Possibilities for sets

An exposition of forcing for philosophers
Draft

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As central as the method of forcing is within set theory, it has yet to be incorporated into the philosopher's toolbox. That strikes me as a shame, since it may well have important applications within philosophy. One barrier is that typical presentations of forcing are overly dry and technical and make it seem inherently bound up with its applications within set theory. The purpose of this note is to try to rectify this. I will show how forcing models can be seen as a special case of a more general class of intentional models that philosophers are already interested in: namely, possibility models.

1 The basic idea

In set theory we use the axioms of ZFC to construct models of those very same axioms. The most well-known example is Gödel's model, L, comprising the so-called constructible sets. In ZFC, we can prove both of each axiom of ZFC that it is is true in L and that the continuum hypothesis (CH) is true in L. We conclude that ZFC does not prove the negation of CH, assuming it's consistent: if it did, it would prove that both CH and its negation hold in L, which is impossible. Think of L as specific way the universe of sets could be: the way it would be if every set were constructible. So the strategy is to find specific ways the universe of sets could be in which the axioms of ZFC are all true but in which some target claim is false, thereby showing that the target claim doesn't follow from those axioms.

As I'm going to present it here, the method of forcing generalises this strategy. Instead of helping us to find specific ways the universe of sets could be—e.g. as it would be if every set were constructible—it helps us to find a range of *possibilities* for the universe of sets, which is to say: a range of non-specific ways the universe of sets could be. We use the axioms of ZFC to construct *possibility* models in which those very same axioms are necessarily true, rather than true simpliciter, but in which some target claim—like CH—is possibly false. As above, we conclude that ZFC does not prove the target claim, assuming it's consistent: if it did, it would prove both that the target claim is necessarily true in the possibility model—because that claim follows from the axioms of ZFC and each of those axioms is necessarily true—and that its negation is possibly false, which is impossible.[1]

[1]Unless otherwise stated, in what follows I will implicitly work in ZFC. Nevertheless, a lot of what I say isn't intrinsically set theoretic. For example, the notion of a possibility model can be naturally formulated in a higher-order logic and its basic properties proved. When we come to forcing possibility models, I'll rely more heavily—though still mostly implicitly—on the background set theory.

2 Possibilities and possibility models

In specifying a range of ways the universe could be, possibility models are like Kripke models. But whereas Kripke models deal with possible worlds, possibility models deal with *possibilities*.[2] Possibilities are like *parts* of possible worlds. Whereas possible worlds are complete specifications of ways the world could be, possibilities are partial specifications. For example, any possible world in which Mae loves George will also either be a possible world in which Mae is a comedian or a possible world in which they aren't. A possibility in which Mae loves George, on the other hand, may make neither claim true. Nevertheless, possibilities can be *extended* to more inclusive possibilities that settle such questions: possibilities in which Mae loves George and works as a comedian, and possibilities in which Mae loves George and doesn't work as a comedian.[3]

A space of possibilities can be modelled by a *partial order*: which is to say a non-empty set together with a reflexive, transitive, and anti-symmetric relation $\leq$ on its members.[4] In particular, we can think of the elements of the partial order as representing the possibilities and its relation $\leq$ as representing extension between them. So, for possibilities p and q, the idea is that $p \leq q$ when p has q as a part; when p includes q; when p extends q. For example, q might be a possibility in which Mae loves George and p a possibility in which Mae loves George and works as a comedian.

Let $\mathbb{P}$ be a partial order. We can obtain a possibility model from $\mathbb{P}$ in two simple steps. First, we specify a domain for the model. For our purposes, the domain can be a proper class.[5] Since our focus here is on set theory, we can think of its elements as *possible sets*, where a possible set is a kind of mathematical object whose identity is determined by what members is has at what possibilities. Possible sets are thus more akin to properties than traditional sets. Second, for our atomic relations, we specify which objects in the domain relate to which relative to what possibilities. Since our focus is on set theory, that means the relations of membership and identity. Formally, then, a possibility model (for the language of first-order set theory, $\mathcal{L}_\in$) is obtained by supplying a domain of objects D and two interpretation functions $I_\in$ and $I_=$ that map ordered pairs of elements of D to subsets of $\mathbb{P}$. For example, suppose that our interpretation function $I_\in$ assigns the set of all possibilities to a pair $\langle x, y\rangle$:

[2] Possibility models were first introduced by Humberstone [1981]. See Holliday [2021] for an up-to-date treatment.

[3] We might think of possible worlds as maximally inclusive possibilities: possibilities that cannot be extended to more inclusive possibilities. Whether there are such possibilities will depend on how rich our space of possibilities is. For some spaces, it may be that all possibilities are extendable and thus that there are no possible worlds in this sense. When every possibility can be extended to a maximally inclusive possibility, the resulting possibility models effectively collapse to Kripke models. The interesting cases are thus those where this fails (see footnote 20 for further discussion). Of course, since classical models are a special case of Kripke models, when there is a single possibility that extends every possibility, the resulting possibility models will effectively collapse to classical models. In section 5 I will offer a more apt notion of possible world for the possibility model approach to forcing.

[4] It is an interesting and important question what happens when the partial order is not a set, but a proper class. As you'll see, there is nothing in the set up that requires us to assume that the partial order is a set, and many of the results below will have analogues when it isn't. If we like, we could even use plural, higher-order, or class quantification, to quantify over propositions. And indeed, there are interesting forcing models, as I'm conceiving them, that are based on proper class-sized possibility spaces. Nevertheless, in the interests of simplicity, I am going to ignore such spaces in this note.

[5] Indeed, when we get to our forcing models, it will be. Since we're implicitly working in ZFC, that means the domain will be definable by some formula in the language of first-order set theory, $\mathcal{L}_\in$. Similarly, for the assignment functions below.

that is, $I_\in(\langle x,y\rangle) = \mathbb{P}$. Then according to that assignment, the claim that x is a member of y is *necessary*. No matter how things had been, x would have been an element of y; x is an element of y according to all possibilities. Conversely, suppose $I_\in$ assigns the empty set of possibilities to $\langle x,y\rangle$: that is, $I_\in(\langle x,y\rangle) = \emptyset$. Then according to that assignment, the claim that x is a member of y is *impossible*. No matter how things had been, x would not have been an element of y; x is an element of y according to no possibilities.

