]
\[ \forall p^n (p^n \in e^n_1 \leftrightarrow [p^n]_{W_A}^{n+1} \in e^{n+1}_1) \land \]
\[ \vdots \]
\[ \forall p^n (p^n \in e^n_r \leftrightarrow [p^n]_{W_A}^{n+1} \in e^{n+1}_r) \]

**Definition 8 (Admissible Frames)** A frame \( \langle W, A \rangle \) is admissible iff for any \( n \in \mathbb{N} \) and \( w^n \in W_A^n \):

- if \( n > 0 \), \( w^n \) refines some world in \( W_A^{n-1} \)
  (i.e. there is some \( w^{n-1} \in W_A^{n-1} \) is such that \( w^{n-1} \rhd_{W_A} w^n \));
- \( w^n \) is refined by some world in \( W_A^{n+1} \)
  (i.e. there is some \( w^{n+1} \in W_A^{n+1} \) is such that \( w^n \rhd_{W_A} w^{n+1} \));

**Proposition 1 (There are admissible frames)** The frame \( \langle W, A \rangle \) is admissible whenever \( A = W^\infty \).

**Proof** For \( n \in \mathbb{N} \), let \( w^n \in W_A^n \). We need to verify two claims:

- if \( n > 0 \), then \( w^n \) refines some world in \( W_A^{n-1} \)
  Since \( n > 0 \), we can let \( w^n = \langle w, e^n_1, \ldots, e^n_r \rangle \). If \( n = 1 \), the result is trivial, since we can let \( w^n = w \). So we may suppose that \( n > 1 \). For each \( i \leq r \) let

  \[ e^{n-2}_i = \{ p^{n-2} \in P_{W_A}^{n-2} : [p^{n-2}]_{W_A}^{n-1} \in e^{n-1}_i \} \]

  Let \( w^{n-1} = \langle w, e^{n-2}_1, \ldots, e^{n-2}_r \rangle \). Since \( A = W^\infty \), \( w^{n-1} \in W_A^{n-1} \).
  In addition, since \( A = W^\infty \), \( P^n_W = P^n_{W_A} \). So:

  \[ \forall p^{n-2} (p^{n-2} \in e^{n-2}_i \leftrightarrow [p^{n-2}]_{W_A}^{n-1} \in e^{n-1}_i) \]

  So it follows from the definition of \( \rhd_{W_A} \) that \( w^{n-1} \rhd_{W_A} w^n \).
• $w^n$ is refined by some world in $W_A^{n+1}$

Suppose, first, that $n = 0$, and let $w^{n+1} = \langle w^n, 0, \ldots, 0 \rangle$. Since $\mathcal{A} = W^\infty$, $w^{n+1} \in W_A^{n+1}$. And it follows immediately from the definition of $\triangleright_{W_A}$ that $w^n \triangleright_{W_A} w^{n+1}$.

Now suppose that $n > 0$ and let $w^n = \langle w, e_1^{n-1}, \ldots, e_r^{n-1} \rangle$. For each $i \leq r$ let

$$e_i^n = \{ [p^{n-1}]^{n} \in P^n_{W_A} : p^{n-1}_W \in e_i^{n-1} \}$$

Let $w^{n+1} = \langle w, e_1^n, \ldots, e_r^n \rangle$. Since $\mathcal{A} = W^\infty$, $w^{n+1} \in W_A^{n+1}$.

In addition, since $\mathcal{A} = W^\infty$, $P^n_W = P^n_{W_A}$. So:

$$\forall p^{n-1} (p^{n-1} \in e_i^{n-1} \iff [p^{n-1}]_{W_A} \in e_i^n)$$

So it follows from the definition of $\triangleright_{W_A}$ that $w^n \triangleright_{W_A} w^{n+1}$.

**Proposition 2 (Injectivity of Refinement)** Fix a frame $\langle W, \mathcal{A} \rangle$. For $v^n, w^n \in W^n_A$ and $w^{n+1} \in W_A^{n+1}$,

$$v^n \triangleright_{W_A} w^{n+1} \land w^n \triangleright_{W_A} w^{n+1} \rightarrow v^n = w^n$$

**Proof** Since the result is trivial if $n = 0$, we assume $n > 0$. Let $w^{n+1} = \langle w, e_1^n, \ldots, e_r^n \rangle$. Since $w^n \triangleright_{W_A} w^{n+1}$, $w^n$ must be $\langle w, e_1^{n-1}, \ldots, e_r^{n-1} \rangle$ for some $e_1^{n-1}, \ldots, e_r^{n-1}$. Since $v^n \triangleright_{W_A} w^{n+1}$, $v^n$ must be $\langle w, f_1^{n-1}, \ldots, f_r^{n-1} \rangle$ for some $f_1^{n-1}, \ldots, f_r^{n-1}$.

Suppose, for reductio, that $v^n \neq w^n$. Then it must be the case that $e_i^{n-1} \neq f_i^{n-1}$ for $i \leq r$. We may assume with no loss of generality that for some $p^{n-1} \in P^{n-1}_{W_A}$, $p^{n-1} \in e_i^{n-1}$ but $p^{n-1} \notin f_i^{n-1}$. Since $w^n \triangleright_{W_A} w^{n+1}$ and $p^{n-1} \in e_i^{n-1}$, it follows from the definition of $\triangleright_{W_A}$ that $[p^{n-1}]_W \in e_i^n$. But since $v^n \triangleright_{W_A} w^{n+1}$ and $p^{n-1} \notin f_i^{n-1}$, it follows from the definition of $\triangleright_{W_A}$ that $[p^{n-1}]_W \notin e_i^n$, which contradicts an earlier assertion.

## 5 Superworlds

**Definition 9 (Superworlds)** Fix a frame $\langle W, \mathcal{A} \rangle$. A superworld $\vec{w}$ of $\langle W, \mathcal{A} \rangle$ is an infinite sequence $\langle w^0, w^1, w^2, \ldots \rangle$ ($w^n \in W^n_A$) such that:

$$w^0 \triangleright_{W_A} w^1 \triangleright_{W_A} w^2 \triangleright_{W_A} \ldots$$

Some additional notation:
• $W_A$ is the set of superworlds of $\langle W, A \rangle$.

• For $\vec{w} \in W_A$, $\vec{w}(n)$ is the $n$th member of $\vec{w}$.

**Proposition 3 (Every world is part of a superworld)** Fix an admissible frame $\langle W, A \rangle$. For any $w^n \in W_A$, there is some $\vec{w} \in W_A$ such that $\vec{w}(n) = w^n$.

**Proof** Since $\langle W, A \rangle$ is admissible, there must be a sequence $\langle v^0, \ldots, v^{n-1}, w^n, v^{n+1}, v^{n+2}, \ldots \rangle$ such that

$v^0 \triangleright_W v^{n-1} \triangleright_W w^n \triangleright_W v^{n+1} \triangleright_W v^{n+2} \triangleright_W \ldots$

**Proposition 4 (No backwards divergence for superworlds)** For $\vec{w}, \vec{v} \in W_A$ and $n, k \in \mathbb{N}$, $\vec{v}(n+k) = \vec{w}(n+k)$ entails $\vec{v}(n) = \vec{w}(n)$.

**Proof** Assume $\vec{v}(n) \neq \vec{w}(n)$. By proposition 2, $\vec{v}(n+1) \neq \vec{w}(n+1)$. Again by proposition 2, $\vec{v}(n+2) \neq \vec{w}(n+2)$. After $k$ iterations of this procedure, we get $\vec{v}(n+k) \neq \vec{w}(n+k)$.

**Definition 10 (Superpropositions)**

• A superproposition $\vec{p}$ of $\langle W, A \rangle$ is a set of superworlds in $W_A$.

• $P_{W_A} = \{ \vec{p} : \vec{p} \subseteq W_A \}$.

• $P^n_{W_A} = \{ \vec{p} \in P_{W_A} : \vec{w}(n) = \vec{v}(n) \rightarrow (\vec{w} \in \vec{p} \leftrightarrow \vec{v} \in \vec{p}) \}$

• For $\vec{p} \in P_{W_A}$, we let $\vec{p}(n) = \{ \vec{w}(n) : \vec{w} \in \vec{p} \}$.

**Proposition 5 (Monotonicity of Superpropositions)** For $n \in \mathbb{N}$, $\vec{p} \in P^n_{W_A} \rightarrow \vec{p} \in P^{n+1}_{W_A}$.

**Proof** Assume $\vec{p} \in P^n_{W_A}$. We suppose $\vec{w}(n+1) = \vec{v}(n+1)$ and $\vec{w} \in \vec{p}$, and we show $\vec{v} \in \vec{p}$. By proposition 4, $\vec{w}(n+1) = \vec{v}(n+1)$ entails $\vec{w}(n) = \vec{v}(n)$. So $\vec{w} \in \vec{p}$ guarantees $\vec{v} \in \vec{p}$.

\[5\text{Note that } \{ \vec{w}(n) : \vec{w} \in \vec{p} \} = \{ v^n \in P^n_{W_A} : \exists \vec{v} \in \vec{p} \left( v^n = \vec{v}(n) \right) \}.\]
Proposition 6 (\(\vec{p}(n)\) is well-behaved, part 1) \(\text{If } \vec{p}, \vec{w} \in P^n_{WA}, \text{ then } \vec{w} \in \vec{p} \iff \vec{w}(n) \in \vec{p}(n).\) 

**Proof** Suppose, first, that \(\vec{w} \in \vec{p}.\) By definition, 
\[
\vec{p}(n) = \{ v^n \in P^n_{WA} : \exists \vec{v} \in \vec{p} : v^n = \vec{v}(n) \}. \]

Since \(\vec{w}\) is a true instance of the following existential:
\[
\exists \vec{v} \in \vec{p} : \vec{w}(n) = \vec{v}(n) \]
we have \(\vec{w}(n) \in \vec{p}(n).\)

Now suppose that \(\vec{w}(n) \in \vec{p}(n).\) By definition, 
\[
\vec{p}(n) = \{ v^n \in P^n_{WA} : \exists \vec{v} \in \vec{p} : v^n = \vec{v}(n) \}. \]
So the fact that \(\vec{w}(n) \in \vec{p}(n)\) entails that there must be some \(\vec{z} \in \vec{p}\) such that \(\vec{z}(n) = \vec{w}(n).\) But since \(\vec{p}, \vec{w} \in P^n_{WA}, \vec{z} \in \vec{p}\) and \(\vec{z}(n) = \vec{w}(n)\) entail that \(\vec{w} \in \vec{p}\).

Proposition 7 (\(\vec{p}(n)\) is well-behaved, part 2) If \(\vec{p}, \vec{q} \in P^n_{WA}, \text{ then } \vec{p}(n) = \vec{q}(n) \text{ entails } \vec{p} = \vec{q}.)

**Proof**
\[
\vec{w} \in \vec{p} \iff \vec{w}(n) \in \vec{p}(n) \quad \text{by proposition 6}
\]
\[
\iff \vec{w}(n) \in \vec{q}(n) \quad \text{since } \vec{p}(n) = \vec{q}(n)
\]
\[
\iff \vec{w} \in \vec{q} \quad \text{by proposition 6}
\]

Proposition 8 (\(\vec{p}(n)\) is well-behaved, part 3) Assume \(\vec{p} \in P^n_{WA}.\) Then:
\[
\vec{p}(n) = p^n \iff \vec{p} = \{ \vec{w} \in WA : \vec{w}(n) \in p^n \}
\]

**Proof**

*Left to right:* We assume \(\vec{p}(n) = p^n,\) and therefore
\[
\{ w^n : \exists \vec{w} \in \vec{p} : w^n = \vec{w}(n) \} = p^n
\]

To verify \(\vec{p} = \{ \vec{w} \in WA : \vec{w}(n) \in p^n \},\) it suffices to check each of the following:

- If \(\vec{v}(n) \in p^n,\) then \(\vec{v} \in \vec{p}\)

Suppose that \(\vec{v}(n) \in p^n.\) By our initial assumption, there is some \(\vec{w} \in \vec{p}\) such that:
\[
\vec{v}(n) = \vec{w}(n)
\]
But since \(\vec{p} \in P^n_{WA},\) this entails
\[
\vec{w} \in \vec{p} \iff \vec{v} \in \vec{p}
\]
which means that we have \(\vec{v} \in \vec{p},\) as desired.
• If $\vec{v}(n) \not\in p^n$, then $\vec{v} \not\in \vec{p}$.

Suppose that $\vec{v}(n) \not\in p^n$. By our initial assumption, every $\vec{w} \in \vec{p}$ is such that:

$$\vec{v}(n) \neq \vec{w}(n)$$

from which it follows that $\vec{v} \not\in \vec{p}$.

**Right to Left:** Assume $\vec{p} = \{\vec{w} \in W_A : \vec{w}(n) \in p^n\}$. By proposition 3:

$$\{\vec{w}(n) : \vec{w}(n) \in p^n\} = p^n$$
equivalently:

$$\{\vec{w}(n) : \vec{w} \in \{\vec{w} \in W_A : \vec{w}(n) \in p^n\}\} = p^n$$

So, by our assumption,

$$\{\vec{w}(n) : \vec{w} \in \vec{p}\} = p^n$$

which is what we want:

$$\vec{p}(n) = p^n$$

**Definition 11 (Superextensions)**

- A superextension $\vec{e}$ of $\langle W, A \rangle$ is a set of superpropositions of $\langle W, A \rangle$.
- $E_{W_A} = \{\vec{e} : \vec{e} \subseteq P_{W_A}\}$.
- $E^n_{W_A} = \{\vec{e} : \vec{e} \subseteq P^n_{W_A}\}$.

**Definition 12 (Superintensions)**

- A superintension $\vec{ı}$ of $\langle W, A \rangle$ is a function from superpropositions of $\langle W, A \rangle$ to superpropositions of $\langle W, A \rangle$.
- $I_{W_A} = \{\vec{ı} : \vec{ı}$ is a function from $P_{W_A}$ into $P_{W_A}\}$.
- $I_n^{n}_{W_A} = \{\vec{ı} \in I_{W_A} : \forall \vec{p} \in P_{W_A} \ (\vec{ı}(\vec{p}) \in P^n_{W_A})\}$.

**Proposition 9 (Monotonicity of Superintensions)** For $n \in \mathbb{N}$, $\vec{ı} \in I^n_{W_A} \rightarrow \vec{ı} \in I^{n+1}_{W_A}$.

**Proof** Let $\vec{ı} \in I^n_{W_A}$ and $\vec{p} \in P_{W_A}$. Since $\vec{ı} \in I^n_{W_A}$, $\vec{ı}(\vec{p}) \in P^n_{W_A}$. So proposition 5 entails that $\vec{ı}(\vec{p}) \in P^{n+1}_{W_A}$.

$$\vec{p}(n+1) = \vec{ı}(n+1)$$

and $\vec{w} \in \vec{p}$, and we show $\vec{v} \in \vec{p}$. By proposition 4,

$$\vec{w}(n+1) = \vec{ı}(n+1)$$

entails $\vec{w}(n) = \vec{ı}(n)$. So $\vec{w} \in \vec{p}$ guarantees $\vec{v} \in \vec{p}$.
6 Extensions for \( Q_i \)

**Definition 13 (Extension Predicate for \( Q_i \))** Fix a frame \( \langle W, A \rangle \). For \( \vec{w} \in W \) and \( n \in \mathbb{N} \), let \( \vec{w}(n+1) = \langle w, e^i_1, \ldots, e^i_r \rangle \). Then:

\[
[W^n A]_{\text{Ext}^n_{Q_i}}(\vec{w}) = \{ \vec{p} \in P^n_{W A} : \exists p^n \in e^i_n (\vec{p}(n) = p^n) \}
\]

**Proposition 10 (Monotonicity of Extensions)** For any \( \vec{w} \in W \) and \( \vec{p} \in P^n_{W A} \),

\[
\vec{p} \in [W^n A]_{\text{Ext}^n_{Q_i}}(\vec{w}) \implies \vec{p} \in [W^{n+1} A]_{\text{Ext}^{n+1}_{Q_i}}(\vec{w})
\]

**Proof** Let \( \vec{w}(n+1) = \langle w, e^i_1, \ldots, e^i_r \rangle \), \( \vec{w}(n+2) = \langle w, e^{i+1}_1, \ldots, e^{i+1}_r \rangle \). Let \( \vec{p} \in [W^n A]_{\text{Ext}^n_{Q_i}}(\vec{w}) \). We verify that \( \vec{p} \) is also in \( [W^{n+1} A]_{\text{Ext}^{n+1}_{Q_i}}(\vec{w}) \).

