Available online at www.sciencedirect.com

SciVerse ScienceDirect

Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 278 (2011) 173–186

www.elsevier.com/locate/entcs

Paraconsistent Modal Logics

Umberto Rivieccio[1]

Research Center for Integrated Science
Japan Advanced Institute of Science and Technology
Nomi, Ishikawa, Japan

Abstract

We introduce a modal expansion of paraconsistent Nelson logic that is also as a generalization of the Belnapian modal logic recently introduced by Odintsov and Wansing. We prove algebraic completeness theorems for both logics, defining and axiomatizing the corresponding algebraic semantics. We provide a representation for these algebras in terms of twist-structures, generalizing a known result on the representation of the algebraic counterpart of paraconsistent Nelson logic.

Keywords: Paraconsistent modal logic, Nelson logic, twist-structure, Belnap logic.

1 Introduction

One of the latest and most challenging trends of research in non-classical logics is the attempt to combine different non-classical approaches together, for instance many-valued and modal logic [14], [15]. Such interaction offers the advantage of dealing with modal notions like belief, knowledge, obligations, in connection with other aspects of reasoning that can be best handled using many-valued logics, for instance vagueness [18], [6] and inconsistency. If the aim is to model human reasoning, it is obvious that all these aspects have to be dealt with at the same time, therefore such study is especially interesting from the point of view of theoretical computer science and AI.

One of the best-known logical systems proposed for handling inconsistent and also partial information is the Belnap-Dunn logic [13], [3], [4]. This system is based on four truth values, which can be thought of as the two classical ones plus two additional values meant to represent, respectively, lack of information and inconsistency (see the famous interpretation proposed by Belnap [3]). Such simple approach, later on generalized by Ginsberg [17] with the notion of *bilattice*, proved to be very

[1] Email: rUmberto@jaist.ac.jp

doi:10.1016/j.entcs.2011.10.014

flexible and has been widely applied in different areas of computer science. In [25] Odintsov and Wansing proposed a modal expansion of the Belnap-Dunn logic that aims at extending Belnap's approach to partiality and inconsistency to the modal setting (see also [22]). In the present work we continue on this line of research but take a more general approach, introducing a logic that can be regarded as a generalization of Odintsov and Wansing's. Our main aim is to introduce a modal expansion of the Belnap-Dunn logic that is somehow minimal (in a sense that will made precise below) and study this system with algebraic logic tools. In this way we obtain some of Odintsov and Wansing's results as special applications of ours and, more importantly, lay a theoretical framework that can be used for the future study of paraconsistent modal logics.

The paper is organized as follows. In Section 2 we introduce paraconsistent Nelson logic, which is the non-modal system on which we will build our paraconsistent modal logics. We reformulate the completeness theorem for this logic in algebraic terms, which will allow us to obtain similar completeness results for the modal expansions that we are going to introduce. In Section 3 we introduce our modal version of paraconsistent Nelson logic. We see that the Belnapian modal logic of [25] can be obtained as an axiomatic strengthening of ours and prove algebraic completeness theorems for both logics. We show how these results can be applied to any extension (i.e., strengthening) of the above-mentioned logics; we introduce and axiomatize classes of algebras that provide algebraic semantics for them. In Section 4 we prove a representation theorem for these algebras that extends known representation results on the algebraic counterpart of paraconsistent Nelson logic. Finally, in Section 5 we mention some open problems and possible lines for future research.

2 Nelson logics

We start by recalling some known results on *paraconsistent Nelson logic* [2], which is the non-modal system that we are going to take as a basis on which to build our modal logic. Our choice is motivated by the fact that within this logic it is possible to combine Belnap's approach to incomplete/inconsistent data with a reasonably strong implication connective, which has essentially all the properties of intuitionistic implication. Modal expansions of Nelson logics have already been considered in the literature, for instance in [23], [24] and [29] (see Section 3). Other choices are of course possible, for instance one could add classical (rather than intuitionistic) implication to the Belnap-Dunn logic, as [26] does (however, the modal counterpart of this logic can be easily obtained as an axiomatic strengthening of ours).

Definition 2.1 *Paraconsistent Nelson logic* $\mathcal{N}4 = \langle \mathbf{Fm}, \vdash_{\mathcal{N}4} \rangle$ is the sentential logic in the language $\{\wedge, \vee, \supset, \neg\}$ defined by the Hilbert-style calculus with axiom schemata:

$(\supset 1)$	$p \supset (q \supset p)$
$(\supset 2)$	$(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$
$(\wedge \supset)$	$(p \wedge q) \supset p \qquad (p \wedge q) \supset q$
$(\supset \wedge)$	$(p \supset q) \supset ((p \supset r) \supset (p \supset (q \wedge r))$
$(\supset \vee)$	$p \supset (p \vee q) \qquad q \supset (p \vee q)$
$(\vee \supset)$	$(p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))$
$(\neg\wedge)$	$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
$(\neg\vee)$	$\neg(p \vee q) \equiv (\neg p \wedge \neg q)$
$(\neg \supset \wedge)$	$\neg(p \supset q) \equiv (p \wedge \neg q)$
$(\neg\neg)$	$p \equiv \neg\neg p$

where $\varphi \equiv \psi$ abbreviates $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$, and with *modus ponens* (MP) as the only inference rule:

$$\frac{p \quad p \supset q}{q}$$

Nelson logic $\mathcal{N}3 = \langle \mathbf{Fm}, \vdash_{\mathcal{N}3} \rangle$ is obtained by adding the following axiom to $\mathcal{N}4$:

$$(\neg \supset) \qquad \neg p \supset (p \supset q).$$

In $\mathcal{N}3$ another unary connective is usually considered, called *intuitionistic negation* $(-)$, as opposed to *strong negation* $(\neg)$. Intuitionistic negation can be defined using the strong one as follows: $-\varphi := \varphi \supset \neg\varphi$. We are also going to use the following abbreviations: $\varphi \to \psi := (\varphi \supset \psi) \wedge (\neg\psi \supset \neg\varphi)$ and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

It is not difficult to prove that $\mathcal{N}4$ is a conservative expansion of the Belnap-Dunn logic (see for instance the axiomatic presentation given in [16]), i.e., the consequence relations of the two logics agree on $\supset$-free formulas. Another interesting comparison (and more useful for our aims) is the following.