It is natural to think of a set of possibilities as a *proposition*: namely, the proposition that one of those possibilities obtains. Our interpretation functions $I_\in$ and $I_=$ can therefore be seen as assignments of propositions to ordered pairs over the domain. And, it turns out, there's a very natural way to extend $I_\in$ and $I_=$ compositionally to assignments of propositions to all claims in $\mathcal{L}_\in$. But first, we need to define a central notion in possibility semantics: namely, that of compossibility. Possibilities are said to be *compossible* if there is a possibility extending each of them: if they can all obtain in a single possibility. So, for example, the possibility that Mae loves George and the possibility that Mae loves Susan are compossible. Possibilities that are not compossible are said to be *incompossible*. So, for example, any possibility in which Mae's hat is red and any possibility in which Mae's hat is green are incompossible. We can think of incompossibility as a kind of *ruling out*. If a possibility is incompossible with some other possibilities, then it rules those possibilities out: it cannot obtain together with any one of them. Any possibility in which Mae's hat is red, for example, rules out all possibilities in which their hat is green. This notion of ruling out is naturally seen as a notion of making *false* (though, of course, it is not the only such notion). So, for example, we can say that a possibility in which Mae's hat is red makes it false that their hat is green. In general, we can say that a possibility p makes a proposition X false when p rules out all the possibilities in X, which is to say: when p is incompossible with all the possibilities in X. I will abbreviate the claim that p is incompossible q as $p \perp q$ and the claim that p is incompossible with all the possibilities in X—equivalently: the claim that p makes X false—as $p \perp X$.

With these notions in place, we can extend the assignment functions $I_\in$ and $I_=$ to the whole language as follows. We let the possibilities for a disjunction be the possibilities for its disjuncts; the possibilities for an existentially quantified claim, the possibilities for its instances; and the possibilities for the negation of ϕ, the possibilities that make it false: which is to say, the possibilities that are incompossible with the possibilities for ϕ. Formally, let $[\![\phi]\!]_\mathcal{M}$ denote the proposition assigned to ϕ by $\mathcal{M}$—the proposition that ϕ—and let $\overline{X} = \{p \in \mathbb{P} : p \perp X\}$. When $\mathcal{M}$ is clear from context, I'll simply write $[\![\phi]\!]$ for $[\![\phi]\!]_M$. Then we have:

- $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$
- $[\![\exists x \phi]\!] = \bigcup_{x \in D} [\![\phi]\!]$
- $[\![\neg\phi]\!] = \overline{[\![\phi]\!]}$

The notions of compossibility, incompossibility, and making false also allow us to define a natural notion of implication between propositions. For propositions $X, Y \subseteq \mathbb{P}$, we say that X implies Y—in symbols, $X \vDash Y$—when it is impossible for an X possibility to obtain whilst Y is false. That is, when there is no possibility that includes an X possibility but is incompossible with all the possibilities in Y.[6] Equivalently: X implies Y when any possibility

[6]Formally: $\neg\exists p \exists q, r(p < q, r \wedge q \in X \wedge r \in \overline{Y})$. In other words, if any possibility containing an X possibility can be extended to contain a Y possibility. Formally: $\forall p \in X \forall q(q < p \rightarrow \exists r < q \exists s \in Y(r < s))$. In other words, when the possibilities making Y false are also possibilities making X false: that is, when $\overline{Y} \subseteq \overline{X}$.

that includes an X possibility does not make Y false.

It's a good exercise to show that the implication relation and the assigned propositions work together in a classical way.[7] In particular, to show that the union $\bigcup \mathcal{X}$ of any set of propositions $\mathcal{X}$ behaves implicationally like their disjunction,[8] and that the set of possibilities incompossible with a proposition X—that is, the possibilities making X false—behaves implicationally and classically like its negation.[9] So, when you're reasoning with propositions in this setting, you can just reason classically in the usual way. One upshot is therefore that the laws of classical first-order logic without identity are preserved under implication.[10]

Theorem 2 Suppose ψ is provable from $\phi_0, ..., \phi_n$ in first-order logic without identity. Then ZFC proves $[\![\phi_0 \wedge ... \wedge \phi_n]\!] \vDash [\![\psi]\!]$ for any possibility model.

By picking a suitable interpretation function $I_=$, we can also ensure that the identity axioms are necessary so that all arguments of classical first-order logic are preserved under implication.[11] In what follows, I will assume that we have done this. With such an assignment in place, we can also simplify our possibility models a little. When the identity axioms are necessary in a possibility model, it is not hard to see that we can, without changing the propositions assigned to claims in the language, identify objects in the domain that are necessarily identical.[12] In other words, we can equally well work with a model in which if $[\![x = y]\!]$ is

[7] As a first step, it is easy to see that implication is reflexive and transitive. Every proposition trivially implies itself and any proposition implying a proposition that implies X already implies X. Formally, $X \vDash X$ and $X \vDash Z$ whenever $X \vDash Y \wedge Y \vDash Z$.

[8] More precisely, each proposition in $\mathcal{X}$ implies $\bigcup \mathcal{X}$ and if each proposition in $\mathcal{X}$ implies some proposition Y, then $\bigcup \mathcal{X}$ already implies Y. Formally, $X \vDash \bigcup \mathcal{X}$ for each $X \in \mathcal{X}$, and $\bigcup \mathcal{X} \vDash Y$ whenever $X \vDash Y$ for each $X \in \mathcal{X}$.

[9] More precisely, the only proposition that implies both X and $\overline{X}$ is the empty set—the absurd proposition—and any proposition that is incompatible in this way with X implies $\overline{X}$. Formally, say that X and Y are *incompatible* if $\forall Z(Z \vDash X \wedge Z \vDash Y \rightarrow Z = \emptyset)$. Then, X and $\overline{X}$ are incompatible and for any proposition Y incompatible with X, $Y \vDash \overline{X}$. Moreover, the implication relation has the classical property that negations of negations of propositions imply those very propositions. Formally: $\overline{\overline{X}} \vDash X$.

[10] More generally, it is not hard to show that, modulo implicational equivalence, the propositions form a complete Boolean algebra.