By the definition of \( [W^n A]_{\text{Ext}^n_{Q_i}}(\vec{w}) \), \( \vec{p} \in P^n_{W A} \) and there is some \( p^n \in e^i_n \) such that \( \vec{p}(n) = p^n \). Since \( \vec{p} \) is in \( P^n_{W A} \), it is also in \( P^{n+1}_w \). So, by the definition of \( [W^{n+1} A]_{\text{Ext}^{n+1}_{Q_i}} \), it suffices to verify each of the following two propositions:

- \( \vec{p}(n+1) = [p^n]^n_{W A} \)
  
  **Proof:** By definition,

\[
[p^n]^{n+1}_{W A} = \{ v^{n+1} : \exists w^n \in p^n (w^n \triangleright_{W A} v^{n+1}) \}
\]

which is equivalent to the following, by proposition 3

\[
[p^n]^{n+1}_{W A} = \{ \vec{v}(n+1) : \vec{v} \in W_A \land \exists w^n \in p^n (w^n \triangleright_{W A} \vec{v}(n+1)) \}
\]

which is equivalent to the following, by proposition 2,

\[
[p^n]^{n+1}_{W A} = \{ \vec{v}(n+1) : \vec{v}(n) \in p^n \}
\]

But we know that \( \vec{p}(n) = p^n \). So:

\[
[p^n]^{n+1}_{W A} = \{ \vec{v}(n+1) : \vec{v}(n) \in \vec{p}(n) \}
\]

which is equivalent to

\[
[p^n]^{n+1}_{W A} = \{ \vec{v}(n+1) : \vec{v}(n) \in \{ w(n) : \vec{w}(n) \in \vec{p} \} \}
\]

equivalently:

\[
[p^n]^{n+1}_{W A} = \{ \vec{v}(n+1) : \vec{v} \in \vec{p} \}
\]

which is what we want:

\[
[p^n]^{n+1}_{W A} = \vec{p}(n+1)
\]
• \( [p^n]^{n+1} \in e^{n+1} \)

Proof: Since \( \bar{w}(n+1) \triangleright_W \bar{w}(n+2) \), we know that \( p^n \in e^n \leftrightarrow [p^n]^{n+1} \in e_i^{n+1} \). So the result is immediate.

Proposition 11 (Conservativity of Extensions) For any \( \bar{w} \in W_A \) and \( \bar{p} \in P^n_{W_A} \ (n \in \mathbb{N}) \),

\[
\bar{p} \in [W^i_{Ext}^{n+1}] (\bar{w}) \rightarrow \bar{p} \in [W^i_{Ext}^n] (\bar{w})
\]

Proof Let \( \bar{w}(n+1) = \langle w, e_1^n, \ldots, e_r^n \rangle \), \( \bar{w}(n+2) = \langle w, e_1^{n+1}, \ldots, e_r^{n+1} \rangle \). Let \( \bar{p} \) be in both \( P^n_{W_A} \) and \( [W^i_{Ext}^{n+1}] (\bar{w}) \). We verify that \( \bar{p} \) is also in \( [W^i_{Ext}^n] (\bar{w}) \).

By the definition of \( [W^i_{Ext}^{n+1}] (\bar{w}) \), there is some \( p^{n+1} \in e_i^{n+1} \) such that \( \bar{p}(n+1) = p^{n+1} \). Let

\[
p^n = \{ w^n \in W^n_A : \exists w^{n+1} \in p^{n+1} (w^n \triangleright_W w^{n+1}) \}
\]

We have \( \bar{p} \in P^n_{W_A} \). So in order to show \( \bar{p} \in [W^i_{Ext}^n] (\bar{w}) \), it suffices to verify each of the following two propositions:

• \( \bar{p}(n) = p^n \)

Proof: By definition,

\[
p^n = \{ w^n \in W^n_A : \exists w^{n+1} \in p^{n+1} (w^n \triangleright_W w^{n+1}) \}
\]

Since \( \bar{p}(n+1) = p^{n+1} \),

\[
p^n = \{ w^n \in W^n_A : \exists w^{n+1} \in \bar{p}(n+1) (w^n \triangleright_W w^{n+1}) \}
\]

which is equivalent to:

\[
p^n = \{ w^n \in W^n_A : \exists w^{n+1} \in \{ \bar{w}(n+1) : \bar{w} \in \bar{p} \} (w^n \triangleright_W w^{n+1}) \}
\]

or, equivalently,

\[
p^n = \{ w^n \in W^n_A : \exists \bar{w} \in \bar{p} (w^n \triangleright_W \bar{w}(n+1)) \}
\]

which is equivalent to the following, by proposition 2,

\[
p^n = \{ w^n \in W^n_A : \exists \bar{w} \in \bar{p} (w^n = \bar{w}(n)) \}
\]
But, by proposition 3, this is equivalent to:

\[ p^n = \{ \vec{v}(n) : \vec{v} \in \mathcal{W}_A \land \exists \vec{w} \in \vec{p} \ (\vec{v}(n) = \vec{w}(n)) \} \]

But we are assuming that that \( \vec{p} \in P^n_{\mathcal{W}_A} \) and, therefore, that, for any \( \vec{w}, \vec{v} \in \mathcal{W}_A \),

\[ \vec{v}(n) = \vec{w}(n) \rightarrow (\vec{v} \in \vec{p} \leftrightarrow \vec{w} \in \vec{p}) \]

which allows us to conclude:

\[ p^n = \{ \vec{v}(n) : \vec{v} \in \vec{p} \} \]

which delivers the desired result:

\[ p^n = \vec{p}(n) \]

\( \bullet \) \( p^n \in e^n_i \)

\textit{Proof:} Since \( \vec{w}(n+1) \triangleright_{\mathcal{W}_A} \vec{w}(n+2) \), we know that \( p^n \in e^n_i \leftrightarrow [p^n]_{\mathcal{W}_A}^{n+1} \in e_i^{n+1} \). So it suffices to show that \( [p^n]_{\mathcal{W}_A}^{n+1} \in e_i^{n+1} \). By definition:

\[ [p^n]_{\mathcal{W}_A}^{n+1} = \{ v^{n+1} : \exists w^n \in p^n (w^n \triangleright_{\mathcal{W}_A} v^{n+1}) \} \]

So, bring in the definition of \( p^n \),

\[ [p^n]_{\mathcal{W}_A}^{n+1} = \{ v^{n+1} : \exists w^n \exists w^{n+1} (w^{n+1} \in p^{n+1} \land w^n \triangleright_{\mathcal{W}_A} w^{n+1} \land w^n \triangleright_{\mathcal{W}_A} v^{n+1}) \} \]

But, since \( \vec{p}(n+1) = p^{n+1} \), we have:

\[ [p^n]_{\mathcal{W}_A}^{n+1} = \{ v^{n+1} : \exists w^n \exists w^{n+1} (w^{n+1} \in \vec{p}(n+1) \land w^n \triangleright_{\mathcal{W}_A} w^{n+1} \land w^n \triangleright_{\mathcal{W}_A} v^{n+1}) \} \]

equivalently:

\[ [p^n]_{\mathcal{W}_A}^{n+1} = \{ v^{n+1} : \exists w^n \exists w^{n+1} (w^{n+1} \in \vec{p}(n+1) \land w^n \triangleright_{\mathcal{W}_A} w^{n+1} \land w^n \triangleright_{\mathcal{W}_A} v^{n+1}) \} \]

Simplifying:

\[ [p^n]_{\mathcal{W}_A}^{n+1} = \{ v^{n+1} : \exists w^n \exists \vec{w} \in \vec{p} (w^n \triangleright_{\mathcal{W}_A} \vec{w}(n+1) \land w^n \triangleright_{\mathcal{W}_A} v^{n+1}) \} \]

which by proposition 2 is equivalent to:

\[ [p^n]_{\mathcal{W}_A}^{n+1} = \{ v^{n+1} : \exists \vec{w} \in \vec{p} (\vec{w}(n) \triangleright_{\mathcal{W}_A} v^{n+1}) \} \]
so proposition 3 gives us

$$[p^n]_{W_A}^{n+1} = \{ \bar{v}(n+1) : \exists \bar{w} \in \bar{p} (\bar{w}(n) \triangleright_{W_A} \bar{v}(n+1)) \}$$

and again by proposition 2,

$$[p^n]_{W_A}^{n+1} = \{ \bar{v}(n+1) : \exists \bar{w} \in \bar{p} (\bar{w}(n) = \bar{v}(n)) \}$$

But we are assuming that that $\bar{p} \in P_{W_A}^n$ and, therefore, that, for any $\bar{w}, \bar{v} \in W_A$,

$$\bar{w}(n) = \bar{v}(n) \rightarrow (\bar{w} \in \bar{p} \iff \bar{v} \in \bar{p})$$

which allows us to conclude:

$$[p^n]_{W_A}^{n+1} = \{ \bar{v}(n+1) : \bar{v} \in \bar{p} \}$$

Or, equivalently,

$$[p^n]_{W_A}^{n+1} = \bar{p}(n+1)$$

which gives us the desired result, since we are assuming that $\bar{p}(n+1) = p^{n+1}$ and $p^{n+1} \in e^{n+1}$.

7 Models

Definition 14 A model is a quadruple $\langle W, A, \bar{\alpha}, k \rangle$, for $\langle W, A \rangle$ an admissible frame, $\bar{\alpha} \in W_A$, and $k \in \mathbb{N}$. (Intuitively, $\bar{\alpha}$ is the actual superworld and $k$ is a level of “resolution” with respect to which truth is to be assessed.)

Definition 15 A variable assignment for $\langle W, A, \bar{\alpha}, k \rangle$ is a function $\sigma$ such that:

- $\sigma(p_i) \in P_{W_A}$;
- $\sigma(pp_i) \subseteq P_{W_A}$ and $\sigma(pp_i) \neq \emptyset$;
- $\sigma(O_i) \in I_{W_A}$.

Definition 16 (Truth at a superworld) Fix a model $\langle W, A, \bar{\alpha}, k \rangle$. For $\phi$ a formula of $\mathcal{L}$, $\bar{w} \in W_A$, and $\sigma$ a variable assignment for $\langle W, A, \bar{\alpha}, k \rangle$, we define the truth of $\phi$ at $\bar{w}$ with respect to $\sigma$ at resolution $k$ (in symbols: $\bar{w} \models_\sigma^k \phi$) using the following recursive clauses:
• \( \bar{w} \models^k p_i \) iff \( \bar{w} \in \sigma(p_i) \);

• \( \bar{w} \models^k O_j p_i \) iff \( \bar{w} \in \sigma(O_j)(\sigma(p_i)) \);

• \( \bar{w} \models^k Q_j p_i \) iff \( \bar{w} \in \sigma(Q_i)(\sigma(p_i)) \);

• \( \bar{w} \models^k \phi = \psi \) iff \( \{ \bar{v} \in W_A : \bar{v} \models^k \phi \} = \{ \bar{v} \in W_A : \bar{v} \models^k \psi \} \);

• \( \bar{w} \models^k \neg \phi \) iff \( \bar{w} \not\models^k \phi \);

• \( \bar{w} \models^k (\phi \land \psi) \) iff \( \bar{w} \models^k \phi \) and \( \bar{w} \models^k \psi \);

• \( \bar{w} \models^k \exists p_i \phi \) iff for some \( \bar{q} \in P_{W_A}, \bar{w} \models^k_{\sigma[q/p_i]} \phi \);

• \( \bar{w} \models^k \exists pp_i \phi \) iff for some \( \bar{A} \subseteq P_{W_A}, \bar{A} \neq \emptyset \) and \( \bar{w} \models^k_{\sigma[\bar{A}/pp_i]} \phi \);

• \( \bar{w} \models^k \forall p_i \phi \) iff for any \( \bar{q} \in P_{W_A}, \bar{w} \models^k_{\sigma[q/p_i]} \phi \);

• \( \bar{w} \models^k \forall pp_i \phi \) iff for any \( \bar{A} \subseteq P_{W_A} \), \( \bar{w} \models^k_{\sigma[\bar{A}/pp_i]} \phi \);

*Proposition 12*

1. \( \bar{w} \models^k \diamond \phi \) iff \( \{ \bar{v} \in W_A : \bar{v} \models^k \phi \} \neq \emptyset \);
2. \( \bar{w} \models^k \Box \phi \) iff \( \{ \bar{v} \in W_A : \bar{v} \models^k \phi \} = W_A \);
3. \( \bar{w} \not\models^k \bot \);
4. \( \bar{w} \models^k (\phi \rightarrow \psi) \) iff: if \( \bar{w} \models^k \phi \), then \( \bar{w} \models^k \psi \);
5. \( \bar{w} \models^k (\phi \leftrightarrow \psi) \) iff: \( \bar{w} \models^k \phi \) iff \( \bar{w} \models^k \psi \);
6. \( \bar{w} \models^k \forall p_i \phi \) iff for any \( \bar{q} \in P_{W_A}, \bar{w} \models^k_{\sigma[q/p_i]} \phi \);
7. \( \bar{w} \models^k \forall pp_i \phi \) iff for any \( \bar{A} \subseteq P_{W_A} \), \( \bar{w} \models^k_{\sigma[\bar{A}/pp_i]} \phi \);
8. $\bar{w} \models^k_{\sigma} \forall \bar{i} \in I^k_{\bar{w}} \iff \bar{w} \models^k_{\sigma[I/O_i]} \phi$;

Proof

1. Recall that $\diamond \phi := \neg (\phi = \bot)$.

   $\bullet \bar{w} \models^k_{\sigma} \phi = \bot \iff \{ \bar{w} : \bar{w} \models^k_{\sigma} \phi \} = \emptyset$

   $\bullet \bar{w} \models^k_{\sigma} \neg (\phi = \bot) \iff \{ \bar{w} : \bar{w} \models^k_{\sigma} \phi \} \neq \emptyset$

2. Recall that $\Box \phi := (\phi = \top)$.

   $\bullet \bar{w} \models^k_{\sigma} \phi = \top \iff \{ \bar{w} : \bar{w} \models^k_{\sigma} \phi \} = \{ \bar{w} \in \bar{w} \models^k_{\sigma} \top \}$

The remaining proofs are trivial.

8 Truth and Validity

Definition 17 An $n$-level variable assignment for $(W, A, \bar{a}, k)$ is a variable assignment $\sigma$ such that:

- $\sigma(p_i) \in P^m_{W_A}$;
- $\sigma(pp_i) \subseteq P^m_{W_A}$ and $\sigma(pp_i) \neq \emptyset$;
- $\sigma(O_i) \in I^n_{W_A}$.

Proposition 13 (Monotonicity of Assignments) For $n, k \in \mathbb{N}$, if $\sigma$ is an $n$-level assignment, it is also a $(n + 1)$-level assignment.

Proof Assume that $\sigma$ is an $n$-level assignment. To show that $\sigma$ is also an $(n + 1)$-level assignment, we need to verify:

- $\sigma(p_i) \in P^{n+1}_{W_A}$

  Proof: Since $\sigma$ is an $n$-level assignment, we have $\sigma(p_i) \in P^m_{W_A}$. So proposition 5 entails $\sigma(p_i) \in P^{n+1}_{W_A}$. 

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Definition 18 (Truth)  For a formula \( \phi \) of \( \mathcal{L} \) to be true at model \( \langle W, A, \vec{\alpha}, k \rangle \) is for it to be the case that \( \vec{\alpha} \models^k \sigma \phi \) for every \( k \)-level assignment \( \sigma \).

Definition 19 (Validity)  For \( \phi \) to be valid (in symbols \( \models \phi \)) is for it to be true at every model.

