Cantor's illusion

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abstract

This analysis shows Cantor's diagonal definition in his 1891 paper was not compatible with his horizontal enumeration of the infinite set M. The diagonal sequence was a counterfeit which he used to produce an apparent exclusion of a single sequence to prove the cardinality of M is greater than the cardinality of the set of integers N.

keywords: Cantor, diagonal, infinite

1. the argument

Translation from Cantor's 1891 paper [1]:

Namely, let m and n be two different characters, and consider a set [Inbegriff] M of elements

\[ E = (x_1, x_2, \ldots, x_v, \ldots) \]

which depend on infinitely many coordinates \( x_1, x_2, \ldots, x_v, \ldots \), and where each of the coordinates is either \( m \) or \( w \). Let \( M \) be the totality [Gesamtheit] of all elements \( E \).

To the elements of \( M \) belong e.g. the following three:

\[ E_I = (m, m, m, m, \ldots), \]
\[ E_{II} = (w, w, w, w, \ldots), \]
\[ E_{III} = (m, w, m, w, \ldots). \]

I maintain now that such a manifold [Mannigfaltigkeit] \( M \) does not have the power of the series 1, 2, 3, ..., \( v \), ....

This follows from the following proposition:

"If \( E_1, E_2, \ldots, E_v, \ldots \) is any simply infinite [einfach unendliche] series of elements of the manifold \( M \), then there always exists an element \( E_0 \) of \( M \), which cannot be connected with any element \( E_v \)."

For proof, let there be

\[ E_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,v}, \ldots) \]
\[ E_2 = (a_{2,1}, a_{2,2}, \ldots, a_{2,v}, \ldots) \]
\[ E_u = (a_{u,1}, a_{u,2}, \ldots, a_{u,v}, \ldots) \]

..........................
where the characters $a_{uv}$ are either $m$ or $w$. Then there is a series $b_1, b_2, \ldots b_v, \ldots$, defined so that $b_v$ is also equal to $m$ or $w$ but is different from $a_{uv}$.

Thus, if $a_{uv} = m$, then $b_v = w$.

Then consider the element

$$E_0 = (b_1, b_2, b_3, \ldots)$$

of $M$, then one sees straight away, that the equation

$$E_0 = E_u$$

cannot be satisfied by any positive integer $u$, otherwise for that $u$ and for all values of $v$.

$$b_v = a_{uv}$$

and so we would in particular have

$$b_u = a_{uu}$$

which through the definition of $b_v$ is impossible. From this proposition it follows immediately that the totality of all elements of $M$ cannot be put into the sequence $[Reihenform]: E_1, E_2, \ldots, E_v, \ldots$ otherwise we would have the contradiction, that a thing $[Ding]$ $E_0$ would be both an element of $M$, but also not an element of $M$.

(end of translation)

2. Cantor's enumeration

The symbols $\{0, 1\}$ will be substituted for $\{m, w\}$ for visual clarity.

Cantor defines an infinite set $M$ consisting of elements $E_n$. Each $E_n$ is an infinite one dimensional horizontal sequence composed of two symbols 0 and 1. He does not specify a rule of formation for sequences, thus they are assumed to result from a random process such as a coin toss. There is one sequence per row, and all sequences are unique differing in one or more positions. He then assigns coordinates to the array of symbols using a two dimensional $(u, v)$ grid.
2.1 orientation

Cantor then defines a diagonal sequence $D$ (red) composed of symbols with coordinates $(u, u)$. The negation of a sequence differs in all positions. Using $D$ as a template, he interchanges all 0's and 1's to produce $E_0$ as the negation of $D$ or (not $D$). He declares, $E_0$ as a horizontal sequence, cannot be in the enumeration since it will conflict with each coordinate $(u, u)$.

2.2 issues
1. A copy of a geometric form inherits the properties of the original, thus $E_0$ should also be a diagonal sequence. Neither D nor $E_0$ are compatible with the horizontal enumeration.