Theorem 1 (ZFC) Modulo implicational equivalence, the empty proposition as the bottom element, union as join, and $\overline{X}$ as complement, are a complete Boolean algebra on the propositions.

There are a number of natural ways to mod out by simply picking particular propositions to play the role of the equivalence class. For example, it is easy to see that the union of all propositions implicationally equivalent to X—$\bigcup\{Y : Y \text{ is implicationally equivalent to} X\}$—is itself implicationally equivalent to X (and thus a member of its equivalence class). It is easy to see that this proposition is simply the proposition containing all possibilities that imply X—$\bigcup\{Y : Y \text{ is implicationally equivalent to} X\} = \{p \in \mathbb{P} : p \vDash X\}$.

[11] The simplest such function takes identity to be absolute, so that when $x = y$, $I_=(\langle x, y\rangle) = \mathbb{P}$ and when $x \neq y$, $I_=(\langle x, y\rangle) = \emptyset$. Unfortunately, this function will not always work for possibility models of set theory. In such models, we want the axiom of extensionality to be necessary. Given the necessity of the identity axioms, this means that $[\![\forall z(z \in x \leftrightarrow z \in y)]\!]$ will have to be implicationally equivalent to $[\![x = y]\!]$. Variation in membership across possibilities will in general therefore require variation in identities. Since the most interesting possibility models will have lots of variation in membership across possibilities, they will typically also have lots of variation in identities. The hardest task in constructing the forcing model described below is precisely in obtaining this link between membership and identity in a way that makes the identity axioms necessary.

[12] More precisely, given a possibility model $\mathcal{M} = \langle \mathbb{P}, D, I_\in, I_= \rangle$ making the identity axioms necessarily true, we can let $\mathcal{M}^=$ be the model whose domain comprises equivalence classes of elements of D under the relation $[\![x = y]\!] \equiv \mathbb{P}$ and whose interpretation functions are defined accordingly. Since these equivalence

necessary, then $x = y$. So, in what follows then, I will assume that a possibility model is as before, but now such that it (1) makes the identity axioms necessary and (2) is such that necessarily identical elements of the domain are really identical.

A natural notion of truth at a possibility is immediate, given our notion of implication. In particular, we can say that a possibility makes a proposition true—or that the proposition is true at the possibility—when the singleton of the possibility implies the proposition. Formally, we can say that p makes X true—or that X is true at p—when $\{p\} \vDash X$, which I will write simply as $p \vDash X$. We can then say that a possibility makes ϕ true—or that ϕ is true at the possibility—when p makes the proposition that ϕ true. Formally, we'll say that p makes ϕ true—or that ϕ is true at p—when $p \vDash [\![\phi]\!]$. Truth at a possibility is the possibility model analogue of truth at a possible world in a Kripke model. But whereas the truth of ϕ at a possible world can be determined by the truth of its subformulas at that world, the truth of ϕ at a possibility will typically only be determined by the truth of its subformulas at *extensions* of that possibility. In particular, whereas a possible world makes $\neg\phi$ true when it fails to make ϕ true, a possibility makes $\neg\phi$ true when it makes ϕ false or rules out ϕs being true: that is, when it is incompossible with all the possibilities that make ϕ true (equivalently: when every possibility extending it fails to make ϕ true). And whereas a possible world makes $\phi \vee \psi$ true when it makes either ϕ or ψ true, a possibility makes $\phi \vee \psi$ true when no possibility extending it makes both ϕ and ψ false or rules out both ϕ and ψ: that is, when no possibility extending is incompossible with all the possibilities that make ϕ true and all the possibilities that make ψ true (equivalently, when every possibility extending it can be further extended to a possibility that makes ϕ true or further extended to a possibility that makes ψ true). In general, we can derive the following clauses for truth at a possibility (extending them to $\forall$, $\wedge$, and $\rightarrow$, which I will take to be defined from $\exists$, $\vee$, and $\neg$ in the usual way).

- $p \vDash \neg\phi \quad \Leftrightarrow \quad p \perp [\![\phi]\!] \quad \Leftrightarrow \quad \neg\exists q \leq p \exists r \in [\![\phi]\!](q \leq r) \quad \Leftrightarrow \quad \forall q \leq p(q \nvDash \phi)$
- $p \vDash \phi \vee \psi \quad \Leftrightarrow \quad \neg\exists q \leq p(q \vDash \neg\phi \wedge w \vDash \neg\psi) \quad \Leftrightarrow \quad \forall q \leq p \exists r \leq q(r \vDash \phi \vee r \vDash \psi)$
- $p \vDash \phi \wedge \psi \quad \Leftrightarrow \quad p \vDash \phi \wedge p \vDash \psi$
- $p \vDash \phi \rightarrow \psi \quad \Leftrightarrow \quad \neg\exists q \leq p(q \vDash \phi \wedge q \vDash \neg\psi) \quad \Leftrightarrow \quad \forall q \leq p(q \vDash \phi \rightarrow q \vDash \psi)$
- $p \vDash \forall x\phi \quad \Leftrightarrow \quad \forall x \in D(p \vDash \phi)$
- $p \vDash \exists x\phi \quad \Leftrightarrow \quad \neg\exists q \leq p \forall x \in D(q \vDash \neg\phi) \quad \Leftrightarrow \quad \forall q \leq p \exists r \leq q \exists x \in D(r \vDash \phi)$

Just as we get used to evaluating the truth of claims at possible worlds by working through examples, so too one can relatively easily get used to evaluating the truth of claims at a possibility by working through examples. The big difference is that even when evaluating non-modal claims at a possibility, we have to consider the possibilities extending it. Let's look at a simple example.

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classes will be proper classes, we can use the Scott trick to obtain sets that play the same role. Then, where $[\![x]\!]$ is the equivalence class for x, the interpretation functions $I'_=$ and $I'_\in$ for $\mathcal{M}^=$ can be defined by: $I'_=([\![x]\!], [\![y]\!]) = \bigcup_{x \in [\![x]\!], y \in [\![y]\!]} [\![x = y]\!]$ and $I'_\in([\![x]\!], [\![y]\!]) = \bigcup_{x \in [\![x]\!], y \in [\![y]\!]} [\![x \in y]\!]$. It is then straightforward to show that in general the proposition assigned by $\mathcal{M}$ to $\phi(\vec{x})$ is implicationally equivalent to the proposition assigned by $\mathcal{M}^=$ to $\phi(\vec{[\![x]\!]})$.