Definition 20  Let \( \mathcal{L}^- \) be the fragment of \( \mathcal{L} \) that excludes \( \uparrow \), \( \downarrow \), and \( Q_1, \ldots, Q_r \).

Proposition 14  \( \phi \in \mathcal{L}^- \) is valid in the present framework if and only if it is valid in a standard higher-order framework.

Proof

Right to Left: Suppose \( \phi \) fails to be valid in the present framework. Then there is some model \( \langle W, A, \vec{\alpha}, k \rangle \) at which \( \phi \) fails to be true. But when the clauses for vocabulary outside \( \mathcal{L}^- \) are ignored, our semantic clauses are totally standard. So \( \phi \) will also fail to be true when \( \langle W, A, \vec{\alpha}, k \rangle \) is thought of as a standard higher-order model.

Left to Right: Suppose \( \phi \) fails to be valid with respect to a standard higher-order model theory. Then it fails to be true according to some standard model. But every standard higher-order model of \( \mathcal{L}^- \) is isomorphic to some model of the form \( \langle W, A, \vec{\alpha}, 0 \rangle \). So \( \phi \) must fail to be true according to some model of the present framework.

9  Substitution

Definition 21 (Notation)

• \( \phi[\psi/p] \) is the result of substituting \( \psi \) for each free occurrence of \( p \) in \( \phi \).
• \( \sigma[\overline{q}/p](\eta) = \begin{cases} 
\sigma(\eta), & \text{if } \eta \neq p \\
\overline{q}, & \text{if } \eta = p 
\end{cases} \)

• We say that \( \psi \) is free for \( p \) in \( \phi \) iff no free variables in \( \psi \) become bound when substituting \( \psi \) for every free occurrence of \( p \) in \( \phi \).

**Proposition 15 (Trivial Substitution)** If \( p \) does not occur free in \( \phi \),

\[
\overline{w} \models^n \phi \leftrightarrow \overline{w} \models^n_{\sigma[\overline{q}/p]} \phi
\]

**Proof** We proceed by induction on the complexity of \( \phi \):

• \( \phi = p_i \).
  Since \( p \) does not occur free in \( \phi \), \( p \neq p_i \). So we have \( \sigma(p_i) = \sigma[\overline{q}/p](p_i) \) and therefore:

\[
\overline{w} \models^n_{\sigma} p_i \leftrightarrow \overline{w} \models^n_{\sigma[\overline{q}/p]} p_i
\]

• \( \phi = Q_j p_i \). If \( n = 0 \), the result is immediate, by the semantic clause for \( Q_j \):

\[
\overline{w} \models^n_{\sigma} Q_j p_i \leftrightarrow \bot \leftrightarrow \overline{w} \models^n_{\sigma[\overline{q}/p]} Q_j p_i
\]

We therefore assume \( n > 0 \). Since \( p \) does not occur free in \( \phi \), \( p \neq p_i \). So we have \( \sigma(p_i) = \sigma[\overline{q}/p](p_i) \) and therefore:

\[
\overline{w} \models^n_{\sigma} Q_j p_i \leftrightarrow \sigma(p_i) \in \left[ \text{Ext}_{Q_j}^{n-1} \right] \leftrightarrow \sigma[\overline{q}/p](p_i) \in \left[ \text{Ext}_{Q_j}^{n-1} \right] \leftrightarrow \overline{w} \models^n_{\sigma[\overline{q}/p]} Q_j p_i
\]

• \( \phi \) is \( O_j p_i \)
  Since \( p \) does not occur free in \( \phi \), \( p \neq p_i \). So we have \( \sigma(p_i) = \sigma[\overline{q}/p](p_i) \) and therefore:

\[
\overline{w} \models^n_{\sigma} O_j p_i \leftrightarrow \sigma(p_i) \in \sigma(O_j) \leftrightarrow \sigma[\overline{q}/p](p_i) \in \sigma[\overline{q}/p](O_j) \leftrightarrow \overline{w} \models^n_{\sigma[\overline{q}/p]} O_j p_i
\]

• \( \phi \) is \( p_i \prec pp_j \)
  Since \( p \) does not occur free in \( \phi \), \( p \neq p_i \). So we have \( \sigma(p_i) = \sigma[\overline{q}/p](p_i) \) and therefore:

\[
\sigma(p_i) \in \sigma(pp_j) \leftrightarrow \sigma[\overline{q}/p](p_i) \in \sigma[\overline{q}/p](pp_j)
\]

from which the result follows by the semantic clause for \( \prec \).
• \( \phi \) is \( \theta = \xi \)

Since \( p \) does not occur free in \( \phi \), it must not occur free in \( \theta \) or \( \xi \). So, by inductive hypothesis:

\[
\vec{w} \models_{\sigma}^{n} \theta \leftrightarrow \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \theta
\]

\[
\vec{w} \models_{\sigma}^{n} \xi \leftrightarrow \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \xi
\]

and therefore

\[
\{ \vec{w} : \vec{w} \models_{\sigma}^{n} \theta \} \leftrightarrow \{ \vec{w} : \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \theta \}
\]

\[
\{ \vec{w} : \vec{w} \models_{\sigma}^{n} \xi \} \leftrightarrow \{ \vec{w} : \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \xi \}
\]

So we have:

\[
\{ \vec{w} : \vec{w} \models_{\sigma}^{n} \theta \} = \{ \vec{w} : \vec{w} \models_{\sigma}^{n} \xi \} \leftrightarrow \{ \vec{w} : \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \theta \} = \{ \vec{w} : \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \xi \}
\]

from which the result follows by the semantic clause for \( = \).

• \( \phi \) is \( \neg \theta \)

Since \( p \) does not occur free in \( \phi \), it must not occur free in \( \theta \). So, by inductive hypothesis:

\[
\vec{w} \models_{\sigma}^{n} \theta \leftrightarrow \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \theta
\]

and therefore

\[
\vec{w} \not\models_{\sigma}^{n} \theta \leftrightarrow \vec{w} \not\models_{\sigma[\vec{q}/p]}^{n} \theta
\]

from which the result follows by the semantic clause for \( \neg \).

• \( \phi \) is \( (\theta \land \xi) \)

Since \( p \) does not occur free in \( \phi \), it must not occur free in \( \theta \) or \( \xi \). So, by inductive hypothesis:

\[
\vec{w} \models_{\sigma}^{n} \theta \leftrightarrow \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \theta
\]

\[
\vec{w} \models_{\sigma}^{n} \xi \leftrightarrow \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \xi
\]

So we have:

\[
(\vec{w} \models_{\sigma}^{n} \theta \land \vec{w} \models_{\sigma}^{n} \xi) \leftrightarrow \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \theta \land \vec{w} \models_{\sigma[\vec{q}/p]}^{n} \xi
\]

from which the result follows by the semantic clause for \( \land \).
• \( \phi \) is \( \exists p_i \theta \)

By the semantic clause for \( \exists \):

\[
\bar{w} \models_n^\sigma \exists p_i \theta \iff \exists \bar{r} \in P^\sigma_{W_A} \bar{w} \models_n^\sigma \exists \bar{r}/\bar{p}_i
\]

\[
\bar{w} \models_n^\sigma[\bar{q}/\bar{p}] \exists p_i \theta \iff \exists \bar{r} \in P^\sigma_{W_A} \bar{w} \models_n^\sigma[\bar{q}/\bar{p}][\bar{r}/\bar{p}_i]
\]

There are two cases:

- Suppose \( p = p_i \). Then \( \sigma[\bar{r}/\bar{p}_i] = \sigma[\bar{q}/\bar{p}][\bar{r}/\bar{p}_i] \). So, merging the above biconditionals gives us:

\[
\bar{w} \models_n^\sigma \exists p_i \theta \iff \bar{w} \models_n^\sigma[\bar{q}/\bar{p}] \exists p_i \theta
\]

which is what we want.

- Suppose \( p \neq p_i \). Then the fact that \( p \) does not occur free in \( \psi \) entails that it does not occur free in \( \theta \). So, by inductive hypothesis:

\[
\bar{w} \models_n^\sigma[\bar{r}/\bar{p}_i] \theta \iff \bar{w} \models_n^\sigma[\bar{q}/\bar{p}][\bar{r}/\bar{p}_i] \theta
\]

But since \( p \neq p_i \), \( \sigma[\bar{r}/\bar{p}_i][\bar{q}/\bar{p}] = \sigma[\bar{q}/\bar{p}][\bar{r}/\bar{p}_i] \). So we have:

\[
\bar{w} \models_n^\sigma[\bar{r}/\bar{p}_i] \theta \iff \bar{w} \models_n^\sigma[\bar{q}/\bar{p}][\bar{r}/\bar{p}_i] \theta
\]

So, merging the above biconditionals gives us:

\[
\bar{w} \models_n^\sigma \exists p_i \theta \iff \bar{w} \models_n^\sigma[\bar{q}/\bar{p}] \exists p_i \theta
\]

which is what we want.

• \( \phi \) is \( \exists pp_i \theta \) or \( \exists O_i \theta \)

Analogous to the second case of the preceding item.

• \( \phi \) is \( \uparrow \theta \)

Since \( p \) does not occur free in \( \phi \), it must not occur free in \( \theta \). So, by inductive hypothesis:

\[
\bar{w} \models_n^{\sigma+1} \theta \iff \bar{w} \models_n^{\sigma+1}[\bar{q}/\bar{p}] \theta
\]

But, by the semantic clause for \( \uparrow \):

\[
\bar{w} \models_n^{\sigma} (\uparrow \theta) \iff \bar{w} \models_n^{\sigma+1} \theta
\]

\[
\bar{w} \models_n^{\sigma[\bar{q}/\bar{p}]} (\uparrow \theta) \iff \bar{w} \models_n^{\sigma+1}[\bar{q}/\bar{p}] \theta
\]

So the result is immediate.

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• φ is ↓ θ

Suppose, first that \( n = 0 \). Then:

\[ \vec{w} \models_\sigma^n (\downarrow \theta) \leftrightarrow \vec{w} \models_\sigma^n \theta \]

\[ \vec{w} \models_\sigma^{n[\vec{q}/p]} (\downarrow \theta) \leftrightarrow \vec{w} \models_\sigma^{n[\vec{q}/p]} \theta \]

Since \( p \) does not occur free in \( \phi \), it must not occur free in \( \theta \). So by inductive hypothesis:

\[ \vec{w} \models_\sigma^n \theta \leftrightarrow \vec{w} \models_\sigma^{n[\vec{q}/p]} \theta \]

So the result is immediate.

Since \( p \) does not occur free in \( \phi \), it must not occur free in \( \theta \). So, by inductive hypothesis:

\[ \vec{w} \models_{n-1} \theta \leftrightarrow \vec{w} \models_{n-1} \theta \]

But, by the semantic clause for ↓:

\[ \vec{w} \models_\sigma^n (\downarrow \theta) \leftrightarrow \vec{w} \models_{n-1} \theta \]

\[ \vec{w} \models_{n[\vec{q}/p]} (\downarrow \theta) \leftrightarrow \vec{w} \models_{n[\vec{q}/p]} \theta \]

So the result is immediate.

**Proposition 16 (Substitution Principle)** Let \( \phi \) and \( \psi \) be formulas with no free variables in common. For \( \vec{w} \in \mathcal{W}_A \) and \( \vec{q} = \{ \vec{w} : \vec{w} \models_\sigma^n \psi \} \),

\[ \vec{w} \models_\sigma^n \phi[\psi/p] \leftrightarrow \vec{w} \models_{n[\vec{q}/p]} \phi \]

**Proof** If \( p \) does not occur free in \( \phi \), \( \phi[\psi/p] = \phi \), which means that the result is immediate, since by proposition 15, we have:

\[ \vec{w} \models_\sigma^n \phi \leftrightarrow \vec{w} \models_{n[\vec{q}/p]} \phi \]

We shall therefore assume that \( p \) occurs free in \( \phi \). We proceed by induction on the complexity of \( \phi \):
\( \phi = p_i \).

Since \( p \) occurs free in \( \phi \), it must be that \( p_i = p \). So \( \phi = p \) and \( \phi[p/\psi] = \psi \). We can therefore argue as follows:

\[
\overline{w} \models^p_\sigma \psi \iff \overline{w} \in \{ \overline{w} : \overline{w} \models^p_\sigma \psi \}
\]

\[
\overline{w} \models^p_\sigma \psi \iff \overline{w} \in \overline{q}
\]

\[
\overline{w} \models^p_\sigma \psi \iff \overline{w} \models^p_{\sigma[\overline{q}/p]} \psi
\]

\[
\overline{w} \models^p_\sigma \phi[p/\psi] \iff \overline{w} \models^p_{\sigma[\overline{q}/p]} \phi
\]

\( \phi = \mathcal{Q}_j p_i \). If \( n = 0 \), the result is immediate, by the semantic clause for \( \mathcal{Q}_j \):

\[
\overline{w} \models^0_\sigma \mathcal{Q}_j p_i[\psi/p] \iff \bot \iff \overline{w} \models^0_{\sigma[\overline{q}/p]} \mathcal{Q}_j p_i
\]

We therefore assume \( n > 0 \). Since \( p \) occurs free in \( \phi \), it must be the case that \( p = p_i \). So \( \phi = \mathcal{Q}_j p \) and \( \phi[p/\psi] = \mathcal{Q}_j \psi \), which means that \( \psi \) must itself be a variable, which we call \( p_l \). We can therefore argue as follows:

\[
\sigma(p_i) \in \left[ \mathcal{W}^{\text{Ext}^n_{\mathcal{Q}_j}} \right] (\overline{w}) \leftrightarrow \{ \overline{w} : \overline{w} \models^p_\sigma \ p_i \} \in \left[ \mathcal{W}^{\text{Ext}^n_{\mathcal{Q}_j}} \right] (\overline{w})
\]

\[
\sigma(p_i) \in \left[ \mathcal{W}^{\text{Ext}^n_{\mathcal{Q}_j}} \right] (\overline{w}) \leftrightarrow \overline{q} \in \left[ \mathcal{W}^{\text{Ext}^n_{\mathcal{Q}_j}} \right] (\overline{w})
\]

\[
\sigma(p_i) \in \left[ \mathcal{W}^{\text{Ext}^n_{\mathcal{Q}_j}} \right] (\overline{w}) \leftrightarrow \sigma[\overline{q}/p](p) \in \left[ \mathcal{W}^{\text{Ext}^n_{\mathcal{Q}_j}} \right] (\overline{w})
\]

\[
\overline{w} \models^p_\sigma \mathcal{Q}_j p_i \iff \overline{w} \models^p_{\sigma[\overline{q}/p]} \mathcal{Q}_j p
\]

\[
\overline{w} \models^p_\sigma \phi[p/\psi] \iff \overline{w} \models^p_{\sigma[\overline{q}/p]} \phi
\]

\( \phi \) is \( \mathcal{O}_j p_i \).