Remark 2.2 As observed in [20, p. 456], the axioms of Definition 2.1 that do not involve strong negation constitute an axiomatization of *positive logic* [28], the $\{\wedge, \vee, \supset\}$-fragment of intuitionistic logic. Therefore, any derivation that is valid in positive logic is also valid in $\mathcal{N}4$. We will use this fact as a lemma to shorten our proofs.

Odintsov [20] proved that paraconsistent Nelson logic $\mathcal{N}4$ is complete with respect to a class of algebras called *N4-lattices*, defined as follows [20, Definition 5.1].

Definition 2.3 An *N4-lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \supset, \neg \rangle$ such that:

(i) the reduct $\langle A, \wedge, \vee, \neg \rangle$ is a De Morgan lattice, i.e., a distributive lattice equipped with a unary operation $\neg\colon A \to A$ (usually called *negation*) such that $\neg\neg a = a$ and $\neg(a \vee b) = \neg a \wedge \neg b$ for all $a, b \in A$,

(ii) the relation $\preceq$ defined, for all $a, b \in A$, as $a \preceq b$ iff $a \supset b = (a \supset b) \supset (a \supset b)$, is a pre-ordering (i.e., reflexive and transitive),

(iii) the relation $\sim$ defined, for all $a, b \in A$, as $a \sim b$ iff $a \preceq b$ and $b \preceq a$, is a congruence relation w.r.t. $\wedge, \vee, \supset$ and the quotient algebra $\langle A, \wedge, \vee, \supset \rangle / \sim$ is a Brouwerian lattice [2],

(iv) for all $a, b \in A$, $\neg(a \supset b) \sim a \wedge \neg b$,

(v) for all $a, b \in A$, $a \leq b$ iff $a \preceq b$ and $\neg b \leq \neg a$, where $\leq$ is the lattice order of $\mathbf{A}$.

$\mathbf{A}$ is said to be *bounded* if its lattice reduct is bounded. $\mathbf{A}$ is called an *N3-lattice* [3] if it is an N4-lattice such that $\neg a \preceq a \supset b$ for all $a, b \in A$.

N4-lattices form a variety [20, Theorem 6.3], which we denote by $\mathsf{N4Lat}$. This class can be also presented as a variety of residuated lattices (see [8]) whose residuated pair is $(*, \rightarrow)$, where the operation $*$ is defined as $x * y := \neg(x \rightarrow \neg y)$.

N4-lattices are an algebraic semantics (in the sense of [5, Definition 2.2]) for paraconsistent Nelson logic. To formally state this result, let us define a translation $\tau \colon Fm \rightarrow Fm \times Fm$ from formulas (in the language of paraconsistent Nelson logic) into pairs of formulas (i.e., equations) in the same language, as follows: for all $\varphi \in Fm$, $\tau(\varphi) := \varphi \approx \varphi \supset \varphi$. This is extended to any $\Gamma \subseteq Fm$ in the obvious way, i.e., $\tau(\Gamma) := \{\tau(\varphi) : \varphi \in \Gamma\}$. Let us denote by $\vDash_{\mathsf{N4Lat}}$ the equational consequence relation (defined as in [5, p. 13]) determined by the class $\mathsf{N4Lat}$. We have the following:

Theorem 2.4 *For any* $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash_{\mathcal{N}4} \varphi \quad \textit{iff} \quad \tau(\Gamma) \vDash_{\mathsf{N4Lat}} \tau(\varphi).$$

In fact, it is easy to show that $\mathsf{N4Lat}$ is *the equivalent* algebraic semantics [5, Definition 2.8] of paraconsistent Nelson logic. To see this, we define an inverse translation $\rho \colon Fm \times Fm \rightarrow Fm$ that to any equation $\varphi \approx \psi$ in the language of N4-lattices assigns the formula $\rho(\varphi \approx \psi) := \varphi \leftrightarrow \psi$. The following result is then obtained as an immediate consequence of [20, Lemma 6.9].

Theorem 2.5 *For any* $\varphi, \psi \in Fm$,

$$\varphi \approx \psi \;\;\exists\vDash_{\mathsf{N4Lat}}\;\; \varphi \leftrightarrow \psi \approx (\varphi \leftrightarrow \psi) \supset (\varphi \leftrightarrow \psi).$$

Rephrasing the statement of the theorem as follows:

$$\varphi \approx \psi \;\;\exists\vDash_{\mathsf{N4Lat}}\;\; \tau \cdot \rho(\varphi \approx \psi)$$

we immediately obtain the following:

---

[2] A *Brouwerian lattice* is a lattice $\langle L, \sqcap, \sqcup \rangle$ equipped with a binary operation $\backslash$ that satisfies the following condition: for all $a, b, c \in L$, $a \sqcap b \leq c$ if and only if $b \leq a \backslash c$. Brouwerian lattices are precisely the 0-free subreducts of Heyting algebras; they are also known in the literature as *generalized Heyting algebras* [10], *Brouwerian algebras* [12], implicative lattices [20] or *relatively pseudo-complemented lattices* [28]. Note also that some authors call "Brouwerian lattices" structures that are (lattice-theoretic) dual to ours.

[3] The *N-lattices* studied, for instance, in [27] and [32] coincide with our N3-lattices, except for the algebraic language in which they are presented. N-lattices are obtained by adding an additional unary operator (corresponding to intuitionistic negation) to the language of N4-lattices (unlike our N3-lattices, they are not, strictly speaking, a sub-variety of N4-lattices).

Theorem 2.6 *Paraconsistent Nelson logic $\mathcal{N}4$ is algebraizable w.r.t. the variety of N4-lattices with equivalence formula $\varphi \leftrightarrow \psi$ and defining equation $\varphi \approx \varphi \supset \varphi$.*

As a corollary, we obtain the following known result (see [30]) on $\mathcal{N}3$.