Suppose we want to figure out whether the claim that every possible set is a member of some possible set—that is, $\forall x \exists y(x \in y)$—is made true by a possibility p. Since it's a universal quantification, we know that p will make it true just in case it makes each instance true. Formally:

$$p \vDash \forall x \exists y(x \in y) \quad \Leftrightarrow \quad \forall x \in D(p \vDash \exists y(x \in y))$$

To figure out whether $p \vDash \forall x \exists y(x \in y)$, then, we need to figure out whether, for each possible set x in the domain, p makes it true that x is a member of some possible set. Since that's an existential quantification, we know that p will make it true just in case every one of p's extensions can be further extended to make some instance true. Formally:

$$p \vDash \exists y(x \in y) \quad \Leftrightarrow \quad \forall q \leq p \exists r \leq q \exists y \in D(r \vDash x \in y)$$

So, we let $q \leq p$ be an arbitrary possibility extending p and we ask: is there a possibility extending q that makes $x \in y$ true for some possible set y? If we can find such a possibility and possible set for each such $q \leq p$, then we've established that p makes $\exists y(x \in y)$ true for x. Alternatively, if there is a $q \leq p$ for which we can't find such a possibility and possible set, then we've established that p does not make $\exists y(x \in y)$ true (and so does not make $\forall x \exists y(x \in y)$ true either). Of course, even if p doesn't make $\exists y(x \in y)$, that doesn't mean it makes its negation true! There may still be possibilities extending p that makes $\exists y(x \in y)$ true. Possibilities are not possible worlds.

When working in a possibility model, it is natural to move back and forth between individual possibilities and propositions. Propositions allow us a more coarse grained view on the model, whereas possibilities allow us a more fine grained view. When we are dealing with logical relations, it makes sense to focus on propositions and forget the possibilities they contain. When we are dealing with the behaviour of particular possible sets, it makes sense to focus on the possibilities that encode specific information about them. Because we explicitly defined implication in terms of the relations of extension among possibilities, we can indeed move freely back and forth between possibilities and propositions: X implies Y precisely when every possibility in X implies Y, which is precisely when no possibility extending an X possibility rules out Y, which is precisely when every possibility in X is only extended by possibilities compossible with some possibility in Y.

$$X \vDash Y \quad \Leftrightarrow \quad \forall p \in X(p \vDash Y) \quad \Leftrightarrow \quad \neg \exists p \in X \exists q \leq p(q \perp Y) \quad \Leftrightarrow \quad \forall p \in X \forall q \leq p(q \not\perp Y)$$

3 Forcing models as possibility models

As I'm thinking of it here, the method of forcing consists in the construction of very specific possibility models in which the axioms of ZFC are necessary. Given a particular space of possibilities, these possibility models are uniquely characterised by three simple features: Extensionality, Maximality, and Well-foundedness. Before I get to them, I need to define an important notion: namely, that of a *profile*.

Possible sets are objects that have members at various possibilities. So, any function f from some possible sets to propositions gives us an abstract specification of a possible set: namely, a possible set that has something as a member in so far as that thing is equal to one of the possible sets in f's domain in so far as the corresponding proposition is true. A profile is simply such a specification.

Definition 1 A *profile* is a function from possible sets to propositions. Let $X \equiv Y$ abbreviate the claim that propositions X and Y are implicationally equivalent: formally, $X \equiv Y$ abbreviates $X \vDash Y \wedge Y \vDash X$. Then we say that a profile f *defines* a possible set s when, for any possible set y in D:

$$[\![y \in s]\!] \equiv [\![(y = x_0 \wedge f(x_0)) \vee (y = x_1 \wedge f(x_1)) \vee ... \vee (y = x_i \wedge f(x_i)) \vee ...]\!]$$

for $x_i \in \mathsf{dom}(f)$.

Let $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ be a possibility model.

Extensionality simply says that the axiom of extensionality is necessarily true in $\mathcal{M}$. In other words, it says that the proposition assigned to the axiom of extensionality is implicationally equivalent to $\mathbb{P}$. Equivalently: it says that the proposition that x and y have the same elements implies the proposition that they are identical, for any possible sets x and y in D. Formally:

$$[\![\forall z(z \in x \leftrightarrow z \in y)]\!] \vDash [\![x = y]\!]$$

Because our possibility models identify necessarily identical possible sets, it is not hard to see that this gives us a criterion of identity for possible sets according to which possible sets containing the same elements at the same possibilities are identical. Extensionality tells us, in other words, that possible sets are completely characterised by what members they have at what possibilities. Possible sets defined by the same profiles are identical.

Maximality says that $\mathcal{M}$ contains as many possible sets as it can, that the model is as rich as possible. More precisely, it says that *every* profile for $\mathcal{M}$ defines a possible set in $\mathcal{M}$.[13]

When s is by a profile f, membership in s is completely determined by membership for the possible sets in $\mathsf{dom}(f)$. Formally, for each y in D, we have:

$$[\![y \in s]\!] \equiv [\![(y = x_0 \wedge x_0 \in s) \vee (y = x_1 \wedge x_1 \in s) \vee ... \vee (y = x_i \wedge x_i \in s) \vee ...]\!]$$

for $x_i \in \mathsf{dom}(f)$. Once we know the possibilities at which those possible sets in $\mathsf{dom}(f)$ are members of s, in other words, we thereby know the possibilities at which any possible set whatsoever is a member of s. When membership for a possible set s is completely determined by membership for the possible sets in x in this way, I will say that x is a *core* for s.[14] It is a nice exercise to show that x is a core for s just in case $x = \mathsf{dom}(f)$ for some profile f that defines x.