Since \( p \) occurs free in \( \phi \), it must be that \( p_i = p \) and therefore that \( \phi \) is \( \mathcal{O}_j p \) and \( \psi \) is a variable, which we call \( p_i \). We may therefore argue as follows:

\[
\sigma(p_i) \in \sigma(\mathcal{O}_j) \leftrightarrow \{ \overline{w} : \overline{w} \models^p_\sigma \ p_i \} \in \sigma(\mathcal{O}_j)
\]

\[
\sigma(p_i) \in \sigma(\mathcal{O}_j) \leftrightarrow \overline{q} \in \sigma(\mathcal{O}_j)
\]

\[
\sigma(p_i) \in \sigma(\mathcal{O}_j) \leftrightarrow \sigma[\overline{q}/p](p) \in \sigma[\overline{q}/p](\mathcal{O}_j)
\]

\[
\overline{w} \models^p_\sigma \mathcal{O}_j p_i \iff \overline{w} \models^p_{\sigma[\overline{q}/p]} \mathcal{O}_j p
\]

\[
\overline{w} \models^p_\sigma \phi[p/\psi] \iff \overline{w} \models^p_{\sigma[\overline{q}/p]} \phi
\]
• \( \phi \) is \( p_i \prec pp_j \)

Since \( p \) occurs free in \( \phi \), it must be that \( p_i = p \) and therefore that \( \phi \) is \( p \prec pp_j \) and \( \psi \) is a variable, which we call \( p_l \). We may therefore argue as follows:

\[
\sigma(p_i) \in \sigma(pp_j) \iff \{ \bar{w} : \bar{w} \models^n_{\sigma} p_i \} \in \sigma(pp_j)
\]

\[
\sigma(p_i) \in \sigma(pp_j) \iff \bar{q} \in \sigma(pp_j)
\]

\[
\sigma(p_i) \in \sigma(pp_j) \iff \sigma[\bar{q}/p](p) \in \sigma[\bar{q}/p](pp_j)
\]

\[
\bar{w} \models^n_{\sigma} p_i \prec pp_j \iff \bar{w} \models^n_{\sigma[\bar{q}/p]} p \prec pp_j
\]

\[
\bar{w} \models^n_{\sigma} \phi[\psi/p] \iff \bar{w} \models^n_{\sigma[\bar{q}/p]} \phi
\]

• \( \phi \) is \( \theta = \xi \)

Since \( \psi \) is free for \( p \) in \( \phi \), it must also be free for \( p \) in \( \theta \) or \( \xi \). So, by inductive hypothesis:

\[
\bar{w} \models^n_{\sigma} \theta[\psi/p] \iff \bar{w} \models^n_{\sigma[\bar{q}/p]} \theta
\]

\[
\bar{w} \models^n_{\sigma} \xi[\psi/p] \iff \bar{w} \models^n_{\sigma[\bar{q}/p]} \xi
\]

So we can argue as follows:

\[
\{ \bar{w} : \bar{w} \models^n_{\sigma} \theta[\psi/p] \} = \{ \bar{w} : \bar{w} \models^n_{\sigma} \xi[\psi/p] \} \iff \{ \bar{w} : \bar{w} \models^n_{\sigma[\bar{q}/p]} \theta \}
\]

\[
\bar{w} \models^n_{\sigma} (\theta[\psi/p] = \xi[\psi/p]) \iff \bar{w} \models^n_{\sigma[\bar{q}/p]} (\theta = \xi)
\]

\[
\bar{w} \models^n_{\sigma} (\theta = \xi)[\psi/p] \iff \bar{w} \models^n_{\sigma[\bar{q}/p]} (\theta = \xi)
\]

• \( \phi \) is \( \neg \theta \)

Since \( \psi \) is free for \( p \) in \( \phi \), it must also be free for \( p \) in \( \theta \). So, by inductive hypothesis:

\[
\bar{w} \models^n_{\sigma} \theta[\psi/p] \iff \bar{w} \models^n_{\sigma[\bar{q}/p]} \theta
\]

Equivalently:

\[
\bar{w} \not\models^n_{\sigma} \theta[\psi/p] \iff \bar{w} \not\models^n_{\sigma[\bar{q}/p]} \theta
\]

So, by the relevant semantic clause:

\[
\bar{w} \models^n_{\sigma} \neg \theta[\psi/p] \iff \bar{w} \models^n_{\sigma[\bar{q}/p]} \neg \theta
\]
• \( \phi \) is \((\theta \land \xi)\)

Since \( \psi \) is free for \( p \) in \( \phi \), it must also be free for \( p \) in \( \theta \) or \( \xi \). So, by inductive hypothesis:

\[
\bar{w} \models^\sigma_n \theta[p/\psi] \iff \bar{w} \models^\sigma_n [\bar{q}/p] \theta
\]

\[
\bar{w} \models^\sigma_n \xi[p/\psi] \iff \bar{w} \models^\sigma_n [\bar{q}/p] \xi
\]

So we can argue as follows:

\[
(\bar{w} \models^\sigma_n \theta[p/\psi] \land \bar{w} \models^\sigma_n \xi[p/\psi]) \iff (\bar{w} \models^\sigma_n [\bar{q}/p] \theta \land \bar{w} \models^\sigma_n [\bar{q}/p] \xi)
\]

\[
\bar{w} \models^\sigma_n \theta[p/\psi] \land \xi[p/\psi] \iff \bar{w} \models^\sigma_n [\bar{q}/p] (\theta \land \xi)
\]

\[
\bar{w} \models^\sigma_n (\theta \land \xi)[p/\psi] \iff \bar{w} \models^\sigma_n [\bar{q}/p] (\theta \land \xi)
\]

• \( \phi \) is \( \exists p_i \theta \)

By the semantic clause for \( \exists \):

\[
\bar{w} \models^\sigma_n (\exists p_i \theta)[p/\psi] \iff \bar{w} \models^\sigma_n \exists p_i (\theta[p/\psi]) \iff \exists \bar{q}' \in P_{W_A}^n \left( \bar{w} \models^\sigma_n [\bar{q}'/p_i] (\theta[p/\psi]) \right)
\]

Since \( \psi \) is free for \( p \) in \( \phi \), \( p_i \) cannot occur free in \( \psi \). So, by proposition 15:

\[
\bar{w} \models^\sigma_n \psi \iff \bar{w} \models^\sigma_n [\bar{q}'/p_i] \psi
\]

which means that:

\[
\left\{ \bar{w} : \bar{w} \models^\sigma_n [\bar{q}'/p_i] \psi \right\} = \left\{ \bar{w} : \bar{w} \models^\sigma_n \psi \right\} = \bar{q}'
\]

Since \( \psi \) is free for \( p \) in \( \phi \), it must also be free for \( p \) in \( \theta \). So, by inductive hypothesis:

\[
\bar{w} \models^\sigma_n [\bar{q}'/p_i] \theta[p/\psi] \iff \bar{w} \models^\sigma_n [\bar{q}'/p_i][\bar{q}/p] \theta
\]

Since \( p \) occurs free in \( \phi \), \( p \neq p_i \). So

\[
\bar{w} \models^\sigma_n [\bar{q}'/p_i][\bar{q}/p] \theta \iff \bar{w} \models^\sigma_n [\bar{q}/p][\bar{q}'/p_i] \theta
\]

Putting all of this together:

\[
\bar{w} \models^\sigma_n (\exists p_i \theta)[p/\psi] \iff \exists \bar{q}' \in P_{W_A}^n \left( \bar{w} \models^\sigma_n [\bar{q}'/p_i] \theta \right)
\]

But, by the semantic clause for \( \exists \),

\[
\bar{w} \models^\sigma_n [\bar{q}'/p_i] \exists p_i \theta \iff \exists \bar{q}' \in P_{W_A}^n \left( \bar{w} \models^\sigma_n [\bar{q}'/p_i] \theta \right)
\]

So the desired result follows.
• \( \phi \) is \( \exists p \theta \) or \( \exists O \theta \)

Analogous to the preceding case.

• \( \phi \) is \( \uparrow \theta \)

Since \( \psi \) is free for \( p \) in \( \phi \), it must also be free for \( p \) in \( \theta \). So, by inductive hypothesis:

\[
\bar{w} \models_{\sigma}^{n+1} \theta[\psi/p] \iff \bar{w} \models_{\sigma[\bar{q}/p]}^{n+1} \theta
\]

But, by the semantic clause for \( \uparrow \):

\[
\bar{w} \models_{\sigma[\bar{q}/p]}^{n} (\uparrow \theta) \iff \bar{w} \models_{\sigma[\bar{q}/p]}^{n+1} \theta[\psi/p]
\]

So the result is immediate.

• \( \phi \) is \( \downarrow \theta \)

Suppose, first that \( n = 0 \). Then:

\[
\bar{w} \models_{\sigma}^{n} (\downarrow \theta)[\psi/p] \iff \bar{w} \models_{\sigma}^{n} (\theta[\psi/p]) \iff \bar{w} \models_{\sigma}^{n} \theta[\psi/p]
\]

\[
\bar{w} \models_{\sigma[\bar{q}/p]}^{n} \theta \iff \bar{w} \models_{\sigma[\bar{q}/p]}^{n} \theta[\psi/p]
\]

Since \( \psi \) is free for \( p \) in \( \phi \), it must also be free for \( p \) in \( \theta \). So the result follows immediately from our inductive hypothesis:

\[
\bar{w} \models_{\sigma}^{n} \theta[\psi/p] \iff \bar{w} \models_{\sigma[\bar{q}/p]}^{n} \theta
\]

Now suppose \( n > 0 \). Since \( \psi \) is free for \( p \) in \( \phi \), it must also be free for \( p \) in \( \theta \). So, by inductive hypothesis:

\[
\bar{w} \models_{\sigma}^{n-1} \theta[\psi/p] \iff \bar{w} \models_{\sigma[\bar{q}/p]}^{n-1} \theta
\]

But, by the semantic clause for \( \downarrow \):

\[
\bar{w} \models_{\sigma}^{n} (\downarrow \theta)[\psi/p] \iff \bar{w} \models_{\sigma}^{n} (\theta[\psi/p]) \iff \bar{w} \models_{\sigma}^{n-1} \theta[\psi/p]
\]

\[
\bar{w} \models_{\sigma[\bar{q}/p]}^{n} \theta \iff \bar{w} \models_{\sigma[\bar{q}/p]}^{n-1} \theta
\]

So the result is immediate.
Proposition 17 (Validity Substitution) Let $\phi$ have no variables in common with $\psi$ or $\theta$ and suppose that $\models \psi \leftrightarrow \theta$. Then:

$$\vec{w} \models_n^\sigma \phi[\psi/p] \iff \vec{w} \models_n^\sigma \phi[\theta/p]$$

Proof Let $\vec{p} = \{\vec{w} : \vec{w} \models_n^\sigma \psi\}$ and $\vec{q} = \{\vec{w} : \vec{w} \models_n^\sigma \theta\}$. Then, by proposition 16,

$$\vec{w} \models_n^\sigma \phi[\psi/p] \iff \vec{w} \models_n^\sigma_{[\vec{p}/p]} \phi \quad \vec{w} \models_n^\sigma \phi[\theta/p] \iff \vec{w} \models_n^\sigma_{[\vec{q}/p]} \theta$$

But since we have $\models \psi \leftrightarrow \theta$, it must be the case that $\vec{p} = \vec{q}$ and therefore that $\sigma[\vec{p}/p] = \sigma[\vec{q}/p]$, which allows us to conclude:

$$\vec{w} \models_n^\sigma \phi[\psi/p] \iff \vec{w} \models_n^\sigma \phi[\theta/p]$$

10 Comprehension

Definition 22 (Valence)

Intuitively, the valence of a formula $\phi$, relative to a level of resolution $k$, is a syntactically characterized upper bound on the resolution that is needed to describe the proposition expressed by $\phi$, when evaluated externally at resolution $k$ (assuming a variable assignment of level 0).

Formally, for $\phi$ a formula and $l \in \mathbb{N}$, the valence of $\phi$ relative to $k$, written $v_k(\phi)$, is defined recursively, as follows:

- $v_k(\phi) = k$, if $\phi$ is atomic;
- $v_k(\phi = \psi) = 0$
- $v_k(\neg \phi) = v_k(\phi)$;
- $v_k(\phi \land \psi) = \max(v_k(\phi), v_k(\psi))$
- $v_k(\exists p_i \phi) = \max(k, v_k(\phi))$
- $v_k(\exists pp_i \phi) = \max(k, v_k(\phi))$
- $v_k(\exists O_i \phi) = \max(k, v_k(\phi))$
- $v_k(\uparrow \phi) = v_{k+1}(\phi)$;
• $v_k(\downarrow \phi) = \begin{cases} v_{k-1}(\phi), & \text{if } k > 0; \\ v_0(\phi), & \text{if } k = 0. \end{cases}$

**Lemma 1 (Level Lemma)** Let $\phi$ be a formula of $L$. For any $n, m \in \mathbb{N}$, let $\sigma$ be an assignment of level $m$ and let $k = \max(m, v_n(\phi))$. We then have:

$$\{\vec{w} : \vec{w} \models^n_\sigma \phi\} \in P_k^{W_A}$$

**Proof** We proceed by induction on the complexity of $\phi$.

For each of the base cases, we proceed by supposing that $\vec{w}(k) = \vec{v}(k)$ and $\vec{w} \models^n_\sigma \phi$, and verifying that $\vec{v} \models^n_\sigma \phi$.

• $\phi = p_i$. The relevant semantic clause gives us $\vec{w} \in \sigma(p_i)$. Since $\sigma$ is a level-$m$ assignment and $m \leq k$, it is also a level-$k$ assignment. So the fact that $\vec{w}(k) = \vec{v}(k)$ guarantees that we also have $\vec{v} \in \sigma(p_i)$ and therefore $\vec{v} \models^n_\sigma \phi$.

• $\phi = Q_j p_i$. If $n = 0$, the result is immediate, since

$$\vec{w} \models^0_\sigma Q_j p_i \iff \bot \iff \vec{v} \models^0_\sigma Q_j p_i$$

So let us assume that $n > 0$. By the definition of valence, $v_n(Q_j p_i) = n$. So we have $k = \max(m, n)$ and therefore $n \leq k$. Since $\vec{w}(k) = \vec{v}(k)$, it follows that $\vec{w}(n) = \vec{v}(n)$ (by proposition 4). Let $\vec{w}(n) = \vec{v}(n) = \langle w, e_1^{n-1}, \ldots, e_r^{n-1} \rangle$. By the semantic clause for $Q_j p_i$, $\vec{w} \models^n_\sigma \phi$ is equivalent to

$$\sigma(p_i) \in \left[ W_A^{\text{Ext}_Q^{n-1}} \right](\vec{w})$$

which, by the definition of $\left[ W_A^{\text{Ext}_Q^{n-1}} \right](\vec{w})$ is equivalent to

$$\sigma(p_i) \in \{ \vec{p} \in P_{W_A}^n : \exists p^{n-1} \in e_j^{n-1}(\vec{p}(n-1) = p^{n-1}) \}$$

which, by the definition of $\left[ W_A^{\text{Ext}_Q^{n-1}} \right](\vec{v})$ is equivalent to

$$\sigma(p_i) \in \left[ W_A^{\text{Ext}_Q^{n-1}} \right](\vec{v})$$

which is equivalent to $\vec{v} \models^n_\sigma \phi$. 

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• $\phi = O_j p_i$. By the relevant semantic clause, $\bar{w} \models^\sigma_\sigma \phi$ is equivalent to $\bar{w} \in \sigma(O_j)(\sigma(p_i))$. Since $\sigma$ is a level-$m$ assignment and $m \leq k$, it is also a level-$k$ assignment. So the fact that $\bar{v}(k) = \bar{v}(k)$ guarantees that we also have $\bar{v} \in \sigma(O_j)(\sigma(p_i))$ and therefore $\bar{v} \models^\sigma_\sigma O_j p_i$.

• $\phi$ is $(\psi = \theta)$ or $p_i \prec p_j$. The result follows from the fact that $\bar{w} \models^\sigma \phi$ does not depend on $\bar{w}$.

For the remaining cases, we assume our inductive hypothesis for arbitrary $\sigma$, $m$, and $n$:

• $\phi = \neg \psi$. By inductive hypothesis,
  \[ \{ \bar{z} : \bar{z} \models^\sigma_\sigma \psi \} \in P_{W_A}^k \]
  But if a subset of $W_A$ is in $P_{W_A}^k$, then so is its complement. So:
  \[ \{ \bar{z} : \bar{z} \not\models^\sigma_\sigma \psi \} \in P_{W_A}^k \]
  which is what we want.

