Why I am not an Absolutist (Or a First-Orderist)
Supplementary Document

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1 Informal Summary

1. We start with a propositional language $\mathscr{L}^-$ consisting of the following symbols:

Symbol	Notation	Type
propositional variables	$p_1, p_2, \ldots$	$\langle\rangle$
plural variables	$pp_1, pp_2, \ldots$	$\langle\rangle\langle\rangle$
operator variables	$O_1, O_2, \ldots$	$\langle\langle\rangle\rangle$
identity symbol	$=$	$\langle\langle\rangle, \langle\rangle\rangle$
inclusion symbol	$\prec$	$\langle\langle\rangle, \langle\rangle\langle\rangle\rangle$
existential quantifier	$\exists$	$\langle\langle\rangle\rangle$
negation symbol	$\neg$	$\langle\langle\rangle\rangle$
conjunction symbol	$\wedge$	$\langle\langle\rangle, \langle\rangle\rangle$
parentheses	$(,)$	-

We also introduce some abbreviations:

Notation	Abbreviates
$\bot$	$\exists p_1(p_1 \wedge \neg p_1)$
$\Diamond\phi$	$\neg(\phi = \bot)$

2. We enrich $\mathscr{L}^-$ to a language $\mathscr{L}$, by adding the following symbols:

Symbol	Notation	Type
condition constants	$\mathcal{Q}_1, \ldots, \mathcal{Q}_r$	$\langle\langle\rangle\rangle$
resolution increase	$\uparrow$	$\langle\langle\rangle\rangle$
resolution decrease	$\downarrow$	$\langle\langle\rangle\rangle$

   The condition constants are used to express "procedures". Intuitively, a procedure $\mathcal{Q}$ might be used to characterize an operator $O_{\mathscr{C}}$, relative to a space of propositions.

   I'll say more about the arrows below.

3. We work with a hierarchy of sets of "worlds", of increasing levels of resolution:

   - For W a non-empty set,
     - $W^0 = W$
     - $P_W^n = \wp(W^n)$
     - $W^{n+1} = \{\langle w, e_1^n, \ldots, e_r^n\rangle : w \in W \wedge e_i^n \subseteq P_W^n\}$ [1]

4. This allows us to define "superworlds":

   - A **superworld** is a sequence $\langle w^0, w^1, w^2, \ldots\rangle$ such that:
     - $w^k \in W_W^k$
     - each w^k is "refined" by w^{k+1}. [2]

[1] Intuitively, e_i^n is the extension of $\mathcal{Q}_i$ at world $\langle w, e_1^n, \ldots, e_r^n\rangle \in W^{n+1}$.

[2] Intuitively, w^k is refined by w^{k+1} when it agrees about the extension of each $\mathcal{Q}_i$ as far as propositions in P_W^n are concerned.

- Superworlds are assessed *at a given level of resolution.*
    - A superworld $\langle w^0, w^1, w^2, \ldots \rangle$ assessed at resolution level k behaves like w^k.

5. The arrows, $\uparrow$ and $\downarrow$
    - $\uparrow$ increases by 1 the level of resolution with respect to which superworlds are assessed.
    - $\downarrow$ decreases by 1 the level of resolution with respect to which superworlds are assessed. [3]

6. The result is a well-behaved system:
    - One gets standard axioms, when attention is restricted to $\mathscr{L}^-$.
    - One gets sensible axioms for the general case, including a nice comprehension principle.

7. One gets a system that does not encourage lapsing into nonsense
    - If logical space is genuinely open-ended, talking about "all possible refinements" is problematic. (For example, it can lead to revenge issues.) But having $\uparrow$ and $\downarrow$ instead of $\Diamond$ allows us to stay well within the range of sense.

2 The language

Definition 1 *$\mathscr{L}$ is a language built from the following symbols:*

- *the propositional variables $p_1, p_2, \ldots,$ which are of type $\langle\rangle$;*
- *the plural propositional variables $pp_1, pp_2, \ldots,$ which are of type $\langle\rangle\langle\rangle$;*
- *the propositional identity symbol, $=$, which is of type $\langle\langle\rangle, \langle\rangle\rangle$;*
- *the propositional inclusion relation $\prec$, which is of type $\langle\langle\rangle, \langle\rangle\langle\rangle\rangle$*
- *the operator variables $O_1, O_2, \ldots,$ which are of type $\langle\langle\rangle\rangle$;*

[3] Unless it is already 0, in which case $\downarrow$ does nothing.

- *for $r > 0$, the indefinitely extensible constants $\mathcal{Q}_1, \ldots, \mathcal{Q}_r$, which are of type $\langle\langle\rangle\rangle$;*
- *the existential quantifier, $\exists$, which binds variables of any type;*
- *the negation symbol, $\neg$, the conjunction symbol, $\wedge$, and parentheses;*
- *the refinement operator, $\uparrow$, and unrefinement operator, $\downarrow$, which are of type $\langle\langle\rangle\rangle$.*

Definition 2 *The expressions "$\bot$", "$\top$", "$\forall$", "$\vee$", "$\rightarrow$", and "$\leftrightarrow$" are defined, in the usual way. In addition:*

- $\Diamond\phi := \bot \neq \phi \qquad \Box\phi := \phi = \top$
- $\phi \gg \psi := (\phi = (\phi \wedge \psi))$

Definition 3 *The formulas of $\mathscr{L}$ are defined recursively, in the obvious way. A sentence is a formula in which every occurrence of a variable is bound by a quantifier.*

3 Some Results

Here are some results, which presuppose that attention is restricted to "natural" models:

For

- $E^+ := \exists p(Op \wedge p)$
- $E^- := \exists p(Op \wedge \neg p)$

Prior $\models O(E^-) \rightarrow (E^+ \wedge E^-)$

Extensional Prior $\models \forall p[(p \leftrightarrow E^-) \rightarrow (Op \rightarrow (E^+ \wedge E^-))]$

An immediate consequence of Prior is:

Modal Prior $\models \neg\forall p \Diamond \forall q(Oq \leftrightarrow (q = p))$

But we can also show:

Modal Prior Next: $\models \forall p {\uparrow} \Diamond \forall q(\mathcal{Q}_i q \leftrightarrow (q = p))$

or, equivalently:

Modal Prior Next: $\models \forall p \Diamond \forall q({\uparrow}\mathcal{Q}_i q \leftrightarrow (q = p))$

There are obvious generalizations of Modal Prior and Modal Prior Next:

Kaplan $\models \neg \forall pp \Diamond \forall q(Oq \leftrightarrow (q \prec pp))$

Kaplan Next $\models \forall pp {\uparrow} \Diamond \forall q(\mathcal{Q}_i q \leftrightarrow q \prec pp)$

or, equivalently:

Kaplan Next: $\models \forall pp \Diamond \forall q({\uparrow}\mathcal{Q}_i q \leftrightarrow q \prec pp)$

The intensional case yields different results. With no need to restrict to natural models, we have:

$\models {\uparrow} \exists O \Box \exists p({\uparrow}\mathcal{Q}_i p \not\leftrightarrow Op)$

and therefore

$\not\models \forall O \Diamond \forall p({\uparrow}\mathcal{Q}_i p \leftrightarrow Op)$

Regarding Russell-Myhill, we have:

Russell-Myhill $\models \exists O \exists P(Op = Pp \wedge \neg \forall q(Oq \leftrightarrow Pq))$

But also:

Russell-Myhill Next Whenever $\mathcal{Q}_i$ and $\mathcal{Q}_j$ are independent, $\models {\uparrow} (\mathcal{Q}_i p \neq \mathcal{Q}_j p)$

Here is an outline of the behavior of $\uparrow$ and $\downarrow$:

- $\models (\neg {\uparrow} \phi) \leftrightarrow ({\uparrow} \neg \phi)$
- $\models (\Diamond {\uparrow} \phi) \leftrightarrow ({\uparrow} \Diamond \phi)$
- $\models ({\uparrow} \phi \wedge {\uparrow} \psi) \leftrightarrow {\uparrow}(\phi \wedge \psi)$
- $\models ({\uparrow} \phi \ = {\uparrow} \psi) \leftrightarrow {\uparrow}(\phi = \psi)$

- $\models \uparrow(p) \leftrightarrow p$
- $\models \uparrow(p \prec pp) \leftrightarrow p \prec pp$
- $\models \uparrow(Op) \leftrightarrow Op$
- $\models (\uparrow\downarrow\uparrow\phi) \leftrightarrow (\uparrow\uparrow\downarrow\phi)$
- $\models \phi \;\Rightarrow\; \models \uparrow\phi$

Existential Generalization Let ψ be free for p in ϕ. For $k = v_0(\psi)$,[4]

$$\models \phi[\psi/p] \to \uparrow^k \exists p \downarrow^k \phi$$

Comprehension Let $k = v_0(\phi)$ and let p be a variable not occurring free in ϕ. Then:

$$\models \uparrow^k \exists p (p = \downarrow^k \phi)$$

4 Frames

We use a non-empty set of "worlds" W to characterize a hierarchy with one level for each natural number. At level n, we introduce a set of n-level worlds (W^n), a set of n-level propositions (P^n), a set of n-level "extensions" (E^n), and a set of n-level intensions (I^n). (An n-level proposition is a set of n-level worlds; an n-level extension is a set of n-level propositions; an n-level intension is a function from n-level propositions to n-level propositions.) The 0-level worlds are just the members of W. An $(n+1)$-level world w^{n+1} is a sequence consisting of a 0-level world and an n-level extension for each indefinitely extensible constaint $\mathcal{Q}_1, \ldots, \mathcal{Q}_r$. Formally,

Definition 4 (Worlds, propositions, extensions, intensions)
For W a non-empty set,

[4] $\uparrow^k := \underbrace{\uparrow \ldots \uparrow}_{k \text{ times}}$ $\quad \downarrow^k := \underbrace{\downarrow \ldots \downarrow}_{k \text{ times}}$. The **valence** of ψ, $v^0(\psi)$, is a syntactically characterized upper bound on the resolution that is needed to describe the proposition expressed by ψ, when evaluated externally at resolution 0 (assuming a variable assignment of level 0).

- $W^0 = W$
- $P^n_W = \wp(W^n)$
- $E^n_W = \wp(P^n_W)$
- $I^n_W = \{f : P^n_W \to P^n_W\}$
- $W^{n+1} = \{\langle w, e^n_1, \ldots, e^n_r\rangle : w \in W \wedge e^n_i \in E^n_W\}$
- $W^\infty = \bigcup_{n\in\mathbb{N}} W^n$

In some applications we may not want to count some worlds in W^∞ as "in-admissible", on metaphysical grounds. We therefore introduce the following additional definitions:

Definition 5 (Frames) *A **frame** is a pair $\langle W, \mathcal{A}\rangle$, where W is a non-empty set and $\mathcal{A} \subseteq W^\infty$.*

Definition 6 (Admissible Worlds) *For $\langle W, \mathcal{A}\rangle$ a frame, we let:*

- $W^0_{\mathcal{A}} = W$
- $W^{n+1}_{\mathcal{A}} = \left\{\langle w, e^n_1, \ldots, e^n_r\rangle \in \mathcal{A} : w \in W \wedge e^n_i \in E^n_{W_{\mathcal{A}}}\right\}$
- $P^n_{W_{\mathcal{A}}} = \wp(W^n_{\mathcal{A}})$
- $E^n_{W_{\mathcal{A}}} = \wp(P^n_{W_{\mathcal{A}}})$
- $I^n_{W_{\mathcal{A}}} = \left\{f : P^n_{W_{\mathcal{A}}} \to P^n_{W_{\mathcal{A}}}\right\}$

Definition 7 (Refinements) *Fix a frame $\langle W, \mathcal{A}\rangle$. Intuitively speaking,*

- *For $w^n \in W^n_{\mathcal{A}}$ and $w^{n+1} \in W^{n+1}_{\mathcal{A}}$, $w^n \rhd_{W_{\mathcal{A}}} w^{n+1}$ states that world w^n is "refined" by world w^{n+1}, relative to $\langle W, \mathcal{A}\rangle$.*
- *For $p^n \in P^n_{W_{\mathcal{A}}}$ and $p^{n+1} \in P^m_{W_{\mathcal{A}}}$, the $(n+1)$-level proposition $[p^n]^{n+1}_{W_{\mathcal{A}}}$ is the set of worlds in $W^{n+1}_{\mathcal{A}}$ that are "refinements" of some world in p^n.*

Formally:

- $w^0 \rhd_{W_{\mathcal{A}}} w^1 \ := \ \exists e^0_1 \ldots e^0_r \in E^0_{W_{\mathcal{A}}}(w^1 = \langle w^0, e^0_1, \ldots, e^0_r\rangle)$

- $[p^n]^{n+1}_{W_\mathcal{A}} \;:=\; \left\{w^{n+1} \in W^{n+1}_\mathcal{A} : \exists w^n \in p^n(w^n \rhd_{W_\mathcal{A}} w^{n+1})\right\}$

-
$$\begin{aligned} w^{n+1} \rhd_{W_\mathcal{A}} w^{n+2} \quad := \quad & \exists w \in W \quad \exists e^n_1 \ldots e^n_r \in E^n_{W_\mathcal{A}} \quad \exists e^{n+1}_1 \ldots e^n_r \in E^{n+1}_{W_\mathcal{A}} \\ & \Big(w^{n+1} = \langle w, e^n_1, \ldots, e^n_r \rangle \wedge w^{n+1} = \left\langle w, e^{n+1}_1, \ldots, e^{n+1}_r \right\rangle \wedge \\ & \quad \forall p^n \left(p^n \in e^n_1 \leftrightarrow [p^n]^{n+1}_{W_\mathcal{A}} \in e^{n+1}_1\right) \wedge \\ & \quad \vdots \\ & \quad \forall p^n \left(p^n \in e^n_r \leftrightarrow [p^n]^{n+1}_{W_\mathcal{A}} \in e^{n+1}_r\right) \Big) \end{aligned}$$

Definition 8 (Admissible Frames) *A frame $\langle W, \mathcal{A} \rangle$ is* ***admissible*** *iff for any $n \in \mathbb{N}$ and $w^n \in W^n_\mathcal{A}$:*

- *if $n > 0$, w^n refines some world in $W^{n-1}_\mathcal{A}$*
  (i.e. there is some $w^{n-1} \in W^{n-1}_\mathcal{A}$ is such that $w^{n-1} \rhd_{W_\mathcal{A}} w^n$);
- *w^n is refined by some world in $W^{n+1}_\mathcal{A}$*
  (i.e. there is some $w^{n+1} \in W^{n+1}_\mathcal{A}$ is such that $w^n \rhd_{W_\mathcal{A}} w^{n+1}$);

Proposition 1 (There are admissible frames) *The frame $\langle W, \mathcal{A} \rangle$ is admissible whenever $\mathcal{A} = W^\infty$.*

Proof For $n \in \mathbb{N}$, let $w^n \in W^n_{W_\mathcal{A}}$. We need to verify two claims:

- if $n > 0$, then w^n refines some world in $W^{n-1}_\mathcal{A}$

  Since $n > 0$, we can let $w^n = \left\langle w, e^{n-1}_1, \ldots, w^{n-1}_r \right\rangle$. If $n = 1$, the result is trivial, since we can let $w^n = w$. So we may suppose that $n > 1$. For each $i \leq r$ let

  $$e^{n-2}_i = \left\{p^{n-2} \in P^{n-2}_{W_\mathcal{A}} : [p^{n-2}]^{n-1}_{W_\mathcal{A}} \in e^{n-1}_i\right\}$$

  Let $w^{n-1} = \left\langle w, e^{n-2}_1, \ldots, w^{n-2}_r \right\rangle$. Since $\mathcal{A} = W^\infty$, $w^{n-1} \in W^{n-1}_\mathcal{A}$.

  In addition, since $\mathcal{A} = W^\infty$, $P^n_W = P^n_{W_\mathcal{A}}$. So:

  $$\forall p^{n-2} \left(p^{n-2} \in e^{n-2}_i \leftrightarrow [p^{n-2}]^{n-1}_{W_\mathcal{A}} \in e^{n-1}_i\right)\Big)$$

  So it follows from the definition of $\rhd_{W_\mathcal{A}}$ that $w^{n-1} \rhd_{W_\mathcal{A}} w^n$.

- w^n is refined by some world in $W_{\mathcal{A}}^{n+1}$

  Suppose, first, that $n = 0$, and let $w^{n+1} = \langle w^n, \emptyset, \ldots, \emptyset \rangle$. Since $\mathcal{A} = W^\infty$, $w^{n+1} \in W_{\mathcal{A}}^{n+1}$. And it follows immediately from the definition of $\rhd_{W_{\mathcal{A}}}$that $w^n \rhd_{W_{\mathcal{A}}} w^{n+1}$.

  Now suppose that $n > 0$ and let $w^n = \left\langle w, e_1^{n-1}, \ldots, e_r^{n-1} \right\rangle$. For each $i \leq r$ let

  $$e_i^n = \left\{ [p^{n-1}]^n \in P_{W_{\mathcal{A}}}^n : p_{W_{\mathcal{A}}}^{n-1} \in e_i^{n-1} \right\}$$

  Let $w^{n+1} = \langle w, e_1^n, \ldots, w_r^n \rangle$. Since $\mathcal{A} = W^\infty$, $w^{n+1} \in W_{\mathcal{A}}^{n+1}$.

  In addition, since $\mathcal{A} = W^\infty$, $P_W^n = P_{W_{\mathcal{A}}}^n$. So:

  $$\forall p^{n-1} \left(p^{n-1} \in e_i^{n-1} \leftrightarrow [p^{n-1}]_{W_{\mathcal{A}}}^n \in e_i^n \right) \Big)$$

  So it follows from the definition of $\rhd_{W_{\mathcal{A}}}$ that $w^n \rhd_{W_{\mathcal{A}}} w^{n+1}$.

Proposition 2 (Injectivity of Refinement) *Fix a frame $\langle W, \mathcal{A} \rangle$. For $v^n, w^n \in W_{\mathcal{A}}^n$ and $w^{n+1} \in W_{\mathcal{A}}^{n+1}$,*

$$v^n \rhd_{W_{\mathcal{A}}} w^{n+1} \wedge w^n \rhd_{W_{\mathcal{A}}} w^{n+1} \rightarrow v^n = w^n$$

Proof Since the result is trivial if $n = 0$, we assume $n > 0$. Let $w^{n+1} = \langle w, e_1^n, \ldots, e_r^n \rangle$. Since $w^n \rhd_{W_{\mathcal{A}}} w^{n+1}$, w^n must be $\left\langle w, e_1^{n-1}, \ldots, e_r^{n-1} \right\rangle$ for some $e_1^{n-1}, \ldots, e_r^{n-1}$. Since $v^n \rhd_{W_{\mathcal{A}}} w^{n+1}$,v^n must be $\left\langle w, f_1^{n-1}, \ldots, f_r^{n-1} \right\rangle$ for some $f_1^{n-1}, \ldots, f_r^{n-1}$.