Well-foundedness says that membership can be determined in a well-founded way. It particular, it says that there is some way of assigning cores to possible sets such that there is no sequence of possible sets $x_0, x_1, ..., x_n, ...$ such that x_0 has a core containing x_1, x_1 has a core containing x_2, x_2 has a core containing x_3, and so on. More precisely, it says that there is a (possibly proper class-sized) function F from D to cores such that there is no ω-sequence of possible sets $x_0, ..., x_n, ...$ for which $F(x_i) \ni x_{i+1}$ for all $i \in \omega$. An example may help to illustrate. By Maximality, there is a possible set, $s_\emptyset$, that never contains anything: it is necessarily empty. Now suppose we have two incompossible possibilities p and q. Then, by

[13]Compare this with the requirement on classical models that for any subset x of the domain, there is a set in the model whose elements are precisely the members of x.

[14]Ordinary sets effectively have one core: namely, themselves. Possible sets can have many.

Maximality, there will be two possible sets s_1 and s_2 such that s_1 contains $s_\emptyset$ in so far as p is the case (and otherwise contains nothing) and such that s_2 contains $s_\emptyset$ in so far as q is the case (and otherwise contains nothing). So, by Extensionality, according to p, $s_\emptyset$ is an element of s_1 and $s_2 = s_\emptyset$ and according to q, $s_\emptyset$ is an element of s_2 and $s_1 = s_\emptyset$. Now there are two importantly different ways of describing s_1 and s_2. On one description, s_1 has $\{s_2\}$ as a core and s_2 has $\{s_1\}$ as a core. To be an element of s_1 is to be equal to s_2 in so far as p is the case and to be an element of s_2 is to be equal to s_1 in so far as q is the case. Accordingly, relative to this assignment of cores, they are non-well-founded: s_1 is a member of the core $\{s_1\}$ of s_2 and s_2 is a member of the core $\{s_2\}$ of s_1. On another description, they both have $\{s_\emptyset\}$ as a core. To be an element of s_1 is to be equal to $s_\emptyset$ in so far as p is the case and to be an element of s_2 is to be equal to $s_\emptyset$ in so far as q is the case. Accordingly, relative to this assignment of cores, they are well-founded: each has $\{s_\emptyset\}$ as its core, and $s_\emptyset$ has nothing in its core, since its core is $\emptyset$. Well-foundedness says that the possible sets are well-founded under *some* such description, under some uniform allocation of cores to possible sets.

We can prove in ZFC that over any space of possibilities there is *exactly one* possibility model satisfying Extensionality, Maximality, and Well-foundedness, up to isomorphism.

Theorem 3 (ZFC) For any partial order $\mathbb{P}$, there is a (proper class-sized) possibility model $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ satisfying Extensionality, Maximality, and Well-foundedness.

Theorem 4 (ZFC) Let $\mathcal{M} = \langle D, \mathbb{P}, I_=, I_\in \rangle$ and $\mathcal{M}' = \langle D', \mathbb{P}, I'_=, I'_\in \rangle$ be possibility models satisfying Extensionality, Maximality, and Well-foundedness. Then, $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic.[15] In other words, let $[\![\phi]\!]_\mathcal{M}$ be the proposition assigned to ϕ according to $\mathcal{M}$. Then, there is a one-one function j from D to D' such that:

$$[\![\phi(\vec{x})]\!]_\mathcal{M} \equiv [\![\phi(j(\vec{x}))]\!]_{\mathcal{M}'}$$

Since the model is unique up to isomorphism, I will refer to it as the *the forcing possibility model* (for $\mathbb{P}$).

It's a nice exercise to verify that in any possibility model satisfying Extensionality, Maximality, and Well-foundedness—and thus in any forcing possibility model—the axioms of ZFC are necessary.

Theorem 5 If ϕ is an axiom of ZFC, then ZFC proves that in any forcing possibility model $\mathcal{M}$, $[\![\phi]\!] \equiv \mathbb{P}$.

This means that just as we can freely use classical logical inferences when reasoning with propositions, or when reasoning about what is true at a possibility, we can now also freely use

[15] *Proof sketch:* Let $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ and $\mathcal{M}' = \langle \mathbb{P}, D', I'_=, I'_\in \rangle$ be possibility models satisfying Extensionality, Maximality, and Well-foundedness and let F and F' witness their respective well-foundedness. Now, by a simple induction on the well-founded relation "$x \in F(y)$" we can use Extensionality and Maximality to recursively build a one-one function π from D to D' for which $[\![x = y]\!]_\mathcal{M} \equiv [\![\pi(x) = \pi(y)]\!]_{\mathcal{M}'}$ and $[\![x \in y]\!]_\mathcal{M} \equiv [\![\pi(x) \in \pi(y)]\!]_{\mathcal{M}'}$. This, in turn, extends via the compositional clauses for assigning propositions to show that $[\![\phi(\vec{x})]\!]_\mathcal{M} \equiv [\![\phi(\pi(\vec{x}))]\!]_{\mathcal{M}'}$ for any ϕ $\Box$. This theorem corresponds to the internal categoricity result in ZFC which says that any two extensional (set-like) well-founded classical models are isomorphic when, for any subset of their domains, there's a set in the model with precisely the same elements.

the theorems of ZFC. For example, if p makes it true that a possible set x is a well-ordered relation, then p will also make it true that x is isomorphic with some ordinal.

For many of the axioms of ZFC, we establish their necessary truth by using $\mathsf{Maximality}$ to find a suitable possible set. For example, given any two possible sets s and s', we can use $\mathsf{Maximality}$ to obtain the possible set that contains both at all possibilities, thus ensuring that the axiom of pairing holds at all possibilities.[16] And depending on which space of possibilities we work with, we can use also use $\mathsf{Maximality}$ to obtain much more exotic possible sets. For example, there is a $\mathbb{P}$ relative to which we can use $\mathsf{Maximality}$ to obtain a possible set that leads to the necessary failure of the continuum hypothesis: that is, $[\![\mathsf{CH}]\!] \equiv \emptyset$.

