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MINIMAL NEGATION IN THE TERNARY RELATIONAL SEMANTICS

A b s t r a c t. Minimal Negation is defined within the basic positive relevance logic in the relational ternary semantics: $B+$. Thus, by defining a number of subminimal negations in the $B+$ context, principles of weak negation are shown to be isolable. Complete ternary semantics are offered for minimal negation in $B+$. Certain forms of reductio are conjectured to be undefinable (in ternary frames) without extending the positive logic. Complete semantics for such kinds of reductio in a properly extended positive logic are offered.

1. Introduction

Captatio benevolentiae

Consider any positive propositional logic L+ with the binary connectives $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$ and the propositional falsity constant F. Define $\neg A =_{def}$

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$A \to F$. Then, negation can in principle be defined in $L+$. For instance, if $L+$ contains

(i) $(A \to (B \to C)) \to ((A \wedge B) \to C))$

as a theorem, then

(ii) $(A \to \neg B) \to \neg(A \wedge B)[(A \to (B \to F)) \to ((A \wedge B) \to F)]$

is a negation theorem of $L+$. Obviously, the more powerful the positive logic is, the stronger the negation defined in it will get. What about the converse? For example, can a positive logic lacking (i) still contain (ii) as a theorem without turning $L+$ into a radical different positive logic?

Introduction

Minimal negation is the "positive" negation corresponding to the positive fragment of intuitionistic propositional logic $I+$. It was defined by Kolmogorov in [7] and Johansson in [6] along the lines commented above. Thus, what is really essential in minimal negation is the positive negation corresponding to $I\to$ (the implicative fragment of $I+$), characterized by the presence of weak double negation $[A \to \neg\neg A]$, weak contraposition $[(A \to B) \to (\neg B \to \neg A)]$ and weak reductio $[(A \to \neg A) \to \neg A]$. Now, in order to introduce minimal negation in the ternary relational semantics, we stay at the basic semantical level. Therefore, we introduce minimal negation within the context of $B+$.

The logic $B+$ deserves to be called a basic positive relevance logic to the effect that the set of its theorems is exactly what is required for the Routley-Meyer type positive relational semantics to work at its fundamental level. So, $B+$ is complete with respect to the basic structures of these semantics. In fact, $B+$ is a basic logic with respect to other semantical perspectives, as shown by Meyer & Routley (in [13]) and Dunn & Meyer (in [4]) and even with respect to other formal calculi, as Lambek Calculus (see, for example, [16]). Now, once we have shown how to introduce minimal negation in $B+$, we have shown how to introduce this type of negation in any logic definable with the ternary relational semantics.

Our purpose in this paper is twofold: (a) we introduce minimal negation in $B+$ following the historical trend commented above. That is, we

define the logic $B_{+,F}$, which is a definitional extension of $B+$, with the falsity constant F, and (b) we answer the question "can minimal negation be defined in weaker implicative (positive) logics than $I\rightarrow$ $(I+)$ in the context of the relational ternary semantics?". And in this question we understand "minimal negation" as that defined by weak double negation, weak contraposition and weak reductio. [9] defines it in the positive fragment of the logic of Relevance R, [10] in contractionless intuitionistic logic, and [11] in Anderson & Belnap's minimal positive logic (see [1]). This paper significantly improves these previous results with the introduction of minimal negation in such a extremely weak logic as the basic positive logic $B+$ (see [2], [14], [16]).

Different negation extensions merge from $B+$ with different modelizations of negation by means of the $*$ operator (e.g.,[2], 4-valued semantics ([14],[15]) or Mares' strategy ([8]) involving the addition of

$$\vdash A \rightarrow B \Rightarrow \vdash \neg B \rightarrow \neg A$$

and eventually,

$$\neg\neg A \rightarrow A$$

But our concern here remains below these extensions, since we shall present, in addition to minimal negation, some varieties of and perspectives on subminimal negation. Obviously, and basically for the same reasons that make De Morgan or Boolean extensions non trivial (see [12] and [16] for general results concerning those negation extensions and their limits), the extension of positive logics (weaker than $I^{\rightarrow}$) with minimal negations is not trivial either. Actually, we show how to introduce minimal negation (in the sense of (b)) in any positive logic between $B+$ and $R+$.