• $\phi = (\psi \land \theta)$. For $k' = \max(m, v_n(\psi))$ and $k'' = \max(m, v_n(\theta))$ our inductive hypothesis gives us:
  \[ \{ \bar{z} : \bar{z} \models^\sigma_\sigma \psi \} \in P_{W_A}^{k'} \quad \{ \bar{w} : \bar{w} \models^\sigma_\sigma \theta \} \in P_{W_A}^{k''} \]
  Let $k^* = \max(k', k'')$. By proposition 5, we have:
  \[ \{ \bar{z} : \bar{z} \models^\sigma_\sigma \psi \}, \{ \bar{w} : \bar{w} \models^\sigma_\sigma \theta \} \in P_{W_A}^{k^*} \]
  Now recall that $k = \max(m, v_n(\psi \land \theta))$. By the definition of valence, $v_n(\psi \land \theta) = \max(v_n(\psi), v_n(\theta))$. So:
  \[
  k = \max(m, \max(v_n(\psi), v_n(\theta))) \\
  = \max(\max(m, v_n(\psi)), \max(m, v_n(\theta))) \\
  = \max(k', k'') \\
  = k^*
  \]
  We therefore have:
  \[ \{ \bar{z} : \bar{z} \models^\sigma_\sigma \psi \}, \{ \bar{w} : \bar{w} \models^\sigma_\sigma \theta \} \in P_{W_A}^k \]
  But if two subsets of $W_A$ are in $P_{W_A}^k$, then so is their intersection. So:
  \[ \{ \bar{z} : \bar{z} \models^\sigma_\sigma \psi \land \theta \} \in P_{W_A}^k \]
  which is what we want.
\( \phi = \exists p_i \psi \). Let \( \vec{w}(k) = \vec{v}(k) \) and assume \( \vec{w} \models^n_\sigma \phi \). By the semantic clause for \( \exists \), we know that for some \( \vec{q} \in P^n_{\mathcal{A}} \), \( \vec{w} \models^n_{\sigma[\vec{q}/p_i]} \psi \). Since \( \sigma \) is an assignment of level \( m \), \( \sigma[\vec{q}/p_i] \) is an assignment of level \( \max(n, m) \). Let \( k' = \max(\max(n, m), v_n(\psi)) \). Our inductive hypothesis gives us:

\[
\{ \vec{z} : \vec{z} \models^n_{\sigma[\vec{q}/p_i]} \psi \} \in P^{k'}_{\mathcal{A}}
\]

But, by the definition of valence, \( v_n(\exists p_i \psi) = \max(n, v_n(\psi)) \). So

\[
k = \max(m, v_n(\exists p_i \psi)) = \max(m, \max(n, v_n(\psi))) = \max(m, v_n(\psi)) = \max(\max(m, n), v_n(\psi)) = k'
\]

We therefore have:

\[
\{ \vec{z} : \vec{z} \models^n_{\sigma[\vec{q}/p_i]} \psi \} \in P^{k}_{\mathcal{A}}
\]

Since \( \vec{w}(k) = \vec{v}(k) \), this means that \( \vec{w} \models^n_{\sigma[\vec{q}/p_i]} \psi \) entails \( \vec{v} \models^n_{\sigma[\vec{q}/p_i]} \psi \). In other words: we know that for some \( \vec{q} \in P^n_{\mathcal{A}} \), \( \vec{v} \models^n_{\sigma[\vec{q}/p_i]} \psi \). So, by the semantic clause for \( \exists \), \( \vec{v} \models^n \exists p_i \psi \).

\( \phi = \exists pp_i \psi \) or \( \phi = \exists O_j \psi \). Analogous to previous case.

\( \phi = \uparrow \psi \). Let \( \vec{w}(k) = \vec{v}(k) \) and assume that \( \vec{w} \models^n_{\sigma} \psi \). By the semantic clause for \( \uparrow \), we have \( \vec{w} \models^{n+1}_{\sigma} \psi \).

By the definition of valence, \( v_n(\uparrow \psi) = v_{n+1}(\psi) \) and therefore \( k = \max(m, v_n(\uparrow \psi)) = \max(m, v_{n+1}(\psi)) \). So our inductive hypothesis gives us:

\[
\{ \vec{z} : \vec{z} \models^{n+1}_{\sigma} \psi \} \in P^{k}_{\mathcal{A}}
\]

So the fact that \( \vec{w}(k) = \vec{v}(k) \) gives us \( \vec{v} \models^{n+1}_{\sigma} \psi \) and therefore \( \vec{v} \models^{n}_{\sigma} \uparrow \psi \), which is what we wanted.

\( \phi = \downarrow \psi \). Let \( \vec{w}(k) = \vec{v}(k) \) and assume that \( \vec{w} \models^n_{\sigma} \downarrow \psi \). We show that \( \vec{v} \models^{n}_{\sigma} \downarrow \psi \).

First, suppose \( n = 0 \). By the semantic clause for \( \downarrow \),

\[
\vec{w} \models^{n}_{\sigma} \downarrow \psi \iff \vec{w} \models^{0}_{\sigma} \psi
\]
So we have \( \vec{w} \models_0 \psi \). The definition of valance gives us \( v_0(\downarrow \psi) = v_0(\psi) \) and therefore \( k = \max(m, v_0(\uparrow \psi)) = \max(m, v_0(\psi)) \). So our inductive hypothesis gives us:

\[
\{ \vec{z} : \vec{z} \models_0 \psi \} \in P^k_{\mathcal{W}_A}
\]

So the fact that \( \vec{w}(k) = \vec{v}(k) \) gives us \( \vec{v} \models_0 \psi \) and therefore \( \vec{v} \models_0 \downarrow \psi \), which is what we wanted.

Now suppose \( n > 0 \). By the semantic clause for \( \downarrow \),

\[
\vec{w} \models^n_{\sigma} \psi \iff \vec{w} \models^{n-1}_{\sigma} \psi
\]

So we have \( \vec{w} \models^{n-1}_{\sigma} \psi \). Since \( n > 0 \), the definition of valance gives us \( v_n(\downarrow \psi) = v_{n-1}(\psi) \) and therefore \( k = \max(m, v_n(\downarrow \psi)) = \max(m, v_{n-1}(\psi)) \).

So our inductive hypothesis gives us:

\[
\{ \vec{z} : \vec{z} \models^{n-1}_{\sigma} \psi \} \in P^k_{\mathcal{W}_A}
\]

So the fact that \( \vec{w}(k) = \vec{v}(k) \) gives us \( \vec{v} \models^{n-1}_{\sigma} \psi \) and therefore \( \vec{v} \models^n_{\sigma} \downarrow \psi \), which is what we wanted.

**Proposition 18 (Level Advance)** For any \( k \in \mathbb{N} \) and formula \( \phi \),

\[
v_{k+1}(\phi) = v_k(\phi) \lor v_{k+1}(\phi) = v_k(\phi) + 1
\]

**Proof** We proceed by induction on the complexity of \( \phi \):

- \( \phi \) atomic
  
  Then \( v_{k+1}(\phi) = k + 1 \) and \( v_k(\phi) = k \). So the result is immediate.

- \( \phi \) is \( \psi = \theta \)
  
  Then \( v_{k+1}(\phi) = 0 = v_k(\phi) = k \). So the result is immediate.

- \( \phi \) is \( \neg \psi \)

  By the definition of valence,

  \[
v_{k+1}(\neg \psi) = v_{k+1}(\psi) \quad v_k(\neg \psi) = v_k(\psi)
  \]

  And, by inductive hypothesis:

  \[
v_{k+1}(\psi) = v_k(\psi) \lor v_{k+1}(\psi) = v_k(\psi) + 1
  \]

  So the result is immediate.
• φ is ψ ∧ θ

By the definition of valence,

\[ v_k(\psi \land \theta) = \max(v_k(\psi), v_k(\theta)) \]

\[ v_{k+1}(\psi \land \theta) = \max(v_{k+1}(\psi), v_{k+1}(\theta)) \]

And by inductive hypothesis:

\[ v_{k+1}(\psi) = v_k(\psi) \lor v_{k+1}(\psi) = v_k(\psi) + 1 \]

\[ v_{k+1}(\theta) = v_k(\theta) \lor v_{k+1}(\theta) = v_k(\theta) + 1 \]

Assume, with no loss of generality, that \( v_k(\psi) \geq v_k(\theta) \). So

\[ v_k(\psi \land \theta) = \max(v_k(\psi), v_k(\theta)) = v_k(\psi) \]

If \( v_{k+1}(\psi) = v_k(\psi) + 1 \), it follows from our inductive hypotheses that

\[ v_{k+1}(\psi \land \theta) = \max(v_{k+1}(\psi), v_{k+1}(\theta)) = v_k(\psi) + 1 = v_k(\psi \land \theta) + 1 \]

which gives us what we want.

So we may assume both \( v_k(\psi) \geq v_k(\theta) \) and \( v_{k+1}(\psi) = v_k(\psi) \). If \( v_{k+1}(\theta) = v_k(\theta) \), it follows from our inductive hypotheses that

\[ v_{k+1}(\psi \land \theta) = \max(v_{k+1}(\psi), v_{k+1}(\theta)) = v_{k+1}(\psi) = v_k(\psi) = v_k(\psi \land \theta) \]

which, again gives us what we want.

So we may assume \( v_k(\psi) \geq v_k(\theta) \), \( v_{k+1}(\psi) = v_k(\psi) \), and \( v_{k+1}(\theta) = v_k(\theta) + 1 \). Since \( v_k(\psi) \geq v_k(\theta) \) and \( v_{k+1}(\psi) = v_k(\psi) \), our inductive hypothesis entails that are only two remaining options:

- \( v_{k+1}(\psi) \geq v_{k+1}(\theta) \), in which case it follows from our inductive hypotheses that

\[ v_{k+1}(\psi \land \theta) = \max(v_{k+1}(\psi), v_{k+1}(\theta)) = v_{k+1}(\psi) = v_k(\psi) = v_k(\psi \land \theta) \]

which gives us what we want.

- \( v_{k+1}(\theta) = v_{k+1}(\psi) + 1 \) (and therefore \( v_k(\psi) = v_k(\theta) \)). So we have:

\[ v_{k+1}(\psi \land \theta) = \max(v_{k+1}(\psi), v_{k+1}(\theta)) = v_{k+1}(\theta) = v_k(\theta) + 1 = v_k(\psi) + 1 = v_k(\psi \land \theta) + 1 \]

which gives us what we want.
• $\phi$ is $\exists p \psi$

By the definition of valence,

$$v_k(\exists p \psi) = \max(k, v_k(\psi)) \quad v_{k+1}(\exists p \psi) = \max(k + 1, v_{k+1}(\psi))$$

And by inductive hypothesis:

$$v_{k+1}(\psi) = v_k(\psi) \lor v_{k+1}(\psi) = v_k(\psi) + 1$$

Suppose first that $k \geq v_k(\psi)$, and therefore:

$$v_k(\exists p \psi) = \max(k, v_k(\psi)) = k.$$ 

By our inductive hypothesis, it must be the case that $k + 1 \geq v_{k+1}(\psi)$. So we have

$$v_{k+1}(\exists p \psi) = \max(k + 1, v_{k+1}(\psi)) = k + 1 = v_k(\exists p \psi) + 1$$

which gives us what we want.

Now suppose $v_k(\psi) > k$, and therefore:

$$v_k(\exists p \psi) = \max(k, v_k(\psi)) = v_k(\psi).$$

By our inductive hypothesis, it must be the case that $v_{k+1}(\psi) \geq k + 1$. So we have

$$v_{k+1}(\exists p \psi) = \max(k + 1, v_{k+1}(\psi)) = v_{k+1}(\psi)$$

By our inductive hypothesis, this means that:

$$v_{k+1}(\exists p \psi) = v_k(\psi) \lor v_{k+1}(\exists p \psi) = v_k(\psi) + 1$$

Since $v_k(\exists p \psi) = v_k(\psi)$, this gives us what we want.

• $\phi$ is $\exists p p \psi$ or $\exists O \psi$

Analogous to preceding case

• $\phi$ is $\triangledown \psi$

By the definition of valence,

$$v_k(\triangledown \psi) = v_{k+1}(\psi) \quad v_{k+1}(\triangledown \psi) = v_{k+2}(\psi)$$
And by inductive hypothesis:
\[ v_{k+2}(\psi) = v_{k+1}(\psi) \lor v_{k+2}(\psi) = v_{k+1}(\psi) + 1 \]

Putting the two together gives us what we want:
\[ v_{k+1}(\uparrow \psi) = v_k(\uparrow \psi) \lor v_{k+1}(\uparrow \psi) = v_k(\uparrow \psi) + 1 \]

• \( \phi \) is \( \downarrow \psi \)

Suppose, first, that \( k = 0 \). Then, by the definition of valence,
\[ v_k(\downarrow \psi) = v_k(\psi) \quad v_{k+1}(\downarrow \psi) = v_k(\psi) \]

which gives us what we want.

Now suppose that \( k > 0 \). By the definition of valence,
\[ v_k(\downarrow \psi) = v_{k-1}(\psi) \quad v_{k+1}(\downarrow \psi) = v_k(\psi) \]

And by inductive hypothesis:
\[ v_k(\psi) = v_{k-1}(\psi) \lor v_k(\psi) = v_{k-1}(\psi) + 1 \]

Putting the two together gives us what we want:
\[ v_{k+1}(\downarrow \psi) = v_k(\downarrow \psi) \lor v_{k+1}(\downarrow \psi) = v_k(\downarrow \psi) + 1 \]

**Proposition 19 (Level Advance Corollary)** *For any formula \( \phi \) and \( k \in \mathbb{N} \),
\[
v_0(\psi) \leq v_k(\phi) \leq v_0(\phi) + k
\]

**Proof** By proposition 18,
\[
\begin{align*}
v_0(\phi) & \leq v_1(\phi) \leq v_0(\phi) + 1 \\
v_1(\phi) & \leq v_2(\phi) \leq v_1(\phi) + 1 \\
& \vdots \\
v_{k-1}(\phi) & \leq v_k(\phi) \leq v_{k-1}(\phi) + 1
\end{align*}
\]

which together entail
\[
v_0(\psi) \leq v_k(\phi) \leq v_0(\phi) + k
\]
Definition 23
\[ \uparrow^k := \uparrow \ldots \uparrow \quad \downarrow^k := \downarrow \ldots \downarrow \]

**Theorem 1 (Existential Generalization)** Let \( \phi \) and \( \psi \) be such that \( \psi \) is free for \( p \) in \( \phi \). For \( k = v_0(\psi) \),

\[ \models \phi[\psi/p] \rightarrow \uparrow^k \exists p \downarrow^k \phi \]

**Proof** Fix a model \( \langle W, A, \vec{\alpha}, n \rangle \). It suffices to verify the following for an arbitrary \( n \)-level assignment \( \sigma \):

\[ \vec{\alpha} \models^n_\sigma \phi[\psi/p] \rightarrow \uparrow^k \exists p \downarrow^k \phi \]

We assume \( \vec{\alpha} \models^n_\sigma \phi[\psi/p] \) and show \( \vec{\alpha} \models^n_\sigma \uparrow^k \exists p \downarrow^k \phi \). For \( l = \max(n, v_n(\psi)) \), lemma 1 gives us:

\[ \{ \vec{w} : \vec{w} \models^n_\sigma \psi \} \in P_{W,A}^l \]

Note that it must be the case that \( l \leq (n+k) \): if \( l = n \) the result is immediate; and if \( l = v_n(\psi) \), we can use proposition 19 to show:

\[ l = v_n(\psi) \leq v_0(\psi) + n = k + n \]

So, by proposition 5, we have:

\[ \{ \vec{w} : \vec{w} \models^n_\sigma \psi \} \in P_{W,A}^{n+k} \]

Accordingly, there exists \( \vec{q} \in P_{W,A}^{n+k} \) such that

\[ \vec{q} = \{ \vec{w} : \vec{w} \models^n_\sigma \psi \} \]

By proposition 16,

\[ \vec{\alpha} \models^n_\sigma \phi[\psi/p] \leftrightarrow \vec{\alpha} \models^n_{\sigma[\vec{q}/p]} \phi \]

So, by our initial assumption:

\[ \vec{\alpha} \models^n_{\sigma[\vec{q}/p]} \phi \]

which is equivalent to the following, by the semantic clause for \( \downarrow^k \):

\[ \vec{\alpha} \models^{n+k}_n \downarrow^k \phi \]
Since \( \vec{q} \in P_{W_A}^{n+k} \), the semantic clause for \( \exists \) entails that
\[
\vec{\alpha} \models_{\sigma}^{n+k} \exists p \downarrow^k \phi
\]
which gives us our desired conclusion, by the semantic clause for \( \uparrow \),
\[
\vec{\alpha} \models_{\sigma}^{n} \uparrow^k \exists p \downarrow^k \phi
\]

**Corollary 1 (Comprehension)** For \( \phi \) a formula, let \( k = v_0(\phi) \) and let \( p \) be a variable not occurring free in \( \phi \). Then:
1. \( \models \uparrow^k \exists p \downarrow^k (p = \phi) \)
2. \( \models \uparrow^k \exists p (p = \downarrow^k \phi) \)

**Proof**
Since \( p \) does not occur free in \( \phi \), \( \phi \) is free for \( p \) in \( p = \phi \). So, by Theorem 1,
\[
\models (p = \phi)[\phi/p] \rightarrow \uparrow^k \exists p \downarrow^k (p = \phi)
\]
Since \( (p = \phi)[\phi/p] = (\phi = \phi) \), part 1 follows immediately by the semantic clauses for \( = \) and \( \rightarrow \).