In addition to the interesting and exotic possible sets, there will also always be a wide range of boring possible sets: sets for which membership is fundamentally a necessary matter. And it turns out these boring possible sets are isomorphic to the sets. The empty set, for example, corresponds to the possible set, $s_\emptyset$, that is necessarily empty; the singleton of the empty set corresponds to the possible set $s_{\{\emptyset\}}$, that necessarily contains $s_\emptyset$ and nothing else; and so on.[17] This means that in addition to *de dicto* necessity claims, we can also make sense of *de re* necessity claims about particular sets, via the corresponding boring possible sets. For example, let s_ω be the boring possible set corresponding to ω and $s_{\mathcal{P}(\omega)}$ the boring possible set corresponding to its powerset. Then there is a space of possibilities $\mathbb{P}$ over which it is necessary in the forcing model that there is some subset of ω *not* in $\mathcal{P}(\omega)$, which is to say $[\![\exists y \subseteq s_\omega(y \notin s_{\mathcal{P}(\omega)})]\!] \equiv \mathbb{P}$. In general, if x is any infinite set and y is any set, there is a space of possibilities $\mathbb{P}$ over which it is necessary in the forcing model that there is some subset of x not in y, which is to say $[\![\exists z \subseteq s_x(z \notin s_y)]\!] \equiv \mathbb{P}$. Similarly, given any set x, there is a $\mathbb{P}$ over which x is necessarily countable in the forcing model, which is to say $[\![s_x \text{ is countable}]\!] \equiv \mathbb{P}$.

Perhaps unsurprisingly, when we restrict our quantifiers solely to these boring possible sets, we get claims that agree with V.[18] In particular, we have the following theorem.

Theorem 6 (ZFC) Let ϕ be Δ_0 with parameters among $\vec{x}$. Then $\phi(\vec{x})$ is true in V just in case $[\![\phi(\vec{s_x})]\!] \equiv \mathbb{P}$.

So, how *do* we ensure that there are the relevant kinds of interesting and exotic possible sets, like those that lead to a failure of the continuum hypothesis? In general, how do choose possibility spaces so that the resulting forcing possibility models that make some target claim possibly false? Unfortunately, there is no general recipe here. But there are two things we typically do. First, by thinking about the target claim, we try to come up with a kind of possible set we'd need to invalidate it. For example, in the case of CH, at the very least, we'd need a possible set that acts as a function from some relevantly large set, say s_{ω_2}, into the powerset of ω. Call this our *wish list item*. We then try to find a rich enough space of possibilities that allows us to obtain our wish list item using $\mathsf{Maximality}$. Again, there is no general recipe for this. But one approach is try to reverse engineer the space of possibilities from our wish list item. Here's an example.

[16] Where that axiom is formulated as the claim that for any two sets, some set contains both.

[17] In general, we can define the notion of a *transitively boring* possible set in a series of stages. In particular, the transitively boring possible sets at stage $\alpha+1$ are all and only the possible sets specified by profiles that are functions from transitively boring possible sets at stage α to the propositions $\emptyset$ and $\mathbb{P}$. We can then recursively associate a unique transitively boring possible set with each set in the obvious way.

[18] Formally, let $[\![\phi]\!]^*$ be like $[\![\phi]\!]$ except that $[\![\exists x\phi]\!]^*$ is the union of instances for the boring possible sets: that is, $[\![\exists x\phi(x)]\!]^* = \bigcup_{x\in V}[\![\phi(s_x)]\!]^*$. Then, we can prove that: $V \vDash \phi(\vec{x}) \leftrightarrow [\![\phi(\vec{s_x})]\!]^* \equiv \mathbb{P}$.

Suppose we want a forcing possibility model in which it is necessary that there is a *new* subset of ω. That is, we want a possible set s that necessarily disagrees on s_ω with every s_y for $y \subseteq \omega$. That's our wish list item. If there were such a possible set, then every possibility p could be extended to witness the disagreement. So, for any p, there will be a $q \leq p$ such that either $q \vDash s_n \in s$ for some $n \notin y$ or $q \vDash s_n \notin s$ for some $n \in y$. So, effectively, as we move down the partial order, we will keep adding information about what elements of ω are and aren't in s. But because we always have to be able to extend so as to disagree with any given y, we can't reach a possibility that is completely opinionated about what elements of ω are and aren't in s.[19] So, elements of the partial order need to encode an increasing but limited amount of information about what elements of ω are and aren't in s. One partial order that has this kind of behaviour "built" in is the set of finite functions from ω to $0, 1$. We can think of each such function as "saying" that s contains s_n, if $f(n) = 1$, or fails to contain s_n, if $f(n) = 0$. Because it only has an opinion about finitely many elements of ω, we can always extend it so that is says s differs from any given y. So, to obtain our desired possible set s, we take the profile that maps s_n to $\{f : \omega \to \{0, 1\} : |\mathsf{dom}(f)| < \omega \wedge f(n) = 1\}$. It is then a simple exercise to check that the corresponding possible set behaves as required.

The second thing we typically do when choosing a suitable possibility space is to try to ensure that it is not *too* rich. For example, some partial orders that allow us to obtain a possible set that behaves like a function from s_{ω_2} into the powerset of ω are so rich that they give us, via $\mathsf{Maximality}$, a function from s_ω onto s_{ω_2}: that is, a possible set witnessing the countability of s_{ω_2}. In other words, they give us a forcing possibility model in which although there is a function from s_{ω_2} into the powerset of s_ω, s_{ω_2} is no longer relevantly large! As a consequence, such partial orders need not result in forcing possibility models invalidating CH. So, how do we ensure that our possibility space is relevantly constrained? Again, there is no general recipe for this. But an example will help to illustrate one important way in which partial orders can be relevantly restrictive. Consider the possibility p that Mae is wearing a green hat. That possibility can be extended to more specific possibilities concerning the shade of the hat. For as many shades of green as there are, there will be possibilities extending p at which Mae's hat is that shade of green. So the set of possibilities where Mae's hat is some shade of green is quite large: let's say it has size 2^ω. The proposition that Mae's hat is some shade of green, in other words, is quite large. The proposition that it is green, on the other hand, is very small: let's say it has size 1. But those propositions are implicationally equivalent! In so far as we're interested in the proposition that Mae's hat is some shade of green, we could equally well have worked with a much smaller proposition. The *countable chain condition* simply says that every proposition is equivalent to some countable sub-proposition.[20] So although the partial order might be rich enough to have uncountably many distinct possibilities that make ϕ true, it is sufficiently impoverished that those possibilities are equivalent to only countably many of them. One important upshot of this restrictive assumption is precisely that it does not allow for possible sets that change the cardinalities of the boring possible sets. More precisely, when $\mathbb{P}$ satisfies the countable chain condition, κ is the cardinality of x in V just in case $[\![s_\kappa \text{ is the cardinality of } s_x]\!] \equiv \mathbb{P}$.[21]

[19] If there were such a possibility p, then $\{n \in \omega : p \vDash s_n \in s\}$ would be a set that p cannot be extended so that s disagrees with it.