The point of defining negations in weak positive logics also lies in the general strategy beyond any particular result. Consider any logic, no matter how weak it is, if it contains at least $B+$. We will show how to treat F so as to obtain either minimal or indeed other exemplars of the spectrum of negations. Interestingly, this strategy allows for the axiomatical and semantical isolation of different principles of negation. Moreover, fine-grained varieties of subminimal negation arise naturally in this setting, which offers a (fragment of) a kind of microscopical companion to [3] (or [5]). We shall work with ternary relational frames, as they are particularly apt to our "microscopical" approach.

The paper is organized in the following way: §2 presents $B+$, recalls some useful semantical facts in a general way and briefly reviews semantic consistency and completeness. §3 introduces the logic B_{+F}. §§4,5 define two deductively equivalent logics endowing $B+$ with reductio-free minimal negation. §6 extends $B+$ with full minimal negation. §7 conjectures the need of extending the positive logic to introduce stronger reductio axioms. §§8,9 endow with both minimal negation and reductio the properly extended positive logic. §10 briefly considers subminimal extensions and summarizes in a diagram the main deductive relations between logics studied in the paper.

2. The logic $B+$

$B+$ is axiomatized with

A1. $A \to A$

A2. $(A \wedge B) \to A \quad (A \wedge B) \to B$

A3. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

A4. $A \to (A \vee B) \quad B \to (A \vee B)$

A5. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

A6. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

The rules of derivation are

Modus ponens: If $\vdash A$ and $\vdash A \to B$, then $\vdash B$

Adjunction: If $\vdash A$ and $\vdash B$, then $\vdash A \wedge B$

Suffixing: If $\vdash A \to B$, then $\vdash (B \to C) \to (A \to C)$

Prefixing: If $\vdash B \to C$, then $\vdash (A \to B) \to (A \to C)$

The following formulae (useful in the proof of the completeness theorem) are derivable:

T1. $(A \wedge B) \to (B \wedge A)$

T2. $((A \vee B) \wedge (C \wedge D)) \rightarrow ((A \wedge C) \vee (B \wedge D))$

T3. $((A \rightarrow C) \vee (B \rightarrow D)) \rightarrow ((A \wedge B) \rightarrow (C \vee D))$

T4. $((A \rightarrow C) \wedge (B \rightarrow D)) \rightarrow ((A \wedge B) \rightarrow (C \wedge D))$

T5. $((A \rightarrow C) \wedge (B \rightarrow D)) \rightarrow ((A \vee B) \rightarrow (C \vee D))$

A $B+$ model is a quadruple $\langle K, O, R, \models \rangle$ where K is a set, O a subset of K and R a ternary relation on K subject to the following definitions and postulates for all $a, b, c, d \in K$ with quantifiers ranging over K:

d1. $a \leq b =_{def} \exists x[x \in O \text{ and } Rxab]$

d2. $R^2abcd =_{def} \exists x[Rabx \text{ and } Rxcd]$

P1. $a \leq a$

P2. $a \leq b$ and $Rbcd \Rightarrow Racd$

$\models$ is a valuation relation from K to the sentences of $B+$ satisfying the following conditions for all propositional variables p, wffs A, B and $a, b, c \in K$:

(i) $a \models p$ and $a \leq b \Rightarrow b \models p$

(ii) $a \models A \vee B$ iff $a \models A$ or $a \models B$

(iii) $a \models A \wedge B$ iff $a \models A$ and $a \models B$

(iv) $a \models A \rightarrow B$ iff for all $b, c \in K$, $Rabc$ and $b \models A \Rightarrow c \models B$

A formula is valid ($\models_{B+} A$) iff $a \models A$ for all $a \in O$ in all models. P1, d1 and simple induction on (i) prove:

Theorem 2.1. (*Semantic consistency of* $B+$) *If* $\vdash_{B+} A$, $\models_{B+} A$

Let K^T be the set of all theories (sets of formulas of $B+$ closed under adjunction and provable entailment) and R^T be defined on K^T as follows: for all formulas A, B and $a, b, c, d \in K^T$, $R^T abc$ iff if $A \rightarrow B \in a$ and $A \in b$, then $B \in c$. Further, let K^C be the set of prime theories (a theory a is prime if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$), O^C the set of all regular

prime theories ($\underline{a}$ is regular if it contains all theorems of $B+$), and R^C the restriction of R^T to K^C. Finally, let $\models^C$ be defined as follows: for any wff A and $a \in K^C$, $a \models^C A$ iff $A \in a$. Then, the $B+$ canonical model is the quadruple $\langle K^C, O^C, R^C, \models^C \rangle$. We now sketch a proof of the completeness theorem, recording a series of later useful lemmas whose proofs can be found (or easily derived from) in, e.g. [2], [11] or [16]:

