A Derivation of the Quantum Mechanical
Momentum Operator in the Position
Representation

Ryan D. Reece

September 23, 2006

Abstract

I show that the momentum operator in quantum mechanics, in
the position representation, commonly known to be a derivative with
respect to a spacial \( x \)-coordinate, can be derived by identifying mo-
mentum as the generator of space translations.

1 Translation Operator

Given an eigenstate of position \( |\vec{x}\rangle \), with eigenvalue \( x \), we define a Translation Operator, \( T(\vec{a}) \), which transforms an eigenstate of position to another
eigenstate of position, with the eigenvalue increased by \( \vec{a} \).

\[
T(\vec{a}) |\vec{x}\rangle \equiv |\vec{x} + \vec{a}\rangle
\]

By the following argument, we note that the adjoint of \( T(\vec{a}) \) moves a state
backward. It transforms an eigenstate of position to another eigenstate of position, with the eigenvalue decreased by \( \vec{a} \).

\[
\langle \vec{x}' | T(\vec{a}) | \vec{x} \rangle = \langle \vec{x}' | \vec{x} + \vec{a} \rangle = \delta((\vec{x} + \vec{a}) - \vec{x}') = \delta(\vec{x} - (\vec{x}' - \vec{a})) = \langle \vec{x}' - \vec{a} | \vec{x} \rangle
\]

\[
\Rightarrow \langle \vec{x}' | T(\vec{a}) \rangle = \langle \vec{x}' - \vec{a} | \vec{x} \rangle
\]
\[ T^\dagger(\vec{a}) |\vec{x}'\rangle = |\vec{x}' - \vec{a}\rangle \quad (7) \]

Note that if we translate forwards by some amount, it is the same as translating backwards by negative that amount.

\[ T(\vec{a}) = T^\dagger(-\vec{a}) \quad (8) \]

If we translate a state forwards and then backwards by the same amount, the state remains unchanged. This implies that the translation operator is unitary.

\[ T^\dagger(\vec{a}) T(\vec{a}) |\vec{x}\rangle = |\vec{x}\rangle \quad (9) \]
\[ \Rightarrow T^\dagger(\vec{a}) = T^{-1}(\vec{a}) \quad (10) \]

Any unitary operator can be written as

\[ T(\vec{a}) = e^{-i\vec{K} \cdot \vec{a}} \quad (11) \]

\[ 1 = T^\dagger(\vec{a}) T(\vec{a}) \quad (12) \]
\[ = e^{i\vec{K} \cdot \vec{a}} e^{-i\vec{K} \cdot \vec{a}} \quad (13) \]
\[ = e^{i(\vec{K} \cdot \vec{a} - \vec{K} \cdot \vec{a})} \quad (14) \]
\[ \Rightarrow \vec{K} = \vec{K}^\dagger \quad (15) \]

Where evidently, \( \vec{K} \) must be hermitian. In general, when writing a unitary operator this way, the operators \( \vec{K} \) are known as the generators of whatever unitary operator one is expressing, in this case: translation.

## 2 Eigenstates of \( \vec{K} \)

Let us call the eigenstates of \( \vec{K} \), which are also eigenstates of \( T(\vec{a}), |\vec{k}\rangle \).

\[ \vec{K} |\vec{k}\rangle = \vec{k} |\vec{k}\rangle \quad \text{and} \quad T(\vec{a}) |\vec{k}\rangle = e^{-i\vec{k} \cdot \vec{a}} |\vec{k}\rangle \quad (16) \]

Let us consider the position projection of the translation operator acting on an eigenstate of translation. Letting the translation operator, operate to the right, we have

\[ \langle \vec{x}|T(\vec{a})|\vec{k}\rangle = e^{-i\vec{k} \cdot \vec{a}} \langle \vec{x}|\vec{k}\rangle \quad (17) \]
\[ = e^{-i\vec{k} \cdot \vec{a}} \psi_{\vec{k}}(\vec{x}) \quad (18) \]
where we have defined the wavefunction to be

$$\psi_k(\vec{x}) = \langle \vec{x} | \vec{k} \rangle$$  \hspace{1cm} (19)

Now consider the same projection, replacing $T(\vec{a})$ with $T^\dagger(-\vec{a})$, and letting it operate to the left.