[20] The usual definition of the countable chain condition is that every set of pair-wise incompossible possibilities is countable—that is, there is no uncountable set of pair-wise incompossible possibilities. It is a nice exercise to show that these two definitions are equivalent.

[21] The core of the proof involves showing that ordinals cannot change their co-finalities. To see how it works,

Forcing arguments are typically a matter of finding partial order that has the right balance richness—enough to result in new and potentially interesting possible sets—and restrictiveness—enough to ensure that the new possible sets fulfil their potential.

4 An iterative conception of possible sets

The simplest way to construct a forcing possibility model is iteratively.[22] Here's the general idea. We first construct its domain in a series of stages, corresponding to the usual construction of the V_αs. We start at stage 0 with no possible sets whatsoever. Then, the possible sets at stage 1 are all and only the functions from possible sets at stage 0 to propositions. Since there is nothing at stage 0, that means the only possible set at stage 1 is the empty set: the function that maps nothing to no propositions. The possible sets at stage 2 are all and only the functions from possible sets at stage 1 to propositions. And so on. In general, successor stages $\alpha+1$ comprise all and only the functions from things at stage α to propositions, limit stages λ comprise all and only the things at previous stages, and the possible sets of the model are the things at some stage or other.

In this construction, the possible sets effectively have their profiles "built in": a function at some stage *is* a profile. So, assuming we can construct a suitable assignment function for identity, $I_=$, we can explicitly define an assignment function for membership, $I_\in$, by letting $I_\in(\langle f, g\rangle)$ be:

$$[\![(f = z_0 \wedge g(z_0)) \vee (f = z_1 \wedge g(z_1)) \vee ... \vee (f = z_i \wedge g(z_i)) \vee ...]\!]$$

for $z_i \in \mathsf{dom}(g)$. Which would immediately guarantee both $\mathsf{Maximality}$ and $\mathsf{Well\text{-}foundedness}$, where $\mathsf{dom}(f)$ is the assigned core for f. So the only non-trivial task is to construct a suitable assignment function for identity that will guarantee both $\mathsf{Extensionality}$ and the identity axioms. We do this recursively along the stages. In particular, once we've defined $I_=$ for pairs of possible sets at stage α, we define $[\![f = g]\!]$, for f, g at stage $\alpha+1$, to be the proposition that for every x in the domain of f, in so far as $f(x)$ is true, x is identical to something y in the domain of g in so far as $g(y)$ is true, and vice versa.[23] We then have to carefully check that this, together with the resulting definition of $I_\in$, indeed validates $\mathsf{Extensionality}$ and the identity axioms.

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suppose we have two infinite ordinals α and β and β has co-finality greater than α: that is, there is no function from α that takes values unbounded in β. Now suppose for contradiction that there is a possible set s coding a function from α unbounded in β. Let $\delta \leq \alpha$. The proposition that s sends s_δ to something in β is just the union of its instances: either it sends s_δ to s_0 or s_1 or... What the countable chain condition tells us is that countable many of its instances are equivalent to all of them. In other words, there is a countable sequence $\gamma_0, \gamma_1, \gamma_2...$ of elements of β such that in so far as s sends s_δ to something in β it either sends it to s_{γ_0} or s_{γ_1} or s_{γ_2} or... Since β has co-finality greater than α and α is infinite, it has co-finality greater than ω. So let λ_δ be the least upper bound of $\gamma_0, \gamma_1, \gamma_2, ...$ in β. Then it will be necessary that in so far as s maps s_δ to something in β, it maps it to something below λ_δ. Now, because β has co-finality greater than α the least upper bound, λ, of all the λ_δ's for $\delta \leq \alpha$ will be less than β. And it will be necessary that in so far as s maps something in s_α to something in s_β, it takes values $\leq \lambda$. But that's inconsistent with the claim that it takes values unbounded in β.

[22] This exactly mirrors the construction of $\mathbb{P}$-names and the forcing relation in standard presentations of forcing. See, for example, Kunen [1971].

[23] Formally: we set $[\![f = g]\!] = [\![(f(z_0) \to [(z_0 = y_0 \wedge g(y_0)) \vee (z_0 = y_1 \wedge g(y_1)) \vee ...] \wedge (f(z_1) \to [(z_1 = y_0 \wedge g(y_0)) \vee (z_1 = y_1 \wedge g(y_1)) \vee ...] \wedge ... \wedge (g(y_0) \to [(y_0 = z_0 \wedge f(z_0)) \vee (y_0 = z_1 \wedge f(z_1)) \vee ...] \wedge (g(y_1) \to [(y_1 = z_0 \wedge f(z_0)) \vee (y_1 = z_1 \wedge f(z_1)) \vee ...] \wedge ...]\!]$. An induction then shows that the assignment functions at each stage agree with one another and so that their union is an assignment function.

This iterative structure isn't simply a quirk of this particular construction. It is an essential feature of forcing possibility models in general, and provides us with a natural iterative way to think about them. To see this, let $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ be a forcing possibility model. We can divide D into a series of stages. Stage 0 will be the empty set. Stage 1 will comprise all the possible sets in $\mathcal{M}$ that have as a core some set of possible at stage 0. Since there is nothing at stage 0, that means the only possible set in $\mathcal{M}$ at stage 1 is $s_\emptyset$. Stage 2 will comprise all the possible sets in $\mathcal{M}$ that have as a core some set of possible sets at stage 1. And so on. In general, successor stages $\alpha + 1$ comprise all and only the possible sets in $\mathcal{M}$ that have as a core a set of possible sets at stage α, and limit stages λ comprise all and only the things at previous stages. More precisely, we let:[24]

- $\mathcal{M}_0 = \emptyset$
- $\mathcal{M}_{\alpha+1} = \{x \in D : \exists y \subseteq \mathcal{M}_\alpha (y \text{ is a core for } x)\}$
- $\mathcal{M}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{M}_\alpha$

A simple induction on the well-founded relation given by Well-foundedness then shows that every possible set in $\mathcal{M}$ must occur in some $\mathcal{M}_\alpha$: $D = \bigcup M_\alpha$. Conversely, if every possible set is in some $\mathcal{M}_\alpha$, then we get a witness to Well-foundedness: if x first occurs at stage $\alpha + 1$, we can assign it $\mathcal{M}_\alpha$ as a core.[25]

Theorem 7 (ZFC) Let $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ be a possibility model satisfying Extensionality. Then, $\mathcal{M}$ satisfies Well-foundedness just in case $D = \bigcup M_\alpha$.