Lemma 2.1. *Let A be any wff, $a \in K^T$ and $A \notin a$. Then, $A \notin x$ for some $x \in K^C$ such that $a \subseteq x$.*

Lemma 2.2. *Let $R^T abc$, $a, b \in K^T, c \in K^C$. Then, $R^T xbc$ for some $x \in K^C$ such that $a \subseteq x$.*

Lemma 2.3. *Let $R^T abc$, $a, b \in K^T$, $c \in K^C$. Then, $R^T axc$ for some $x \in K^C$ such that $b \subseteq x$.*

Lemma 2.4. *If $\nvdash_{B+} A$ there is some $x \in O^C$ such that $A \notin x$.*

Lemma 2.5. *Let $a, b \in K^T$. The set $x = \{B : \exists A(A \to B \in a$ and $A \in b)\}$ is a theory and $R^T abx$.*

Lemma 2.6. *$a \leq^C b$ iff $a \subseteq b$*

Lemma 2.7. *The canonical postulates hold in the $B+$ canonical model.*

Lemma 2.8. *$\models^C$ is a valuation relation satisfying conditions (i)-(iv) above.*

Lemma 2.9. *The canonical $B+$ model is in fact a model.*

From Lemmas 2.4 and 2.9 we have,

Theorem 2.2. (*Completeness of $B+$*) *If $\models_{B+} A$, then $\vdash_{B+} A$*

3. The logic $B_{+,F}$

In order to define the logic $B_{+,F}$, we add to the sentential language of $B+$ the propositional falsity constant F together with the definition $\neg A =_{def} A \to F$. For example, we note that the following schemes are provable in $B_{+,F}$

T6. If $\vdash A \to B$, then $\vdash \neg B \to \neg A$

T7. If $\vdash \neg B$, then $\vdash (A \to B) \to \neg A$

T8. $\vdash \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

T9. $\vdash (\neg A \vee \neg B) \to \neg(A \wedge B)$

A B_{+F} model is a quintuple $\langle K, O, S, R, \models\rangle$ where $\langle K, O, R, \models\rangle$ is a $B+$ model and S a subset of K such that $S \cap O \neq \Phi$. The following clauses are also added:

(v) $a \leq b$ and $a \models F \Rightarrow b \models F$

(vi) $a \models F$ iff $a \notin S$

$\models_{B+,F} A$ (A is $B_{+,F}$ valid) iff $a \models A$ for all $a \in O$ in all models.

We note that F is not valid: let $a \in S \cap O$. Then, $a \neq F$. But $a \in O$, so $\not\models B_{+,F} A$.

Theorem 3.1 (*semantic consistency of* $B_{+,F}$).

Proof. Immediate by Theorem 2.1. □

We define the $B_{+,F}$ canonical model as the quintuple

$$\langle K^C, O^C, S^C, R^C, \models^C\rangle$$

where $\langle K^C, O^C, R^C, \models^C\rangle$ is the $B+$ canonical model and S^C is interpreted as as the set of all consistent theories. A theory a is consistent iff $F \notin a$.

Lemma 3.1. *$S^C \cap O^C$ is not empty.*

Proof. As $\not\models_{B+,F} F$, by Theorem 3.1, we have $\nvdash_{B+,F} F$, i.e., $F \notin B_{+,F}$. Since $B_{+,F}$ is a theory, Lemma 2.1 applies and there is some $x \in K^C$ such

that $B_{+F} \subseteq x$ and $F \notin x$. Thus x is consistent and $x \in O^C$. Therefore, $x \in S^C$. □

Lemma 3.2. *Clauses (v) and (vi) hold in the canonical model.*

Proof. Lemmas 2.6 and 3.1 respectively. □

Lemma 3.3. *The $B_{+,F}$ canonical model is indeed a* $\mathrm{B}_{+,F}$ *model.*

Proof. Lemmas 2.9, 3.1 and 3.2. □

Theorem 3.2. *(Completeness of $B_{+,F}$). If $\models_{B+,F} A$, $\vdash_{B+,F} A$.*