Suppose, for reductio, that $v^n \neq w^n$. Then it must be the case that $e_i^{n-1} \neq f_i^{n-1}$ for $i \leq r$. We may assume with no loss of generality that for some $p^{n-1} \in P_{W_{\mathcal{A}}}^{n-1}$, $p^{n-1} \in e_i^{n-1}$ but $p^{n-1} \notin f_i^{n-1}$. Since $w^n \rhd_{W_{\mathcal{A}}} w^{n+1}$ and $p^{n-1} \in e_i^{n-1}$, it follows from the definition of $\rhd_{W_{\mathcal{A}}}$ that $[p^{n-1}]^n \in e_i^n$. But since $v^n \rhd_{W_{\mathcal{A}}} w^{n+1}$ and $p^{n-1} \notin f_i^{n-1}$, it follows from the definition of $\rhd_{W_{\mathcal{A}}}$ that $[p^{n-1}]^n \notin e_i^n$, which contradicts an earlier assertion.

5 Superworlds

Definition 9 (Superworlds) *Fix a frame $\langle W, \mathcal{A} \rangle$. A **superworld** $\vec{w}$ of $\langle W, \mathcal{A} \rangle$ is an infinite sequence $\langle w^0, w^1, w^2, \ldots \rangle$ ($w^n \in W_{\mathcal{A}}^n$) such that:*

$$w^0 \rhd_{W_{\mathcal{A}}} w^1 \rhd_{W_{\mathcal{A}}} w^2 \rhd_{W_{\mathcal{A}}} \ldots$$

Some additional notation:

- *$\mathcal{W}_\mathcal{A}$ is the set of superworlds of $\langle W, \mathcal{A}\rangle$.*
- *For $\vec{w} \in \mathcal{W}_\mathcal{A}$, $\vec{w}(n)$ is the nth member of $\vec{w}$.*

Proposition 3 (Every world is part of a superworld) *Fix an admissible frame $\langle W, \mathcal{A}\rangle$. For any $w^n \in W^n_\mathcal{A}$, there is some $\vec{w} \in \mathcal{W}_\mathcal{A}$ such that $\vec{w}(n) = w^n$.*

Proof Since $\langle W, \mathcal{A}\rangle$ is admissible, there must be a sequence

$$\left\langle v^0, \ldots, v^{n-1}, w^n, v^{n+1}, v^{n+2}, \ldots \right\rangle$$

such that

$$v^0 \rhd_{W_\mathcal{A}} v^{n-1} \rhd_{W_\mathcal{A}} w^n \rhd_{W_\mathcal{A}} v^{n+1} \rhd_{W_\mathcal{A}} v^{n+1} \rhd_{W_\mathcal{A}} \ldots$$

Proposition 4 (No backwards divergence for superworlds) *For $\vec{w}, \vec{v} \in \mathcal{W}_\mathcal{A}$ and $n, k \in \mathbb{N}$, $\vec{v}(n+k) = \vec{w}(n+k)$ entails $\vec{v}(n) = \vec{w}(n)$.*

Proof Assume $\vec{v}(n) \neq \vec{w}(n)$. By proposition 2, $\vec{v}(n+1) \neq \vec{w}(n+1)$. Again by proposition 2, $\vec{v}(n+2) \neq \vec{w}(n+2)$. After k iterations of this procedure, we get $\vec{v}(n+k) \neq \vec{w}(n+k)$.

Definition 10 (Superpropositions)

- *A superproposition $\vec{p}$ of $\langle W, \mathcal{A}\rangle$ is a set of superworlds in $\mathcal{W}_\mathcal{A}$.*
- $P_{\mathcal{W}_\mathcal{A}} = \{\vec{p} : \vec{p} \subseteq \mathcal{W}_\mathcal{A}\}$.
- $P^n_{\mathcal{W}_\mathcal{A}} = \{\vec{p} \in P_{\mathcal{W}_\mathcal{A}} : \vec{w}(n) = \vec{v}(n) \to (\vec{w} \in \vec{p} \leftrightarrow \vec{v} \in \vec{p})\}$
- *For $\vec{p} \in P_{\mathcal{W}_\mathcal{A}}$, we let $\vec{p}(n) = \{\vec{w}(n) : \vec{w} \in \vec{p}\}$.*[5]

Proposition 5 (Monotonicity of Superpropositions) *For $n \in \mathbb{N}$, $\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}} \to \vec{p} \in P^{n+1}_{\mathcal{W}_\mathcal{A}}$.*

Proof Assume $\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}}$. We suppose $\vec{w}(n+1) = \vec{v}(n+1)$ and $\vec{w} \in \vec{p}$, and we show $\vec{v} \in \vec{p}$. By proposition 4, $\vec{w}(n+1) = \vec{v}(n+1)$ entails $\vec{w}(n) = \vec{v}(n)$. So $\vec{w} \in \vec{p}$ guarantees $\vec{v} \in \vec{p}$.

[5] Note that $\{\vec{w}(n) : \vec{w} \in \vec{p}\} = \left\{v^n \in P^n_{W_\mathcal{A}} : \exists \vec{v} \in \vec{p}(v^n = \vec{v}(n))\right\}$.

Proposition 6 ($\vec{p}(n)$ is well-behaved, part 1) *If* $\vec{p}, \in P^n_{\mathcal{W}_\mathcal{A}}$, *then* $\vec{w} \in \vec{p} \leftrightarrow \vec{w}(n) \in \vec{p}(n)$.

Proof Suppose, first, that $\vec{w} \in \vec{p}$. By definition, $\vec{p}(n) = \{v^n \in P^n_{W_\mathcal{A}} : \exists \vec{v} \in \vec{p}(v^n = \vec{v}(n))\}$. Since $\vec{w}$ is a true instance of the following existential:

$$\exists \vec{v} \in \vec{p}(\vec{w}(n) = \vec{v}(n))$$

we have $\vec{w}(n) \in \vec{p}(n)$.

Now suppose that $\vec{w}(n) \in \vec{p}(n)$. By definition, $\vec{p}(n) = \{v^n \in P^n_{W_\mathcal{A}} : \exists \vec{v} \in \vec{p}(v^n = \vec{v}(n))\}$. So the fact that $\vec{w}(n) \in \vec{p}(n)$ entails that there must be some $\vec{z} \in \vec{p}$ such that $\vec{z}(n) = \vec{w}(n)$. But since $\vec{p}, \in P^n_{\mathcal{W}_\mathcal{A}}$, $\vec{z} \in \vec{p}$ and $\vec{z}(n) = \vec{w}(n)$ entail that $\vec{w} \in \vec{p}$.

Proposition 7 ($\vec{p}(n)$ is well-behaved, part 2) *If* $\vec{p}, \vec{q} \in P^n_{\mathcal{W}_\mathcal{A}}$, *then* $\vec{p}(n) = \vec{q}(n)$ *entails* $\vec{p} = \vec{q}$.

Proof

$$\begin{aligned} \vec{w} \in \vec{p} &\leftrightarrow \vec{w}(n) \in \vec{p}(n) && \text{by proposition 6} \\ &\leftrightarrow \vec{w}(n) \in \vec{q}(n) && \text{since } \vec{p}(n) = \vec{q}(n) \\ &\leftrightarrow \vec{w} \in \vec{q} && \text{by proposition 6} \end{aligned}$$

Proposition 8 ($\vec{p}(n)$ is well-behaved, part 3) *Assume* $\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}}$. *Then:*

$$\vec{p}(n) = p^n \leftrightarrow \vec{p} = \{\vec{w} \in \mathcal{W}_\mathcal{A} : \vec{w}(n) \in p^n\}$$

Proof

Left to right: We assume $\vec{p}(n) = p^n$, and therefore

$$\{w^n : \exists \vec{w} \in \vec{p}(w^n = \vec{w}(n))\} = p^n$$

To verify $\vec{p} = \{\vec{w} \in \mathcal{W}_\mathcal{A} : \vec{w}(n) \in p^n\}$, it suffices to check each of the following:

- If $\vec{v}(n) \in p^n$, then $\vec{v} \in \vec{p}$

  Suppose that $\vec{v}(n) \in p^n$. By our initial assumption, there is some $\vec{w} \in \vec{p}$ such that:

  $$\vec{v}(n) = \vec{w}(n)$$

  But since $\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}}$, this entails

  $$\vec{w} \in \vec{p} \leftrightarrow \vec{v} \in \vec{p}$$

  which means that we have $\vec{v} \in \vec{p}$, as desired.

- If $\vec{v}(n) \notin p^n$, then $\vec{v} \notin \vec{p}$.

  Suppose that $\vec{v}(n) \notin p^n$. By our initial assumption, every $\vec{w} \in \vec{p}$ is such that:

  $$\vec{v}(n) \neq \vec{w}(n)$$

  from which it follows that $\vec{v} \notin \vec{p}$.

Right to Left: Assume $\vec{p} = \{\vec{w} \in \mathcal{W}_{\mathcal{A}} : \vec{w}(n) \in p^n\}$. By proposition 3:

$$\{\vec{w}(n) : \vec{w}(n) \in p^n\} = p^n$$

equivalently:

$$\{\vec{w}(n) : \vec{w} \in \{\vec{w} \in \mathcal{W}_{\mathcal{A}} : \vec{w}(n) \in p^n\}\} = p^n$$

So, by our assumption,

$$\{\vec{w}(n) : \vec{w} \in \vec{p}\} = p^n$$

which is what we want:

$$\vec{p}(n) = p^n$$

Definition 11 (Superextensions)

- *A superextension $\vec{e}$ of $\langle W, \mathcal{A}\rangle$ is a set of superpropositions of $\langle W, \mathcal{A}\rangle$.*
- $E_{\mathcal{W}_{\mathcal{A}}} = \{\vec{e} : \vec{e} \subseteq P_{\mathcal{W}_{\mathcal{A}}}\}$.
- $E^n_{\mathcal{W}_{\mathcal{A}}} = \left\{\vec{e} : \vec{e} \subseteq P^n_{\mathcal{W}_{\mathcal{A}}}\right\}$.

Definition 12 (Superintensions)

- *A superintension $\vec{i}$ of $\langle W, \mathcal{A}\rangle$ is a function from superpropositions of $\langle W, \mathcal{A}\rangle$ to superpropositions of $\langle W, \mathcal{A}\rangle$.*
- $I_{\mathcal{W}_{\mathcal{A}}} = \{\vec{\imath} : \vec{\imath}$ *is a function from* $P_{\mathcal{W}_{\mathcal{A}}}$ *into* $P_{\mathcal{W}_{\mathcal{A}}}\}$.
- $I^n_{\mathcal{W}_{\mathcal{A}}} = \left\{\vec{\imath} \in I_{\mathcal{W}_{\mathcal{A}}} : \forall \vec{p} \in P_{\mathcal{W}_{\mathcal{A}}} \ \left(\vec{\imath}(\vec{p}) \in P^n_{\mathcal{W}_{\mathcal{A}}}\right)\right\}$.

Proposition 9 (Monotonicity of Superintensions) *For $n \in \mathbb{N}$, $\vec{\imath} \in I^n_{\mathcal{W}_{\mathcal{A}}} \rightarrow \vec{\imath} \in I^{n+1}_{\mathcal{W}_{\mathcal{A}}}$.*

Proof *Let $\vec{\imath} \in I^n_{\mathcal{W}_{\mathcal{A}}}$ and $\vec{p} \in P_{\mathcal{W}_{\mathcal{A}}}$. Since $\vec{\imath} \in I^n_{\mathcal{W}_{\mathcal{A}}}$, $\vec{\imath}(\vec{p}) \in P^n_{\mathcal{W}_{\mathcal{A}}}$. So proposition 5 entails that $\vec{\imath}(\vec{p}) \in P^{n+1}_{\mathcal{W}_{\mathcal{A}}}$*

$\vec{p}(n+1) = \vec{v}(n+1)$ and $\vec{w} \in \vec{p}$, and we show $\vec{v} \in \vec{p}$. By proposition 4, $\vec{w}(n+1) = \vec{v}(n+1)$ entails $\vec{w}(n) = \vec{v}(n)$. So $\vec{w} \in \vec{p}$ guarantees $\vec{v} \in \vec{p}$.

6 Extensions for $\mathcal{Q}_i$

Definition 13 (Extension Predicate for $\mathcal{Q}_i$) *Fix a frame $\langle W, \mathcal{A}\rangle$. For $\vec{w} \in \mathcal{W}_{\mathcal{A}}$ and $n \in \mathbb{N}$, let $\vec{w}(n+1) = \langle w, e_1^n, \ldots, e_r^n\rangle$. Then:*

$$\left[{}^{\mathcal{W}}_{\mathcal{A}} Ext^n_{\mathcal{Q}_i}\right](\vec{w}) = \left\{\vec{p} \in P^n_{\mathcal{W}_{\mathcal{A}}} : \exists p^n \in e_i^n(\vec{p}(n) = p^n)\right\}$$

Proposition 10 (Monotonicity of Extensions) *For any $\vec{w} \in \mathcal{W}_{\mathcal{A}}$ and $\vec{p} \in P_{\mathcal{W}_{\mathcal{A}}}$,*

$$\vec{p} \in \left[{}^{\mathcal{W}}_{\mathcal{A}} Ext^n_{\mathcal{Q}_i}\right](\vec{w}) \to \vec{p} \in \left[{}^{\mathcal{W}}_{\mathcal{A}} Ext^{n+1}_{\mathcal{Q}_i}\right](\vec{w})$$

Proof Let $\vec{w}(n+1) = \langle w, e_1^n, \ldots, e_r^n\rangle$, $\vec{w}(n+2) = \left\langle w, e_1^{n+1}, \ldots, e_r^{n+1}\right\rangle$. Let $\vec{p} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^n_{\mathcal{Q}_i}\right](\vec{w})$. We verify that $\vec{p}$ is also in $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n+1}_{\mathcal{Q}_i}\right](\vec{w})$.

By the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^n_{\mathcal{Q}_i}\right](\vec{w})$, $\vec{p} \in P^n_{\mathcal{W}_{\mathcal{A}}}$ and there is some $p^n \in e_i^n$ such that $\vec{p}(n) = p^n$. Since $\vec{p}$ is in $P^n_{\mathcal{W}_{\mathcal{A}}}$, it is also in $P^{n+1}_w a$. So, by the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n+1}_{\mathcal{Q}_i}\right]$, it suffices to verify each of the following two propositions:

- $\vec{p}(n+1) = [p^n]^{n+1}_{W_{\mathcal{A}}}$

  Proof: By definition,

  $$[p^n]^{n+1}_{W_{\mathcal{A}}} = \left\{v^{n+1} : \exists w^n \in p^n(w^n \rhd_{W_{\mathcal{A}}} v^{n+1})\right\}$$

  which is equivalent to the following, by proposition 3

  $$[p^n]^{n+1}_{W_{\mathcal{A}}} = \{\vec{v}(n+1) : \vec{v} \in \mathcal{W}_{\mathcal{A}} \wedge \exists w^n \in p^n(w^n \rhd_{W_{\mathcal{A}}} \vec{v}(n+1))\}$$

  which is equivalent to the following, by proposition 2,

  $$[p^n]^{n+1}_{W_{\mathcal{A}}} = \{\vec{v}(n+1) : \vec{v}(n) \in p^n\}$$

  But we know that $\vec{p}(n) = p^n$. So:

  $$[p^n]^{n+1}_{W_{\mathcal{A}}} = \{\vec{v}(n+1) : \vec{v}(n) \in \vec{p}(n)\}$$

  which is equivalent to

  $$[p^n]^{n+1}_{W_{\mathcal{A}}} = \{\vec{v}(n+1) : \vec{v}(n) \in \{\vec{w}(n) : \vec{w} \in \vec{p}\}\}$$

  equivalently:

  $$[p^n]^{n+1}_{W_{\mathcal{A}}} = \{\vec{v}(n+1) : \vec{v} \in \vec{p}\}$$

  which is what we want:

  $$[p^n]^{n+1}_{W_{\mathcal{A}}} = \vec{p}(n+1)$$

- $[p^n]^{n+1} \in e^{n+1}$

  Proof: Since $\vec{w}(n+1) \rhd_{W_\mathcal{A}} \vec{w}(n+2)$, we know that $p^n \in e_i^n \leftrightarrow [p^n]_{\mathcal{W}_\mathcal{A}}^{n+1} \in e_i^{n+1}$. So the result is immediate.

Proposition 11 (Conservativity of Extensions) *For any* $\vec{w} \in \mathcal{W}_\mathcal{A}$ *and* $\vec{p} \in P_{\mathcal{W}_\mathcal{A}}^n$ $(n \in \mathbb{N})$,

$$\vec{p} \in \left[{}^{\mathcal{W}}_{\mathcal{A}} Ext_{\mathcal{Q}_i}^{n+1}\right](\vec{w}) \rightarrow \vec{p} \in \left[{}^{\mathcal{W}}_{\mathcal{A}} Ext_{\mathcal{Q}_i}^{n}\right](\vec{w})$$

Proof Let $\vec{w}(n+1) = \langle w, e_1^n, \ldots, e_r^n \rangle$, $\vec{w}(n+2) = \left\langle w, e_1^{n+1}, \ldots, e_r^{n+1} \right\rangle$. Let $\vec{p}$ be in both $P_{\mathcal{W}_\mathcal{A}}^n$ and $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}_{\mathcal{Q}_i}^{n+1}\right](\vec{w})$. We verify that $\vec{p}$ is also in $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}_{\mathcal{Q}_i}^{n}\right](\vec{w})$.