To verify part 2, fix a model \( (W, A, \vec{\alpha}, n) \). It suffices to verify the following for an arbitrary \( n \)-level assignment \( \sigma \):
\[
\vec{\alpha} \models_{\sigma}^{n} \uparrow^k \exists p (p = \downarrow^k \phi)
\]
By part 1, we know that:
\[
\vec{\alpha} \models_{\sigma}^{n} \uparrow^k \exists p \downarrow^k (p = \phi)
\]
which, by the semantic clause for \( \uparrow \), is equivalent to:
\[
\vec{\alpha} \models_{\sigma}^{n+k} \exists p \downarrow^k (p = \phi)
\]
So, by the semantic clause for \( \exists \), there is some \( \vec{q} \in P_{W_A}^{n+k} \) such that:
\[
\vec{\alpha} \models_{\sigma[\vec{q}/p]}^{n+k} \downarrow^k (p = \phi)
\]
which, by the semantic clause for \( \downarrow \), is equivalent to:
\[
\vec{\alpha} \models_{\sigma[\vec{q}/p]}^{n} p = \phi
\]

which, by the semantic clause for \(=\), is equivalent to:
\[
\{ \vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} p \} = \{ \vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi \}
\]
which is just
\[
\vec{q} = \{ \vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi \}
\]
which, by the semantic clause for \(\downarrow\), is equivalent to:
\[
\vec{q} = \{ \vec{w} : \vec{w} \models^{n+k}_{\sigma[\vec{q}/p]} \downarrow^k\phi \}
\]
which is just
\[
\{ \vec{w} : \vec{w} \models^{n+k}_{\sigma[\vec{q}/p]} p \} = \{ \vec{w} : \vec{w} \models^{n+k}_{\sigma[\vec{q}/p]} \downarrow^k\phi \}
\]
which, by the semantic clause for \(=\), is equivalent to:
\[
\vec{\alpha} \models^{n+k}_{\sigma[\vec{q}/p]} p = \downarrow^k\phi
\]
Since \(\vec{q} \in P^{n+k}_{W_A}\), the semantic clause for \(\exists\) entails that this is equivalent to:
\[
\vec{\alpha} \models^{n+k}_{\sigma} \exists p (p = \downarrow^k\phi)
\]
which, by the semantic clause for \(\uparrow\), is equivalent to:
\[
\vec{\alpha} \models^{n}_{\sigma} \uparrow^k \exists p (p = \downarrow^k\phi)
\]

**Proposition 20 (Non-triviality)** There is a frame \(\langle W, A \rangle\), a level-\(n\) assignment \(\sigma\) \((n \in \mathbb{N})\), and a formula \(\phi\) of \(\mathcal{L}\) such that
\[
\{ \vec{w} : \vec{w} \models^n_{\sigma} \phi \} \notin P^n_{W_A}\]

**Proof** Let \(W = \{0\}\) and \(A = W^\infty\). Let \(w^1 = \left(0, \{0\}, \ldots, \{0\}\right)\) and \(v^1 = \left(0, \emptyset, \ldots, \emptyset\right)\). Let \(\vec{w}\) and \(\vec{v}\) be such that \(\vec{w}(1) = w^1\) and \(\vec{v}(1) = v^1\). Let \(\sigma\) be a level-0 assignment such that \(\sigma(p_1) = W_A\), and let \(\phi = \uparrow Q_1(p_1)\). Our semantic clauses then entail:
\[
\vec{w} \models^0_{\sigma} \uparrow Q_1(p_1) \iff \vec{w} \models^1_{\sigma} Q_1(p_1) \iff W_A \models_{[\mathcal{A}_1^N \text{Ext}^0_{Q_1}]} (\vec{w})
\]
But by the definition of $[W_A^{\text{Ext}_{Q_1}}]$ and the fact that $w^1 = \left(0, \{0\}, \ldots, \{0\}\right)$:

$$p \in [W_A^{\text{Ext}_{Q_1}}] (\vec{w}) \iff p \in P_{W_A}^0 \land \exists p^0 \in \{0\} \left(p(0) = p^0\right)$$

$$\iff p \in P_{W_A}^0 \land \vec{p}(0) = \emptyset$$

$$\iff p = \emptyset$$

So we have $\vec{w} \models_{\sigma}^0 Q_1(p_1)$. In contrast, we don’t have $\vec{v} \models_{\sigma}^0 Q_1(p_1)$. For, again by our semantic clauses,

$$\vec{v} \models_{\sigma}^0 Q_1(p_1) \leftrightarrow \vec{v} \models_{\sigma}^1 Q_1(p_1) \leftrightarrow W_A \in [W_A^{\text{Ext}_{Q_1}}] (\vec{v})$$

And we know from the definition of $[W_A^{\text{Ext}_{Q_1}}]$ and the fact that $v^1 = \left(0, \emptyset, \ldots, \emptyset\right)$ that

$$p \in [W_A^{\text{Ext}_{Q_1}}] (\vec{v}) \iff \bot$$

Since $\vec{w}(0) = \vec{v}(0) = 0$, we may conclude that

$$\{ \vec{z} : \vec{z} \models_{\sigma}^0 Q_1(p_1) \} \notin P_{W_A}^0$$

11 Axioms and Rules

Proposition 21 (Quantifiers)

1. Universal instantiation (propositional): $\models \forall p(\phi) \rightarrow \phi$

2. Universal instantiation (plural): $\models \forall pp(\phi) \rightarrow \phi$

3. Universal instantiation (intensional): $\models \forall O(\phi) \rightarrow \phi$

4. Existential generalization (propositional): $\models \phi \rightarrow \exists p \phi$.

5. Existential generalization (plural): $\models \phi \rightarrow \exists pp \phi$.

6. Existential generalization (intensional): $\models \phi \rightarrow \exists O \phi$.

Proof
1. **Universal Instantiation** (we focus on the propositional case; the others are analogous)

Fix a model \( \langle W, A, \bar{\alpha}, n \rangle \). For an arbitrary \( n \)-level assignment \( \sigma \), we assume \( \bar{\alpha} \models_\sigma^n \forall p(\phi) \) and show \( \bar{\alpha} \models_\sigma^n \phi \). Using the (derived) semantic clause for \( \forall \), our assumption entails that for any \( \bar{q} \in P^n_{W,A} \):

\[
\bar{\alpha} \models_\sigma^n \phi
\]

So this is true, in particular, when \( \bar{q} = \sigma(p) \) and therefore \( \sigma = \sigma[\bar{q}/p] \), which means that we have:

\[
\bar{\alpha} \models_\sigma^n \phi
\]

as desired.

4. **Existential Generalization** (we focus on the propositional case; the others are analogous)

Fix a model \( \langle W, A, \bar{\alpha}, n \rangle \). For an arbitrary \( n \)-level assignment \( \sigma \), we assume \( \bar{\alpha} \models_\sigma^n \phi \) and show \( \bar{\alpha} \models_\sigma^n \exists p \phi \). By the semantic clause for \( \exists \), it therefore suffices to verify that for some \( \bar{q} \in P^n_{W,A} \):

\[
\bar{\alpha} \models_\sigma^n \exists_{\bar{q}/p} \phi
\]

Let \( \bar{q} = \sigma(p) \). Accordingly, \( \sigma = \sigma[\bar{q}/p] \). So all we need to verify is

\[
\bar{\alpha} \models_\sigma^n \phi
\]

which is precisely what we had assumed.

**Proposition 22 (Rules)**

1. **Modus Ponens**: if \( \models \phi \) and \( \models \phi \rightarrow \psi \), then \( \models \psi \).
2. **Universal generalization (propositional)**: if \( \models \phi \), then \( \models \forall p \phi \).
3. **Universal generalization (plural)**: if \( \models \phi \), then \( \models \forall pp \phi \).
4. **Universal generalization (intensional)** if \( \models \phi \), then \( \models \forall O \phi \).
5. **Existential generalization (propositional)** if \( \models \phi \rightarrow \exists p \phi \).
6. Existential generalization (plural): if $\models \phi \rightarrow \models \exists p \phi$.

7. Existential generalization (intensional): if $\models \phi \rightarrow \models \exists O \phi$.

8. Next Introduction: if $\models \phi$, then $\models \uparrow \phi$.

9. Necessitation: if $\models \phi$, then $\models \Box \phi$.

Proof

1. Modus Ponens

   Fix a model $\langle W, A, \vec{\alpha}, n \rangle$. It suffices to verify the following for an arbitrary $n$-level assignment $\sigma$: if $\vec{\alpha} \models^n_\sigma \phi$ and $\vec{\alpha} \models^n_\sigma \phi \rightarrow \psi$, then $\vec{\alpha} \models^n_\sigma \psi$, which follows immediately from the (derived) semantic clause for $\rightarrow$.

2. Universal Generalization (we focus on the propositional case; the others are analogous)

   Assume $\models \phi$ and fix a model $\langle W, A, \vec{\alpha}, n \rangle$. It suffices to verify the following for an arbitrary $n$-level assignment $\sigma$: $\vec{\alpha} \models^n_{\sigma} \forall p \phi$. By the (derived) semantic clause for $\forall$, it therefore suffices to verify that for any $\vec{q} \in P^n_{W_A}$:

   $\vec{\alpha} \models^n_{\sigma[\vec{q}/p]} \phi$

   But this is an immediate consequence of $\models \phi$, since $\sigma[\vec{q}/p]$ is an assignment of level $n$.

5. Next Introduction

   Assume $\models \phi$ and fix a model $\langle W, A, \vec{\alpha}, n \rangle$. It suffices to verify the following for an arbitrary $n$-level assignment $\sigma$: $\vec{\alpha} \models^n_{\sigma} \uparrow \phi$. By the semantic clause for $\uparrow$ it therefore suffices to verify:

   $\vec{\alpha} \models^{n+1}_{\sigma} \phi$

   But since $\sigma$ is a level-$n$ assignment, proposition 13 entails that it is also a level-$(n + 1)$ assignment. So the result is an immediate consequence of $\models \phi$.

6. Necessitation

   Assume $\models \phi$ and fix a model $\langle W, A, \vec{\alpha}, n \rangle$. It suffices to verify the following for an arbitrary $n$-level assignment $\sigma$: $\vec{\alpha} \models^n_{\sigma} \Box \phi$. 

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By the (derived) semantic clause for □ it therefore suffices to verify that, for arbitrary \( \vec{w} \in W_A \):

\[
\vec{w} \models^n \phi
\]

which follows immediately from \( \models \phi \).

**Proposition 23 (The behavior of ↑)**

1. \( \models (\neg \uparrow \phi) \leftrightarrow (\uparrow \neg \phi) \)
2. \( \models (\Box \uparrow \phi) \leftrightarrow (\uparrow \Box \phi) \)
3. \( \models (\uparrow \phi \land \uparrow \psi) \leftrightarrow (\uparrow (\phi \land \psi)) \)
4. \( \models (\uparrow \phi = \uparrow \psi) \leftrightarrow (\phi = \psi) \)
5. \( \models \uparrow (p) \leftrightarrow p \)
6. \( \models \uparrow (p < pp) \leftrightarrow p < pp \)
7. \( \models \uparrow (Op) \leftrightarrow Op \)
8. \( \models (\downarrow \downarrow \phi) \leftrightarrow (\downarrow \downarrow \phi) \)

**Proof** Fix a model \( \langle W, A, \vec{\alpha}, n \rangle \). For an arbitrary \( n \)-level assignment \( \sigma \):

1. \( \models (\neg \uparrow \phi) \leftrightarrow (\uparrow \neg \phi) \)
   \[
   \vec{\alpha} \models_{\sigma}^{n+1} \phi \leftrightarrow \vec{\alpha} \not\models_{\sigma}^{n+1} \phi \\
   \vec{\alpha} \not\models_{\sigma}^{n+1} \phi \leftrightarrow \vec{\alpha} \models_{\sigma}^{n+1} \neg \phi \\
   \vec{\alpha} \models_{\sigma}^{n} \neg \uparrow \phi \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} \uparrow \neg \phi
   \]
2. \( \models (\Box \uparrow \phi) \leftrightarrow (\uparrow \Box \phi) \)
   \[
   \{ \vec{w} : \vec{w} \models_{\sigma}^{n+1} \phi \} \neq \emptyset \leftrightarrow \{ \vec{w} : \vec{w} \models_{\sigma}^{n+1} \phi \} \neq \emptyset \\
   \{ \vec{w} : \vec{w} \models_{\sigma}^{n} \uparrow \phi \} \neq \emptyset \leftrightarrow \vec{\alpha} \models_{\sigma}^{n+1} \Box \phi \\
   \vec{\alpha} \models_{\sigma}^{n} \Box \uparrow \phi \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} \uparrow \Box \phi
   \]
3. \( \models (\uparrow \phi \land \uparrow \psi) \leftrightarrow \uparrow (\phi \land \psi) \)

\[ (\vec{\alpha} \models_{\sigma}^{n+1} \phi \land \vec{\alpha} \models_{\sigma}^{n+1} \psi) \leftrightarrow (\vec{\alpha} \models_{\sigma}^{n+1} \phi \land \vec{\alpha} \models_{\sigma}^{n+1} \psi) \]

\[ (\vec{\alpha} \models_{\sigma}^{n} (\uparrow \phi \land \uparrow \psi) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n+1} (\phi \land \psi) \]

\[ \vec{\alpha} \models_{\sigma}^{n} (\uparrow \phi \land \uparrow \psi) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} (\phi \land \psi) \]

4. \( \models (\uparrow \phi = \uparrow \psi) \leftrightarrow \uparrow (\phi = \psi) \)

\[ \{ \vec{w} : \vec{w} \models_{\sigma}^{n+1} \phi \} = \{ \vec{w} : \vec{w} \models_{\sigma}^{n+1} \psi \} \leftrightarrow \{ \vec{w} : \vec{w} \models_{\sigma}^{n+1} \phi \} = \{ \vec{w} : \vec{w} \models_{\sigma}^{n+1} \psi \} \]

\[ \{ \vec{w} : \vec{w} \models_{\sigma}^{n} \uparrow \phi \} = \{ \vec{w} : \vec{w} \models_{\sigma}^{n} \uparrow \psi \} \leftrightarrow \vec{\alpha} \models_{\sigma}^{n+1} \phi = \psi \]

\[ \vec{\alpha} \models_{\sigma}^{n} (\uparrow \phi = \uparrow \psi) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} (\phi = \psi) \]

5. \( \models \uparrow (p) \leftrightarrow p \)

\[ \vec{\alpha} \in \sigma(p) \leftrightarrow \vec{\alpha} \in \sigma(p) \]

\[ \vec{\alpha} \models_{\sigma}^{n+1} p \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} p \]

\[ \vec{\alpha} \models_{\sigma}^{n} \uparrow (p) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} p \]

6. \( \models \uparrow (p < pp) \leftrightarrow p < pp \)

\[ \sigma(p) \in \sigma(pp) \leftrightarrow \sigma(p) \in \sigma(pp) \]

\[ \vec{\alpha} \models_{\sigma}^{n+1} (p < pp) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} p < pp \]

\[ \vec{\alpha} \models_{\sigma}^{n} \uparrow (p < pp) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} p < pp \]