2. There is an inconsistency in Cantor’s sequence definition. The horizontal sequences were formed independently of each other, and entered randomly in the enumeration. D was formed using a specific rule of formation dependent on one element from each horizontal sequence and could only be a qualified sequence in a diagonal enumeration as in fig.2. If the enumeration consisted of diagonal sequences, there would be no interference of D and $E_0$ since they are parallel. In the original enumeration all horizontal sequences were parallel and did not interact. At this point Cantor is comparing two different enumerations, a diagonal form with a horizontal form. Both forms cannot coexist in the same enumeration without interference.
Fig. 3 eliminates the clutter of a full enumeration to emphasize the relation of a diagonal and horizontal form. As shown the diagonal D could exist anywhere in the enumeration since duplicates cannot be detected with a single comparison such as coordinate (6, 6). If u6 was replaced with E0 then a conflict would appear at coordinate (6, 6), which can't be 0 and 1 simultaneously. Since the sequences are formed from two symbols, there are two subsets M0 and M1, one containing sequence S, the other containing its negation (not S). If D is a member of M0 then by symmetry E0 is a member of M1, making both members of M.

3. refutation

For this purpose the symbols \{0, 1\} are substituted for \{m, w\}, for visual clarity. A sequence or string is represented as s.
Fig. 4 is a basic flow chart for forming any $s$ in the process of generating a binary tree graph $T$, a model that represents the Cantor set $M$ in terms of sets and subsets.

Any $s$ must begin with 0 or 1. The set $M$ can be divided into two subsets $M_0$ and $M_1$. Each selection is independent of all others, and $T$ contains copies of itself at every branch, thus the perpetual loop in fig. 4. The following sample is an array of symbols using Cantor's coordinate system $(v, u)$ for column and row. Each $s$ has no last $v$ and the list has no last row.

$0111...$
$1000...$
$0011...$
$1100...$
$0001...$

$D=00101...$
$E_0=11010...$
Fig. 6 tracks the path of D with row numbers from the sample on the right. As a sequence, it is not a contiguous path in the tree, but jumps between subsets M0 and M1 which is not possible. A path must continuously progress in v remaining in its initial subset for its entire existence. Each element of D is already assigned to a horizontal s.
D is the counterfeit for the existing path C, 3rd from the top in column 4.
Fig. 7 has a mirror axis ma. Any s can be rotated 180° about ma to form its negation (not s). The beginning of C and E₀ are shown in red. In the tree graph the spacing of branches was decreasing for the purpose of confining the illustration to a single page.
A more realistic perspective is shown in fig.8 with an exponential growth rate of $2^v$ for both M0 and M1, with the \((u,v)\) plane of each graph spaced apart in 3D space.

\[ \begin{array}{c}
\text{T} \\
\text{1} \\
\text{0} \\
\text{1} \\
\end{array} \quad \ldots \\
\begin{array}{c}
\text{0} \\
\text{1} \\
\text{0} \\
\end{array} \\
\begin{array}{c}
\text{0} \\
\text{1} \\
\text{0} \\
\end{array} \\
\begin{array}{c}
\text{0} \\
\text{1} \\
\text{0} \\
\text{1} \\
\end{array} \\
\begin{array}{c}
\text{0} \\
\text{1} \\
\text{0} \\
\text{1} \\
\end{array} \\
\begin{array}{c}
\text{0} \\
\text{1} \\
\text{0} \\
\text{1} \\
\end{array} \\
\ldots \\
\end{array} \]

\( \text{fig.9} \)

\( C=00101.. \)
\( E_0=11010.. \)

C determines which subsets are excluded in forming \( E_0 \).
Position 1 can't be 0 which excludes subset M0.
Position 2 can't be 0 which excludes subset M10.
Position 3 can't be 1 which excludes subset M111.
Since there is no last selection, the final subset containing \( E_0 \) cannot be determined, but \( E_0 \) is definitely in subset M1, since it is determined by position 1.

**Conclusion**

1. The diagonal \( D \) cannot be formed using the flow chart in fig.4.
2. The tree graph in fig.7 shows \( C \) and \( E_0 \) do not intersect, being members of different subsets. This contradicts Cantor's declaration of a missing \( E_0 \) in section 2.1.
3. The set \( N \) cannot be exhausted, which is the source for \( u \) and \( v \).
4. Cantor's contradiction, that a thing cannot be in two different locations simultaneously, is a logical truth. The question then becomes which location is correct. Since there is access to the beginning of a sequence, the first symbol determines which subset.
5. Cantor's argument uses misdirection in the form of the diagonal \( D \). This paper shows \( E_0 \) must be a member of \( T \).

**Reference**

[1] THE LOGIC MUSEUM  Copyright © E.D.Buckner 2005