By the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}_{\mathcal{Q}_i}^{n+1}\right](\vec{w})$, there is some $p^{n+1} \in e_i^{n+1}$ such that $\vec{p}(n+1) = p^{n+1}$. Let

$$p^n = \left\{ w^n \in W_\mathcal{A}^n : \exists w^{n+1} \in p^{n+1}(w^n \rhd_{W_\mathcal{A}} w^{n+1}) \right\}$$

We have $\vec{p} \in P_{\mathcal{W}_\mathcal{A}}^n$. So in order to show $\vec{p} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}_{\mathcal{Q}_i}^{n}\right](\vec{w})$, it suffices to verify each of the following two propositions:

- $\vec{p}(n) = p^n$

  Proof: By definition,

  $$p^n = \left\{ w^n \in W_\mathcal{A}^n : \exists w^{n+1} \in p^{n+1}(w^n \rhd_{W_\mathcal{A}} w^{n+1}) \right\}$$

  Since $\vec{p}(n+1) = p^{n+1}$,

  $$p^n = \left\{ w^n \in W_\mathcal{A}^n : \exists w^{n+1} \in \vec{p}(n+1)(w^n \rhd_{W_\mathcal{A}} w^{n+1}) \right\}$$

  which is equivalent to:

  $$p^n = \left\{ w^n \in W_\mathcal{A}^n : \exists w^{n+1} \in \{\vec{w}(n+1) : \vec{w} \in \vec{p}\} (w^n \rhd_{W_\mathcal{A}} w^{n+1}) \right\}$$

  or, equivalently,

  $$p^n = \{ w^n \in W_\mathcal{A}^n : \exists \vec{w} \in \vec{p}\ (w^n \rhd_{W_\mathcal{A}} \vec{w}(n+1)) \}$$

  which is equivalent to the following, by proposition 2,

  $$p^n = \{ w^n \in W_\mathcal{A}^n : \exists \vec{w} \in \vec{p}\ (w^n = \vec{w}(n)) \}$$

But, by proposition 3, this is equivalent to:

$$p^n = \{\vec{v}(n) : \vec{v} \in \mathcal{W}_{\mathcal{A}} \wedge \exists \vec{w} \in \vec{p}\ (\vec{v}(n) = \vec{w}(n))\}$$

But we are assuming that that $\vec{p} \in P^n_{\mathcal{W}_{\mathcal{A}}}$ and, therefore, that, for any $\vec{w}, \vec{v} \in \mathcal{W}_{\mathcal{A}}$,

$$\vec{v}(n) = \vec{w}(n) \to (\vec{v} \in \vec{p} \leftrightarrow \vec{w} \in \vec{p})$$

which allows us to conclude:

$$p^n = \{\vec{v}(n) : \vec{v} \in \vec{p}\}$$

which delivers the desired result:

$$p^n = \vec{p}(n)$$

- $p^n \in e^n_i$

  Proof: Since $\vec{w}(n+1) \rhd_{W_{\mathcal{A}}} \vec{w}(n+2)$, we know that $p^n \in e^n_i \leftrightarrow [p^n]^{n+1}_{\mathcal{W}_{\mathcal{A}}} \in e^{n+1}_i$. So it suffices to show that $[p^n]^{n+1}_{\mathcal{W}_{\mathcal{A}}} \in e^{n+1}_i$. By definition:

$$[p^n]^{n+1}_{W_{\mathcal{A}}} = \left\{v^{n+1} : \exists w^n \in p^n(w^n \rhd_{W_{\mathcal{A}}} v^{n+1})\right\}$$

  So, brining in the definition of p^n,

$$[p^n]^{n+1}_{W_{\mathcal{A}}} = \left\{v^{n+1} : \exists w^n \exists w^{n+1}(w^{n+1} \in p^{n+1} \wedge w^n \rhd_{W_{\mathcal{A}}} w^{n+1} \wedge w^n \rhd_{W_{\mathcal{A}}} v^{n+1})\right\}$$

  But, since $\vec{p}(n+1) = p^{n+1}$, we have:

$$[p^n]^{n+1}_{W_{\mathcal{A}}} = \left\{v^{n+1} : \exists w^n \exists w^{n+1}(w^{n+1} \in \vec{p}(n+1) \wedge w^n \rhd_{W_{\mathcal{A}}} w^{n+1} \wedge w^n \rhd_{W_{\mathcal{A}}} v^{n+1})\right\}$$

  equivalently:

$$[p^n]^{n+1}_{W_{\mathcal{A}}} = \left\{v^{n+1} : \exists w^n \exists w^{n+1}(w^{n+1} \in \{\vec{z}(n+1) : \vec{z} \in \vec{p}\} \wedge w^n \rhd_{W_{\mathcal{A}}} w^{n+1} \wedge w^n \rhd_{W_{\mathcal{A}}} v^{n+1})\right\}$$

  Simplifying:

$$[p^n]^{n+1}_{W_{\mathcal{A}}} = \left\{v^{n+1} : \exists w^n \exists \vec{w} \in \vec{p}\ (w^n \rhd_{W_{\mathcal{A}}} \vec{w}(n+1) \wedge w^n \rhd_{W_{\mathcal{A}}} v^{n+1})\right\}$$

  which by proposition 2 is equivalent to:

$$[p^n]^{n+1}_{W_{\mathcal{A}}} = \left\{v^{n+1} : \exists \vec{w} \in \vec{p}\ (\vec{w}(n) \rhd_{W_{\mathcal{A}}} v^{n+1})\right\}$$

so proposition 3 gives us

$$[p^n]^{n+1}_{W_\mathcal{A}} = \{\vec{v}(n+1) : \exists \vec{w} \in \vec{p}\ (\vec{w}(n) \rhd_{W_\mathcal{A}} \vec{v}(n+1))\}$$

and again by proposition 2,

$$[p^n]^{n+1}_{W_\mathcal{A}} = \{\vec{v}(n+1) : \exists \vec{w} \in \vec{p}\ (\vec{w}(n) = \vec{v}(n))\}$$

But we are assuming that that $\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}}$ and, therefore, that, for any $\vec{w}, \vec{v} \in \mathcal{W}_\mathcal{A}$,

$$\vec{w}(n) = \vec{v}(n) \to (\vec{w} \in \vec{p} \leftrightarrow \vec{v} \in \vec{p})$$

which allows us to conclude:

$$[p^n]^{n+1}_{W_\mathcal{A}} = \{\vec{v}(n+1) : \vec{v} \in \vec{p}\}$$

Or, equivalently,

$$[p^n]^{n+1}_{W_\mathcal{A}} = \vec{p}(n+1)$$

which gives us the desired result, since we are assuming that $\vec{p}(n+1) = p^{n+1}$ and $p^{n+1} \in e^{n+1}_i$.

7 Models

Definition 14 *A **model** is a quadruple $\langle W, \mathcal{A}, \vec{\alpha}, k\rangle$, for $\langle W, \mathcal{A}\rangle$ an admissible frame, $\vec{\alpha} \in \mathcal{W}_\mathcal{A}$, and $k \in \mathbb{N}$. (Intuitively, $\vec{\alpha}$ is the actual superworld and k is a level of "resolution" with respect to which truth is to be assessed.)*

Definition 15 *A **variable assignment** for $\langle W, \mathcal{A}, \vec{\alpha}, k\rangle$ is a function σ such that:*

- $\sigma(p_i) \in P_{\mathcal{W}_\mathcal{A}}$;
- $\sigma(pp_i) \subseteq P_{\mathcal{W}_\mathcal{A}}$ *and* $\sigma(pp_i) \neq \emptyset$;
- $\sigma(O_i) \in I_{\mathcal{W}_\mathcal{A}}$.

Definition 16 (Truth at a superworld) *Fix a model $\langle W, \mathcal{A}, \vec{\alpha}, k\rangle$. For ϕ a formula of $\mathscr{L}$, $\vec{w} \in \mathcal{W}_\mathcal{A}$, and σ a variable assignment for $\langle W, \mathcal{A}, \vec{\alpha}, k\rangle$, we define the **truth** of ϕ at $\vec{w}$ with respect to σ at resolution k (in symbols: $\vec{w} \models^k_\sigma \phi$) using the following recursive clauses:*

- $\vec{w} \models^k_\sigma p_i$ *iff* $\vec{w} \in \sigma(p_i)$;
- $\vec{w} \models^k_\sigma \mathcal{Q}_j p_i$ *iff* $\begin{cases} \sigma(p_i) \in \left[{}^{\mathcal{W}}_{\mathcal{A}} Ext^{k-1}_{\mathcal{Q}_j}\right](\vec{w}), & \text{if } k > 0 \\ \bot, & \text{if } k = 0 \end{cases}$
- $\vec{w} \models^k_\sigma O_j p_i$ *iff* $\vec{w} \in \sigma(O_j)(\sigma(p_i))$;
- $\vec{w} \models^k_\sigma p_i \prec pp_j$ *iff* $\sigma(p_i) \in \sigma(pp_j)$;
- $\vec{w} \models^k_\sigma \phi = \psi$ *iff* $\left\{\vec{v} \in \mathcal{W}_{\mathcal{A}} : \vec{v} \models^k_\sigma \phi\right\} = \left\{\vec{v} \in \mathcal{W}_{\mathcal{A}} : \vec{v} \models^k_\sigma \psi\right\}$
- $\vec{w} \models^k_\sigma \neg\phi$ *iff* $\vec{w} \not\models^k_\sigma \phi$;
- $\vec{w} \models^k_\sigma (\phi \wedge \psi)$ *iff* $\vec{w} \models^k_\sigma \phi$ *and* $\vec{w} \models^k_\sigma \psi$;
- $\vec{w} \models^k_\sigma \exists p_i \phi$ *iff for some* $\vec{q} \in P^k_{\mathcal{W}_{\mathcal{A}}}$, $\vec{w} \models^k_{\sigma[\vec{q}/p_i]} \phi$;
- $\vec{w} \models^k_\sigma \exists pp_i \phi$ *iff for some* $\vec{A} \subseteq P^k_{\mathcal{W}_{\mathcal{A}}}$, $\vec{A} \neq \emptyset$ *and* $\vec{w} \models^k_{\sigma[\vec{A}/pp_i]} \phi$;
- $\vec{w} \models^k_\sigma \exists O_j \phi$ *iff for some* $\vec{\imath} \in I^k_{\mathcal{W}_{\mathcal{A}}}$, $\vec{w} \models^k_{\sigma[\vec{\imath}/O_j]} \phi$;
- $\vec{w} \models^k_\sigma \uparrow \phi$ *iff* $\vec{w} \models^{k+1}_\sigma \phi$;
- $\vec{w} \models^k_\sigma \downarrow \phi$ *iff* $\begin{cases} \models^{k-1}_\sigma \phi, & \text{if } k > 0 \\ \models^0_\sigma \phi, & \text{if } k = 0 \end{cases}$

Proposition 12

1. $\vec{w} \models^k_\sigma \Diamond\phi$ *iff* $\left\{\vec{v} \in \mathcal{W}_{\mathcal{A}} : \vec{v} \models^k_\sigma \phi\right\} \neq \emptyset$;
2. $\vec{w} \models^k_\sigma \Box\phi$ *iff* $\left\{\vec{v} \in \mathcal{W}_{\mathcal{A}} : \vec{v} \models^k_\sigma \phi\right\} = \mathcal{W}_{\mathcal{A}}$;
3. $\vec{w} \not\models^k_\sigma \bot$;
4. $\vec{w} \models^k_\sigma (\phi \rightarrow \psi)$ *iff: if* $\vec{w} \models^k_\sigma \phi$, *then* $\vec{w} \models^k_\sigma \psi$;
5. $\vec{w} \models^k_\sigma (\phi \leftrightarrow \psi)$ *iff:* $\vec{w} \models^k_\sigma \phi$ *iff* $\vec{w} \models^k_\sigma \psi$;
6. $\vec{w} \models^k_\sigma \forall p_i \phi$ *iff for any* $\vec{q} \in P^k_{\mathcal{W}_{\mathcal{A}}}$, $\vec{w} \models^k_{\sigma[\vec{q}/p_i]} \phi$;
7. $\vec{w} \models^k_\sigma \forall pp_i \phi$ *iff for any* $\vec{A} \subseteq P^k_{\mathcal{W}_{\mathcal{A}}}$, $\vec{w} \models^k_{\sigma[\vec{A}k/pp_i]} \phi$;

8. $\vec{w} \models^k_\sigma \forall O_i \phi$ *iff for any* $\vec{\imath} \in I^k_{\mathcal{W}}$, $\vec{w} \models^k_{\sigma[\vec{\imath}/O_i]} \phi$*;*

Proof

1. Recall that $\Diamond\phi := \neg(\phi = \bot)$.
   - $\vec{w} \models^k_\sigma \phi = \bot$ iff $\{\vec{w} : \vec{w} \models^k_\sigma \phi\} = \{\vec{w} \in : \vec{w} \models^k_\sigma \bot\}$ iff $\{\vec{w} : w^k \models^k_\sigma \phi\} = \emptyset$
   - $\vec{w} \models^k_\sigma \neg(\phi = \bot)$ iff $\{\vec{w} : \vec{w} \models^k_\sigma \phi\} \neq \emptyset$
2. Recall that $\Box\phi := (\phi = \top)$.
   - $\vec{w} \models^k_\sigma \phi = \top$ iff $\{\vec{w} : \vec{w} \models^k_\sigma \phi\} = \{\vec{w} \in : \vec{w} \models^k_\sigma \top\}$ iff $\{\vec{w} : w^k \models^k_\sigma \phi\} = \mathcal{W}_{\mathcal{A}}$.

   The remaining proofs are trivial.

8 Truth and Validity

Definition 17 *An* ***n-level variable assignment*** *for* $\langle W, \mathcal{A}, \vec{\alpha}, k\rangle$ *is a variable assignment* σ *such that:*

- $\sigma(p_i) \in P^n_{\mathcal{W}_{\mathcal{A}}}$*;*
- $\sigma(pp_i) \subseteq P^n_{\mathcal{W}_{\mathcal{A}}}$ *and* $\sigma(pp_i) \neq \emptyset$*;*
- $\sigma(O_i) \in I^n_{\mathcal{W}_{\mathcal{A}}}$.

Proposition 13 (Monotonicity of Assignments) *For* $n, k \in \mathbb{N}$*, if* σ *is an* n*-level assignment, it is also a* $(n+1)$*-level assignment.*

Proof Assume that σ is an n-level assignment. To show that σ is also an $(n+1)$-level assignment, we need to verify:

- $\sigma(p_i) \in P^{n+1}_{\mathcal{W}_{\mathcal{A}}}$

  Proof: Since σ is an n-level assignment, we have $\sigma(p_i) \in P^n_{\mathcal{W}_{\mathcal{A}}}$. So proposition 5 entails $\sigma(p_i) \in P^{n+1}_{\mathcal{W}_{\mathcal{A}}}$.

- $\sigma(pp_i) \subseteq P^{n+1}_{\mathcal{W}_\mathcal{A}}$

  Proof: Since σ is an n-level assignment, we have $\sigma(pp_i) \subseteq P^{n}_{\mathcal{W}_\mathcal{A}}$. So, for each $\vec{q} \in \sigma(pp_i)$, proposition 5 entails $\vec{q} \in P^{n+1}_{\mathcal{W}_\mathcal{A}}$. So $\sigma(pp_i) \subseteq P^{n+1}_{\mathcal{W}_\mathcal{A}}$.

- $\sigma(O_i) \in I^{n+1}_{\mathcal{W}_\mathcal{A}}$

  Proof: Since σ is an n-level assignment, we have $\sigma(O_i) \in I^{n}_{\mathcal{W}_\mathcal{A}}$. So proposition 9 entails $\sigma(O_i) \in I^{n+1}_{\mathcal{W}_\mathcal{A}}$.

Definition 18 (Truth) *For a formula ϕ of $\mathscr{L}$ to be **true** at model $\langle W, \mathcal{A}, \vec{\alpha}, k\rangle$ is for it to be the case that $\vec{\alpha} \models^k_\sigma \phi$ for every k-level assignment σ.*

Definition 19 (Validity) *For ϕ to be **valid** (in symbols $\models \phi$) is for it to be true at every model.*

Definition 20 *Let $\mathscr{L}^-$ be the fragment of $\mathscr{L}$ that excludes $\uparrow$, $\downarrow$, and $\mathcal{Q}_1, \ldots, \mathcal{Q}_r$.*

Proposition 14 *$\phi \in \mathscr{L}^-$ is valid in the present framework if and only if it is valid in a standard higher-order framework.*

Proof

Right to Left: Suppose ϕ fails to be valid in the present framework. Then there is some model $\langle W, \mathcal{A}, \vec{\alpha}, k\rangle$ at which ϕ fails to be true. But when the clauses for vocabulary outside $\mathscr{L}^-1$ are ignored, our semantic clauses are totally standard. So ϕ will also fail to be true when $\langle W, \mathcal{A}, \vec{\alpha}, k\rangle$ is thought of as a standard higher-order model.

Left to Right: Suppose ϕ fails to be valid with respect to a standard higher-order model theory. Then it fails to be true according to some standard model. But every standard higher-order model of $\mathscr{L}^-$ is isomorphic to some model of the form $\langle W, \mathcal{A}, \vec{\alpha}, 0\rangle$. So ϕ must fail to be true according to some model of the present framework.

9 Substitution

Definition 21 (Notation)

- *$\phi[\psi/p]$ is the result of substituting ψ for each free occurrence of p in ϕ.*

- $\sigma[\vec{q}/p](\eta) = \begin{cases} \sigma(\eta), & \textit{if } \eta \neq p \\ \vec{q}, & \textit{if } \eta = p \end{cases}$

- *We say that* ***ψ is free for p in ϕ*** *iff no free variables in ψ become bound when substituting ψ for every free occurrence of p in ϕ.*

Proposition 15 (Trivial Substitution) *If p does not occur free in ϕ,*

$$\vec{w} \models^n_\sigma \phi \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi$$

Proof We proceed by induction on the complexity of ϕ:

- $\phi = p_i$.

  Since p does not occur free in ϕ, $p \neq p_i$. So we have $\sigma(p_i) = \sigma[\vec{q}/p](p_i)$ and therefore:
  $$\vec{w} \models^n_\sigma p_i \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} p_i$$

- $\phi = \mathcal{Q}_j p_i$. If $n = 0$, the result is immediate, by the semantic clause for $\mathcal{Q}_j$:
  $$\vec{w} \models^0_\sigma \mathcal{Q}_j p_i \leftrightarrow \bot \leftrightarrow \vec{w} \models^0_{\sigma[\vec{q}/p]} \mathcal{Q}_j p_i$$

  We therefore assume $n > 0$. Since p does not occur free in ϕ, $p \neq p_i$. So we have $\sigma(p_i) = \sigma[\vec{q}/p](p_i)$ and therefore:

  $$\vec{w} \models^n_\sigma \mathcal{Q}_j p_i \leftrightarrow \sigma(p_i) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right] \leftrightarrow \sigma[\vec{q}/p](p_i) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \mathcal{Q}_j p_i$$

- ϕ is $O_j p_i$

  Since p does not occur free in ϕ, $p \neq p_i$. So we have $\sigma(p_i) = \sigma[\vec{q}/p](p_i)$ and therefore:

  $$\vec{w} \models^n_\sigma O_j p_i \leftrightarrow \sigma(p_i) \in \sigma(O_j) \leftrightarrow \sigma[\vec{q}/p](p_i) \in \sigma[\vec{q}/p](O_j) \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} O_j p_i$$

- ϕ is $p_i \prec pp_j$

  Since p does not occur free in ϕ, $p \neq p_i$. So we have $\sigma(p_i) = \sigma[\vec{q}/p](p_i)$ and therefore:

  $$\sigma(p_i) \in \sigma(pp_j) \leftrightarrow \sigma[\vec{q}/p](p_i) \in \sigma[\vec{q}/p](pp_j)$$

  from which the result follows by the semantic clause for $\prec$.