Corollary 2.7 *Nelson logic $\mathcal{N}3$ is algebraizable w.r.t. the variety of N3-lattices with equivalence formula $\varphi \leftrightarrow \psi$ and defining equation $\varphi \approx \varphi \supset \varphi$.*

3 Modal Nelson logics

We are now going to introduce and study our modal version of (paraconsistent) Nelson logic. We start with a syntactical definition of the logic, then look for an appropriate algebraic semantics. As mentioned above, our criterion in the choice of the logic is to define a system as general as possible (that includes all the Belnapian modal logics considered in [25] as special cases) but still well-behaved from an algebraic point of view. In fact, the logic introduced in the following definition is somehow minimal in the sense that rules like ($\Box$1) and ($\Box$2) are needed (although they could be weakened, as we will see) if one wants to obtain an algebraic completeness result like Theorem 3.6.

Definition 3.1 *Modal* $\mathcal{N}4$ is the sentential logic $\mathcal{MN}4 = \langle \mathbf{Fm}, \vdash_{\mathcal{MN}4} \rangle$ in the language $\{\wedge, \vee, \supset, \neg, \Box\}$ defined by the axioms and rule of the Hilbert-style calculus for $\mathcal{N}4$ of Definition 2.1 plus the following monotonicity rules:

$$(\Box 1) \quad \frac{p \supset q}{\Box p \supset \Box q} \qquad\qquad (\Box 2) \quad \frac{\neg p \supset \neg q}{\neg \Box p \supset \neg \Box q.}$$

Modal $\mathcal{N}3$ is the axiomatic extension of $\mathcal{MN}4$ obtained by adding the axiom $\neg p \supset (p \supset q)$.

Notice that the consequence relation of $\mathcal{MN}4$ is a global one, in the sense that it is obtained by adding rules (rather than only axioms) to its non-modal basis. We will use the following abbreviation: $\Diamond\varphi := \neg\Box\neg\varphi$. Our definition of $\Diamond$ reflects the assumption that strong negation has an almost classical behaviour (as happens in non-modal Nelson logics) also with respect to the modalities. Such assumption is equivalent to the property called *formal duality* in [24] (see also [23] and [29]), where it is shown that there are natural modal expansions of Nelson logics that do not satisfy it. These expansions are neither stronger nor weaker than $\mathcal{MN}4$ because they satisfy additional axioms that fail in our logic. The study of the relation between such logics and ours constitutes an interesting problem for future research.

On the other hand, the Belnapian modal logic $\mathcal{BK}$ introduced in [25] can be seen as a language expansion (and an axiomatic strengthening) of $\mathcal{MN}4$. The language of $\mathcal{BK}$ is obtained by adding a falsum constant $\bot$ to the language of $\mathcal{MN}4$. $\mathcal{BK}$ can be axiomatized [25, Theorem 4] by adding the following axioms to our presentation of $\mathcal{MN}4$:

$(\supset 3)$	$((p \supset q) \supset p) \supset p$
$(\bot)$	$\neg\bot$
$(\bot \supset)$	$\bot \supset p$
(K1)	$(\Box p \wedge \Box q) \supset \Box(p \wedge q)$
(K2)	$\Box(p \supset p)$
$(-\Box)$	$-\Box p \equiv \Diamond - p$
$(-\Diamond)$	$-\Diamond p \equiv \Box - p$

where $-\varphi$ abbreviates $\varphi \supset \bot$. Let us notice that the first three axioms $(\supset 3)$, $(\bot)$ and $(\bot \supset)$ ensure that all the theorems of classical non-modal logic in the language $\{\wedge, \vee, \supset, \bot\}$ are also theorems of $\mathcal{BK}$, while (K1) and (K2) tell us that $\mathcal{BK}$ is a (many-valued) *normal* modal logic. The last two axioms obviously deal with the interaction of the two modalities with the two negations (recall that $\Diamond$ is by definition $\neg\Box\neg$), but their meaning will became clearer when we consider the algebraic counterpart of $\mathcal{BK}$.

In [25] some axiomatic extensions of $\mathcal{BK}$ are introduced as Belnapian counterparts of well-known systems of classical modal logic, for instance:

$$\begin{aligned} \mathcal{B}3\mathcal{K} &:= \mathcal{BK} + \neg p \supset (p \supset q) \\ \mathcal{BS}4 &:= \mathcal{BK} + \{\Box p \supset p,\ \Box p \supset \Box\Box p\} \\ \mathcal{B}3\mathcal{S}4 &:= \mathcal{B}3\mathcal{K} + \{\Box p \supset p,\ \Box p \supset \Box\Box p\}. \end{aligned}$$

Our next aim is to obtain an algebraic completeness theorem for $\mathcal{MN}4$ (and, as a corollary, for all the expansions/extensions of $\mathcal{MN}4$ mentioned above). We will need the following lemma.

Lemma 3.2 *For all $\varphi, \psi \in Fm$, $\varphi \leftrightarrow \psi \vdash_{\mathcal{MN}4} \Box\varphi \leftrightarrow \Box\psi$.*

Proof. Using Remark 2.2 it is easy to prove that the formula $\varphi \leftrightarrow \psi$ is interderivable in $\mathcal{N}4$ (thus, a fortiori, in $\mathcal{MN}4$) with the set of formulas $\Gamma = \{\varphi \supset \psi, \psi \supset \varphi, \neg\varphi \supset \neg\psi, \neg\psi \supset \neg\varphi\}$. Thus, to prove the lemma it is sufficient to show that $\Gamma \vdash_{\mathcal{MN}4} \Box\varphi \supset \Box\psi$, $\Gamma \vdash_{\mathcal{MN}4} \Box\psi \supset \Box\varphi$, $\Gamma \vdash_{\mathcal{MN}4} \neg\Box\varphi \supset \neg\Box\psi$, $\Gamma \vdash_{\mathcal{MN}4} \neg\Box\psi \supset \neg\Box\varphi$. The first two derivations follow easily from rule $(\Box 1)$, while for the latter two we apply $(\Box 2)$. □

