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Frege and Peano on definitions

Abstract. Frege and Peano started in 1896 a debate where they contrasted the respective conceptions on the theory and practice of mathematical definitions. Which was (if any) the influence of the Frege-Peano debate on the conceptions by the two authors on the theme of defining in mathematics and which was the role played by this debate in the broader context of their scientific interaction?[1]

Historical data

Gottlob Frege (1848-1925) and Giuseppe Peano (1858-1932) approximately lived in the same years and was prominent pioneers of the then emerging discipline nowadays knew as Mathematical Logic.
To the best of my knowledge, no meeting between Peano and Frege is reported [2]. However, evidence of a non-episodic interaction is provided by two notes published by Peano on Frege's work, two others by Frege on Peano's and by letters from Peano to Frege and unpublished writings found in Frege's *Nachlass* (see the Appendix for a, hopefully complete, list and a tentative chronology).
Documentary evidence dates from 1891 (a draft of a letter from Frege to Peano which is likely to be his first, answering to a previous dispatch of writings by Peano) to 1903. There are evident gaps in the correspondence which suggest that in that period contacts were more frequent they appear now. Moreover, nothing indicates in a definite way that any contact stopped in 1903, even though from this time onward the interests of the two author seem to diverge.
The scientific production of the two authors mostly overlaps on the following themes:

- Creating and developing a symbolic language for mathematics (a *ideography*).
- The analysis of the general logic laws used in mathematical reasoning.

[1]Part of the present note elaborates on data collected during my doctoral studies in the years 2007-2009 at the Department of Mathematics "Giuseppe Peano" of Torino. I wish to thank Clara Silvia Roero for her invaluable contribution in making facts about Peano's work and life suitably available to scholars and for introducing me to the methodology of the History of mathematics.

[2]Both Frege and Peano appear in the "Comité d'Honneur" of the First International Congress of Philosophy (Paris, 1900), but Peano attended the Congress while Frege did not.

- Foundations of arithmetics, analysis and geometry, with particular emphasis on the analysis of the concept of natural number.

- Theory and practice of defining in mathematics.

Public and private discussions between Frege and Peano mostly deal with contrasting the respective ideographies and principles of defining. In the following paragraphs I will focus on the latter topic.

The debate on defining

"Defining in mathematics" is a central theme for both the authors. They devote to this topic entire paragraphs in their main works and a number of other published or unpublished writings throughout their scientific production.
Frege mainly gives a systematic account of his theory of definitions in the two parts of *Grundgesetze der Arithmetik* (1893, 1903): in the first one he describes the role he assigns to defining inside his symbolic system; in the second part he states his *Principles of definition*. In between this two moments, the discussions with Peano and Hilbert take place.
Peano's writings on defining alternate between theorical statements on definitions and applications to the definitions of a series of fundamental mathematical notions (from the *Area of a surface* (1890) to *Limit* (1913)). The main expositions of his ideas on the topic are in *Notations de Logique Mathématique* (1894), in the framework of his symbolic language, and *Le definizioni in matematica* (1911), more in general. The debate Frege-Peano on defining lies between Peano's symbolic and conceptual exposition of the matter, as did for Frege.
The debate on defining also plays a central role in contrasting Frege's ideography with Peano's: against Nidditch's diminishing appraisal ([7], p. 108) there are the appreciations by Peano (quoted by Nidditch himself) and Frege: "What is at stake here [the canons of definition] is perhaps the deepest difference between the two concept-scripts" ([4], p. 152).
Frege's *Begriffsschrift* (1879) and *Grundgesetze I* (1893) on one side and Peano's *Notations* (1894) and *Formulaire de mathématiques, vol. 1* (1895) on the other are the works the two authors mainly refer to in discussing about defining in mathematics. In these works we can retrace some shared principles which form the common ground for the subsequent debate:

- Definitions are nominal.

- Defining is an act of willing, not an act of judgement.

- The purpose of defining is to *abbreviate* sentences (The principles of non-creativity and of eliminability are implicitely assumed).

- A definition has to have the form of an equation of two *homogeneous* terms: the *definiens* and the *definiendum*.