- ϕ is $\theta = \xi$

  Since p does not occur free in ϕ, it must not occur free in θ or ξ. So, by inductive hypothesis:

  $$\vec{w} \models^n_\sigma \theta \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta$$

  $$\vec{w} \models^n_\sigma \xi \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \xi$$

  and therefore

  $$\{\vec{w} : \vec{w} \models^n_\sigma \theta\} \leftrightarrow \left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta\right\}$$

  $$\{\vec{w} : \vec{w} \models^n_\sigma \xi\} \leftrightarrow \left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} \xi\right\}$$

  So we have:

  $$\{\vec{w} : \vec{w} \models^n_\sigma \theta\} = \{\vec{w} : \vec{w} \models^n_\sigma \xi\} \leftrightarrow \left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta\right\} = \left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} \xi\right\}$$

  from which the result follows by the semantic clause for $=$.

- ϕ is $\neg\theta$

  Since p does not occur free in ϕ, it must not occur free in θ. So, by inductive hypothesis:

  $$\vec{w} \models^n_\sigma \theta \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta$$

  and therefore

  $$\vec{w} \not\models^n_\sigma \theta \leftrightarrow \vec{w} \not\models^n_{\sigma[\vec{q}/p]} \theta$$

  from which the result follows by the semantic clause for $\neg$.

- ϕ is $(\theta \wedge \xi)$

  Since p does not occur free in ϕ, it must not occur free in θ or ξ. So, by inductive hypothesis:

  $$\vec{w} \models^n_\sigma \theta \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta$$

  $$\vec{w} \models^n_\sigma \xi \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \xi$$

  So we have:

  $$(\vec{w} \models^n_\sigma \theta \wedge \vec{w} \models^n_\sigma \xi) \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta \wedge \vec{w} \models^n_{\sigma[\vec{q}/p]} \xi\,()$$

  from which the result follows by the semantic clause for $\wedge$.

- ϕ is $\exists p_i\theta$

  By the semantic clause for $\exists$:

$$\vec{w} \models^n_\sigma \exists p_i\theta \leftrightarrow \exists \vec{r} \in P^n_{\mathcal{W}_\mathcal{A}}\vec{w} \models^n_{\sigma[\vec{r}/p_i]}$$

$$\vec{w} \models^n_{\sigma[\vec{q}/p]} \exists p_i\theta \leftrightarrow \exists \vec{r} \in P^n_{\mathcal{W}_\mathcal{A}}\vec{w} \models^n_{\sigma[\vec{q}/p][\vec{r}/p_i]}$$

  There are two cases:

  - Suppose $p = p_i$. Then $\sigma[\vec{r}/p_i] = \sigma[\vec{q}/p][\vec{r}/p_i]$. So, merging the above biconditionals gives us:

$$\vec{w} \models^n_\sigma \exists p_i\theta \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \exists p_i\theta$$

    which is what we want.

  - Suppose $p \neq p_i$. Then the fact that p does not occur free in ψ entails that it does not occur free in θ. So, by inductive hypothesis:

$$\vec{w} \models^n_{\sigma[\vec{r}/p_i]} \theta \leftrightarrow \vec{w} \models^n_{\sigma[\vec{r}/p_i][\vec{q}/p]} \theta$$

    But since $p \neq p_i$, $\sigma[\vec{r}/p_i][\vec{q}/p] = \sigma[\vec{q}/p][\vec{r}/p_i]$. So we have:

$$\vec{w} \models^n_{\sigma[\vec{r}/p_i]} \theta \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p][\vec{r}/p_i]} \theta$$

    So, merging the above biconditionals gives us:

$$\vec{w} \models^n_\sigma \exists p_i\theta \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \exists p_i\theta$$

    which is what we want.

- ϕ is $\exists pp_i\theta$ or $\exists O_i\theta$

  Analogous to the second case of the preceding item.

- ϕ is $\uparrow \theta$

  Since p does not occur free in ϕ, it must not occur free in θ. So, by inductive hypothesis:

$$\vec{w} \models^{n+1}_\sigma \theta \leftrightarrow \vec{w} \models^{n+1}_{\sigma[\vec{q}/p]} \theta$$

  But, by the semantic clause for $\uparrow$:

$$\vec{w} \models^n_\sigma (\uparrow\theta) \leftrightarrow \vec{w} \models^{n+1}_\sigma \theta$$

$$\vec{w} \models^n_{\sigma[\vec{q}/p]} (\uparrow\theta) \leftrightarrow \vec{w} \models^{n+1}_{\sigma[\vec{q}/p]} \theta$$

  So the result is immediate.

- ϕ is $\downarrow \theta$

  Suppose, first that $n = 0$. Then:

$$\vec{w} \models^n_\sigma (\downarrow\theta) \leftrightarrow \vec{w} \models^n_\sigma \theta$$

$$\vec{w} \models^n_{\sigma[\vec{q}/p]} (\downarrow\theta) \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta$$

  Since p does not occur free in ϕ, it must not occur free in θ. So by inductive hypothesis:

$$\vec{w} \models^n_\sigma \theta \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta$$

  So the result is immediate.

  Since p does not occur free in ϕ, it must not occur free in θ. So, by inductive hypothesis:

$$\vec{w} \models^{n-1}_\sigma \theta \leftrightarrow \vec{w} \models^{n-1}_{\sigma[\vec{q}/p]} \theta$$

  But, by the semantic clause for $\downarrow$:

$$\vec{w} \models^n_\sigma (\downarrow\theta) \leftrightarrow \vec{w} \models^{n-1}_\sigma \theta$$

$$\vec{w} \models^n_{\sigma[\vec{q}/p]} (\downarrow\theta) \leftrightarrow \vec{w} \models^{n-1}_{\sigma[\vec{q}/p]} \theta$$

  So the result is immediate.

Proposition 16 (Substitution Principle) *Let ϕ and ψ be formulas with no free variables in common. For $\vec{w} \in \mathcal{W}_\mathcal{A}$ and $\vec{q} = \{\vec{w} : \vec{w} \models^n_\sigma \psi\}$,*

$$\vec{w} \models^n_\sigma \phi[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi$$

Proof If p does not occur free in ϕ, $\phi[\psi/p] = \phi$, which means that the result is immediate, since by proposition 15, we have:

$$\vec{w} \models^n_\sigma \phi \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi$$

We shall therefore assume that p occurs free in ϕ. We proceed by induction on the complexity of ϕ:

- $\phi = p_i$.

  Since p occurs free in ϕ, it must be that $p_i = p$. So $\phi = p$ and $\phi[\psi/p] = \psi$. We can therefore argue as follows:

$$\vec{w} \models^n_\sigma \psi \leftrightarrow \vec{w} \in \{\vec{w} : \vec{w} \models^n_\sigma \psi\}$$

$$\vec{w} \models^n_\sigma \psi \leftrightarrow \vec{w} \in \vec{q}$$

$$\vec{w} \models^n_\sigma \psi \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} p$$

$$\vec{w} \models^n_\sigma \phi[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi$$

- $\phi = \mathcal{Q}_j p_i$. If $n = 0$, the result is immediate, by the semantic clause for $\mathcal{Q}_j$:

$$\vec{w} \models^0_\sigma \mathcal{Q}_j p_i[\psi/p] \leftrightarrow \bot \leftrightarrow \vec{w} \models^0_{\sigma[\vec{q}/p]} \mathcal{Q}_j p_i$$

  We therefore assume $n > 0$. Since p occurs free in ϕ, it must be the case that $p = p_i$. So $\phi = \mathcal{Q}_j p$ and $\phi[\psi/p] = \mathcal{Q}_j \psi$, which means that ψ must itself be a variable, which we call p_l. We can therefore argue as follows:

$$\sigma(p_l) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right](\vec{w}) \leftrightarrow \{\vec{w} : \vec{w} \models^n_\sigma p_l\} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right](\vec{w})$$

$$\sigma(p_l) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right](\vec{w}) \leftrightarrow \vec{q} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right](\vec{w})$$

$$\sigma(p_l) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right](\vec{w}) \leftrightarrow \sigma[\vec{q}/p](p) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right](\vec{w})$$

$$\vec{w} \models^n_\sigma \mathcal{Q}_j p_l \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \mathcal{Q}_j p$$

$$\vec{w} \models^n_\sigma \phi[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi$$

- ϕ is $O_j p_i$

  Since p occurs free in ϕ, it must be that $p_i = p$ and therefore that ϕ is $O_j p$ and ψ is a variable, which we call p_l. We may therefore argue as follows:

$$\sigma(p_l) \in \sigma(O_j) \leftrightarrow \{\vec{w} : \vec{w} \models^n_\sigma p_l\} \in \sigma(O_j)$$

$$\sigma(p_l) \in \sigma(O_j) \leftrightarrow \vec{q} \in \sigma(O_j)$$

$$\sigma(p_l) \in \sigma(O_j) \leftrightarrow \sigma[\vec{q}/p](p) \in \sigma[\vec{q}/p](O_j)$$

$$\vec{w} \models^n_\sigma O_j p_l \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} O_j p$$

$$\vec{w} \models^n_\sigma \phi[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi$$

- ϕ is $p_i \prec pp_j$

  Since p occurs free in ϕ, it must be that $p_i = p$ and therefore that ϕ is $p \prec pp_j$ and ψ is a variable, which we call p_l. We may therefore argue as follows:

$$\sigma(p_l) \in \sigma(pp_j) \leftrightarrow \{\vec{w} : \vec{w} \models^n_\sigma \ p_l\} \in \sigma(pp_j)$$

$$\sigma(p_l) \in \sigma(pp_j) \leftrightarrow \vec{q} \in \sigma(pp_j)$$

$$\sigma(p_l) \in \sigma(pp_j) \leftrightarrow \sigma[\vec{q}/p](p) \in \sigma[\vec{q}/p](pp_j)$$

$$\vec{w} \models^n_\sigma p_l \prec pp_j \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} p \prec pp_j$$

$$\vec{w} \models^n_\sigma \phi[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi$$

- ϕ is $\theta = \xi$

  Since ψ is free for p in ϕ, it must also be free for p in θ or ξ. So, by inductive hypothesis:

$$\vec{w} \models^n_\sigma \theta[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta$$

$$\vec{w} \models^n_\sigma \xi[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \xi$$

  So we can argue as follows:

$$\{\vec{w} : \vec{w} \models^n_\sigma \theta[\psi/p]\} = \{\vec{w} : \vec{w} \models^n_\sigma \xi[\psi/p]\} \leftrightarrow \left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta\right\} = \left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} \xi\right\}$$

$$\vec{w} \models^n_\sigma (\theta[\psi/p] = \xi[\psi/p]) \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} (\theta = \xi)$$

$$\vec{w} \models^n_\sigma (\theta = \xi)[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} (\theta = \xi)$$

- ϕ is $\neg\theta$

  Since ψ is free for p in ϕ, it must also be free for p in θ. So, by inductive hypothesis:

$$\vec{w} \models^n_\sigma \theta[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta$$

  Equivalently:

$$\vec{w} \not\models^n_\sigma \theta[\psi/p] \leftrightarrow \vec{w} \not\models^n_{\sigma[\vec{q}/p]} \theta$$

  So, by the relevant semantic clause:

$$\vec{w} \models^n_\sigma \neg\theta[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \neg\theta$$

- ϕ is $(\theta \wedge \xi)$

  Since ψ is free for p in ϕ, it must also be free for p in θ or ξ. So, by inductive hypothesis:

$$\vec{w} \models^n_\sigma \theta[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta$$
$$\vec{w} \models^n_\sigma \xi[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \xi$$

  So we can argue as follows:

$$(\vec{w} \models^n_\sigma \theta[\psi/p] \wedge \vec{w} \models^n_\sigma \xi[\psi/p]) \leftrightarrow \left(\vec{w} \models^n_{\sigma[\vec{q}/p]} \theta \wedge \vec{w} \models^n_{\sigma[\vec{q}/p]} \xi\right)$$
$$\vec{w} \models^n_\sigma \theta[\psi/p] \wedge \xi[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} (\theta \wedge \xi)$$
$$\vec{w} \models^n_\sigma (\theta \wedge \xi)[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} (\theta \wedge \xi)$$

- ϕ is $\exists p_i \theta$

  By the semantic clause for $\exists$:

$$\vec{w} \models^n_\sigma (\exists p_i \theta)[\psi/p] \leftrightarrow \vec{w} \models^n_\sigma \exists p_i(\theta[\psi/p]) \leftrightarrow \exists \vec{q'} \in P^n_{\mathcal{W}_\mathcal{A}} \left(\vec{w} \models^n_{\sigma[\vec{q'}/p_i]} (\theta[\psi/p])\right)$$

  Since ψ is free for p in ϕ, p_i cannot occur free in ψ. So, by proposition 15:

$$\vec{w} \models^n_\sigma \psi \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q'}/p_i]} \psi$$

  which means that:

$$\left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{q'}/p_i]} \psi\right\} = \{\vec{w} : \vec{w} \models^n_\sigma \psi\} = \vec{q}$$

  Since ψ is free for p in ϕ, it must also be free for p in θ. So, by inductive hypothesis:

$$\vec{w} \models^n_{\sigma[\vec{q'}/p_i]} \theta[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q'}/p_i][\vec{q}/p]} \theta$$

  Since p occurs free in ϕ, $p \neq p_i$. So

$$\vec{w} \models^n_{\sigma[\vec{q'}/p_i][\vec{q}/p]} \theta \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p][\vec{q'}/p_i]} \theta$$

  Putting all of this together:

$$\vec{w} \models^n_\sigma (\exists p_i \theta)[\psi/p] \leftrightarrow \exists \vec{q'} \in P^n_{\mathcal{W}_\mathcal{A}} \left(\vec{w} \models^n_{\sigma[\vec{q}/p][\vec{q'}/p_i]} \theta\right)$$

  But, by the semantic clause for $\exists$,

$$\vec{w} \models^n_{\sigma[\vec{q}/p]} \exists p_i \theta \leftrightarrow \exists \vec{q'} \in P^n_{\mathcal{W}_\mathcal{A}} \left(\vec{w} \models^n_{\sigma[\vec{q}/p][\vec{q'}/p_i]} \theta\right)$$

  So the desired result follows.

- ϕ is $\exists pp_i\theta$ or $\exists O_i\theta$

  Analogous to the preceding case.

- ϕ is $\uparrow\theta$

  Since ψ is free for p in ϕ, it must also be free for p in θ. So, by inductive hypothesis:
  $$\vec{w} \models^{n+1}_{\sigma} \theta[\psi/p] \leftrightarrow \vec{w} \models^{n+1}_{\sigma[\vec{q}/p]} \theta$$
  But, by the semantic clause for $\uparrow$:
  $$\vec{w} \models^{n}_{\sigma} (\uparrow\theta)[\psi/p] \leftrightarrow \vec{w} \models^{n}_{\sigma}\uparrow(\theta[\psi/p]) \leftrightarrow \vec{w} \models^{n+1}_{\sigma} \theta[\psi/p]$$
  $$\vec{w} \models^{n}_{\sigma[\vec{q}/p]} (\uparrow\theta) \leftrightarrow \vec{w} \models^{n+1}_{\sigma[\vec{q}/p]} \theta$$
  So the result is immediate.

- ϕ is $\downarrow\theta$

  Suppose, first that $n = 0$. Then:
  $$\vec{w} \models^{n}_{\sigma} (\downarrow\theta)[\psi/p] \leftrightarrow \vec{w} \models^{n}_{\sigma}\downarrow(\theta[\psi/p]) \leftrightarrow \vec{w} \models^{n}_{\sigma} \theta[\psi/p]$$
  $$\vec{w} \models^{n}_{\sigma[\vec{q}/p]} (\downarrow\theta) \leftrightarrow \vec{w} \models^{n}_{\sigma[\vec{q}/p]} \theta$$
  Since ψ is free for p in ϕ, it must also be free for p in θ. So the result follows immediately from our inductive hypothesis:
  $$\vec{w} \models^{n}_{\sigma} \theta[\psi/p] \leftrightarrow \vec{w} \models^{n}_{\sigma[\vec{q}/p]} \theta$$
  Now suppose $n > 0$. Since ψ is free for p in ϕ, it must also be free for p in θ. So, by inductive hypothesis:
  $$\vec{w} \models^{n-1}_{\sigma} \theta[\psi/p] \leftrightarrow \vec{w} \models^{n-1}_{\sigma[\vec{q}/p]} \theta$$
  But, by the semantic clause for $\downarrow$:
  $$\vec{w} \models^{n}_{\sigma} (\downarrow\theta)[\psi/p] \leftrightarrow \vec{w} \models^{n}_{\sigma}\downarrow(\theta[\psi/p]) \leftrightarrow \vec{w} \models^{n-1}_{\sigma} \theta[\psi/p]$$
  $$\vec{w} \models^{n}_{\sigma[\vec{q}/p]} (\downarrow\theta) \leftrightarrow \vec{w} \models^{n-1}_{\sigma[\vec{q}/p]} \theta$$
  So the result is immediate.