An easy consequence of [5, Theorem 4.7] is that any logic (in our case $\mathcal{MN}4$) obtained by adding new connectives to an algebraizable logic (i.e., $\mathcal{N}4$) is also algebraizable (with the same translations), provided the new connectives satisfy item (iv) of [5, Theorem 4.7], which corresponds to our Lemma 3.2. The same reasoning applies to $\mathcal{BK}$ and its extensions, as these logics are obtained from $\mathcal{MN}4$ by adding only one constant (which obviously satisfies [5, Theorem 4.7 (iv)]) and axioms. Thus, we immediately have the following:

Theorem 3.3 *The logics $\mathcal{MN}4$ and $\mathcal{BK}$ (and all the extensions of these two logics) are algebraizable with equivalence formula $\varphi \leftrightarrow \psi$ and defining equation $\varphi \approx \varphi \supset \varphi$.*

We are now going to define algebras that will be proven to be the equivalent algebraic semantics of $\mathcal{MN}4$ and $\mathcal{BK}$.

Definition 3.4 An *MN4-lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \supset, \neg, \Box \rangle$ such that the reduct $\langle A, \wedge, \vee, \supset, \neg \rangle$ is an N4-lattice and, for all $a, b \in A$,

(Q1) if $a \preceq b$, then $\Box a \preceq \Box b$

(Q2) if $\neg a \preceq \neg b$, then $\neg \Box a \preceq \neg \Box b$.

Following the notation adopted for the logic, we write $\Diamond a$ as a shorthand for $\neg \Box \neg a$.

Definition 3.5 A *BK-lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \supset, \neg, \Box, \bot \rangle$ such that the reduct $\langle A, \wedge, \vee, \supset, \neg, \Box \rangle$ is an MN4-lattice with a distinguished element $\bot \in A$ and, for all $a, b \in A$,

(E1) $(a \supset b) \supset a \preceq a$

(E2) $\neg \bot = \neg \bot \supset \neg \bot$

(E3) $\bot \preceq a$

(E4) $\Box a \wedge \Box b \preceq \Box(a \wedge b)$

(E5) $\Box(a \supset a) = \Box(a \supset a) \supset \Box(a \supset a)$

(E6) $-\Box a \sim \Diamond - a$

(E7) $-\Diamond a \sim \Box - a$

where $a \sim b$ abbreviates the two equalities $a \preceq b$ and $b \preceq a$, while $-a$ abbreviates $a \supset \bot$.

It is easy to check that our BK-lattices coincide with those introduced in [22] to provide an algebraic semantics for the logic $\mathcal{BK}$ (this follows from Theorem 3.6 together with the results proved at the end of Section 5 of [22]). Notice that in the previous definition (E2) and (E3) could be equivalently replaced by the assumption that the lattice reduct of $\mathbf{A}$ be bounded (the bottom and top elements are, respectively, $\bot$ and $\neg\bot$).

The above classes of algebras, which we denote by MN4Lat and BKLat, are by definition quasi-varieties (we are going to prove that BKLat is, in fact, a variety). The reader may have noticed that the presentations given in Definitions 3.4 and 3.5 are obtained by simply applying our translation τ to the modal axioms and rules of $\mathcal{MN}4$ and $\mathcal{BK}$. This procedure was introduced by Blok and Pigozzi [5, Theorem 2.17] as an algorithm to axiomatize the equivalent algebraic semantics of any algebraizable logic. Thus, taking into account Theorem 2.6, the following result is immediate.

Theorem 3.6 *$\mathcal{MN}4$ is algebraizable w.r.t. the class* MN4Lat *of MN4-lattices and $\mathcal{BK}$ is algebraizable w.r.t. the class* BKLat *of BK-lattices.*

It follows from [5, Corollary 4.9] that all the extensions of $\mathcal{MN}4$ and $\mathcal{BK}$, in particular the ones considered in [25] (i.e., $\mathcal{B}3\mathcal{K}$, $\mathcal{BS}4$ and $\mathcal{B}3\mathcal{S}4$), are also algebraizable w.r.t. sub-quasi-varieties of BK-lattices that can be axiomatized by adding the τ-translation of the additional axioms and rules. We will provide some information about these classes of algebras in the next section, using the more concrete description of MN4-lattices given by the twist-structure representation.

4 Representation of MN4-lattices

An interesting feature of some algebras related to non-classical logics (not only N4-lattices but also bilattices [7] and some residuated lattices [31]) is that they can be represented using so-called *twist-stuctures*. Such representation is very convenient, as it allows to solve many problems concerning these algebras by working on more traditional and better-known structures (such as Boolean or Heyting algebras). Another advantage is that the twist-structure construction can be used to introduce new algebras (and corresponding logics) that, while sharing some desired features of, e.g., Boolean algebras, are suitable for paraconsistent reasoning (see for instance [19]).

It is well-known that N3-lattices can be represented via twist-strcutures (see for instance [32]), while for N4-lattices such result has been more recently obtained by Odintsov [20]. In this section we are going to see how Odintsov's construction can be extended to obtain a similar representation of MN4-lattices.

Let $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash \rangle$ be a Brouwerian lattice (as defined in Footnote 2). A *full twist-structure over* $\mathbf{L}$ is an algebra $\mathbf{L}^{\bowtie} = \langle L \times L, \wedge, \vee, \supset, \neg \rangle$ with operations defined, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L \times L$, as follows:

$$
\begin{aligned}
\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle &:= \langle a_1 \sqcap b_1,\ a_2 \sqcup b_2 \rangle \\
\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle &:= \langle a_1 \sqcup b_1,\ a_2 \sqcap b_2 \rangle \\
\langle a_1, a_2 \rangle \supset \langle b_1, b_2 \rangle &:= \langle a_1 \backslash b_1,\ a_1 \sqcap b_2 \rangle \\
\neg \langle a_1, a_2 \rangle &:= \langle a_2,\ a_1 \rangle .
\end{aligned}
$$

A *twist-structure over* $\mathbf{L}$ is a subalgebra $\mathbf{A}$ (w.r.t. to the language $\{\wedge, \vee, \supset, \neg\}$) of the full twist-structure $\mathbf{L}^{\bowtie}$ such that $\pi_1(A) = L$, where $\pi_1(A) = \{a_1 \in L : \langle a_1, a_2 \rangle \in A\}$.