$$\langle \vec{x} | T(\vec{a}) | \vec{k} \rangle = \langle \vec{x} | T^\dagger(-\vec{a}) | \vec{k} \rangle$$  \hspace{1cm} (20)

$$\langle \vec{x} | T(\vec{a}) | \vec{k} \rangle = \langle \vec{x} - \vec{a} | \vec{k} \rangle$$  \hspace{1cm} (21)

$$\langle \vec{x} - \vec{a} | \vec{k} \rangle = \psi_k(\vec{x} - \vec{a})$$  \hspace{1cm} (22)

Equating the two methods, we have

$$\psi_k(\vec{x} - \vec{a}) = e^{-i\vec{k} \cdot \vec{a}} \psi_k(\vec{x})$$  \hspace{1cm} (23)

Letting $\vec{x} = 0$, and $\vec{a} = -\vec{y}$, we recognize that this gives plane wave solutions for the wavefunction.

$$\psi_k(\vec{y}) = \psi_k(0) e^{i\vec{k} \cdot \vec{y}}$$  \hspace{1cm} (24)

As hypothesized by de Broglie, and first experimentally verified by electron diffraction, a particle in an eigenstate of momentum has a wavefunction with a wavevector, $\vec{k}$, related to its momentum $\vec{p}$ by

$$\vec{p} = \hbar \vec{k}$$  \hspace{1cm} (25)

This means that the $\vec{K}$ operator that we have been discussing is indeed the wavevector operator. We can now write the translation operator as

$$T(\vec{a}) = e^{-i\vec{P} \vec{a}/\hbar}$$  \hspace{1cm} (26)

Aside from the constant, $\hbar$, momentum is the generator of translation.

### 3 Matrix Elements of $\vec{P}$ in the $|\vec{x}\rangle$ Basis

For simplicity, let us now consider translation in only one dimension.

$$T(a) = e^{-iaP/\hbar}$$  \hspace{1cm} (27)

The following clever manipulation reveals how to write the momentum operator in terms of the translation operator.

$$\frac{\partial}{\partial a} \bigg|_{a=0} T(a) = -i \frac{\hbar}{P}$$  \hspace{1cm} (28)
\[ P = i\hbar \frac{\partial}{\partial a} \bigg|_{a=0} T(a) \]  

(29)

We should now ask what the matrix elements are of the momentum operator in the position basis.

\[ \langle x' | P | x \rangle = i\hbar \frac{\partial}{\partial a} \bigg|_{a=0} \langle x' | T(a) | x \rangle \]  

\[ = i\hbar \frac{\partial}{\partial a} \bigg|_{a=0} \delta(x + a - x') \]  

\[ = i\hbar \delta'(x - x') \]  

(30) \hspace{2cm} (31) \hspace{2cm} (32)

4 \quad \vec{P} \text{ Acting on a Wavefunction}

We should now take a digression to investigate what is meaning of this derivative of a delta function, \( \delta'(x) \). We integrate by parts, a \( \delta'(x - y) \) acting on some arbitrary function, \( f(x) \). Note that the boundary term is zero because \( \delta(x - y) \) is zero on the boundary, provided a boundary of integration is not at position \( y \).

\[ \int \delta'(x - y) f(x) \, dx = 0 - \int \delta(x - y) f'(x) \, dx \]  

\[ = -f'(y) \]  

(33) \hspace{2cm} (34)

Evidently, the derivative of a delta function is sort of a tool for evaluating the derivative of some function at a certain point.

Now we may ask how we can represent the momentum operator in the position basis. Because the number of states in the position basis are uncountably infinite, a matrix representation would be awkward. We see by the following argument that there is a much more elegant way of writing the momentum operator.

Consider the momentum operator acting on the wavefunction of some
state state $|\psi\rangle$.

$$P \psi(x) = \langle x | P | \psi \rangle$$

$$= \int \langle x | P | x' \rangle \langle x' | \psi \rangle \, dx'$$

$$= i\hbar \int \delta'(x' - x) \psi(x') \, dx'$$

$$= -i\hbar \frac{\partial \psi(x')}{\partial x'} \bigg|_{x' = x}$$

$$= -i\hbar \frac{\partial \psi(x)}{\partial x}$$

$$\therefore P \rightarrow -i\hbar \frac{\partial}{\partial x}$$