Proposition 17 (Validity Substitution) *Let ϕ have no variables in common with ψ or θ and suppose that $\models \psi \leftrightarrow \theta$. Then:*

$$\vec{w} \models^n_\sigma \phi[\psi/p] \leftrightarrow \vec{w} \models^n_\sigma \phi[\theta/p]$$

Proof Let $\vec{p} = \{\vec{w} : \vec{w} \models^n_\sigma \psi\}$ and $\vec{q} = \{\vec{w} : \vec{w} \models^n_\sigma \theta\}$. Then, by proposition 16,

$$\vec{w} \models^n_\sigma \phi[\psi/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{p}/p]} \phi \qquad \vec{w} \models^n_\sigma \phi[\theta/p] \leftrightarrow \vec{w} \models^n_{\sigma[\vec{q}/p]} \theta$$

But since we have $\models \psi \leftrightarrow \theta$, it must be the case that $\vec{p} = \vec{q}$ and therefore that $\sigma[\vec{p}/p] = \sigma[\vec{q}/p]$, which allows us to conclude:

$$\vec{w} \models^n_\sigma \phi[\psi/p] \leftrightarrow \vec{w} \models^n_\sigma \phi[\theta/p]$$

10 Comprehension

Definition 22 (Valence)

Intuitively, the valence of a formula ϕ, relative to a level of resolution k, is a syntactically characterized upper bound on the resolution that is needed to describe the proposition expressed by ϕ, when evaluated externally at resolution k (assuming a variable assignment of level 0).

Formally, for ϕ a formula and $l \in \mathbb{N}$, the ***valence*** *of ϕ relative to k, written $v_k(\phi)$, is defined recursively, as follows:*

- $v_k(\phi) = k$, *if ϕ is atomic;*
- $v_k(\phi = \psi) = 0$
- $v_k(\neg\phi) = v_k(\phi)$*;*
- $v_k(\phi \wedge \psi) = \max(v_k(\phi), v_k(\psi))$
- $v_k(\exists p_i \phi) = \max(k, v_k(\phi))$
- $v_k(\exists pp_i \phi) = \max(k, v_k(\phi))$
- $v_k(\exists O_i \phi) = \max(k, v_k(\phi))$
- $v_k(\uparrow\phi) = v_{k+1}(\phi)$*;*

- $v_k(\downarrow\phi) = \begin{cases} v_{k-1}(\phi), & \textit{if } k > 0; \\ v_0(\phi), & \textit{if } k = 0. \end{cases}$

Lemma 1 (Level Lemma) *Let ϕ be a formula of $\mathcal{L}$. For any $n, m \in \mathbb{N}$, let σ be an assignment of level m and let $k = \max(m, v_n(\phi))$. We then have:*

$$\{\vec{w} : \vec{w} \models^n_\sigma \phi\} \in P^k_{\mathcal{W}_\mathcal{A}}$$

Proof We proceed by induction on the complexity of ϕ.

For each of the base cases, we proceed by supposing that $\vec{w}(k) = \vec{v}(k)$ and $\vec{w} \models^n_\sigma \phi$, and verifying that $\vec{v} \models^n_\sigma \phi$.

- $\phi = p_i$. The relevant semantic clause gives us $\vec{w} \in \sigma(p_i)$. Since σ is a level-m assignment and $m \leq k$, it is also a level-k assignment. So the fact that $\vec{w}(k) = \vec{v}(k)$ guarantees that we also have $\vec{v} \in \sigma(p_i)$ and therefore $\vec{v} \models^m_\sigma \phi$.

- $\phi = \mathcal{Q}_j p_i$. If $n = 0$, the result is immediate, since

$$\vec{w} \models^0_\sigma \mathcal{Q}_j p_i \leftrightarrow \bot \leftrightarrow \vec{v} \models^0_\sigma \mathcal{Q}_j p_i$$

So let us assume that $n > 0$. By the definition of valence, $v_n(\mathcal{Q}_j p_i) = n$. So we have $k = \max(m, n)$ and therefore $n \leq k$. Since $\vec{w}(k) = \vec{v}(k)$, it follows that $\vec{w}(n) = \vec{v}(n)$ (by proposition 4). Let $\vec{w}(n) = \vec{v}(n) = \langle w, e^{n-1}_1, \ldots, e^{n-1}_r \rangle$. By the semantic clause for $\mathcal{Q}_j p_i$, $\vec{w} \models^n_\sigma \phi$ is equivalent to

$$\sigma(p_i) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right](\vec{w})$$

which, by the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right](\vec{w})$ is equivalent to

$$\sigma(p_i) \in \left\{\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}} : \exists p^{n-1} \in e^{n-1}_j(\vec{p}(n-1) = p^{n-1})\right\}$$

which, by the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right](\vec{v})$ is equivalent to

$$\sigma(p_i) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_j}\right](\vec{v})$$

which is equivalent to $\vec{v} \models^n_\sigma \phi$.

- $\phi = O_j p_i$. By the relevant semantic clause, $\vec{w} \models^n_\sigma \phi$ is equivalent to $\vec{w} \in \sigma(O_j)(\sigma(p_i))$. Since σ is a level-m assignment and $m \leq k$, it is also a level-k assignment. So the fact that $\vec{w}(k) = \vec{v}(k)$ guarantees that we also have $\vec{v} \in \sigma(O_j)(\sigma(p_i))$ and therefore $\vec{v} \models^n_\sigma O_j p_i$.

- ϕ is $(\psi = \theta)$ or $p_i \prec pp_j$. The result follows from the fact that $\vec{w} \models^n_\sigma \phi$ does not depend on $\vec{w}$.

For the remaining cases, we assume our inductive hypothesis for arbitrary σ, m, and n:

- $\phi = \neg\psi$. By inductive hypothesis,

  $$\{\vec{z} : \vec{z} \models^n_\sigma \psi\} \in P^k_{\mathcal{W}_\mathcal{A}}$$

  But if a subset of $\mathcal{W}_\mathcal{A}$is in $P^k_{\mathcal{W}_\mathcal{A}}$, then so is its complement. So:

  $$\{\vec{z} : \vec{z} \not\models^n_\sigma \psi\} \in P^k_{\mathcal{W}_\mathcal{A}}$$

  which is what we want.

- $\phi = (\psi \wedge \theta)$. For $k' = \max(m, v_n(\psi))$ and $k'' = \max(m, v_n(\theta))$ our inductive hypothesis gives us:

  $$\{\vec{z} : \vec{z} \models^n_\sigma \psi\} \in P^{k'}_{\mathcal{W}_\mathcal{A}} \qquad \{\vec{w} : \vec{w} \models^n_\sigma \theta\} \in P^{k''}_{\mathcal{W}_\mathcal{A}}$$

  Let $k^* = \max(k', k'')$. By proposition 5, we have:

  $$\{\vec{z} : \vec{z} \models^n_\sigma \psi\}, \{\vec{w} : \vec{w} \models^n_\sigma \theta\} \in P^{k^*}_{\mathcal{W}_\mathcal{A}}$$

  Now recall that $k = \max(m, v_n(\psi \wedge \theta))$. By the definition of valence, $v_n(\psi \wedge \theta) = \max(v_n(\psi), v_n(\theta))$. So:

  $$\begin{aligned} k &= \max(m, \max(v_n(\psi), v_n(\theta))) \\ &= \max(\max(m, v_n(\psi)), \max(m, v_n(\theta))) \\ &= \max(k', k'') \\ &= k^* \end{aligned}$$

  We therefore have:

  $$\{\vec{z} : \vec{z} \models^n_\sigma \psi\}, \{\vec{w} : \vec{w} \models^n_\sigma \theta\} \in P^k_{\mathcal{W}_\mathcal{A}}$$

  But if two subsets of $\mathcal{W}_\mathcal{A}$are in $P^k_{\mathcal{W}_\mathcal{A}}$, then so is their intersection. So:

  $$\{\vec{z} : \vec{z} \models^n_\sigma \psi \wedge \theta\} \in P^k_{\mathcal{W}_\mathcal{A}}$$

  which is what we want.

- $\phi = \exists p_i \psi$. Let $\vec{w}(k) = \vec{v}(k)$ and assume $\vec{w} \models^n_\sigma \phi$. By the semantic clause for $\exists$, we know that for some $\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}}$, $\vec{w} \models^n_{\sigma[\vec{q}/p_i]} \psi$. Since σ is an assignment of level m, $\sigma[\vec{q}/p_i]$ is an assignment of level $\max(n, m)$. Let $k' = \max(\max(n, m), v_n(\psi))$. Our inductive hypothesis gives us:

  $$\left\{\vec{z} : \vec{z} \models^n_{\sigma[\vec{q}/p_i]} \psi\right\} \in P^{k'}_{\mathcal{W}_\mathcal{A}}$$

  But, by the definition of valence, $v_n(\exists p_i \psi) = \max(n, v_n(\psi))$. So

  $$\begin{aligned} k &= \max(m, v_n(\exists p_i \psi)) \\ &= \max(m, \max(n, v_n(\psi))) \\ &= \max(m, n, v_n(\psi)) \\ &= \max(\max(m, n), v_n(\psi)) \\ &= k' \end{aligned}$$

  We therefore have:

  $$\left\{\vec{z} : \vec{z} \models^n_{\sigma[\vec{q}/p_i]} \psi\right\} \in P^{k}_{\mathcal{W}_\mathcal{A}}$$

  Since $\vec{w}(k) = \vec{v}(k)$, this means that $\vec{w} \models^n_{\sigma[\vec{q}/p_i]} \psi$ entails $\vec{v} \models^n_{\sigma[\vec{q}/p_i]} \psi$. In other words: we know that for some $\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}}$, $\vec{v} \models^n_{\sigma[\vec{q}/p_i]} \psi$. So, by the semantic clause for $\exists$, $\vec{v} \models^n_\sigma \exists p_i \psi$.

- $\phi = \exists pp_i \psi$ or $\phi = \exists O_j \psi$. Analogous to previous case.

- $\phi = \uparrow \psi$ Let $\vec{w}(k) = \vec{v}(k)$ and assume that $\vec{w} \models^n_\sigma \uparrow \psi$. By the semantic clause for $\uparrow$, we have $\vec{w} \models^{n+1}_\sigma \psi$.

  By the definition of valence, $v_n(\uparrow \psi) = v_{n+1}(\psi)$ and therefore $k = \max(m, v_n(\uparrow \psi)) = \max(m, v_{n+1}(\psi))$. So our inductive hypothesis gives us:

  $$\left\{\vec{z} : \vec{z} \models^{n+1}_\sigma \psi\right\} \in P^{k}_{\mathcal{W}_\mathcal{A}}$$

  So the fact that $\vec{w}(k) = \vec{v}(k)$ gives us $\vec{v} \models^{n+1}_\sigma \psi$ and therefore $\vec{v} \models^n_\sigma \uparrow \psi$, which is what we wanted.

- $\phi = \downarrow \psi$. Let $\vec{w}(k) = \vec{v}(k)$ and assume that $\vec{w} \models^n_\sigma \downarrow \psi$. We show that $\vec{v} \models^n_\sigma \downarrow \psi$.

  First, suppose $n = 0$. By the semantic clause for $\downarrow$,

  $$\vec{w} \models^n_\sigma \downarrow \psi \leftrightarrow \vec{w} \models^0_\sigma \psi$$

So we have $\vec{w} \models^0_\sigma \psi$. The definition of valance gives us $v_0(\downarrow\psi) = v_0(\psi)$ and therefore $k = \max(m, v_0(\uparrow\psi)) = \max(m, v_0(\psi))$. So our inductive hypothesis gives us:

$$\left\{\vec{z} : \vec{z} \models^0_\sigma \psi\right\} \in P^k_{\mathcal{W}_\mathcal{A}}$$

So the fact that $\vec{w}(k) = \vec{v}(k)$ gives us $\vec{v} \models^0_\sigma \psi$ and therefore $\vec{v} \models^0_\sigma \downarrow\psi$, which is what we wanted.

Now suppose $n > 0$. By the semantic clause for $\downarrow$,

$$\vec{w} \models^n_\sigma \downarrow\psi \leftrightarrow \vec{w} \models^{n-1}_\sigma \psi$$

So we have $\vec{w} \models^{n-1}_\sigma \psi$. Since $n > 0$, the definition of valence gives us $v_n(\downarrow\psi) = v_{n-1}(\psi)$ and therefore $k = \max(m, v_n(\downarrow\psi)) = \max(m, v_{n-1}(\psi))$. So our inductive hypothesis gives us:

$$\left\{\vec{z} : \vec{z} \models^{n-1}_\sigma \psi\right\} \in P^k_{\mathcal{W}_\mathcal{A}}$$

So the fact that $\vec{w}(k) = \vec{v}(k)$ gives us $\vec{v} \models^{n-1}_\sigma \psi$ and therefore $\vec{v} \models^n_\sigma \downarrow\psi$, which is what we wanted.

Proposition 18 (Level Advance) *For any $k \in \mathbb{N}$ and formula ϕ,*

$$v_{k+1}(\phi) = v_k(\phi) \vee v_{k+1}(\phi) = v_k(\phi) + 1$$

Proof We proceed by induction on the complexity of ϕ:

- ϕ atomic

  Then $v_{k+1}(\phi) = k + 1$ and $v_k(\phi) = k$. So the result is immediate.

- ϕ is $\psi = \theta$

  Then $v_{k+1}(\phi) = 0 = v_k(\phi) = k$. So the result is immediate.

- ϕ is $\neg\psi$

  By the definition of valence,

  $$v_{k+1}(\neg\psi) = v_{k+1}(\psi) \qquad v_k(\neg\psi) = v_k(\psi)$$

  And, by inductive hypothesis:

  $$v_{k+1}(\psi) = v_k(\psi) \vee v_{k+1}(\psi) = v_k(\psi) + 1$$

  So the result is immediate.

- ϕ is $\psi \wedge \theta$

  By the definition of valence,

  $$v_k(\psi \wedge \theta) = \max(v_k(\psi), v_k(\theta))$$

  $$v_{k+1}(\psi \wedge \theta) = \max(v_{k+1}(\psi), v_{k+1}(\theta))$$

  And by inductive hypothesis:

  $$v_{k+1}(\psi) = v_k(\psi) \vee v_{k+1}(\psi) = v_k(\psi) + 1$$

  $$v_{k+1}(\theta) = v_k(\theta) \vee v_{k+1}(\theta) = v_k(\theta) + 1$$

  Assume, with no loss of generality, that $v_k(\psi) \geq v_k(\theta)$. So

  $$v_k(\psi \wedge \theta) = \max(v_k(\psi), v_k(\theta)) = v_k(\psi)$$

  If $v_{k+1}(\psi) = v_k(\psi) + 1$, it follows from our inductive hypotheses that

  $$v_{k+1}(\psi \wedge \theta) = \max(v_{k+1}(\psi), v_{k+1}(\theta)) = v_{k+1}(\psi) = v_k(\psi) + 1 = v_k(\psi \wedge \theta) + 1$$

  which gives us what we want.

  So we may assume both $v_k(\psi) \geq v_k(\theta)$ and $v_{k+1}(\psi) = v_k(\psi)$. If $v_{k+1}(\theta) = v_k(\theta)$, it follows from our inductive hypotheses that

  $$v_{k+1}(\psi \wedge \theta) = \max(v_{k+1}(\psi), v_{k+1}(\theta)) = v_{k+1}(\psi) = v_k(\psi) = v_k(\psi \wedge \theta)$$

  which, again gives us what we want.

  So we may assume $v_k(\psi) \geq v_k(\theta)$, $v_{k+1}(\psi) = v_k(\psi)$, and $v_{k+1}(\theta) = v_k(\theta) + 1$. Since $v_k(\psi) \geq v_k(\theta)$ and $v_{k+1}(\psi) = v_k(\psi)$, our inductive hypothesis entails that are only two remaining options:

  - $v_{k+1}(\psi) \geq v_{k+1}(\theta)$, in which case it follows from our inductive hypotheses that

    $$v_{k+1}(\psi \wedge \theta) = \max(v_{k+1}(\psi), v_{k+1}(\theta)) = v_{k+1}(\psi) = v_k(\psi) = v_k(\psi \wedge \theta)$$

    which gives us what we want.
  - $v_{k+1}(\theta) = v_{k+1}(\psi) + 1$ (and therefore $v_k(\psi) = v_k(\theta)$). So we have:

    $$v_{k+1}(\psi \wedge \theta) = \max(v_{k+1}(\psi), v_{k+1}(\theta)) = v_{k+1}(\theta) = v_k(\theta) + 1 = v_k(\psi) + 1 = v_k(\psi \wedge \theta) + 1$$

    which gives us what we want.

- ϕ is $\exists p\psi$

  By the definition of valence,

  $$v_k(\exists p\psi) = \max(k, v_k(\psi)) \qquad v_{k+1}(\exists p\psi) = \max(k+1, v_{k+1}(\psi))$$

  And by inductive hypothesis:

  $$v_{k+1}(\psi) = v_k(\psi) \vee v_{k+1}(\psi) = v_k(\psi) + 1$$

  Suppose first that $k \geq v_k(\psi)$, and therefore:

  $$v_k(\exists p\psi) = \max(k, v_k(\psi)) = k.$$

  By our inductive hypothesis, it must be the case that $k+1 \geq v_{k+1}(\psi)$. So we have

  $$v_{k+1}(\exists p\psi) = \max(k+1, v_{k+1}(\psi)) = k+1 = v_k(\exists p\psi) + 1$$

  which gives us what we want.

  Now suppose $v_k(\psi) > k$, and therefore:

  $$v_k(\exists p\psi) = \max(k, v_k(\psi)) = v_k(\psi).$$

  By our inductive hypothesis, it must be the case that $v_{k+1}(\psi) \geq k+1$. So we have

  $$v_{k+1}(\exists p\psi) = \max(k+1, v_{k+1}(\psi)) = v_{k+1}(\psi)$$

  By our inductive hypothesis, this means that:

  $$v_{k+1}(\exists p\psi) = v_k(\psi) \vee v_{k+1}(\exists p\psi) = v_k(\psi) + 1$$

  Since $v_k(\exists p\psi) = v_k(\psi)$, this gives us what we want.

- ϕ is $\exists pp\psi$ or $\exists O\psi$

  Analogous to preceding case

- ϕ is $\uparrow\psi$

  By the definition of valence,

  $$v_k(\uparrow\psi) = v_{k+1}(\psi) \qquad v_{k+1}(\uparrow\psi) = v_{k+2}(\psi)$$

And by inductive hypothesis:

$$v_{k+2}(\psi) = v_{k+1}(\psi) \vee v_{k+2}(\psi) = v_{k+1}(\psi) + 1$$

Putting the two together gives us what we want:

$$v_{k+1}(\uparrow\psi) = v_k(\uparrow\psi) \vee v_{k+1}(\uparrow\psi) = v_k(\uparrow\psi) + 1$$

- ϕ is $\downarrow\psi$

  Suppose, first, that $k = 0$. Then, by the definition of valence,

  $$v_k(\downarrow\psi) = v_k(\psi) \qquad v_{k+1}(\downarrow\psi) = v_k(\psi)$$

  which gives us what we want.