We can now notice that the name "twist-structure" refers to the fact that the first component of each binary operation is defined as in a direct product, while the second one is somehow twisted.

It is easy to check that any twist-structure is an N4-lattice. The lattice order is given, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in A$, by $\langle a_1, a_2 \rangle \leq \langle b_1, b_2 \rangle$ iff $a_1 \sqsubseteq b_1$ and $b_2 \sqsubseteq a_2$, where $\sqsubseteq$ is the lattice order of the Brouwerian lattice $\mathbf{L}$. Thus, by the assumption that $\pi_1(A) = L$, we have that if $\mathbf{A}$ is bounded, then its bottom element is $\langle 0, 1 \rangle$ and its top element is $\langle 1, 0 \rangle$, where 0 and 1 are, respectively, the bottom and top element of $\mathbf{L}$.

Recall that, by Definition 2.3 (iii), for any N4-lattice $\mathbf{A} = \langle A, \wedge, \vee, \supset, \neg \rangle$, the quotient algebra $\langle A, \wedge, \vee, \supset \rangle / {\sim}$ is a Brouwerian lattice. Letting $\mathbf{A}^* := \langle A, \wedge, \vee, \supset \rangle / {\sim}$, we can state Odintsov's representation result [20, Proposition 5.3] as follows.

Theorem 4.1 *Any N4-lattice* $\mathbf{A}$ *is isomorphic to a twist-structure over* $\mathbf{A}^*$ *through the map* $\iota\colon \mathbf{A} \to A/{\sim} \times A/{\sim}$ *defined as* $\iota(a) := \langle [a], [\neg a] \rangle$*, where* $[a]$ *denotes the equivalence class of* $a \in A$ *modulo* $\sim$*.*

Our next aim is to obtain a similar result for MN4-lattices. To this end we are going to extend the twist-structure construction as follows.

Given a lattice $\mathbf{L}$ with associated order $\sqsubseteq$, we will say that a function $f\colon L \to L$ is a *modal operator on* $\mathbf{L}$ if it is monotone, i.e., if $a \sqsubseteq b$ implies $f(a) \sqsubseteq f(b)$ for all $a, b \in L$.

Definition 4.2 Let $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, f, g \rangle$ be a Brouwerian lattice with modal operators f, g. The algebra $\mathbf{L}^{\bowtie} = \langle L \times L, \wedge, \vee, \supset, \neg, \Box \rangle$ is defined as follows:

- (i) the reduct $\langle L \times L, \wedge, \vee, \supset, \neg \rangle$ is the full twist-structure over $\langle L, \sqcap, \sqcup, \backslash \rangle$,
- (ii) the operation $\Box\colon L \times L \to L \times L$ is defined, for all $\langle a_1, a_2 \rangle \in L \times L$, as $\Box \langle a_1, a_2 \rangle := \langle f(a_1), g(a_2) \rangle$.

Notice that there is no requirement on the interaction between f and g. For instance, it can happen that $f = g$. Our construction is obviously a generalization of (and was inspired by) the one introduced in [25, Definition 7]. In fact, if we add the additional requirement that $\mathbf{L}$ be a *modal Boolean algebra*[4], then we obtain precisely Odintsov and Wansing's construction.

Proposition 4.3 *For any Brouwerian lattice with modal operators* $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, f, g \rangle$*, the algebra* $\mathbf{L}^{\bowtie}$ *constructed as in Definition 4.2 is an MN4-lattice.*

Proof. We know from [20, Proposition 5.2] that the $\{\wedge, \vee, \supset, \neg\}$-reduct of $\mathbf{L}^{\bowtie}$ is an N4-lattice. It only remains to check that (Q1) and (Q2) of Definition 3.4 are satisfied. By [20, Proposition 5.2 (a)] we have that, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L \times L$, $\langle a_1, a_2 \rangle \preceq \langle b_1, b_2 \rangle$ iff $a_1 \sqsubseteq b_1$, where $\sqsubseteq$ is the lattice order of $\mathbf{L}$. So, if $\langle a_1, a_2 \rangle \preceq \langle b_1, b_2 \rangle$, then $a_1 \sqsubseteq b_1$ and, by the monotonicity of f, we obtain $f(a_1) \sqsubseteq f(b_1)$, which is equivalent to $\Box \langle a_1, a_2 \rangle \preceq \Box \langle b_1, b_2 \rangle$. A similar argument, using the monotonicity of g, shows that (Q2) is also satisfied. $\Box$

Proposition 4.4 *Let* $\mathbf{A} = \langle A, \wedge, \vee, \supset, \neg, \Box \rangle$ *be an MN4-lattice. Then:*

- *(i) the relation* $\sim$*, defined according to Definition 2.3 (iii), is compatible with the operations* $\Box$ *and* $\Diamond$
- *(ii) the quotient algebra* $\mathbf{A}^* = \langle A, \wedge, \vee, \supset, \Box, \Diamond \rangle / {\sim}$ *is a Brouwerian lattice with modal operators* $\Box$ *and* $\Diamond$*.*

Proof. (i). Compatibility with $\Box$ follows immediately from (Q1). The case of $\Diamond$ is also easily proved using (Q2).
(ii). By Definition 2.3 (iii), the quotient algebra $\langle A, \wedge, \vee, \supset \rangle / {\sim}$ is a Brouwerian

[4] A *modal Boolean algebra* or simply *modal algebra* [9] is an algebra $\langle L, \sqcap, \sqcup, \backslash, f, g, 0, 1 \rangle$ such that the reduct $\langle L, \sqcap, \sqcup, \backslash, 0, 1 \rangle$ is a Boolean algebra and the modal operators satisfy, for all $a, b \in L$: $f(a \sqcap b) = f(a) \sqcap f(b)$, $f(1) = 1$ and $f(a) = g(a')'$, where a' denotes the Boolean complement of a (definable as $a \backslash 0$).