7. \( \models \uparrow (Op) \leftrightarrow Op \)

\[ \vec{\alpha} \in \sigma(O)(\sigma(p)) \leftrightarrow \vec{\alpha} \in \sigma(O)(\sigma(p)) \]

\[ \vec{\alpha} \models_{\sigma}^{n+1} (Op) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} Op \]

\[ \vec{\alpha} \models_{\sigma}^{n} \uparrow (Op) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} Op \]
8. \( \models \left( \uparrow \downarrow \phi \right) \iff \left( \uparrow \uparrow \downarrow \phi \right) \)

\[
\begin{align*}
\bar{\alpha} \models_{\sigma} n^{+1} & \iff \bar{\alpha} \models_{\sigma} n + 1 \\
\bar{\alpha} \models_{\sigma} (\uparrow \phi) & \iff \bar{\alpha} \models_{\sigma} n + 2 (\downarrow \phi) \\
\bar{\alpha} \models_{\sigma} n + 1 (\downarrow \uparrow \phi) & \iff \bar{\alpha} \models_{\sigma} n (\uparrow \downarrow \phi) \\
\end{align*}
\]

12. **The behavior of \( Q \)**

**Definition 24** A natural model is a model \( \langle W, A, \bar{\alpha}, n \rangle \) such that \( A = W^\infty \).

**Proposition 24 (Non-functionality of Refinement)** Fix a natural model \( \langle W, A, \bar{\alpha}, n \rangle \). For any \( w^n \in W^n \), there are and \( w^{n+1}, v^{n+1} \in W^{n+1} \) such that \( v^{n+1} \neq w^{n+1} \) but

\[
w^n \triangleright_{W_A} w^{n+1} \land w^n \triangleright_{W_A} v^{n+1}
\]

**Proof** Suppose, first, that \( n = 0 \) and therefore that \( w^n = w \in W \). Let \( e^0 = \emptyset \) and \( f^0 = \{ W \} \). For \( i \) such that \( 1 < i \leq r \), let \( e^0_i = f^0_i = \emptyset \). Let \( w^{n+1} = \langle w, e^0_1, \ldots, w^0_r \rangle \) and \( v^{n+1} = \langle w, f^0_1, \ldots, f^0_r \rangle \). Since \( \langle W, A, \bar{\alpha}, n \rangle \) is a natural model, \( w^{n+1}, v^{n+1} \in W^{n+1} \). And since \( e_i^0 \neq f_i^0 \), \( w^{n+1} \neq v^{n+1} \). But it follows from the definition of \( \triangleright_{W_A} \) that

\[
w^n \triangleright_{W_A} w^{n+1} \land w^n \triangleright_{W_A} v^{n+1}
\]

Now suppose that \( n > 0 \) and let \( w^n = \langle w, e^1_1, \ldots, e^n_{n-1} \rangle \). Since \( \langle W, A, \bar{\alpha}, n \rangle \) is a natural model (and therefore \( A = W^\infty \)), it follows from Cantor’s Theorem that \( |P_{W_A}^{n-1}| > |P_{W_A}^n| \). So there must be some \( p^n \in P_{W_A}^n \) that is not identical to \( [p^{n-1}]_{W_A}^n \) for \( p^{n-1} \in P_{W_A}^{n-1} \). For each \( i \leq r \), let \( f_i^n = \{ [p^{n-1}]_{W_A}^n : p^{n-1} \in e_i^{n-1} \} \). Let \( e_1^n = f_1^n \cup \{ p^n \} \), and for \( i \) such that \( 1 < i \leq r \), let \( e_i^n = f_i^n \). Let \( w^{n+1} = \langle w, e_1^n, \ldots, w^n \rangle \) and \( v^{n+1} = \langle w, f_1^n, \ldots, f^n \rangle \). Since \( \langle W, A, \bar{\alpha}, n \rangle \) is a natural model, \( w^{n+1}, v^{n+1} \in W_A^{n+1} \). And since \( e_1^n \neq f_1^n \), \( w^{n+1} \neq v^{n+1} \). But it follows from the definition of \( \triangleright_{W_A} \) that

\[
w^n \triangleright_{W_A} w^{n+1} \land w^n \triangleright_{W_A} v^{n+1}
\]
Proposition 25 (Non-triviality of the Superproposition Hierarchy)
Fix a natural model \( \langle W, A, \vec{\alpha}, n \rangle \). For any \( n \in \mathbb{N} \), there is a super-proposition \( \vec{q} \) such that \( \vec{q} \in P_{W,A}^{n+1} \) but \( \vec{q} \notin P_{W,A}^n \).

Proof: Let \( \vec{w} \) be an arbitrary world in \( W_A \). By proposition 24, there are \( v_{n+1}, z_{n+1} \in W_{A}^{n+1} \) such that \( v_{n+1} \neq z_{n+1} \) but
\[
\vec{w}(n + 1) \triangleright W_A v_{n+1} \land \vec{w}(n) \triangleright W_A z_{n+1}
\]
By proposition 3, \( w_{n+1} \) and \( v_{n+1} \) we may assume that there are superworlds \( \vec{v} \) and \( \vec{z} \) such that \( \vec{v}(n + 1) = v_{n+1} \) and \( \vec{z}(n + 1) = z_{n+1} \) and therefore such that \( \vec{v}(n + 1) \neq \vec{z}(n + 1) \). And by proposition 2, \( \vec{v}(n) = \vec{w}(n) = \vec{z}(n) \).

Let \( \vec{q} = \{ y \in W_A : \vec{y}(n+1) = v(n+1) \} \). Trivially, \( \vec{q} \in P_{W,A}^n \). But \( \vec{q} \notin P_{W,A}^n \), since \( \vec{z} \notin \vec{q} \) even though \( \vec{v}(n) = \vec{z}(n) \).

Proposition 26 (Prior and Kaplan) When attention is restricted to natural models:

1. No Same Level: \( = \exists p \neg Q,p \)
2. Kaplan Next: \( = \forall p \uparrow \diamond \forall q (Q,q \leftrightarrow q < pp) \)
3. Kaplan Next: \( = \forall pp \diamond \forall q (\uparrow Q,q \leftrightarrow q < pp) \)
4. Modal Prior Next: \( = \forall p \uparrow \diamond \forall q (Q,q \leftrightarrow q = p) \)
5. Modal Prior Next: \( = \forall p \diamond \forall q (\uparrow Q,q \leftrightarrow q = p) \)

Proof

1. \( = \exists p \neg Q,p \)
Fix a natural model \( \langle W, A, \vec{\alpha}, n \rangle \). Suppose, first, that \( n = 0 \) and let \( \sigma \) be an arbitrary \( n \)-level assignment. By the semantic clause for \( Q \)
\[
\vec{\alpha} \not\models^{n}_\sigma Q,p
\]
So, by the semantic clause for \( \neg \),
\[
\vec{\alpha} \models^{n}_\sigma \neg Q,p
\]
So, by existential generalization (proposition 21),

$$\bar{\alpha} \models^n \exists \bar{p} \bar{\neg} \bar{Q_i} \bar{p}$$

Now assume $n > 0$ and let $\bar{\bar{q}}$ be in $P^n_{\mathcal{W}_A}$ but not $P^{n-1}_{\mathcal{W}_A}$ (since $(W, A, \bar{\alpha}, n)$ is a natural model, proposition 25 entails that such a $\bar{\bar{q}}$ must exist). Suppose, for reductio, that for some $n$-level assignment $\sigma$, $\bar{\alpha} \models^n \bar{Q_i} \bar{p}$. By the semantic clause for $\bar{Q_i}$,

$$\bar{\bar{q}} \in [\mathcal{W}^n_{\mathcal{A}^\text{Ext}}_{\bar{Q_i}}] (\bar{\alpha})$$

Now let $\bar{\bar{\bar{q}}} = \langle w, e_1^{n-1}, \ldots, e_r^{n-1} \rangle$. By the definition of $[\mathcal{W}^n_{\mathcal{A}^\text{Ext}}_{\bar{Q_i}}]$, $\bar{\bar{\bar{q}}} \in [\mathcal{W}^n_{\mathcal{A}^\text{Ext}}_{\bar{Q_i}}] (\bar{\alpha}) \leftrightarrow (\bar{\bar{q}} \in P^{n-1}_{\mathcal{W}_A} \land \exists p^{n-1} \in e_i^{n-1}(q(n-1) = p^{n-1}))$

So we have $\bar{\bar{q}} \in P^{n-1}_{\mathcal{W}_A}$, which contradicts an earlier assumption. It follows that for every $n$-level assignment $\sigma$: $\bar{\alpha} \not\models^n \bar{Q_i} \bar{p}$

So we can get the desired result by replicating the reasoning we deployed in the case $n = 0$.

2. $\models \forall \bar{p} \bar{\bar{p}} \downarrow \diamond \forall \bar{q}(\bar{Q_i} \bar{q} \leftrightarrow q < \bar{p} \bar{p})$

Fix a natural model $(W, A, \bar{\alpha}, n)$. Fix arbitrary $\bar{v} \in W$ and $\bar{\bar{B}} \subseteq P^n_{\mathcal{W}_A}$ ($\bar{\bar{B}} \neq \emptyset$) and let

$$B^n = \{ p^n \in P^n_{W_A} : \exists \bar{p} \in \bar{\bar{B}} (p^n = \bar{p}(n)) \} \quad v^{n+1} = \langle \bar{v}, e_1^n, \ldots, e_r^n \rangle$$

where $e_j^n = B^n$ for each $j \neq r$. Since $(W, A, \bar{\alpha}, n)$ is a natural model, $v^{n+1} \in W^{n+1}_A$. So, by proposition 3, there is a superworld $\bar{\bar{v}} \in \mathcal{W}_A$ such that $\bar{v}(n + 1) = v^{n+1}$.

Now pick an arbitrary $\bar{\bar{q}} \in P^{n+1}_{W_A}$. We verify:

$$\bar{\bar{q}} \in [\mathcal{W}^n_{\mathcal{A}^\text{Ext}}_{\bar{Q_i}}] (\bar{\bar{v}}) \leftrightarrow \bar{\bar{q}} \in \bar{\bar{B}}$$

Assume $\bar{\bar{q}} \in [\mathcal{W}^n_{\mathcal{A}^\text{Ext}}_{\bar{Q_i}}] (\bar{\bar{v}})$. By the definition of $[\mathcal{W}^n_{\mathcal{A}^\text{Ext}}_{\bar{Q_i}}]$, we have:

$$\bar{\bar{q}} \in P^n_{W_A} \land \exists p^n \in e_i^n(\bar{q}(n) = p^n)$$
and therefore
\[ \exists p^n \in B^n(q(n) = p^n) \]
So, by the definition of \( B^n \):
\[ \exists p^n \in \left\{ p^n \in P^n_{\mathcal{W}} : \exists \vec{p} \in \vec{B}(p^n = \vec{p}(n)) \right\} (q(n) = p^n) \]
equivalently
\[ \exists p^n \in P^n_{\mathcal{W}A} \exists \vec{p} \in \vec{B}(p^n = \vec{p}(n) \land q(n) = p^n) \]
We may therefore fix \( p^n \in P^n_{\mathcal{W}A} \) and \( \vec{p} \in \vec{B} \) such that
\[ p^n = \vec{p}(n) \land q(n) = p^n \]
and therefore
\[ \vec{p}(n) = q(n). \]
But since \( q \in P^n_{\mathcal{W}A} \) and \( \vec{p} \in \vec{B} \) (and therefore \( \vec{p} \in P^n_{\mathcal{W}A} \)), proposition 7 entails:
\[ \vec{p} = q \]
which is what we wanted.

\[ \leftarrow \]
Assume \( q \in \vec{B} \). Since \( \vec{B} \subseteq P^n_{\mathcal{W}A} \), \( q \in P^n_{\mathcal{W}A} \). So our assumption is equivalent to:
\[ \exists \vec{p} \in \vec{B}(\vec{p}(n) = q(n)) \]
which is equivalent to:
\[ \exists p^n \in P^n_{\mathcal{W}A} \exists \vec{p} \in \vec{B}(p^n = \vec{p}(n) \land q(n) = p^n) \]
and therefore
\[ \exists p^n \in \left\{ p^n \in P^n_{\mathcal{W}A} : \exists \vec{p} \in \vec{B}(p^n = \vec{p}(n)) \right\} (q(n) = p^n) \]
which, by the definition of \( B^n \), is equivalent to:
\[ \exists p^n \in B^n(q(n) = p^n) \]
which is equivalent to
\[ \exists p^n \in e^n_i(q(n) = p^n) \]
Since $\vec{q} \in P^n_{W_A}$, we may conclude:
$$\vec{q} \in P^n_{W_A} \land \exists p^n \in e^n_i(\vec{q}(n)) = p^n$$
which gives us what we want, by the definition of $[W^A_{\text{Ext}}_Q^n]$:
$$\vec{q} \in [W^A_{\text{Ext}}_Q^n](\vec{v})$$
We have shown that for arbitrary $\vec{q} \in P^{n+1}_{W_A}$ and $\vec{B} \subseteq P^n_{W_A}$ ($\vec{B} \neq \emptyset$),
$$\vec{q} \in [W^A_{\text{Ext}}_Q^n](\vec{v}) \leftrightarrow \vec{q} \in \vec{B}$$
So, by the semantic clause for $Q$ and $\prec$, we have the following for an arbitrary level $n$ assignment $\sigma$:
$$\vec{v} \models^{n+1}_{\sigma[\vec{B}/pp][\vec{q}/p]} Qp \leftrightarrow p \prec pp$$
But since $\vec{q}$ was an arbitrary member of $P^{n+1}_{W_A}$, the (derived) semantic clause for $\forall$ gives us:
$$\vec{v} \models^{n+1}_{\sigma[\vec{B}/pp]} \forall p(Qp \leftrightarrow p \prec pp)$$
Since $\vec{v} \in W_A$, this gives us:
$$\{ \vec{w} : \vec{w} \models^{n+1}_{\sigma[\vec{B}/pp]} \forall p(Qp \leftrightarrow p \prec pp) \} \neq \emptyset$$
So, by the (derived) semantic clause for $\Diamond$,
$$\vec{\alpha} \models^{n+1}_{\sigma[\vec{B}/pp]} \Diamond \forall p(Qp \leftrightarrow p \prec pp)$$
So, by the semantic clause for $\uparrow$,
$$\vec{\alpha} \models^n_{\sigma[\vec{B}/pp]} \uparrow \Diamond \forall p(Qp \leftrightarrow p \prec pp)$$
But since $\vec{B} \subseteq P^n_{W_A}$ was chosen arbitrarily, the (derived) semantic clause for $\forall$ gives us:
$$\vec{\alpha} \models^n \forall pp \uparrow \Diamond \forall p(Qp \leftrightarrow p \prec pp)$$
which is what we wanted.

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3. \( \models \forall pp \forall q (\uparrow (Q, q) \leftrightarrow q \prec pp) \)

Fix a natural model \( \langle W, \mathcal{A}, \vec{\alpha}, n \rangle \). Fix arbitrary \( v \in W \) and \( \vec{B} \subseteq P^*_W \) \( (\vec{B} \neq \emptyset) \) and define \( \vec{v} \) as in the previous case. As in the previous case, we can show for arbitrary \( \vec{q} \in P^*_W \)

\[ \vec{q} \in [\mathcal{W}^{\text{Ext}^n_Q}] (\vec{v}) \leftrightarrow \vec{q} \in \vec{B} \]

So, by the semantic clause for \( Q \) and \( \prec \), we have the following for an arbitrary level \( n \) assignment \( \sigma \):

\[ \vec{v} \models_{\sigma[\vec{B}/pp][\vec{q}/p]}^{n+1} Qp \leftrightarrow \vec{v} \models_{\sigma[\vec{B}/pp][\vec{q}/p]}^n p \prec pp \]

So, by the semantic clause for \( \uparrow \),

\[ \vec{v} \models_{\sigma[\vec{B}/pp][\vec{q}/p]}^n (Qp) \leftrightarrow \vec{v} \models_{\sigma[\vec{B}/pp][\vec{q}/p]}^n p \prec pp \]

and therefore

\[ \vec{v} \models_{\sigma[\vec{B}/pp][\vec{q}/p]}^n (Qp) \leftrightarrow p \prec pp \]

But since \( \vec{q} \) was chosen arbitrarily from \( P^*_W \), proposition 5 guarantees that the result also holds when \( \vec{q} \) is chosen arbitrarily from \( P^n_W \). So the (derived) semantic clause for \( \forall \) gives us:

\[ \vec{v} \models_{\sigma[\vec{B}/pp]}^n \forall p( (Qp) \leftrightarrow p \prec pp) \]

Since \( \vec{v} \in \mathcal{W}_A \), this gives us:

\[ \{ \vec{w} : \vec{w} \models_{\sigma[\vec{B}/pp]}^n \forall p( (Qp) \leftrightarrow p \prec pp) \} \neq \emptyset \]

So, by the (derived) semantic clause for \( \Diamond \),

\[ \vec{\alpha} \models_{\sigma[\vec{B}/pp]}^n \forall p( (Qp) \leftrightarrow p \prec pp) \]

But since \( \vec{B} \subseteq P^n_{\mathcal{W}_A} \) was chosen arbitrarily, the (derived) semantic clause for \( \forall \) gives us:

\[ \vec{\alpha} \models_{\sigma}^{n} \forall pp \forall p( (Qp) \leftrightarrow p \prec pp) \]

which is what we wanted.
4. $\models \forall p \uparrow \forall q (Q_i q \leftrightarrow q < pp)$
Analogous to the proof of more general result.