  Now suppose that $k > 0$. By the definition of valence,

  $$v_k(\downarrow\psi) = v_{k-1}(\psi) \qquad v_{k+1}(\downarrow\psi) = v_k(\psi)$$

  And by inductive hypothesis:

  $$v_k(\psi) = v_{k-1}(\psi) \vee v_k(\psi) = v_{k-1}(\psi) + 1$$

  Putting the two together gives us what we want:

  $$v_{k+1}(\downarrow\psi) = v_k(\downarrow\psi) \vee v_{k+1}(\downarrow\psi) = v_k(\downarrow\psi) + 1$$

Proposition 19 (Level Advance Corollary) *For any formula ϕ and $k \in \mathbb{N}$,*

$$v_0(\psi) \leq v_k(\phi) \leq v_0(\phi) + k$$

Proof By proposition 18,

$$\begin{array}{rcccl} v_0(\phi) & \leq & v_1(\phi) & \leq & v_0(\phi) + 1 \\ v_1(\phi) & \leq & v_2(\phi) & \leq & v_1(\phi) + 1 \\ & & \vdots & & \\ v_{k-1}(\phi) & \leq & v_k(\phi) & \leq & v_{k-1}(\phi) + 1 \end{array}$$

which together entail

$$v_0(\psi) \leq v_k(\phi) \leq v_0(\phi) + k$$

Definition 23

$$\uparrow^k := \underbrace{\uparrow \dots \uparrow}_{k \ times} \qquad \downarrow^k := \underbrace{\downarrow \dots \downarrow}_{k \ times}$$

Theorem 1 (Existential Generalization) *Let ϕ and ψ be such that ψ is free for p in ϕ. For $k = v_0(\psi)$,*

$$\models \phi[\psi/p] \rightarrow \uparrow^k \exists p \downarrow^k \phi$$

Proof Fix a model $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$. It suffices to verify the following for an arbitrary n-level assignment σ:

$$\vec{\alpha} \models^n_\sigma \phi[\psi/p] \rightarrow \uparrow^k \exists p \downarrow^k \phi$$

We assume $\vec{\alpha} \models^n_\sigma \phi[\psi/p]$ and show $\vec{\alpha} \models^n_\sigma \uparrow^k \exists p \downarrow^k \phi$. For $l = \max(n, v_n(\psi))$, lemma 1 gives us:

$$\{\vec{w} : \vec{w} \models^n_\sigma \psi\} \in P^l_{\mathcal{W}_\mathcal{A}}$$

Note that it must be the case that $l \leq (n+k)$: if $l = n$ the result is immediate; and if $l = v_n(\psi)$, we can use proposition 19 to show:

$$l = v_n(\psi) \leq v_0(\psi) + n = k + n$$

So, by proposition 5, we have:

$$\{\vec{w} : \vec{w} \models^n_\sigma \psi\} \in P^{n+k}_{\mathcal{W}_\mathcal{A}}$$

Accordingly, there exists $\vec{q} \in P^{n+k}_{\mathcal{W}_\mathcal{A}}$ such that

$$\vec{q} = \{\vec{w} : \vec{w} \models^n_\sigma \ \psi\}$$

By proposition 16,

$$\vec{\alpha} \models^n_\sigma \phi[\psi/p] \leftrightarrow \vec{\alpha} \models^n_{\sigma[\vec{q}/p]} \phi$$

So, by our initial assumption:

$$\vec{\alpha} \models^n_{\sigma[\vec{q}/p]} \phi$$

which is equivalent to the following, by the semantic clause for $\downarrow$:

$$\vec{\alpha} \models^{n+k}_{\sigma[\vec{q}/p]} \downarrow^k \phi$$

Since $\vec{q} \in P^{n+k}_{\mathcal{W}_\mathcal{A}}$, the semantic clause for $\exists$ entails that

$$\vec{\alpha} \models^{n+k}_{\sigma} \exists p \downarrow^k \phi$$

which gives us our desired conclusion, by the semantic clause for $\uparrow$,

$$\vec{\alpha} \models^{n}_{\sigma} \uparrow^k \exists p \downarrow^k \phi$$

Corollary 1 (Comprehension) *For ϕ a formula, let $k = v_0(\phi)$ and let p be a variable not occurring free in ϕ. Then:*

1. $\models \uparrow^k \exists p \downarrow^k (p = \phi)$
2. $\models \uparrow^k \exists p (p = \downarrow^k \phi)$

Proof Since p does not occur free in ϕ, ϕ is free for p in $p = \phi$. So, by Theorem 1,

$$\models (p = \phi)[\phi/p] \to \uparrow^k \exists p \downarrow^k (p = \phi)$$

Since $(p = \phi)[\phi/p] = (\phi = \phi)$, part 1 follows immediately by the semantic clauses for $=$ and $\to$.

To verify part 2, fix a model $\langle W, \mathcal{A}, \vec{\alpha}, n \rangle$. It suffices to verify the following for an arbitrary n-level assignment σ:

$$\vec{\alpha} \models^{n}_{\sigma} \uparrow^k \exists p (p = \downarrow^k \phi)$$

By part 1, we know that:

$$\vec{\alpha} \models^{n}_{\sigma} \uparrow^k \exists p \downarrow^k (p = \phi)$$

which, by the semantic clause for $\uparrow$, is equivalent to:

$$\vec{\alpha} \models^{n+k}_{\sigma} \exists p \downarrow^k (p = \phi)$$

So, by the semantic clause for $\exists$, there is some $\vec{q} \in P^{n+k}_{\mathcal{W}_\mathcal{A}}$ such that:

$$\vec{\alpha} \models^{n+k}_{\sigma[\vec{q}/p]} \downarrow^k (p = \phi)$$

which, by the semantic clause for $\downarrow$, is equivalent to:

$$\vec{\alpha} \models^{n}_{\sigma[\vec{q}/p]} p = \phi$$

which, by the semantic clause for $=$, is equivalent to:

$$\left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} p\right\} = \left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi\right\}$$

which is just

$$\vec{q} = \left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{q}/p]} \phi\right\}$$

which, by the semantic clause for $\downarrow$, is equivalent to:

$$\vec{q} = \left\{\vec{w} : \vec{w} \models^{n+k}_{\sigma[\vec{q}/p]} \downarrow^k \phi\right\}$$

which is just

$$\left\{\vec{w} : \vec{w} \models^{n+k}_{\sigma[\vec{q}/p]} p\right\} = \left\{\vec{w} : \vec{w} \models^{n+k}_{\sigma[\vec{q}/p]} \downarrow^k \phi\right\}$$

which, by the semantic clause for $=$, is equivalent to:

$$\vec{\alpha} \models^{n+k}_{\sigma[\vec{q}/p]} p = \downarrow^k \phi$$

Since $\vec{q} \in P^{n+k}_{\mathcal{W}_{\mathcal{A}}}$, the semantic clause for $\exists$ entails that this is equivalent to:

$$\vec{\alpha} \models^{n+k}_{\sigma} \exists p(p = \downarrow^k \phi)$$

which, by the semantic clause for $\uparrow$, is equivalent to:

$$\vec{\alpha} \models^{n}_{\sigma} \uparrow^k \exists p(p = \downarrow^k \phi)$$

Proposition 20 (Non-triviality) *There is a frame $\langle W, \mathcal{A} \rangle$, a level-$n$ assignment σ ($n \in \mathbb{N}$), and a formula ϕ of $\mathscr{L}$ such that*

$$\{\vec{w} : \vec{w} \models^n_\sigma \phi\} \notin P^n_{\mathcal{W}_{\mathcal{A}}}$$

Proof Let $W = \{0\}$ and $\mathcal{A} = W^\infty$. Let $w^1 = \left\langle 0, \underbrace{\{\{0\}\}, \ldots, \{\{0\}\}}_{r \text{ times}} \right\rangle$ and $v^1 = \left\langle 0, \underbrace{\emptyset, \ldots, \emptyset}_{r \text{ times}} \right\rangle$. Let $\vec{w}$ and $\vec{v}$ be such that $\vec{w}(1) = w^1$ and $\vec{v}(1) = v^1$. Let σ be a level-0 assignment such that $\sigma(p_1) = \mathcal{W}_{\mathcal{A}}$, and let $\phi = \uparrow \mathcal{Q}_1(p_1)$. Our semantic clauses then entail:

$$\vec{w} \models^0_\sigma \uparrow \mathcal{Q}_1(p_1) \leftrightarrow \vec{w} \models^1_\sigma \mathcal{Q}_1(p_1) \leftrightarrow \mathcal{W}_{\mathcal{A}} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^0_{\mathcal{Q}_1}\right](\vec{w})$$

But by the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^0_{\mathcal{Q}_1}\right]$ and the fact that $w^1 = \left\langle 0, \underbrace{\{\{0\}\}, \ldots, \{\{0\}\}}_{r \text{ times}} \right\rangle$:

$$\begin{array}{rcl} \vec{p} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^0_{\mathcal{Q}_1}\right](\vec{w}) & \leftrightarrow & \vec{p} \in P^0_{\mathcal{W}_\mathcal{A}} \wedge \exists p^0 \in \{\{0\}\}\,(\vec{p}(0) = p^0) \\ & \leftrightarrow & \vec{p} \in P^0_{\mathcal{W}_\mathcal{A}} \wedge \vec{p}(0) = \{0\} \\ & \leftrightarrow & \vec{p} = \mathcal{W}_\mathcal{A} \end{array}$$

So we have $\vec{w} \models^0_\sigma\!\uparrow \mathcal{Q}_1(p_1)$. In contrast, we don't have $\vec{v} \models^0_\sigma\!\uparrow \mathcal{Q}_1(p_1)$. For, again by our semantic clauses,

$$\vec{v} \models^0_\sigma\!\uparrow \mathcal{Q}_1(p_1) \leftrightarrow \vec{v} \models^1_\sigma \mathcal{Q}_1(p_1) \leftrightarrow \mathcal{W}_\mathcal{A} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^0_{\mathcal{Q}_1}\right](\vec{v})$$

And we know from the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^0_{\mathcal{Q}_1}\right]$ and the fact that $v^1 = \left\langle 0, \underbrace{\emptyset, \ldots, \emptyset}_{r \text{ times}} \right\rangle$ that

$$\vec{p} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^0_{\mathcal{Q}_1}\right](\vec{v}) \quad \leftrightarrow \quad \bot$$

Since $\vec{w}(0) = \vec{v}(0) = 0$, we may conclude that

$$\left\{\vec{z} : \vec{z} \models^0_\sigma\!\uparrow \mathcal{Q}_1(p_1)\right\} \notin P^0_{\mathcal{W}_\mathcal{A}}$$

11 Axioms and Rules

Proposition 21 (Quantifiers)

1. *Universal instantiation (propositional):* $\models \forall p(\phi) \to \phi$
2. *Universal instantiation (plural):* $\models \forall pp(\phi) \to \phi$
3. *Universal instantiation (intensional):* $\models \forall O(\phi) \to \phi$
4. *Existential generalization (propositional):* $\models \phi \to \exists p\ \phi$.
5. *Existential generalization (plural):* $\models \phi \to \models \exists pp\ \phi$.
6. *Existential generalization (intensional):* $\models \phi \to \models \exists O\ \phi$.

Proof

1. *Universal Instantiation* (we focus on the propositional case; the others are analogous)

   Fix a model $\langle W, \mathcal{A}, \vec{\alpha}, n \rangle$. For an arbitrary n-level assignment σ, we assume $\vec{\alpha} \models^n_\sigma \forall p(\phi)$ and show $\vec{\alpha} \models^n_\sigma \phi$. Using the (derived) semantic clause for $\forall$, our assumption entails that for any $\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}}$:

   $$\vec{\alpha} \models^n_{\sigma[\vec{q}/p]} \phi$$

   So this is true, in particular, when $\vec{q} = \sigma(p)$ and therefore $\sigma = \sigma[\vec{q}/p]$, which means that we have:

   $$\vec{\alpha} \models^n_\sigma \phi$$

   as desired.

4. *Existential Generalization* (we focus on the propositional case; the others are analogous)

   Fix a model $\langle W, \mathcal{A}, \vec{\alpha}, n \rangle$. For an arbitrary n-level assignment σ, we assume $\vec{\alpha} \models^n_\sigma \phi$ and show $\vec{\alpha} \models^n_\sigma \exists p\phi$. By the semantic clause for $\exists$, it therefore suffices to verify that for some $\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}}$:

   $$\vec{\alpha} \models^n_{\sigma[\vec{q}/p]} \phi$$

   Let $\vec{q} = \sigma(p)$. Accordingly, $\sigma = \sigma[\vec{q}/p]$. So all we need to verify is

   $$\vec{\alpha} \models^n_\sigma \phi$$

   which is precisely what we had assumed.

Proposition 22 (Rules)

1. *Modus Ponens: if* $\models \phi$ *and* $\models \phi \to \psi$*, then* $\models \psi$*.*
2. *Universal generalization (propositional): if* $\models \phi$*, then* $\models \forall p\ \phi$*.*
3. *Universal generalization (plural): if* $\models \phi$*, then* $\models \forall pp\ \phi$*.*
4. *Universal generalization (intensional) if* $\models \phi$*, then* $\models \forall O\ \phi$*.*
5. *Existential generalization (propositional): if* $\models \phi \to \exists p\ \phi$*.*

6. *Existential generalization (plural): if* $\models \phi \rightarrow \models \exists pp\ \phi$.

7. *Existential generalization (intensional): if* $\models \phi \rightarrow \models \exists O\ \phi$.

8. *Next Introduction: if* $\models \phi$*, then* $\models \uparrow \phi$.

9. *Necessitation: if* $\models \phi$*, then* $\models \Box\phi$.

Proof

1. *Modus Ponens*

   Fix a model $\langle W, \mathcal{A}, \vec{\alpha}, n \rangle$. It suffices to verify the following for an arbitrary n-level assignment σ: if $\vec{\alpha} \models^n_\sigma \phi$ and $\vec{\alpha} \models^n_\sigma \phi \to \psi$, then $\vec{\alpha} \models^n_\sigma \psi$, which follows immediately from the (derived) semantic clause for $\to$.

2. *Universal Generalization* (we focus on the propositional case; the others are analogous)

   Assume $\models \phi$ and fix a model $\langle W, \mathcal{A}, \vec{\alpha}, n \rangle$. It suffices to verify the following for an arbitrary n-level assignment σ: $\vec{\alpha} \models^n_\sigma \forall p \phi$. By the (derived) semantic clause for $\forall$, it therefore suffices to verify that for any $\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}}$:

   $$\vec{\alpha} \models^n_{\sigma[\vec{q}/p]} \phi$$

   But this is an immediate consequence of $\models \phi$, since $\sigma[\vec{q}/p]$ is an assignment of level n.

5. *Next Introduction*

   Assume $\models \phi$ and fix a model $\langle W, \mathcal{A}, \vec{\alpha}, n \rangle$. It suffices to verify the following for an arbitrary n-level assignment σ: $\vec{\alpha} \models^n_\sigma \uparrow \phi$. By the semantic clause for $\uparrow$ it therefore suffices to verify:

   $$\vec{\alpha} \models^{n+1}_\sigma \phi$$

   But since σ is a level-n assignment, proposition 13 entails that it is also a level-$(n+1)$ assignment. So the result is an immediate consequence of $\models \phi$.

6. *Necessitation* Assume $\models \phi$ and fix a model $\langle W, \mathcal{A}, \vec{\alpha}, n \rangle$. It suffices to verify the following for an arbitrary n-level assignment σ: $\vec{\alpha} \models^n_\sigma \Box\phi$.

By the (derived) semantic clause for $\Box$ it therefore suffices to verify that, for arbitrary $\vec{w} \in \mathcal{W}_{\mathcal{A}}$:

$$\vec{w} \models^n_\sigma \phi$$

which follows immediately from $\models \phi$.