lattice. It only remains to check monotonicity of $\Box$ and $\Diamond$. By Theorem 4.1 together with [20, Proposition 5.2 (a)], we know that, for all $a, b \in A$, the condition that $a \preceq b$ is equivalent to $[a] \sqsubseteq [b]$, where $\sqsubseteq$ is the lattice order of the quotient lattice $\langle A, \wedge, \vee, \supset \rangle / \sim$. So, if $[a] \sqsubseteq [b]$, then $a \preceq b$, which by (Q1) implies $\Box a \preceq \Box b$. The latter, using the fact that $\sim$ is compatible with $\Box$, implies $[\Box a] = \Box[a] \sqsubseteq \Box[b] = [\Box b]$. Monotonicity of $\Diamond$ can be proved in the same way. $\Box$

Theorem 4.5 *Any MN4-lattice* $\mathbf{A}$ *is isomorphic to a twist-structure over* $\mathbf{A}^*$*, defined as in Definition 4.2, through the map* $\iota \colon \mathbf{A} \to A/\sim \times A/\sim$ *given by* $\iota(a) := \langle [a], [\neg a] \rangle$ *for all* $a \in A$*.*

Proof. We know by By Theorem 4.1 that the map ι is an isomorphism between the corresponding N4-lattice reducts. It only remains to check that ι is a homomorphism with respect to $\Box$, i.e., that $\iota(\Box a) = \Box' \iota(a)$ for all $a \in A$, where $\Box'$ denotes the operation defined on the twist-structure according to Definition 4.2 (ii). We have:

$$\begin{aligned}
\iota(\Box a) &= \langle [\Box a], [\neg \Box a] \rangle & \\
&= \langle [\Box a], [\neg \Box \neg\neg a] \rangle & \text{by double negation law} \\
&= \langle [\Box a], [\Diamond \neg a] \rangle & \text{by definition} \\
&= \langle \Box [a], \Diamond [\neg a] \rangle & \text{by Proposition 4.4} \\
&= \Box' \langle [a], [\neg a] \rangle & \\
&= \Box' \iota(a). &
\end{aligned}$$

$\Box$

Using the above algebraic results, we can now be a bit more precise on the "minimality" of $\mathcal{MN}4$ mentioned in the Introduction. As we have seen in Section 3, it follows from [5, Theorem 4.7] that the weakest algebraizable expansion of paraconsistent Nelson logic (with just one basic modality $\Box$) is obtained by adding to any axiomatic presentation of paraconsistent Nelson logic the rule of Lemma 3.2:

$$\varphi \leftrightarrow \psi \vdash \Box\varphi \leftrightarrow \Box\psi. \tag{1}$$

However, it is easy to check that any twist-structure over a Brouwerian lattice with operators constructed according to Definition 4.2, even if we drop any requirement on the operators (in our case, monotonicity), will satisfy the following conditions: (i) if $a \sim b$, then $\Box a \sim \Box b$, (ii) if $\neg a \sim \neg b$, then $\neg \Box a \sim \neg \Box b$. On a logical level, these correspond to the rules:

$$\frac{p \equiv q}{\Box p \supset \Box q} \qquad\qquad \frac{\neg p \equiv \neg q}{\neg \Box p \supset \neg \Box q}$$

which are obviously stronger than (1), but still slightly weaker than the ones used in Definition 3.1 to introduce our logic $\mathcal{MN}4$.

We are now going to use Theorem 4.5 to obtain more information on BK-lattices and to compare the abstract presentation of this class of algebras given by Definition 3.5 with the one based on twist-structures introduced in [25, Definition 7].

Let **A** be an MN4-lattice. By Theorem 4.5, we may assume that **A** is a twist-structure over its associated Brouwerian lattice with modal operators $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, f, g \rangle$, i.e., $A \subseteq L \times L$. Suppose there is an element $\langle a_1, a_2 \rangle \in A$ that satisfies (E2) and (E3) of Definition 3.5. Then, by (E2), we have $\langle a_2, a_1 \rangle = \neg \langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle \supset \langle a_2, a_1 \rangle = \langle 1, a_1 \sqcap a_2 \rangle$, where 1 is the top element of **L**. Thus, $a_2 = 1$. (E3) tells us that, for any $\langle b_1, b_2 \rangle \in A$, we have $a_1 \sqsubseteq b_1$, where $\sqsubseteq$ is the lattice order of **L**. The assumption that $\pi_1(A) = L$ allows us to conclude that a_1 is the bottom element of **L** (which we denote by 0), therefore, $\langle a_1, a_2 \rangle = \langle 0, 1 \rangle$. We see then that (E2) and (E3) correspond exactly to the requirement that the lattice reduct of **A** be bounded (in such case the associated Brouwerian lattice is also bounded, i.e., it is in fact a Heyting algebra).

In a similar way, it is easy to check that (E1) holds if and only if the associated Brouwerian lattice **L** satisfies Peirce's law: $(x \backslash y) \backslash x = x$. Such algebras are known as *generalized Boolean algebras* [1] or *classical implicative lattices* [11] and correspond to the 0-free subreducts of Boolean algebras. Thus, (E1), (E2) and (E3) taken together hold if and only if **L** is a Boolean algebra.

(E6) and (E7) are the identities that define the interaction of the two modal operators in $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, f, g \rangle$. For example, one instance of (E6) is $-\Box \langle a_1, a_2 \rangle \preceq \Diamond - \langle a_1, a_2 \rangle$, which means that $f(a_1)' \sqsubseteq g(a_1')$, where $' \colon L \to L$ is the Boolean complement operation. Together with (E7), this means that the two modal operators are interdefinable in the classical way.

So, (E1) to (E7) all together say that **L** is a modal Boolean algebra. From these considerations the following result immediately follows.