Proof. Note that an analogue of Lemma 2.4 is immediate for $B_{+,F.}$ Thus, Theorem 3.2 follows by Lemma 3.3. □

4. $B+$ with minimal negation but without reductio: the logic Bm

We add to B_{+F} the axiom

A7. $(A \to (B \to F)) \to (B \to (A \to F))$

Note that, for example, in addition to T6-T9, the following theorems are provable in Bm:

T10. $(A \to \neg B) \to (B \to \neg A)$

T11. $A \to \neg\neg A$

T12. $(A \to B) \to (\neg B \to \neg A)$

T13. $\neg\neg\neg A \to \neg A$

T14. $(\neg A \wedge \neg B) \to \neg(A \vee B)$

Models for Bm are defined similarly to those for B_{+F} but with the addition of the postulate:

P3. R^2abcd and $d \in S \Rightarrow \exists x \exists y (Racx, Rxby$ and $y \in S)$

$\models_{Bm} A$ (A is Bm valid) iff $a \models A$ for all $a \in O$ in all models.

We prove

Theorem 4.1. (*Semantic consistency of* Bm) *If* $\vdash_{Bm} A$, *then* $\models_{Bm} A$.

Proof. Given Theorem 2.1, we have to prove that A7 is valid. Use P3. □

We define the Bm canonical model as the quintuple

$$\langle K^C, O^C, S^C, R^C, \models^C \rangle,$$

where $\langle K^C, O^C, R^C, \models^C \rangle$ is a $B+$ canonical model and S^C is interpreted as the set of all consistent theories. A theory a is consistent iff the negation of a theorem does not belong to a.

Lemma 4.1. $F \in a$ *iff* $\underline{a}$ *is inconsistent.*

Proof. Suppose $F \in a$. By T7, $(F \to F) \to F \in a$. Thus, a is inconsistent because it contains the negation of a theorem. Suppose now a is inconsistent. Then, $A \to F \in a$ (A is a theorem). By T7, $(A \to F) \to F$ is a theorem. So, $F \in a$. □

Lemma 4.2. Let $R^{T2}abcd$, $a, b, c, d \in K^T$ *and* d *consistent. Then, there is some* x *in* K^C *and some* y *in* S^C *such that* $R^T acx$ *and* $R^T xby$.

Proof. Suppose $a, b, c, d \in K^T$ and d consistent. Suppose further $R^{T2}abcd$, i.e., $R^T abx$ and $R^T xcd$ for some $x \in K^T$, d being consistent. Define [cf. Lemma 2.5.] the theory $u = \{B : \exists A(A \to B \in a \text{ and } A \in c)\}$ such that $R^T acu$. Next, define the theory $w = \{B : \exists A(A \to B \in u \text{ and } A \in b)\}$ such that $R^T ubw$. We first prove that w is consistent. Suppose it is not. Then, $F \in w$ [Lemma 4.1.]. By definitions of u and w, $A \to (B \to F) \in a$, $A \in c$ and $B \in b$. By A7, $B \to (A \to F) \in a$. Given $R^T abx$, $A \to F \in x$. Given $R^T xcd$, $F \in d$, contradicting the hypothesis.

Summing up, we have $u, w \in K^T$ with w consistent, $R^T acu$ and $R^T ubw$. As $F \notin w$, Lemma 2.1 applies and there is some $y \in K^C$ such that $w \subseteq y$ and $F \notin y$ (hence y is consistent). By definitions, $R^T uby$. By Lemma 2.2, there is some x in K^C satisfying $R^T xby$ and $u \subseteq x$. As $R^T acu$, $R^T acx$

follows from definitions. Therefore, we have $x, y \in K^C$ ($y \in S^C$) such that $R^T acx$ and $R^T xby$, which was to be proved. □

Lemma 4.3. *The canonical version of P3* [*that is,* $R^{C2}abcd$ *and* $d \in S^C \Rightarrow \exists x \exists y (R^C acx$ *and* $R^C xby$ *and* $y \in S^C)$] *holds in the Bm canonical model.*

Proof. Lemma 4.2. □

Lemma 4.4. *The Bm canonical model is indeed a Bm model.*

Proof. Lemmas 2.9, 3.1, 4.3 and 3.2. □

Now we can prove

Theorem 4.2. (*Completeness of Bm*)*If* $\models_{Bm} A$, $\vdash_{Bm} A$.