Proposition 23 (The behavior of $\uparrow$)

1. $\models (\neg \uparrow \phi) \leftrightarrow (\uparrow \neg \phi)$
2. $\models (\Diamond \uparrow \phi) \leftrightarrow (\uparrow \Diamond \phi)$
3. $\models (\uparrow \phi \wedge \uparrow \psi) \leftrightarrow \uparrow (\phi \wedge \psi)$
4. $\models (\uparrow \phi \ = \ \uparrow \psi) \leftrightarrow \uparrow (\phi = \psi)$
5. $\models \uparrow (p) \leftrightarrow p$
6. $\models \uparrow (p \prec pp) \leftrightarrow p \prec pp$
7. $\models \uparrow (Op) \leftrightarrow Op$
8. $\models (\uparrow\downarrow\uparrow \phi) \leftrightarrow (\uparrow\uparrow\downarrow \phi)$

Proof Fix a model $\langle W, \mathcal{A}, \vec{\alpha}, n \rangle$. For an arbitrary n-level assignment σ:

1. $\models (\neg \uparrow \phi) \leftrightarrow (\uparrow \neg \phi)$

$$\vec{\alpha} \not\models^{n+1}_\sigma \phi \leftrightarrow \vec{\alpha} \not\models^{n+1}_\sigma \phi$$
$$\vec{\alpha} \not\models^{n}_\sigma \uparrow \phi \leftrightarrow \vec{\alpha} \models^{n+1}_\sigma \neg \phi$$
$$\vec{\alpha} \models^{n}_\sigma \neg \uparrow \phi \leftrightarrow \vec{\alpha} \models^{n}_\sigma \uparrow \neg \phi$$

2. $\models (\Diamond \uparrow \phi) \leftrightarrow (\uparrow \Diamond \phi)$

$$\left\{\vec{w} : \vec{w} \models^{n+1}_\sigma \phi\right\} \neq \emptyset \leftrightarrow \left\{\vec{w} : \vec{w} \models^{n+1}_\sigma \phi\right\} \neq \emptyset$$
$$\{\vec{w} : \vec{w} \models^{n}_\sigma \uparrow \phi\} \neq \emptyset \leftrightarrow \vec{\alpha} \models^{n+1}_\sigma \Diamond \phi$$
$$\vec{\alpha} \models^{n}_\sigma \Diamond \uparrow \phi \leftrightarrow \vec{\alpha} \models^{n}_\sigma \uparrow \Diamond \phi$$

3. $\models (\uparrow\phi \wedge \uparrow\psi) \leftrightarrow \uparrow(\phi \wedge \psi)$

$$(\vec{\alpha} \models_{\sigma}^{n+1} \phi \ \wedge \vec{\alpha} \models_{\sigma}^{n+1} \psi) \leftrightarrow (\vec{\alpha} \models_{\sigma}^{n+1} \phi \wedge \vec{\alpha} \models_{\sigma}^{n+1} \psi)$$
$$(\vec{\alpha} \models_{\sigma}^{n} \uparrow\phi \ \wedge \vec{\alpha} \models_{\sigma}^{n} \uparrow\psi) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n+1} (\phi \wedge \psi)$$
$$\vec{\alpha} \models_{\sigma}^{n} (\uparrow\phi \wedge \uparrow\psi) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} \uparrow(\phi \wedge \psi)$$

4. $\models (\uparrow\phi \ = \uparrow\psi) \leftrightarrow \uparrow(\phi = \psi)$

$$\left\{\vec{w} : \vec{w} \models_{\sigma}^{n+1} \phi\right\} = \left\{\vec{w} : \vec{w} \models_{\sigma}^{n+1} \psi\right\} \leftrightarrow \left\{\vec{w} : \vec{w} \models_{\sigma}^{n+1} \phi\right\} = \left\{\vec{w} : \vec{w} \models_{\sigma}^{n+1} \psi\right\}$$
$$\{\vec{w} : \vec{w} \models_{\sigma}^{n} \uparrow\phi\} = \{\vec{w} : \vec{w} \models_{\sigma}^{n} \uparrow\psi\} \leftrightarrow \vec{\alpha} \models_{\sigma}^{n+1} \phi = \psi$$
$$\vec{\alpha} \models_{\sigma}^{n} (\uparrow\phi \ = \uparrow\psi) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} \uparrow(\phi = \psi)$$

5. $\models \uparrow(p) \leftrightarrow p$

$$\vec{\alpha} \in \sigma(p) \leftrightarrow \vec{\alpha} \in \sigma(p)$$
$$\vec{\alpha} \models_{\sigma}^{n+1} p \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} p$$
$$\vec{\alpha} \models_{\sigma}^{n} \uparrow(p) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} p$$

6. $\models \uparrow(p \prec pp) \leftrightarrow p \prec pp$

$$\sigma(p) \in \sigma(pp) \leftrightarrow \sigma(p) \in \sigma(pp)$$
$$\vec{\alpha} \models_{\sigma}^{n+1} (p \prec pp) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} p \prec pp$$
$$\vec{\alpha} \models_{\sigma}^{n} \uparrow(p \prec pp) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} p \prec pp$$

7. $\models \uparrow(Op) \leftrightarrow Op$

$$\vec{\alpha} \in \sigma(O)(\sigma(p)) \leftrightarrow \vec{\alpha} \in \sigma(O)(\sigma(p))$$
$$\vec{\alpha} \models_{\sigma}^{n+1} (Op) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} Op$$
$$\vec{\alpha} \models_{\sigma}^{n} \uparrow(Op) \leftrightarrow \vec{\alpha} \models_{\sigma}^{n} Op$$

8. $\models (\uparrow\downarrow\uparrow\phi) \leftrightarrow (\uparrow\uparrow\downarrow\phi)$

$$\vec{\alpha} \models^{n+1}_{\sigma} \phi \leftrightarrow \vec{\alpha} \models^{n+1}_{\sigma} \phi$$
$$\vec{\alpha} \models^{n}_{\sigma} (\uparrow\phi) \leftrightarrow \vec{\alpha} \models^{n+2}_{\sigma} (\downarrow\phi)$$
$$\vec{\alpha} \models^{n+1}_{\sigma} (\downarrow\uparrow\phi) \leftrightarrow \vec{\alpha} \models^{n+1}_{\sigma} (\uparrow\downarrow\phi)$$
$$\vec{\alpha} \models^{n}_{\sigma} (\uparrow\downarrow\uparrow\phi) \leftrightarrow \vec{\alpha} \models^{n}_{\sigma} (\uparrow\uparrow\downarrow\phi)$$

12 The behavior of $\mathcal{Q}$

Definition 24 *A* ***natural model*** *is a model $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$ such that $\mathcal{A} = W^{\infty}$.*

Proposition 24 (Non-functionality of Refinement) *Fix a natural model $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$. For any $w^n \in W^n_{\mathcal{A}}$, there are and $w^{n+1}, v^{n+1} \in W^{n+1}_{\mathcal{A}}$ such that $v^{n+1} \neq w^{n+1}$ but*

$$w^n \rhd_{W_{\mathcal{A}}} w^{n+1} \wedge w^n \rhd_{W_{\mathcal{A}}} v^{n+1}$$

Proof Suppose, first, that $n = 0$ and therefore that $w^n = w \in W$. Let $e^0_1 = \emptyset$ and $f^0_1 = \{W\}$. For i such that $1 < i \leq r$, let $e^0_i = f^0_i = \emptyset$. Let $w^{n+1} = \langle w, e^0_1, \ldots, w^0_r\rangle$ and $v^{n+1} = \langle w, f^0_1, \ldots, f^0_r\rangle$. Since $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$ is a natural model, $w^{n+1}, v^{n+1} \in W^{n+1}_{\mathcal{A}}$. And since $e^0_1 \neq f^0_1$, $w^{n+1} \neq v^{n+1}$. But it follows from the definition of $\rhd_{W_{\mathcal{A}}}$that

$$w^n \rhd_{W_{\mathcal{A}}} w^{n+1} \wedge w^n \rhd_{W_{\mathcal{A}}} v^{n+1}$$

Now suppose that $n > 0$ and let $w^n = \left\langle w, e^{n-1}_1, \ldots, e^{n-1}_r\right\rangle$. Since $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$ is a natural model (and therefore $\mathcal{A} = W^{\infty}$), it follows from Cantor's Theorem that $|P^{n-1}_{W_{\mathcal{A}}}| > |P^{n}_{W_{\mathcal{A}}}|$. So there must be some $p^n \in P^n_{W_{\mathcal{A}}}$ that is not identical to $[p^{n-1}]^n_{W_{\mathcal{A}}}$ for $p^{n-1} \in P^{n-1}_{W_{\mathcal{A}}}$. For each $i \leq r$, let $f^n_i = \left\{[p^{n-1}]^n_{W_{\mathcal{A}}} : p^{n-1} \in e^{n-1}_i\right\}$. Let $e^n_1 = f^n_1 \cup \{p^n\}$, and for i such that $1 < i \leq r$, let $e^n_i = f^n_i$. Let $w^{n+1} = \langle w, e^n_1, \ldots, w^n_r\rangle$ and $v^{n+1} = \langle w, f^n_1, \ldots, f^n_r\rangle$. Since $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$ is a natural model, $w^{n+1}, v^{n+1} \in W^{n+1}_{\mathcal{A}}$. And since $e^n_1 \neq f^n_1$, $w^{n+1} \neq v^{n+1}$. But it follows from the definition of $\rhd_{W_{\mathcal{A}}}$that

$$w^n \rhd_{W_{\mathcal{A}}} w^{n+1} \wedge w^n \rhd_{W_{\mathcal{A}}} v^{n+1}$$

Proposition 25 (Non-triviality of the Superproposition Hierarchy) *Fix a natural model $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$. For any $n \in \mathbb{N}$, there is a super-proposition $\vec{q}$ such that $\vec{q} \in P^{n+1}$ but $\vec{q} \notin P^n_{\mathcal{W}_\mathcal{A}}$.*

Proof Let $\vec{w}$ be an arbitrary world in $\mathcal{W}_\mathcal{A}$. By proposition 24, there are $v^{n+1}, z^{n+1} \in W^{n+1}_\mathcal{A}$ such that $v^{n+1} \neq z^{n+1}$ but

$$\vec{w}(n) \rhd_{W_\mathcal{A}} v^{n+1} \wedge \vec{w}(n) \rhd_{W_\mathcal{A}} z^{n+1}$$

By proposition 3, w^{n+1} and v^{n+1} we may assume that there are superworlds $\vec{v}$ and $\vec{z}$ such that $\vec{v}(n+1) = v^{n+1}$ and $\vec{z}(n+1) = z^{n+1}$ and therefore such that $\vec{v}(n+1) \neq \vec{z}(n+1)$. And by proposition 2, $\vec{v}(n) = \vec{w}(n) = \vec{z}(n)$.

Let $\vec{q} = \{\vec{y} \in \mathcal{W}_\mathcal{A} : \vec{y}(n+1) = v(n+1)\}$. Trivially, $\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}}$. But $\vec{q} \notin P^n_{\mathcal{W}_\mathcal{A}}$, since $\vec{z} \notin \vec{q}$ even though $\vec{v}(n) = \vec{z}(n)$.

Proposition 26 (Prior and Kaplan) *When attention is restricted to natural models:*

1. ***No Same Level:*** $\models \exists p \neg \mathcal{Q}_i p$
2. ***Kaplan Next:*** $\models \forall pp {\uparrow} \Diamond \forall q (\mathcal{Q}_i q \leftrightarrow q \prec pp)$
3. ***Kaplan Next:*** $\models \forall pp \Diamond \forall q ({\uparrow} \mathcal{Q}_i q \leftrightarrow q \prec pp)$
4. ***Modal Prior Next:*** $\models \forall p {\uparrow} \Diamond \forall q (\mathcal{Q}_i q \leftrightarrow q = p)$
5. ***Modal Prior Next:*** $\models \forall p \Diamond \forall q ({\uparrow} \mathcal{Q}_i q \leftrightarrow q = p)$

Proof

1. $\models \exists p \neg \mathcal{Q}_i p$

   Fix a natural model $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$. Suppose, first, that $n = 0$ and let σ be an arbitrary n-level assignment. By the semantic clause for $\mathcal{Q}$

   $$\vec{\alpha} \not\models^n_\sigma \mathcal{Q}_i p$$

   So, by the semantic clause for $\neg$,

   $$\vec{\alpha} \models^n_\sigma \neg \mathcal{Q}_i p$$

So, by existential generalization (proposition 21),

$$\vec{\alpha} \models^n_\sigma \exists p \neg \mathcal{Q}_i p$$

Now assume $n > 0$ and let $\vec{q}$ be in $P^n_{\mathcal{W}_\mathcal{A}}$ but not $P^{n-1}_{\mathcal{W}_\mathcal{A}}$ (since $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$ is a natural model, proposition 25 entails that such a $\vec{q}$ must exist). Suppose, for *reductio*, that for some n-level assignment σ, $\vec{\alpha} \models^n_\sigma \mathcal{Q}_i p$. By the semantic clause for $\mathcal{Q}_i$,

$$\vec{q} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_i}\right](\vec{\alpha})$$

Now let let $\vec{\alpha}(n) = \left\langle w, e^{n-1}_1, \ldots, e^{n-1}_r \right\rangle$. By the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_i}\right]$,

$$\vec{q} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n-1}_{\mathcal{Q}_i}\right](\vec{\alpha}) \leftrightarrow \left(\vec{q} \in P^{n-1}_{\mathcal{W}_\mathcal{A}} \wedge \exists p^{n-1} \in e^{n-1}_i(\vec{q}(n-1) = p^{n-1})\right)$$

So we have $\vec{q} \in P^{n-1}_{\mathcal{W}_\mathcal{A}}$, which contradicts an earlier assumption. It follows that for every n-level assignment σ:

$$\vec{\alpha} \not\models^n_\sigma \mathcal{Q}_i p$$

So we can get the desired result by replicating the reasoning we deployed in the case $n = 0$.

2. $\models \forall pp \uparrow \Diamond \forall q(\mathcal{Q}_i q \leftrightarrow q \prec pp)$

   Fix a natural model $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$. Fix arbitrary $v \in W$ and $\vec{B} \subseteq P^n_{\mathcal{W}_\mathcal{A}}$ ($\vec{B} \neq \emptyset$) and let

   $$B^n = \left\{p^n \in P^n_{W_\mathcal{A}} : \exists \vec{p} \in \vec{B}(p^n = \vec{p}(n))\right\} \qquad v^{n+1} = \langle v, e^n_1, \ldots, e^n_r\rangle$$

   where $e^n_j = B^n$ for each $j \neq r$. Since $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$ is a natural model, $v^{n+1} \in W^{n+1}_\mathcal{A}$. So, by proposition 3, there is a superworld $\vec{v} \in \mathcal{W}_\mathcal{A}$ such that $\vec{v}(n+1) = v^{n+1}$.

   Now pick an arbitrary $\vec{q} \in P^{n+1}_{\mathcal{W}_\mathcal{A}}$. We verify:

   $$\vec{q} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n}_{\mathcal{Q}_i}\right](\vec{v}) \leftrightarrow \vec{q} \in \vec{B}$$

   - $\rightarrow$

     Assume $\vec{q} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n}_{\mathcal{Q}_i}\right](\vec{v})$. By the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n}_{\mathcal{Q}_i}\right]$, we have:

     $$\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}} \wedge \exists p^n \in e^n_i(\vec{q}(n) = p^n)$$

and therefore

$$\exists p^n \in B^n(\vec{q}(n) = p^n)$$

So, by the definition of B^n:

$$\exists p^n \in \left\{p^n \in P^n_{W_\mathcal{A}} : \exists \vec{p} \in \vec{B}(p^n = \vec{p}(n))\right\}(\vec{q}(n) = p^n)$$

equivalently

$$\exists p^n \in P^n_{W_\mathcal{A}} \exists \vec{p} \in \vec{B}(p^n = \vec{p}(n) \wedge \vec{q}(n) = p^n)$$

We may therefore fix $p^n \in P^n_{W_\mathcal{A}}$ and $\vec{p} \in \vec{B}$ such that

$$p^n = \vec{p}(n) \wedge \vec{q}(n) = p^n$$

and therefore

$$\vec{p}(n) = \vec{q}(n).$$

But since $\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}}$ and $\vec{p} \in \vec{B}$ (and therefore $\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}}$), proposition 7 entails:

$$\vec{p} = \vec{q}$$

which is what we wanted.

- $\leftarrow$

  Assume $\vec{q} \in \vec{B}$. Since $\vec{B} \subseteq P^n_{\mathcal{W}_\mathcal{A}}$, $\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}}$. So our assumption is equivalent to:

$$\exists \vec{p} \in \vec{B}(\vec{p}(n) = \vec{q}(n))$$

which is equivalen to:

$$\exists p^n \in P^n_{W_\mathcal{A}} \exists \vec{p} \in \vec{B}(p^n = \vec{p}(n) \wedge \vec{q}(n) = p^n)$$

and therefore

$$\exists p^n \in \left\{p^n \in P^n_{W_\mathcal{A}} : \exists \vec{p} \in \vec{B}(p^n = \vec{p}(n))\right\}(\vec{q}(n) = p^n)$$

which, by the definition of B^n, is equivalent to:

$$\exists p^n \in B^n(\vec{q}(n) = p^n)$$

which is equivalent to

$$\exists p^n \in e^n_i(\vec{q}(n) = p^n)$$

Since $\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}}$, we may conclude:

$$\vec{q} \in P^n_{\mathcal{W}_\mathcal{A}} \wedge \exists p^n \in e^n_i(\vec{q}(n) = p^n)$$

which gives us what we want, by the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^n_{\mathcal{Q}_i}\right]$:

$$\vec{q} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^n_{\mathcal{Q}_i}\right](\vec{v})$$

We have shown that for arbitrary $\vec{q} \in P^{n+1}_{\mathcal{W}_\mathcal{A}}$ and $\vec{B} \subseteq P^n_{\mathcal{W}_\mathcal{A}}$ ($\vec{B} \neq \emptyset$),

$$\vec{q} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^n_{\mathcal{Q}_i}\right](\vec{v}) \leftrightarrow \vec{q} \in \vec{B}$$

So, by the semantic clause for $\mathcal{Q}$ and $\prec$, we have the following for an arbitrary level n assignment σ:

$$\vec{v} \models^{n+1}_{\sigma[\vec{B}/pp][\vec{q}/p]} \mathcal{Q}p \leftrightarrow p \prec pp$$

But since $\vec{q}$ was an arbitrary member of $P^{n+1}_{\mathcal{W}_\mathcal{A}}$, the (derived) semantic clause for $\forall$ gives us:

$$\vec{v} \models^{n+1}_{\sigma[\vec{B}/pp]} \forall p(\mathcal{Q}p \leftrightarrow p \prec pp)$$

Since $\vec{v} \in \mathcal{W}_\mathcal{A}$, this gives us:

$$\left\{\vec{w} : \vec{w} \models^{n+1}_{\sigma[\vec{B}/pp]} \forall p(\mathcal{Q}p \leftrightarrow p \prec pp)\right\} \neq \emptyset$$

So, by the (derived) semantic clause for $\Diamond$,

$$\vec{\alpha} \models^{n+1}_{\sigma[\vec{B}/pp]} \Diamond\forall p(\mathcal{Q}p \leftrightarrow p \prec pp)$$

So, by the semantic clause for $\uparrow$,

$$\vec{\alpha} \models^{n}_{\sigma[\vec{B}/pp]} \uparrow\Diamond\forall p(\mathcal{Q}p \leftrightarrow p \prec pp)$$

But since $\vec{B} \subseteq P^n_{\mathcal{W}_\mathcal{A}}$ was chosen arbitrarily, the (derived) semantic clause for $\forall$ gives us:

$$\vec{\alpha} \models^{n}_{\sigma} \forall pp\uparrow\Diamond\forall p(\mathcal{Q}p \leftrightarrow p \prec pp)$$

which is what we wanted.