Theorem 4.6 *Any BK-lattice (defined as in Definition 3.5) is isomorphic to a twist-structure over a modal Boolean algebra defined as in [25, Definition 7].*

The previous result can easily be extended to obtain an abstract presentation of the class of BK-lattices that correspond to Odintsov and Wansing's logic $\mathcal{BS}4$, which are built as twist-structures over *topological Boolean algebras* (i.e., modal algebras that additionally satisfy $\Box a \sqsubseteq a$ and $\Box\Box a = \Box a$).

We are now going to use Theorem 4.6 to show that the class of BK-lattices, introduced as a quasi-variety (Definitions 3.4 and 3.5), is in fact a variety. We will need the following lemmas.

Lemma 4.7 *Any BK-lattice satisfies the following equation:*

$$\Box(x \supset y) \preceq \Box x \supset \Box y \tag{2}$$

Proof. Using the twist-structure representation of BK-lattices and [20, Proposition 5.2 (a)], one readily sees that proving (2) amounts to checking that, in any modal Boolean algebra **B**, for all $a, b \in B$, it holds that $f(a \backslash b) \sqsubseteq f(a) \backslash f(b)$. By residuation, we have that the former inequality is equivalent to $f(a) \sqcap f(a \backslash b) \sqsubseteq f(b)$. Recall that any Boolean algebra (in fact, even any Brouwerian lattice) satisfies $x \sqcap (x \backslash y) = x \sqcap y$. Now we use the equation $f(x \sqcap y) = f(x) \sqcap f(y)$ to obtain $f(a) \sqcap f(a \backslash b) = f(a \sqcap (a \backslash b)) = f(a \sqcap b) = f(a) \sqcap f(b) \sqsubseteq f(b)$, which concludes our proof. $\Box$

We are now able to prove that BK-lattices are equationally axiomatizable, i.e., they form a variety.

Lemma 4.8 *Let* $\mathbf{A} = \langle A, \wedge, \vee, \supset, \neg, \Box, \bot \rangle$ *be an algebra such that the reduct* $\langle A, \wedge, \vee, \supset, \neg \rangle$ *is an N4-lattice and the following equations are satisfied:*

(i) (E1) to (E7) of Definition 3.5

(ii) (2) of Lemma 4.7.

Then quasi-equations (Q1) and (Q2) of Definition 3.4 are also satisfied, i.e., $\mathbf{A}$ *is a BK-lattice.*

Proof. To prove that $\mathbf{A}$ satisfies (Q1), let $a, b \in A$ be such that $a \preceq b$, i.e., $a \supset b = (a \supset b) \supset (a \supset b)$. By (E5), this means that $\Box(a \supset b) = \Box(a \supset b) \supset \Box(a \supset b)$. Applying (2), we obtain $\Box(a \supset b) \supset \Box(a \supset b) \preceq \Box a \supset \Box b$. Recall that, in any N4-lattice and for all a, b, it holds that $a \supset a \preceq b$ implies $b = b \supset b$ (this can be easily checked using the twist-structure representation of N4-lattices). Thus we obtain $\Box a \supset \Box b = (\Box a \supset \Box b) \supset (\Box a \supset \Box b)$, i.e., $\Box a \preceq \Box b$. To prove (Q2) we are going to use the following property: $a \preceq b$ iff $-b \preceq -a$ (again, using the twist-structure representation, it can be easily checked that this holds in any bounded N4-lattice). Assume $\neg a \preceq \neg b$, i.e., $-\neg b \preceq -\neg a$. Applying (Q1), we obtain $\Box - \neg b \preceq \Box - \neg a$. By (E7), we have $-\Diamond \neg b \preceq \Box - \neg b \preceq \Box - \neg a \preceq -\Diamond \neg a$. Thus, by transitivity of $\preceq$, we obtain $-\Diamond \neg b \preceq -\Diamond \neg a$. As observed above, this implies $\Diamond \neg a \preceq \Diamond \neg b$, i.e., $\neg \Box \neg \neg a \preceq \neg \Box \neg \neg b$. Now, applying the double negation law, we obtain $\neg \Box a \preceq \neg \Box b$. $\Box$

Lemmas 4.7 and 4.8 immediately imply the anticipated result.

Theorem 4.9 *The class of BK-lattices is a variety, axiomatized by the equations that define the variety of N4-lattices plus (E1)–(E7) of Definition 3.5 and (2) of Lemma 4.7.*

Notice that in the proof of Lemma 4.8 only (E5) and (E7) are used, which implies that the class of algebras that have an N4-lattice reduct and satisfy these two equations plus (2) of Lemma 4.7 is also a variety. An interesting question (still unsolved) is whether the class of MN4-lattices is also equationally axiomatizable. In the next section we are going to mention some more open problems and further lines of research.

5 Conclusions and future work

The results presented in the previous sections are obviously just a preliminary study of modal expansions of Nelson (and Belnapian) logics; the most interesting issues are yet to be addressed. For instance, we have only dealt with a global consequence relation associated with these logics. Such choice has allowed us to obtain algebraic completeness results as straightforward applications of the general theory of algebraization of logics; however, the notion of consequence usually considered in modal logic is the local one. The next step is then to investigate the local consequence associated with our logics; this is closely related to the issue of finding an appropriate possible worlds semantics for them (which is likely to be a neighborhood semantics, as our logics are non-normal), as Odintsov and Wansing did for $\mathcal{BK}$ in [25].

As mentioned above, there are modal expansions of paraconsistent Nelson logic that are not comparable to the logic introduced in the present paper. This indicates that the problem of defining a minimal modal expansion of paraconsistent Nelson logic has not yet been solved in full generality. A related issue would then be to determine whether such logic can be endowed with an algebraic semantics that admits some kind of twist-structure representation. These questions will be addressed in a future publication.

In our opinion, another interesting question is whether our approach could be further generalized by considering systems whose non-modal fragment is weaker than paraconsistent Nelson logic, for instance logics with an implication that behaves like that of some substructural (rather than intuitionistic) logic. This problem can perhaps be addressed algebraically, extending the construction introduced in [19] for defining twist-structures over residuated lattices.