Proof. Note that an analogue of Lemma 2.4 is immediate for Bm. Thus, Theorem 4.2 follows by Lemma 4.4. □

5. A semantical alternative

The logic Bm' is $B+$ plus

A8. $A \to ((A \to F) \to F)$

and

A9. $(A \to B) \to ((B \to F) \to (A \to F))$

A Bm' model is just a Bm model but with these two differences: P3 is deleted and the following postulates are added:

P4. $Rabc$ and $c \in S \Rightarrow \exists x (x \in S$ and $Rbax)$

P5. $R^2abcd \Rightarrow \exists x \exists y (Racx$ and $Rbcy$ and $y \in S)$

As for the semantic consistency of Bm', we leave to the reader the proof that A8 (use P4) and A9 (use P5) are valid. Define the Bm' canonical model similarly to the Bm canonical model and note that an analogue of Lemma 4.1. is immediate. The reader can verify:

Lemma 5.1. *P4 and P5 hold in the canonical model.*

Next, we have

Lemma 5.2. *The Bm' canonical model is a Bm' model.*

Proof. Lemmas 2.9, 3.1, 3.2 and 5.1. □

Finally, we prove

Theorem 5.1. (*Completeness of Bm'*) *If $\models Bm'A$, then $\vdash Bm'A$.*

Proof. As an analogue of Lemma 2.4 is immediate, Theorem 5.1 follows by Lemma 5.2. □

Bm and Bm' are syntactically equivalent, as stated by the proposition below:

Lemma 5.3. *Given $B+$, A7 is derivable from A8 and A9. Conversely, A8 and A9 are, given $B+$, derivable from A7.*

The proof is left to the reader.

6. Bm with the reductio axiom: the logic Bmr

We add to Bm the axiom

A10. $(A \to (A \to F)) \to (A \to F)$

and note that, in addition to T6-T12, the following are exemplar theorems and rules of Bmr:

T15. $(A \to \neg A) \to \neg A$

T16. If $\vdash A \to B$, then $\vdash (A \to \neg B) \to \neg A$

T17. If $\vdash A \to \neg B$, then $\vdash (A \to B) \to \neg A$

T18. $(A \to \neg B) \to \neg(A \wedge B)$

T19. $(A \to B) \to \neg(A \wedge \neg B)$

T20. $\neg(A \wedge \neg A)$

T21. $\neg\neg(A \vee \neg A)$

Bmr can alternatively be axiomatized with T16, T17, T18 or T19 instead of A10.

Models for *Bmr* are defined similarly as those for *Bm* but with the addition of the postulate:

P6. $Rabc$ and $c \in S \Rightarrow \exists x \exists y(Rabx$ and $Rxby$ and $y \in S)$

To prove the semantic consistency of *Bmr* with respect to these models, it is enough to verify the validity of A10 by means of P6. Therefore, we have

Theorem 6.1. (*Semantic consistency of Bmr*) *If* $\vdash_{Bmr} A$, *then* $\models_{Bmr} A$.

The *Bmr* canonical model is defined similarly to the corresponding one for *Bm*. An analogue for *Bmr* of Lemma 2.4. is immediate. Then, we prove

Lemma 6.1. *Given* $a, b, c \in K^T$, c *consistent and* $R^T abc$, *then there are* $x \in K^C$, $y \in S^C$ *and* $R^T abx$, $R^T xby$.

Proof. Assume hypothesis and define the theory [cf. Lemma 2.5] $u = \{B : \exists A(A \to B) \in a$ and $A \in b)\}$ such that $R^T abu$, and the theory $w = \{B : \exists A(A \to B) \in u$ and $A \in b)\}$ satisfying $R^T ubw$. Suppose for reductio w is inconsistent. Then, $F \in w$ [Lemma 4.1]. By definition of w, $B \to F \in u$, $B \in b$. By definition of u, $A \to (B \to F) \in a$, $A \in b$. Then, by T17, $(A \wedge B) \to F \in a$. But since $R^T abc$ and $A \wedge B \in b$ $[A, B \in b]$, $F \in c$, contradicting the consistency of c. Therefore, w is consistent. Now, we use