3. $\models \forall pp \Diamond \forall q (\uparrow(\mathcal{Q}_i q) \leftrightarrow q \prec pp)$

Fix a natural model $\langle W, \mathcal{A}, \vec{\alpha}, n \rangle$. Fix arbitrary $v \in W$ and $\vec{B} \subseteq P^n_{\mathcal{W}_\mathcal{A}}$ ($\vec{B} \neq \emptyset$) and define $\vec{v}$ as in the previous case. As in the previous case, we can show for arbitrary $\vec{q} \in P^{n+1}_{\mathcal{W}_\mathcal{A}}$

$$\vec{q} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^n_{\mathcal{Q}_i}\right](\vec{v}) \leftrightarrow \vec{q} \in \vec{B}$$

So, by the semantic clause for $\mathcal{Q}$ and $\prec$, we have the following for an arbitrary level n assignment σ:

$$\vec{v} \models^{n+1}_{\sigma[\vec{B}/pp][\vec{q}/p]} \mathcal{Q}p \leftrightarrow \vec{v} \models^n_{\sigma[\vec{B}/pp][\vec{q}/p]} p \prec pp$$

So, by the semantic clause for $\uparrow$,

$$\vec{v} \models^n_{\sigma[\vec{B}/pp][\vec{q}/p]} \uparrow(\mathcal{Q}p) \leftrightarrow \vec{v} \models^n_{\sigma[\vec{B}/pp][\vec{q}/p]} p \prec pp$$

and therefore

$$\vec{v} \models^n_{\sigma[\vec{B}/pp][\vec{q}/p]} \uparrow(\mathcal{Q}p) \leftrightarrow p \prec pp$$

But since $\vec{q}$ was chosen arbitrarily from $P^{n+1}_{\mathcal{W}_\mathcal{A}}$, proposition 5 guarantees that the result also holds when $\vec{q}$ is chosen arbitrarily from $P^n_{\mathcal{W}_\mathcal{A}}$. So the (derived) semantic clause for $\forall$ gives us:

$$\vec{v} \models^n_{\sigma[\vec{B}/pp]} \forall p(\uparrow(\mathcal{Q}p) \leftrightarrow p \prec pp)$$

Since $\vec{v} \in \mathcal{W}_\mathcal{A}$, this gives us:

$$\left\{\vec{w} : \vec{w} \models^n_{\sigma[\vec{B}/pp]} \forall p(\uparrow(\mathcal{Q}p) \leftrightarrow p \prec pp)\right\} \neq \emptyset$$

So, by the (derived) semantic clause for $\Diamond$,

$$\vec{\alpha} \models^n_{\sigma[\vec{B}/pp]} \Diamond \forall p(\uparrow(\mathcal{Q}p) \leftrightarrow p \prec pp)$$

But since $\vec{B} \subseteq P^n_{\mathcal{W}_\mathcal{A}}$ was chosen arbitrarily, the (derived) semantic clause for $\forall$ gives us:

$$\vec{\alpha} \models^n_\sigma \forall pp \Diamond \forall p(\uparrow(\mathcal{Q}p) \leftrightarrow p \prec pp)$$

which is what we wanted.

4. $\models \forall pp {\uparrow} \Diamond \forall q(\mathcal{Q}_i q \leftrightarrow q \prec pp)$

   Analogous to the proof of more general result.

5. $\models \forall p \Diamond \forall q({\uparrow}\mathcal{Q}_i q \leftrightarrow q = p)$

   Analogous to the proof of more general result.

Definition 25 *Fix a model $\langle W, \mathcal{A}, \vec{\alpha}, m\rangle$. $\mathcal{Q}_i$ and $\mathcal{Q}_j$ are **independent** (relative to the relevant model) if and only if, for any $\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}}$ $(n \in \mathbb{N})$, there is $\vec{w} \in \mathcal{W}_\mathcal{A}$ such that*

$$\vec{p} \in \left[{}^{\mathcal{W}}_{\mathcal{A}} Ext^n_{\mathcal{Q}_i}\right](\vec{w}) \leftrightarrow \vec{p} \notin \left[{}^{\mathcal{W}}_{\mathcal{A}} Ext^n_{\mathcal{Q}_j}\right](\vec{w})$$

Proposition 27 (Some models exemplify independence) *Whenever $i \neq j$, $\mathcal{Q}_i$ and $\mathcal{Q}_j$ and independent relative to any natural model.*

Proof Assume, with no loss of generality, that $i = 1$ and $j = 2$. Let $\langle W, \mathcal{A}, \vec{\alpha}, m\rangle$ be a natural model and let $\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}}$ $(n \in \mathbb{N})$. For any $w \in W$, let

$$w^{n+1} = \left\langle w, \{\vec{p}(n)\}, \underbrace{\emptyset, \ldots, \emptyset}_{(r-1)\text{ times}} \right\rangle$$

Since $\langle W, \mathcal{A}, \vec{\alpha}, m\rangle$ is a natural model, $w^{n+1} \in W^{n+1}_\mathcal{A}$. So, by proposition 3, there is $\vec{w} \in \mathcal{W}_\mathcal{A}$ such that $\vec{w}(n+1) = w^{n+1}$. We then have:

$$\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}} \wedge \vec{p}(n) = \vec{p}(n), \quad \neg(\bot)$$
$$\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}} \wedge \exists p^n \in \{\vec{p}(n)\}(\vec{p}(n) = p^n), \quad \neg(\vec{p} \in P^n_{\mathcal{W}_\mathcal{A}} \wedge \exists p^n \in \emptyset(\vec{p}(n) = p^n))$$
$$\vec{p} \in \left[{}^{\mathcal{W}}_{\mathcal{A}} \mathrm{Ext}^n_{\mathcal{Q}_1}\right](\vec{w}), \quad \vec{p} \notin \left[{}^{\mathcal{W}}_{\mathcal{A}} \mathrm{Ext}^n_{\mathcal{Q}_1}\right](\vec{w})$$

Proposition 28 (Russell-Myhill Next) *Whenever $\mathcal{Q}_i$ and $\mathcal{Q}_j$ are independent, $\models {\uparrow}(\mathcal{Q}_i p \neq \mathcal{Q}_j p)$*

Proof Let $\mathcal{Q}_i$ and $\mathcal{Q}_j$ be independent and assume, for *reductio*, that $\not\models {\uparrow} \neg(\mathcal{Q}_i p = \mathcal{Q}_j p)$. By our assumption, there is a model $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$ and an n-level assignment σ such that:

$$\vec{\alpha} \not\models^n_\sigma {\uparrow} \neg(\mathcal{Q}_i p = \mathcal{Q}_j p)$$

which, by proposition 23, is equivalent to:

$$\vec{\alpha} \not\models_\sigma^n \neg \uparrow (\mathcal{Q}_i p = \mathcal{Q}_j p)$$

which, by the semantic clause for $\neg$, is equivalent to:

$$\vec{\alpha} \models_\sigma^n \uparrow (\mathcal{Q}_i p = \mathcal{Q}_j p)$$

which, by the semantic clause for $\uparrow$, is equivalent to:

$$\vec{\alpha} \models_\sigma^{n+1} \mathcal{Q}_i p = \mathcal{Q}_j p$$

which, by the semantic clause for $=$, is equivalent to:

$$\left\{\vec{w} : \vec{w} \models_\sigma^{n+1} \mathcal{Q}_i p\right\} = \left\{\vec{w} : \vec{w} \models_\sigma^{n+1} \mathcal{Q}_j p\right\}$$

So, for any $\vec{w} \in \mathcal{W}_\mathcal{A}$,

$$\vec{w} \models_\sigma^{n+1} \mathcal{Q}_i p \leftrightarrow \vec{w} \models_\sigma^{n+1} \mathcal{Q}_j p$$

So, by the semantic clause for $\mathcal{Q}$, the following holds for any $\vec{w} \in \mathcal{W}_\mathcal{A}$,

$$\sigma(p) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^n_{\mathcal{Q}_1}\right](\vec{w}) \leftrightarrow \sigma(p) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^n_{\mathcal{Q}_2}\right](\vec{w})$$

which contradicst the assumption that $\mathcal{Q}_i$ and $\mathcal{Q}_j$ are independent.

Proposition 29 (Intensional Cases)

1. $\models \uparrow \exists O \Box \exists p (\uparrow(\mathcal{Q}_i p) \not\leftrightarrow Op)$

2. $\not\models \forall O \Diamond \forall p (\uparrow(\mathcal{Q}_i p) \leftrightarrow Op)$

Proof

1. $\models \uparrow \exists O \Box \exists p (\uparrow(\mathcal{Q}_i p) \not\leftrightarrow Op)$
   Fix a model $\langle W, \mathcal{A}, \vec{\alpha}, n\rangle$ and an arbitrary n-level assignment, σ. Let $\vec{\imath} \in I_{\mathcal{W}_\mathcal{A}}$ be defined as follows:

$$\vec{\imath}(\vec{q}) = \left\{\vec{w} \in \mathcal{W}_\mathcal{A} : \vec{w} \not\models^{n+1}_{\sigma[\vec{q}/p]} \mathcal{Q}_i p\right\}$$

   Let us verify that $\vec{\imath} \in I^{n+1}_{\mathcal{W}_\mathcal{A}}$:

We assume $\vec{q} \in P^{n+1}_{\mathcal{W}_\mathcal{A}}$ and show $\vec{\imath}(\vec{q}) \in P^{n+1}_{\mathcal{W}_\mathcal{A}}$. Since $\vec{q} \in P^{n+1}_{\mathcal{W}_\mathcal{A}}$, and since σ is an assignment of level n, $\sigma[\vec{q}/p]$ is an assignment of level $n{+}1$. So Lemma 1 gives us:

$$\left\{\vec{w} \in \mathcal{W}_\mathcal{A} : \vec{w} \not\models^{n+1}_{\sigma} \mathcal{Q}_i p\right\} \in P^{n+1}_{\mathcal{W}_\mathcal{A}}$$

which is what we wanted.

So we know that $\vec{\imath} \in I^{n+1}_{\mathcal{W}_\mathcal{A}}$ and therefore that $\sigma[\vec{\imath}/O]$ is an assignment of level $n+1$.

Choose $\vec{v} \in \mathcal{W}_\mathcal{A}$ arbitrarily and let $\vec{q} \in P^{n}_{\mathcal{W}_\mathcal{A}}$. Then propositions 10 and 11 give us:

$$\vec{q} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n+1}_{\mathcal{Q}_i}\right](\vec{v}) \leftrightarrow \vec{q} \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^{n}_{\mathcal{Q}_i}\right](\vec{v})$$

So, by the semantic clause for $\mathcal{Q}_i$,

$$\vec{v} \models^{n+2}_{\sigma[\vec{\imath}/O][\vec{q}/p]} \mathcal{Q}_i p \leftrightarrow \vec{v} \models^{n+1}_{\sigma[\vec{q}/p]} \mathcal{Q}_i p$$

which is equivalent to:

$$\vec{v} \models^{n+2}_{\sigma[\vec{\imath}/O][\vec{q}/p]} \mathcal{Q}_i p \not\leftrightarrow \vec{v} \not\models^{n+1}_{\sigma[\vec{q}/p]} \mathcal{Q}_i p$$

which is equivalent to:

$$\vec{v} \models^{n+2}_{\sigma[\vec{\imath}/O][\vec{q}/p]} \mathcal{Q}_i p \not\leftrightarrow \vec{v} \in \left\{\vec{w} \in \mathcal{W}_\mathcal{A} : \vec{w} \not\models^{n+1}_{\sigma[\vec{q}/p]} \mathcal{Q}_i p\right\}$$

so, by the definition of $\vec{\imath}$,

$$\vec{v} \models^{n+2}_{\sigma[\vec{\imath}/O][\vec{q}/p]} \mathcal{Q}_i p \not\leftrightarrow \vec{v} \in \vec{\imath}(\vec{q})$$

So, by the semantic clauses for $\uparrow$ and Op,

$$\vec{v} \models^{n+1}_{\sigma[\vec{\imath}/O][\vec{q}/p]} \uparrow(\mathcal{Q}_i p) \not\leftrightarrow \vec{v} \models^{n+1}_{\sigma[\vec{\imath}/O][\vec{q}/p]} Op$$

So, by the semantic clauses for Boolean operators,

$$\vec{v} \models^{n+1}_{\sigma[\vec{\imath}/O][\vec{q}/p]} \uparrow(\mathcal{Q}_i p) \not\leftrightarrow Op$$

Since $\vec{q}$ is in $P^{n}_{\mathcal{W}_\mathcal{A}}$ and therefore in $P^{n+1}_{\mathcal{W}_\mathcal{A}}$, the semantic clause for $\exists$ gives us:

$$\vec{v} \models^{n+1}_{\sigma[\vec{\imath}/O]} \exists p(\uparrow(\mathcal{Q}_i p) \not\leftrightarrow Op)$$

Since $\vec{v} \in \mathcal{W}_\mathcal{A}$ was chosen arbitrarily, this gives us:

$$\left\{\vec{v} : \vec{v} \models^{n+1}_{\sigma[\vec{\imath}/O]} \exists p(\uparrow(\mathcal{Q}_i p) \not\leftrightarrow Op)\right\} = \mathcal{W}_\mathcal{A}$$

So, by the (derived) semantic clause for $\Box$,

$$\vec{\alpha} \models^{n+1}_{\sigma[\vec{\imath}/O]} \Box\exists p(\uparrow(\mathcal{Q}_i p) \not\leftrightarrow Op)$$

But since $\vec{\imath} \in I^{n+1}_{\mathcal{W}_\mathcal{A}}$, the semantic clause for $\exists$ gives us

$$\vec{\alpha} \models^{n+1}_{\sigma} \exists O\Box\exists p(\uparrow(\mathcal{Q}_i p) \not\leftrightarrow Op)$$

So, by the semantic clause for $\uparrow$,

$$\vec{\alpha} \models^{n}_{\sigma} \uparrow\exists O\Box\exists p(\uparrow(\mathcal{Q}_i p) \not\leftrightarrow Op)$$

which is what we wanted.

2. $\not\models \forall O\Diamond\forall p(\uparrow(\mathcal{Q}_i p) \leftrightarrow Op)$
   Suppose otherwise:

   $$\models \forall O\Diamond\forall p(\uparrow(\mathcal{Q}_i p) \leftrightarrow Op)$$

   By proposition 22, this means that:

   $$\models\uparrow \forall O\Diamond\forall p(\uparrow(\mathcal{Q}_i p) \leftrightarrow Op)$$

   But by the previous result, we have

   $$\models \uparrow\exists O\Box\exists p(\uparrow(\mathcal{Q}_i p) \not\leftrightarrow Op)$$

   which is equivalent to:

   $$\models \neg\uparrow\forall O\Diamond\forall p(\uparrow(\mathcal{Q}_i p) \leftrightarrow Op)$$

Proposition 30 (Validity Failures)

- $\not\models \uparrow\phi \rightarrow \phi$

Proof

- Consider a model $\langle W, \mathcal{A}, \vec{\alpha}, 0\rangle$, where $W = \{0\}$, $\mathcal{A} = W^{\infty}$, $w^1 = \left\langle 0, \underbrace{\{\emptyset\}, \ldots, \{\emptyset\}}_{r \text{ times}} \right\rangle$, and $\vec{\alpha}$ is such that $\vec{\alpha}(1) = w^1$. Let σ be an assignment such that $\sigma(p) = \emptyset$. So we have $\sigma(p) \in P^0_{\mathcal{W}_\mathcal{A}}$ and $\sigma(p)(0) = \{\vec{w}(0) : \vec{w} \in \sigma(p)\} = \emptyset$. We verify that $\vec{\alpha} \models^0_\sigma \uparrow \mathcal{Q}_i(p)$ but $\vec{\alpha} \not\models^0_\sigma \mathcal{Q}_i(p)$:

  The latter is an immediate consequence of the semantic clause for $\mathcal{Q}_i$. So it suffices to verify the former. But, trivially,

  $$\exists p^0 \in \{\emptyset\} \, (\emptyset = p^0)$$

  And since $\sigma(p) \in P^0_{\mathcal{W}_\mathcal{A}}$ and $\sigma(p)(0) = \emptyset$, this gives us:

  $$\sigma(p) \in P^0_{\mathcal{W}_\mathcal{A}} \wedge \exists p^0 \in \{\emptyset\} \, (\sigma(p)(0) = p^0)$$

  equivalently,

  $$\sigma(p) \in \left\{ \vec{p} \in P^0_{\mathcal{W}_\mathcal{A}} : \exists p^0 \in \{\emptyset\} \, (\vec{p}(0) = p^0) \right\}$$

  So, by the definition of $\left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^0_{\mathcal{Q}_i}\right]$

  $$\sigma(p) \in \left[{}^{\mathcal{W}}_{\mathcal{A}}\mathrm{Ext}^0_{\mathcal{Q}_i}\right]$$

  So, by the semantic clause for $\mathcal{Q}_i$

  $$\vec{\alpha} \models^1_\sigma \mathcal{Q}_i(p)$$

  So, by the semantic clause for $\uparrow$:

  $$\vec{\alpha} \models^0_\sigma \uparrow \mathcal{Q}_i(p)$$

13 Examples

A proof of *Prior*: $\models OE^- \to (E^+ \wedge E^-)$

- $E^+ := \exists p(Op \wedge p)$
- $E^- := \exists p(Op \wedge \neg p)$

1. OE^- (assumption) [1]

2. $\neg E^-$ (assumption) [2]
3. $\neg\neg\forall p(Op \to p)$ (from 2, by definition) [2]
4. $\forall p(Op \to p)$ (from 3, by Double Negation Elimination) [2]
5. $(OE^- \to E^-)$ (from 4, by Universal Instantiation) [2]
6. E^- (from 1 and 5, by Modus Ponens) [2, 1]
7. E^- (from 6 discharging 2, by Conditional Proof) [1]
8. $(O(E^-) \wedge E^-)$ (from 7 and 1, by Conjunction Introduction) [1]
9. $\exists p(Op \wedge p)$ (from 8, by Existential Generalization) [1]
10. $(E^+ \wedge E^-)$ (from 7 and 9, by Conjunction Introduction) [1]
11. $OE^- \to (E^+ \wedge E^-)$ (from 10, discharging 1, by Conditional Proof)

A proof of *Modal Prior*: $\models \exists p\Box\neg\forall q(Oq \leftrightarrow (q = p))$

1. $\forall q(Oq \leftrightarrow (q = E^-))$ (assumption) [1]
2. $OE^- \leftrightarrow (E^- = E^-))$ (from 1, by UG) [1]
3. OE^- (from 1, by MP and reflexivity of identilty) [1]
4. $OE^- \to (E^+ \wedge E^-)$ (*Prior*) []
5. $E^+ \wedge E^-$ (from 3 and 4 by MP) [1]
6. $\exists p(Op \wedge \neg p)$ (from 5, by conjunction elimination) [1]
7. $(Op \wedge \neg p)$ (from 6, by EI) [1]
8. $Op \leftrightarrow (p = E^-)$ (from 1, by UG) [1]
9. $p = \neg E^-$ (from 7 and 8), by MP and conj. elim.) [1]
10. $\neg E^-$ (from 7 and 9), by identity subs. and conj. elim.) [1]
11. $\neg\forall q(Oq \leftrightarrow (q = E^-))$ (by *reductio*, from 5 and 10, discharging 1) []
12. $\Box\neg\forall q(Oq \leftrightarrow (q = E^-))$ (from 11, by Necessitation) []
13. $\exists p\Box\neg\forall q(Oq \leftrightarrow (q = p))$ (from 12, by Existential Generalization) []