Finally, we believe that MN4-lattices also deserve further study from a purely algebraic point of view. An intriguing problem, to mention but one, is whether the representation of MN4-lattices given by Theorem 4.5 can be refined by giving a characterization of which subsets of full-twist structures correspond to universes of MN4-lattices (see [21, Theorem 3.1] and [22, Section 6], where analogous problems are solved for N4-lattices and BK-lattices).

References

[1] Abad, M., J. P. D. Varela and A. Torrens, *Topological representation for implication algebras*, Algebra Universalis **52** (2004), pp. 39–48.

[2] Almukdad, A. and D. Nelson, *Constructible falsity and inexact predicates*, The Journal of Symbolic Logic **49** (1984), pp. 231–233.

[3] Belnap, N. D., *How a computer should think*, in: G. Ryle, editor, *Contemporary Aspects of Philosophy*, Oriel Press, Boston, 1976 pp. 30–56.

[4] Belnap, Jr., N. D., *A useful four-valued logic*, in: J. M. Dunn and G. Epstein, editors, *Modern uses of multiple-valued logic (Fifth Internat. Sympos., Indiana Univ., Bloomington, Ind., 1975)*, Reidel, Dordrecht, 1977, pp. 5–37. Episteme, Vol. 2.

[5] Blok, W. J. and D. Pigozzi, "Algebraizable logics," Mem. Amer. Math. Soc. **396**, A.M.S., Providence, 1989.

[6] Bou, F., F. Esteva, L. Godo and R. Rodríguez, *On the minimum many-valued modal logic over a finite residuated lattice*, Journal of Logic and Computation (2011), to appear.

[7] Bou, F., R. Jansana and U. Rivieccio, *Varieties of interlaced bilattices*, Algebra Universalis, to appear.

[8] Busaniche, M. and R. Cignoli, *Residuated lattices as an algebraic semantics for paraconsistent Nelson's logic*, Journal of Logic and Computation **19** (2009), pp. 1019–1029.

[9] Chagrov, A. and M. Zakharyaschev, "Modal Logic," Oxford Logic Guides **35**, Oxford University Press, 1997.

[10] Cignoli, R. and A. Torrens, *Free Algebras in Varieties of Glivenko MTL-algebras Satisfying the Equation* $2(x^2) = (2x)^2$, Studia Logica **83** (2006), pp. 157–181.

[11] Curry, H. B., "Foundations of mathematical logic," Dover Publications Inc., New York, 1977, viii+408 pp., corrected reprinting.

[12] Davey, B. A., *Dualities for equational classes of Brouwerian algebras and Heyting algebras*, Transactions of the Americal Mathematical Society **221** (1976), pp. 119–146.

[13] Dunn, J. M., "The algebra of intensional logics," Ph. D. Thesis, University of Pittsburgh (1966).

[14] Fitting, M., *Many-valued modal logics*, Fundamenta Informaticae **15** (1992), pp. 235–254.

[15] Fitting, M., *Many-valued modal logics, II*, Fundamenta Informaticae **17** (1992), pp. 55–73.

[16] Font, J. M., *Belnap's four-valued logic and De Morgan lattices*, Logic Journal of the I.G.P.L. **5** (1997), pp. 413–440.

[17] Ginsberg, M. L., *Multivalued logics: A uniform approach to inference in artificial intelligence*, Computational Intelligence **4** (1988), pp. 265–316.

[18] Hájek, P. and D. Harmancová, *A many-valued modal logic*, in: *Proceedings IPMU'96. Information Processing and Management of Uncertainty in Knowledge-Based Systems*, Universidad de Granada, Granada, 1996, pp. 1021–1024.

[19] Jansana, R. and U. Rivieccio, *Residuated bilattices*, Soft Computing, 2011, DOI 10.1007/s00500-011-0752-x.

[20] Odintsov, S. P., *Algebraic semantics for paraconsistent Nelson's logic*, Journal of Logic and Computation **13** (2003), pp. 453–468.

[21] Odintsov, S. P., *On the representation of N4-lattices*, Studia Logica **76** (2004), pp. 385–405.

[22] Odintsov, S. P. and E.I. Latkin, *BK-lattices. Algebraic semantics for Belnapian modal logics*, Manuscript.

[23] Odintsov, S. P. and H. Wansing, *Inconsistency-tolerant description logic: motivation and basic systems*, in: V. Hendricks and J. Malinowski, editors, *50 Years of Studia Logica*, Kluwer, Dordrecht, 2003, pp. 301–335.

[24] Odintsov, S. P. and H. Wansing, *Constructive Predicate Logic and Constructive Modal Logic. Formal Duality versus Semantical Duality*, in: V. Hendricks et al., editors, *First-Order Logic Revisited*, Logos Verlag, Berlin, 2004 pp. 269–286.

[25] Odintsov, S. P. and H. Wansing, *Modal logic with Belnapian truth values*, Journal of Applied Non-Classical Logics **20** (2010), pp. 279–301.

[26] Pynko, A. P., *Functional completeness and axiomatizability within Belnap's four-valued logic and its expansions*, Journal of Applied Non-Classical Logics **9** (1999), pp. 61–105.

[27] Rasiowa, H., *$\mathcal{N}$-lattices and constructive logic with strong negation*, Fundamenta Mathematicae **46** (1958), pp. 61–80.

[28] Rasiowa, H., "An algebraic approach to non-classical logics," Studies in Logic and the Foundations of Mathematics **78**, North-Holland, Amsterdam, 1974.

[29] Sherkhonov, E., *Modal operators over constructive logic*, Journal of Logic and Computation **18** (2008), pp. 815–829.

[30] Spinks, M. and R. Veroff, *Constructive logic with strong negation is a substructural logic*, Studia Logica **88** (2008), pp. 325–348.

[31] Tsinakis, C. and A. M. Wille, *Minimal varieties of involutive residuated lattices*, Studia Logica **83** (2006), pp. 407–423.

[32] Vakarelov, D., *Notes on $\mathcal{N}$-lattices and constructive logic with strong negation*, Studia Logica **36** (1977), pp. 109–125.