The documents According to the current knowledge, the debate Frege-Peano on defining took place in the years 1896-1897. We extract a tentative chronology from the full documentary evidence of Frege-Peano interaction given in the appendix:

1. (6.07.1896) Frege, *Über die Begriffsschrift des Herrn Peano und meine eigene. Vortrag, gehalten in der ausserordenlichen Sitzung vom 6 Juli 1896*, "Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-Physische Klasse" 48, 1896.

2. (29.09.1896) Letter from Frege to Peano. Frege asks Peano to publish his letter in *Rivista di Matematica*.

3. (14.10.1896) Letter from Peano to Frege.

4. (4.04.1897) Peano (*Studii di logica matematica*, Atti della Reale Accademia delle Scienze di Torino vol. 32, 1896-97) refers to Frege's talk given 6.07.1896.

5. (1896/1897) Draft of a letter from Frege to Peano (undated).

6. (1898) The letter from Frege to Peano (29.09.1896) is published in *Rivista di Matematica* (*Lettera del sig. G. Frege all'Editore*, R.d.M vol. 6, 1898, pp. 53-59).

7. (1898) Peano, *Corrisp.: Risposta* [*a Lettera del sig. G. Frege all'Editore*], R.d.M vol. 6, 1898, pp. 60-61.

8. (1897/1898?) Frege's unpublished *Begründung meiner strengeren Grundsätze des Definierens* contrasts Peano's way of defining with his own.

9. (1903) Frege (*Grundgesetze der Arithmetik, begriffsscrhriftlich abgeleitet, Band II*, Jena, 1903, footnote to §58) refers to Peano, *Risposta* (1898).

The actual sequence of the above-mentioned documents cannot be established with absolute certainty: we do not know if Peano knew the content of Frege's Leipzig talk soon after Frege delivered it (06.07.1896) or when it was published (1897); we do not have elements for dating Frege's unpublished writings; and we do not know if Peano communicated to Frege the final version of his *Risposta* before it was published in *Rivista di Matematica* (1898) together with Frege's letter (29.09.1896).

The debate In his Leipzig conference (06.07.1896) Frege, although contrasting his own view with Peano's, focuses on aspects of the two ideographies which can be more or less suitable in achieving different specific goals, within a (presumably) shared request for rigour in mathematical resoning. He confines to a short paragraph the theme of definitions, which will reveal a strong matter of controversy: "The manner of defining, especially, I find wanting in logical perfection. That the same symbol is explained more than once is almost the rule. Conditional definitions are also very numerous. As against this, I require that each sign be defined just once, and completely, not several times over and in piecemeal fashion; I require that the reference of the defining expression coincide unconditionally with the reference of the defined one." ([2], p. 4)
The debate properly starts with Frege's critiques and remarks on Peano's definitions in the letter of 29.09.1896 (intended by Frege for publishing in Peano's *Rivista di Matematica*). The letter is presented as an answer to Peano's review of *Grundgesetze I* (appeared the year before in *Rivista*) and the declared purpose is to argue against Peano's claim that "the system of the *Formulary* represents a more profound analysis" ([5], p. 113). In particular, the main controverse point is the number of primitive symbols in the two ideographies and, for correctly counting them, Frege is led to carefully examine Peano's definitions of a number of notions.
They remain two direct answers from Peano to Frege's letter. The *Risposta* published in *Rivista* in 1898 and the letter of 14.10.1896 (*Reply*), sent just a couple of weeks after receiving Frege's and conceived also as a reply to Frege's Leipzig talk.
Most of Frege's technical points are accepted by Peano, in his first answer or in the course of the debate, so that the irreducible nut of the disagreement between Frege and Peano on defining can gradually emerge. The problem manifests itself in the presence in Peano's *Formulaire* of *multiple* definitions for the same sign. The two authors discuss this fact sometimes in general terms, sometimes referring to binary operations, like addition, or to relations, like equality, as paradigmatic examples.