Lemmas 2.1 and 2.2 to extend u and w to some $x \in K^C$ ($u \subseteq x$) and some $y \in S^C$ ($w \subseteq y$) such that $R^T abx$ and $R^T xby$, as required. □

Lemma 6.2. *Canonical P6 holds in the Bmr canonical model.*

Proof: Lemma 6.1. □

Lemma 6.3. *The Bmr canonical model is a Bmr model.*

Proof. Lemmas 4.4 and 6.2. □

Finally, we have

Theorem 6.2. *(Completeness of Bmr) If* $\models_{Bmr} A$*, then* $\vdash_{Bmr} A$*.*

Proof. By an analogue of Lemma 2.4 and 6.3. □

7. Note on the reductio axiom

As we have seen in §6, the reductio axiom, i.e.,

T15. $(A \to \neg A) \to \neg A$

or the reductio rules

T16. If $\vdash A \to B$, then $\vdash (A \to \neg B) \to \neg A$

T17. If $\vdash A \to \neg B$, then $\vdash (A \to B) \to \neg A$

are provable in Bmr. But we remark that the reductio theorems corresponding to T16 and T17, that is,

ρ. $(A \to B) \to ((A \to \neg B) \to \neg A)$

and

ρ'. $(A \to \neg B) \to ((A \to B) \to \neg A)$

are not. A simple proof of this fact is the following. Consider the set of matrices below, where designated values are starred and F is assigned the value 1.

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	1	2	2	3
2^*	0	1	2	3
3^*	0	0	1	3

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2^*	0	1	2	2
3^*	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2^*	2	2	2	3
3^*	3	3	3	3

This set verifies Bmr but falsifies $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow F)) \rightarrow (A \rightarrow F))$ (ρ) only when $A = 2$, $B = 1$, and $(A \rightarrow (B \rightarrow F)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow F))$ (ρ') only when $A = B = 2$.

Now, our question is: could ρ and/or ρ' be introduced in Bmr as, e.g., A10 has been introduced or, for example, T16 can be? Our conjecture is that they can't: A11 below (or some instance of it – cf. some lines below-) seems necessary in the proof of the canonical adequacy of the semantical postulates for ρ and ρ'. We establish in what follows a setting for discussing the point.

8. The positive logic $Bp+$ and its minimal negation.

In order to define the logic $Bp+$ [$B+$ with prefixing as a theorem] we add to $B+$ the axiom

A11. $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

Models are defined similarly to $B+$ models but with the addition of the postulate

P7. $R^2abcd \Rightarrow \exists x(Rbcx$ and $Raxd)$

Theorem 8.1. $\vdash_{Bp+} A$ *iff* $\models_{Bp+} A$

Proof. For the semantic consistency of $Bp+$, we have to prove that A11 is valid. Use P7. For its completeness, given Theorem 2.2, clearly we just need to verify that P7 holds in the canonical model. $\square$

Bpm is defined from $Bp+$ as Bm was defined from $B+$. Models for Bpm are exactly as those for Bm, but with the addition of P7.

Theorem 8.2. $\vdash_{Bpm} A$ *iff* $\models_{Bpm} A$

Proof. (a) Semantic consistency. As the proof of Bm: A11 is valid (cf. Theorem 4.1). (b) Completeness. Given Theorem 4.2, we just have to prove the fact that P6 holds in the canonical model. □

9. Bpm with ρ and ρ': the logic $Bpmr$.

We add to Bpm the axiom

A12. $(A \to B) \to ((A \to (B \to F)) \to (A \to F))$

noting that

T22. $(A \to (B \to F)) \to ((A \to B) \to (A \to F))$

becomes a theorem. Moreover, we note that (in addition to T22) T15, T16, T17, T18 or T19 can now be used , among other possibilities, to axiomatize $Bpmr$ instead of A12.

Models for $Bpmr$ are defined as those for Bmr but with the addition of the postulate

P8. $R^2abcd \Rightarrow \exists x \exists y \exists z (Racy$ and $Rbcx$ and $Rxyz$ and $z \in S)$

Theorem 9.1. (*Semantic consistency of* $Bpmr$) *If* $\vdash_{Bpmr} A$, *then* $\models_{Bpmr} A$

Proof. Use Theorem 8.2 and P8 to prove the validity of A12. □

Note. For T22 we use the postulate

P8'. $R^2abcd \Rightarrow \exists x \exists y \exists z (Racy$ and $Rbcx$ and $Ryxz$ and $z \in S)$

P8 and P8' are provably equivalent with P4 [see §5]. For T15, T16, T17, T18 and T19 use P6.