5. $\models \forall p \uparrow \forall q (\uparrow Q_i q \leftrightarrow q = p)$
Analogous to the proof of more general result.

**Definition 25** Fix a model $\langle W, A, \vec{\alpha}, m \rangle$. $Q_i$ and $Q_j$ are **independent** (relative to the relevant model) if and only if, for any $\vec{p} \in P^n_W$ ($n \in \mathbb{N}$), there is $\vec{w} \in W_A$ such that

$$
\vec{p} \in \left[ W \text{Ext}^n_{Q_i} \right] (\vec{w}) \leftrightarrow \vec{p} \not\in \left[ W \text{Ext}^n_{Q_j} \right] (\vec{w})
$$

**Proposition 27 (Some models exemplify independence)** Whenever $i \neq j$, $Q_i$ and $Q_j$ and independent relative to any natural model.

**Proof** Assume, with no loss of generality, that $i = 1$ and $j = 2$. Let $\langle W, A, \vec{\alpha}, m \rangle$ be a natural model and let $\vec{p} \in P^n_W$ ($n \in \mathbb{N}$). For any $w \in W$, let

$$
w^{n+1} = \left\langle w, \{\vec{p}(n)\}, \emptyset, \ldots, \emptyset \right\rangle_{(r-1) \text{ times}}
$$

Since $\langle W, A, \vec{\alpha}, m \rangle$ is a natural model, $w^{n+1} \in W^n_{A_{n+1}}$. So, by proposition 3, there is $\vec{w} \in W_A$ such that $\vec{w}(n+1) = w^{n+1}$. We then have:

$$
\vec{p} \in P^n_W \land \vec{p}(n) = \vec{p}(n), \quad \neg(\bot)
$$

$$
\vec{p} \in P^n_W \land \exists p^n \in \{\vec{p}(n)\} (\vec{p}(n) = p^n), \quad \neg(\vec{p} \in P^n_W \land \exists p^n \in \emptyset(\vec{p}(n) = p^n))
$$

$$
\vec{p} \in \left[ A \text{Ext}^n_{Q_1} \right] (\vec{w}), \quad \vec{p} \not\in \left[ A \text{Ext}^n_{Q_2} \right] (\vec{w})
$$

**Proposition 28 (Russell-Myhill Next)** Whenever $Q_i$ and $Q_j$ are independent, $\models \uparrow (Q_i p \neq Q_j p)$

**Proof** Let $Q_i$ and $Q_j$ be independent and assume, for reductio, that $\not\models \uparrow \neg(\overline{Q}_i p = \overline{Q}_j p)$. By our assumption, there is a model $\langle W, A, \vec{\alpha}, n \rangle$ and an $n$-level assignment $\sigma$ such that:

$$
\vec{\alpha} \not\models_{\sigma} \uparrow \neg(\overline{Q}_i p = \overline{Q}_j p)
$$
which, by proposition 23, is equivalent to:

$$\vec{\alpha} \not\models^n \neg (\mathcal{Q}_i p = \mathcal{Q}_j p)$$

which, by the semantic clause for $$\neg$$, is equivalent to:

$$\vec{\alpha} \models^n (\mathcal{Q}_i p = \mathcal{Q}_j p)$$

which, by the semantic clause for $$\uparrow$$, is equivalent to:

$$\vec{\alpha} \models^{n+1} \mathcal{Q}_i p = \mathcal{Q}_j p$$

which, by the semantic clause for $$=$$, is equivalent to:

$$\{ \vec{w} : \vec{w} \models^{n+1} \mathcal{Q}_i p \} = \{ \vec{w} : \vec{w} \models^{n+1} \mathcal{Q}_j p \}$$

So, for any $$\vec{w} \in \mathcal{W}_A$$,

$$\vec{w} \models^{n+1} \mathcal{Q}_i p \leftrightarrow \vec{w} \models^{n+1} \mathcal{Q}_j p$$

So, by the semantic clause for $$\mathcal{Q}$$, the following holds for any $$\vec{w} \in \mathcal{W}_A$$,

$$\sigma(p) \in \llbracket_{\mathcal{W}_A}^{\mathcal{W}^n_{\text{Ext}}_{\mathcal{Q}_1}} (\vec{w}) \leftrightarrow \sigma(p) \in \llbracket_{\mathcal{W}_A}^{\mathcal{W}^n_{\text{Ext}}_{\mathcal{Q}_2}} (\vec{w})$$

which contradicst the assumption that $$\mathcal{Q}_i$$ and $$\mathcal{Q}_j$$ are independent.

**Proposition 29 (Intensional Cases)**

1. $$\models \uparrow \exists O \Box p (\uparrow(\mathcal{Q}_i p) \not\rightarrow Op)$$
2. $$\not\models \forall O \Diamond \forall p (\uparrow(\mathcal{Q}_i p) \leftrightarrow Op)$$

**Proof**

1. $$\models \uparrow \exists O \Box p (\uparrow(\mathcal{Q}_i p) \not\rightarrow Op)$$

   Fix a model $$\langle \mathcal{W}, \mathcal{A}, \vec{\alpha}, n \rangle$$ and an arbitrary $$n$$-level assignment, $$\sigma$$. Let $$\vec{\tau} \in I_{\mathcal{W}_A}$$ be defined as follows:

$$\vec{\tau}(\vec{q}) = \left\{ \vec{w} \in \mathcal{W}_A : \vec{w} \not\models^{n+1}_{\sigma[\vec{q}/p]} \mathcal{Q}_i p \right\}$$

Let us verify that $$\vec{\tau} \in I_{\mathcal{W}_A}^{n+1}$$:
We assume $\vec{q} \in P_{n+1}^{n+1}$ and show $\vec{r}(\vec{q}) \in P_{n+1}^{n+1}$. Since $\vec{q} \in P_{n+1}^{n+1}$, and since $\sigma$ is an assignment of level $n$, $\sigma[\vec{q}/\overline{p}]$ is an assignment of level $n+1$. So Lemma 1 gives us:

$$\{ \vec{w} \in W_A : \vec{w} \not\models^{n+1}_{\sigma} Q_{i} p \} \in P_{n+1}^{n+1}$$

which is what we wanted.

So we know that $\vec{r} \in I_{n+1}^{n+1}$ and therefore that $\sigma[\vec{r}/O]$ is an assignment of level $n+1$.

Choose $\vec{v} \in W_A$ arbitrarily and let $\vec{q} \in P_{n}^{n}$. Then propositions 10 and 11 give us:

$$\vec{q} \in [W]^{n+1}_{\text{Ext}_{Q_{i}}} (\vec{v}) \leftrightarrow \vec{q} \in [W]^{n}_{\text{Ext}_{Q_{i}}} (\vec{v})$$

So, by the semantic clause for $Q_{i}$,

$$\vec{v} \models^{n+2}_{\sigma[\vec{r}/O][\vec{q}/\overline{p}]} Q_{i} p \leftrightarrow \vec{v} \models^{n+1}_{\sigma[\vec{q}/\overline{p}]} Q_{i} p$$

which is equivalent to:

$$\vec{v} \models^{n+2}_{\sigma[\vec{r}/O][\vec{q}/\overline{p}]} Q_{i} p \not\leftrightarrow \vec{v} \not\models^{n+1}_{\sigma[\vec{q}/\overline{p}]} Q_{i} p$$

which is equivalent to:

$$\vec{v} \models^{n+2}_{\sigma[\vec{r}/O][\vec{q}/\overline{p}]} Q_{i} p \not\leftrightarrow \vec{v} \not\models^{n+1}_{\sigma[\vec{q}/\overline{p}]} Q_{i} p$$

so, by the definition of $\vec{r}$,

$$\vec{v} \models^{n+2}_{\sigma[\vec{r}/O][\vec{q}/\overline{p}]} Q_{i} p \not\leftrightarrow \vec{v} \in \vec{r}(\vec{q})$$

So, by the semantic clauses for $\uparrow$ and $Op$,

$$\vec{v} \models^{n+1}_{\sigma[\vec{r}/O][\vec{q}/\overline{p}]} \uparrow(Q_{i} p) \not\leftrightarrow \vec{v} \models^{n+1}_{\sigma[\vec{r}/O][\vec{q}/\overline{p}]} Op$$

So, by the semantic clauses for Boolean operators,

$$\vec{v} \models^{n+1}_{\sigma[\vec{r}/O][\vec{q}/\overline{p}]} \uparrow(Q_{i} p) \not\leftrightarrow Op$$

Since $\vec{q}$ is in $P_{n}^{n}$ and therefore in $P_{n+1}^{n+1}$, the semantic clause for $\exists$ gives us:

$$\vec{v} \models^{n+1}_{\sigma[\vec{r}/O]} \exists p(\uparrow(Q_{i} p) \not\leftrightarrow Op)$$
Since $\vec{v} \in W_A$ was chosen arbitrarily, this gives us:

$$\left\{ \vec{v} : \vec{v} \models^{n+1}_{\sigma[\vec{v}/\vec{O}]} \exists p(\uparrow(Q_i p) \not\leftrightarrow Op) \right\} = W_A$$

So, by the (derived) semantic clause for $\Box$,

$$\vec{\alpha} \models^{n+1}_{\sigma[\vec{v}/\vec{O}]} \Box \exists p(\uparrow(Q_i p) \not\leftrightarrow Op)$$

But since $\vec{v} \in I^{n}_{W_A}$, the semantic clause for $\exists$ gives us

$$\vec{\alpha} \models^{n+1}_{\sigma} \exists O \Box \exists p(\uparrow(Q_i p) \not\leftrightarrow Op)$$

So, by the semantic clause for $\uparrow$,

$$\vec{\alpha} \models^{n}_{\sigma} \uparrow \exists O \Box \exists p(\uparrow(Q_i p) \not\leftrightarrow Op)$$

which is what we wanted.

2. $\not\models \forall O \Diamond \forall p(\uparrow(Q_i p) \leftrightarrow Op)$

Suppose otherwise:

$$\models \forall O \Diamond \forall p(\uparrow(Q_i p) \leftrightarrow Op)$$

By proposition 22, this means that:

$$\models \uparrow \forall O \Diamond \forall p(\uparrow(Q_i p) \leftrightarrow Op)$$

But by the previous result, we have

$$\models \uparrow \exists O \Box \exists p(\uparrow(Q_i p) \not\leftrightarrow Op)$$

which is equivalent to:

$$\models \neg \uparrow \forall O \Diamond \forall p(\uparrow(Q_i p) \leftrightarrow Op)$$

**Proposition 30 (Validity Failures)**

- $\not\models \uparrow \phi \rightarrow \phi$

**Proof**
Consider a model $\langle W, A, \vec{\alpha}, 0 \rangle$, where $W = \{0\}$, $A = W^\infty$, $w^1 = \langle 0, r \times \emptyset, \ldots, \emptyset \rangle$, and $\vec{\alpha}$ is such that $\vec{\alpha}(1) = w^1$. Let $\sigma$ be an assignment such that $\sigma(p) = \emptyset$. So we have $\sigma(p) \in P^0_{W,A}$ and $\sigma(p)(0) = \{\vec{w}(0) : \vec{w} \in \sigma(p)\} = \emptyset$. We verify that $\vec{\alpha} \models^0 \uparrow Q_i(p)$ but $\vec{\alpha} \not\models^0 Q_i(p)$: The latter is an immediate consequence of the semantic clause for $Q_i$. So it suffices to verify the former. But, trivially,

$$\exists p^0 \in \{\emptyset\} (\emptyset = p^0)$$

And since $\sigma(p) \in P^0_{W,A}$ and $\sigma(p)(0) = \emptyset$, this gives us:

$$\sigma(p) \in P^0_{W,A} \land \exists p^0 \in \{\emptyset\} (\sigma(p)(0) = p^0)$$

equivalently,

$$\sigma(p) \in \{\vec{p} \in P^0_{W,A} : \exists p^0 \in \{\emptyset\} (\vec{p}(0) = p^0)\}$$

So, by the definition of $[A^W_{\alpha}]_{i}$

$$\sigma(p) \in [A^W_{\alpha}]_{i}$$

So, by the semantic clause for $Q_i$

$$\vec{\alpha} \models^1 Q_i(p)$$

So, by the semantic clause for $\uparrow$:

$$\vec{\alpha} \models^0 \uparrow Q_i(p)$$

### 13 Examples

A proof of Prior: \( \models OE^- \rightarrow (E^+ \land E^-) \)

- \( E^+ := \exists p (Op \land p) \)
- \( E^- := \exists p (Op \land \neg p) \)

1. \( OE^- \) (assumption) [1]
2. \( \neg E^- \) (assumption) [2]
3. \( \neg \neg \forall p(\text{Op} \rightarrow p) \) (from 2, by definition) [2]
4. \( \forall p(\text{Op} \rightarrow p) \) (from 3, by Double Negation Elimination) [2]
5. \( (OE^- \rightarrow E^-) \) (from 4, by Universal Instantiation) [2]
6. \( E^- \) (from 1 and 5, by Modus Ponens) [2, 1]
7. \( E^- \) (from 6 discharging 2, by Conditional Proof) [1]
8. \( (O(E^-) \land E^-) \) (from 7 and 1, by Conjunction Introduction) [1]
9. \( \exists p(\text{Op} \land p) \) (from 8, by Existential Generalization) [1]
10. \( (E^+ \land E^-) \) (from 7 and 9, by Conjunction Introduction) [1]
11. \( OE^- \rightarrow (E^+ \land E^-) \) (from 10, discharging 1, by Conditional Proof)

A proof of Modal Prior: \( \models \exists p \square \neg \forall q(Oq \leftrightarrow (q = p)) \)
1. \( \forall q(Oq \leftrightarrow (q = E^-)) \) (assumption) [1]
2. \( OE^- \leftrightarrow (E^- = E^-) \) (from 1, by UG) [1]
3. \( OE^- \) (from 1, by MP and reflexivity of identity) [1]
4. \( OE^- \rightarrow (E^+ \land E^-) \) (Prior) []
5. \( E^+ \land E^- \) (from 3 and 4 by MP) [1]
6. \( \exists p(\text{Op} \land \neg p) \) (from 5, by conjunction elimination) [1]
7. \( (\text{Op} \land \neg p) \) (from 6, by EI) [1]
8. \( Op \leftrightarrow (p = E^-) \) (from 1, by UG) [1]
9. \( p = \neg E^- \) (from 7 and 8), by MP and conj. elim.) [1]
10. \( \neg E^- \) (from 7 and 9), by identity subs. and conj. elim.) [1]
11. \( \neg \forall q(Oq \leftrightarrow (q = E^-)) \) (by reductio, from 5 and 10, discharging 1) []
12. \( \square \neg \forall q(Oq \leftrightarrow (q = E^-)) \) (from 11, by Necessitation) []
13. \( \exists p \square \neg \forall q(Oq \leftrightarrow (q = p)) \) (from 12, by Existential Generalization) []