Frege's argument runs as follows. He requires every symbol to have one and only one meaning for the sake of making inferences under the logical assumption of the *tertium non datur* law. Hence, he claims that multiple definitions of the same symbol assigning different meanings cannot be allowed. Frege says that, in mathematics, apparently non-contradictory multiple definitions can come under two patterns: the first one is represented by *equivalent definitions* the second by *conditional definitions*.
Frege's distinction between sense and denotation makes the nature of equivalent definitions particularly perspicuous: they are definitions which assign to the *definiendum* the same denotation but not the same sense and their equivalence is just a theorem disguised under the form of a multiple definition.
About conditional definitions, Frege simply says that they are not definitions at all, since they are *incomplete*, i.e., they lack assigning a meaning to the *definiendum* in all possible cases.
Peano does not agree with Frege's banish of conditional definitions from mathematics, even though Frege himself envisages the possibility of amending conditional definitions as partial steps in a *definition by cases*, provided we prove the full definition to be complete, i.e., we prove that the partial definitions are compatible and exhaust all possible cases.
Peano's defence of conditional definitions appeals to the fact that there is no limit to the possibility of extending the applicability of an operation, like addition, by means of new definitions, both in principle and due to the unpredictability of the progress of science.[3]
It becomes evident that, even on the way of defining, the disagreement between Frege and Peano is rooted in the different aims the ideographies of the two authors are intended to achieve. Frege was already aware of these differences in his Leipzig conference and Peano, in his *Reply*, confirms Frege's speculations: "As you so well put it, my principal aim is to publish the *Formulary* and not to deal exclusively with logic or with a particular subject" ([5], p. 122).
In an undated draft of a letter to Peano (likely after Peano's *Reply*) Frege agrees to follow his collegue's line of reasoning by discussing the relevance of conditional definitions for mathematical practice. But even on this common ground the two approaches dramatically diverge.
Peano's approach to mathematical practice is naturalistic. He wants to translate in his symbolic language all mathematical ways of expression and reasoning. On the restricted area of definitions, a clear clue of this intention

[3]On different assessments of conditional definitions facing different purposes in Peano and Frege, see also Quine [12, p. 42].

is given by his project of having a *census* of all modes of definitions, a project he sketches in his *Reply* and pursues in later works.
On the other hand, Frege's approach is prescriptive. Even though in his *Draft* to Peano he recognises the difficulties in doing other way mathematicians actually do, he says that "logical requirements must not be suppressed because of practical difficulties" ([5], p. 125), also distinguishing between the moment of discovery and the moment of systematic presentation. The conclusion Frege draws is sharply critical of mathematicians' attitude towards defining: "It is regrettable that there exists no agreement among mathematicians about the principles to be followed in defining. To produce such an agreement would be a worthwhile task for a mathematical congress. Complete lawlessness now prevails in this area, which is indeed convenient for mathematical writers but damaging to their science. There is not even agreement about what defining really is" ([5], p. 129).
Most of Frege's arguments on defining which he refined in discussing with Peano (as well as others more represented in the discussion, on the same subject, with Hilbert) contribute to the mature exposition of the *Principles of definition* contained in the second volume of *Grundgesetze* (1903). In particular, in a footnote to paragraph 58 ([4], pp. 160-161) Frege entirely quotes the part of Peano's *Risposta* which intend to defend mathematicians' habit of conditional defininig and substantially summarises his critiques we have seen above.

Conclusions

The debate on defininig epitomises both the common ground from which departed Frege and Peano's work in mathematical logic (a symbolic language for the regimentation of mathematical modes of expression and reasoning) and their different aims and approaches.
Frege stresses the fact that his *Begriffsschift* was designed for making inferences ("our vernacular languages are also not made for conducting proofs. And it is precisely the defects that spring from this that have been my main reason for setting up a conceptual notation." [5], p. 115), while Peano's ideography is conceived for expressing all the existing mathematics ("even if we regard this ideography as only a graphic symbolism capable of representing in a brief and precise form all the propositions of mathematics, its importance is evident. Further, this criterion of being able to use a symbolism as a language may be used to recognize whether it is complete or not." [10], p. 192).
Following the development of the debate we can observe a shift from an