Canonical models for $Bpmr$ are defined as the corresponding ones for Bmr. Again, an analogue of Lemma 2.4 is immediate for $Bpmr$.

Lemma 9.1. *Let $a, b, c, d \in K^T$ and let d be a consistent theory. If $R^{T2}abcd$, then there are $x, y, z \in K^C$ such that $R^T acy$, $R^T bcx$, $R^T xyz$ and $z \in S^C$.*

Proof. Suppose $R^{T2}abcd$, i.e., $R^T abx$ and $R^T xcd$ with $a, b, c, d \in K^T$ and d consistent.

Define as in Lemma 2.5 the theories $u = \{B : \exists A(A \to B \in a$ and $A \in c)\}$, $w = \{B : \exists A(A \to B) \in b$ and $A \in c)\}$, $v = \{B : \exists A(A \to B \in w$ and $A \in u)\}$ satisfying $R^T acu$, $R^T bcw$ and $R^T wuv$. We prove that v is consistent. Otherwise, $F \in v$ [Lemma 4.1.]. Definitions grant $A \to (B \to F) \in b$, $C \to B \in a$, $A, C \in c$. By A9, $(B \to F) \to (C \to F) \in a$. Since $((B \to F) \to (C \to F)) \to ((A \to (B \to F)) \to (A \to (C \to F)))$ is a theorem [A11], $(A \to (B \to F)) \to (A \to (C \to F)) \in a$. Given $R^T abx$ and $A \to (B \to F) \in b$, necessarily $A \to (C \to F) \in x$. But $(A \to (C \to F)) \to ((A \wedge C) \to F)$ is a theorem [T18]. So, $(A \wedge C) \to F \in x$. As $R^T xcd$ and $A \wedge C \in c$, a fortiori $F \in d$, which contradicts the consistency of d. We conclude that w is consistent. Now, Lemmas 2.1, 2.2 and 2.3 apply and we can define $x, y, z \in K^C$ such that $u \subseteq y$, $w \subseteq x$, $v \subseteq z$ and $R^T acy$, $R^T bcx$, $R^T xyz$ and $z \in S^C$, as required. □

From Lemma 9.1 we deduce:

Lemma 9.2. *Canonical P8 holds in the Bpmr canonical model.*

And from both Lemmas 9.1 and 9.2,

Lemma 9.3. *Any Bpmr canonical model is a Bpmr model.*

Theorem 9.2. (*Completeness of Bpmr*) *If $\models_{Bpmr} A$ then $\vdash_{Bpmr} A$.*

Proof. Analogue of Lemma 4.3 for *Bpmr* and Lemma 9.3. □

Note. The proof that the canonical P8' holds in the *Bpmr* canonical model is similar to that for P8.

10. Four final remarks

A. Bm, Bmr, Bpm and $Bpmr$ can in principle be defined with a negation connective instead of the falsity constant F. See [10] for a general strategy.

B. Given $B+$, weak double negation and contraposition are isolable.

Let Bc [$B+$ with weak contraposition] and Bdn [$B+$ with weak double negation] be the result of adding

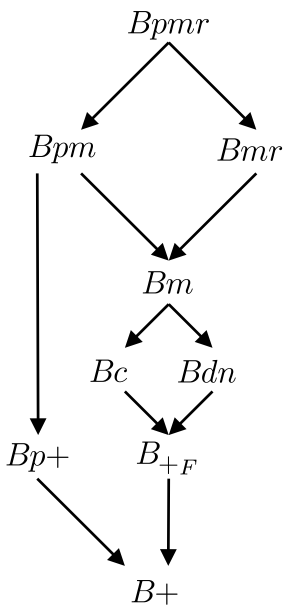
A9. $(A \to B) \to ((B \to F) \to (A \to F))$

and

A8. $A \to ((A \to F) \to F)$

to $B+$, respectively.

C. The relations the logics treated in this paper maintain to each other can be summarized in the following diagram:



D. We have shown how to introduce minimal negation (in the sense of (b) in the introduction) in any logic containing the logic $B+$ (reductio as a rule) and $Bp+$ (reductio as a theorem).

Notes

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