initial focus on the technical aspects of defining inside an ideography to more conceptual and informal concerns about this intellectual task. As the focus moves, the different motivations leading the work of the two authors become more evident and, in the background, the different curricula also play a role (as a matter of fact, Peano was a philosopher not more than Frege was a working mathematician).
In the last, we can say that from the debate on defining re-emerges a fundamental distinction between Frege's and Peano's projected ideographies: while the former is intended as a *tool* in analysing inferences made in the mathematical language, the latter aims to be itself a (symbolic) language in which to *translate* the mathematical discourse.
With some hindsight these differences in methodology and aims between Peano and Frege ideographies were already manifest in the very origin of the respective projects, long before the debate on definitions: "I believe that I can best make the relation of my ideography to ordinary language clear if I compare it to that which the microscope has to the eye." (Frege, [6, p. 6]); "On peut réduire toute théorie en symboles, car tout langage parlé, et toute écriture, est un symbolisme [...] Avec un peu d'habitude on transforme tout de suite les symboles en langage et réciproquement." (Peano, [8, pp. 41-42]).
However, the novelty of their proposals initially made the two authors willing to emphasise similarities against other logical or mathematical approaches. The debate on definitions made Frege and Peano more aware of the impact of their different purposes on the respective ideographies. They initially perceived themselves as competitors selling a same product, but soon they realized that their products was designed for different targets within different projects.

Appendix: Documentary evidence of Frege-Peano interaction

1. (1891) Peano (*Principii di logica matematica*, R.d.M vol. 1, 1891, p. 9, n. 5) refers to Frege, *Begriffsschrift*, 1879.

2. (1891 – 1894) Draft of a letter from Frege to Peano (undated).

3. (30.01.1894) Postcard from Peano to Frege.

4. (10.02.1894) Letter from Peano to Frege.

5. (1894?) Handwritten notes by Frege on the last page of Peano's letter (10.02.1894).

6. (1894) Peano (*Notations de Logique Mathématique. Introduction au Formulaire de Mathématiques*, Turin, 1894, p. 3) mentions Frege in a list of Mathematical Logic authors.

7. (14.08.1895) Postcard from Peano to Frege.

8. Peano's review (*Recens.: Dr. Gottlob Frege, Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet. Erster Band, Jena, 1893*, R.d.M vol. 5, 1895) of Frege's *Grundgesetze I.*

9. (17.09.1895) Frege reads *Über di Begriffsschrift des Herrn Peano und meine eigene* at the Mathematics section of the Congress of Natural Sciences held in Lübeck.

10. (24.10.1895) Postcard from Peano to Frege.

11. (5.04.1896) Postcard from Peano to Frege.

12. (6.07.1896) Frege, *Über die Begriffsschrift des Herrn Peano und meine eigene. Vortrag, gehalten in der ausserordenlichen Sitzung vom 6 Juli 1896*, "Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-Physische Klasse" 48, 1896.

13. (29.09.1896) Letter from Frege to Peano. Frege asks Peano to publish his letter in *Rivista di Matematica.*

14. (3.10.1896) Postcard from Peano to Frege.

15. (14.10.1896) Letter from Peano to Frege.

16. (1896/1897) Draft of a letter from Frege to Peano (undated).

17. (4.04.1897) Peano (*Studii di logica matematica*, Atti della Reale Accademia delle Scienze di Torino vol. 32, 1896-97) refers to Frege's talk given 6.07.1896.

18. (11.08.1897) At the First Congress of Mathematicians, held in Zürich, Peano gives a talk and distributes *Formulaire de Mathématiques, t. II §1, "Logique Mathématique"*, where he credits Frege for some propositions.

19. (1898) The letter from Frege to Peano (29.09.1896) is published in *Rivista di Matematica* (*Lettera del sig. G. Frege all'Editore*, R.d.M vol. 6, 1898, pp. 53-59).

20. (1898) Peano, *Corrisp.: Risposta* [*a Lettera del sig. G. Frege all'Editore*], R.d.M vol. 6, 1898, pp. 60-61.

21. (1897/1898?) Frege's unpublished *Begründung meiner strengeren Grundsätze des Definierens* contrasts Peano's way of defining with his own.

22. (1898/1899?) Frege's unpublished *Logische Mängel in der Mathematik* refers to Peano's answer in R.d.M. vol. 6, 1898.

23. (20.07.1900) Peano (*Formules de logique mathèmatiques*, R.d.M vol. 7, 1900) mentions Frege as the author of an ideography.

24. (7.01.1903) Postcard from Peano to Frege.

25. Frege (*Grundgesetze der Arithmetik, begriffsscrhriftlich abgeleitet, Band II*, Jena, 1903, footnote to §58) refers to Peano, *Risposta* (1898).

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