When every possible set in $\mathcal{M}$ occurs at some stage, Maximality turns out to be equivalent to the claim that any profile whose domain is a set of possible sets at stage α defines a possible set at stage $\alpha + 1$.

Definition 2 $\mathcal{M}_{\alpha+1}$ is *maximal* for $\mathcal{M}_\alpha$ when every profile whose domain is a subset of $\mathcal{M}_\alpha$ defines a possible set in $\mathcal{M}_{\alpha+1}$.

Theorem 8 (ZFC) Let $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ be a possibility model satisfying Extensionality and Well-foundedness. Then, $\mathcal{M}$ satisfies Maximality just in case for all α, $\mathcal{M}_{\alpha+1}$ is maximal for $\mathcal{M}_\alpha$.

Corollary 1 (ZFC) Let $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ be a possibility model satisfying Extensionality. Then, $\mathcal{M}$ satisfies Well-foundedness and Maximality just in case $D = \bigcup M_\alpha$ and for all α, $\mathcal{M}_{\alpha+1}$ is maximal for $\mathcal{M}_\alpha$.

Say that a possibility model $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ is *fully iterative* if $D = \bigcup M_\alpha$ and for all α, $\mathcal{M}_{\alpha+1}$ is maximal for $\mathcal{M}_\alpha$.

[24] At each stage we have set-many possible sets, rather than proper class-many, because if $\mathcal{M}_\alpha$ is set-sized, then there are only set-many profiles based on $\mathcal{M}_\alpha$ and possible sets with the same profiles are identical by Extensionality and the fact that we are working with possibility models in which necessarily identical sets are really identical.

[25] It is easy to see that if $y \subseteq \mathcal{M}_\alpha$ is a core for x, then so too is $\mathcal{M}_\alpha$ itself. Simply consider the profile f that maps each z in y to $[\![z \in x]\!]$ and each $z \in \mathcal{M}_\alpha$ that is not in y to $\emptyset$.

Corollary 2 (ZFC) Let $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ be a possibility model. Then, $\mathcal{M}$ satisfies Extensionality, Well-foundedness, and Maximality just in case it is fully iterative and satisfies Extensionality.

Corollary 3 (ZFC) Let $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ and $\mathcal{M}' = \langle \mathbb{P}, D', I'_=, I'_\in \rangle$ be fully iterative possibility models satisfying Extensionality. Then, they are isomorphic.

This isn't simply a nice alternative way to think of forcing possibility models and their categoricity theorem. It also highlights two interesting further points. First, it leads to a much more general, *quasi-categoricity* result. Second, it shows that if we ultimately want to think of possible sets as a new sui generis kind of mathematical object, then we could see them as governed by an *iterative conception of possible sets*. Let's look at these points in turn.

4.1 Quasi-categoricity

Definition 3 Say that a possibility model $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ it is *partially iterative* if there is some λ for which $D = \bigcup_{\alpha<\lambda} M_\alpha$ and for all $\alpha < \lambda$, $\mathcal{M}_{\alpha+1}$ is maximal for $\mathcal{M}_\alpha$. In any such model $\mathcal{M}$, for any $\beta + 1 < \lambda$, there will be a boring set at stage $\beta + 1$ necessarily containing all the possible sets at stage β. Call this $s_{\mathcal{M}_\beta}$.

Corollary 4 (ZFC) Let $\mathcal{M} = \langle \mathbb{P}, D, I_=, I_\in \rangle$ and $\mathcal{M}' = \langle \mathbb{P}, D', I'_=, I'_\in \rangle$ be possibility models satisfying Extensionality. Suppose they are partially iterative, and let α and α' be the corresponding ordinals. Then, (i) if $\alpha = \alpha'$, $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic, and (ii) if $\alpha < \alpha'$, $\mathcal{M}$ is isomorphic to $\mathcal{M}'_\alpha$ in $\mathcal{M}'$: that is, there is a one-one function j from D to M'_α such that $[\![\phi(\vec{x})]\!]_\mathcal{M} \equiv [\![\phi^{s_{\mathcal{M}_\alpha}}(\vec{j(x)})]\!]_{\mathcal{M}'}$.

4.2 An iterative conception of possible sets

Using these ideas, it is straightforward to modify our possibility model $\mathcal{M}$ to obtain models of an iterative conception of possible sets. First, we add a set of "stages" together with an "earlier than" relation. For simplicity, we can assume that this relation is a well-order. As a result, we can identify the stages with the predecessors of some ordinal α and we can identify the earlier than relation with $\leq$ on those ordinals. Second, we add a relation of "formed at" or simply "at" between the possible sets and the stages. We then impose two obvious constraints on them.

1. We have a principle of Plenitude, which is just a version of the maximality claim we saw earlier. Plenitude says that any profile available at a stage—that is, any function from some possible sets at that stage to propositions—defines a possible set at all subsequent stages.
   - Formally, it says that for any profile f whose domain is a subset of the things in D at stage α, and any stage β later than α, there is some x in D that f defines.
2. We have a principle of Priority, which says that anything at a stage must be defined by a profile available at some prior stage.
   - Formally, it says that for any x in D at stage α, there is some stage $\beta < \alpha$ and a profile f whose domain is a subset of the things in D at β such that f defines x.

Call possibility models extended in this way *iterative possibility models*.

We can think of iterative possibility models as, on the way I'm thinking about it, precisely the standard models of the iterative conception of possible sets. It is not hard to see that the iterative possibility models satisfying Extensionality are precisely the partially iterative possibility models satisfying Extensionality.

Theorem 9 (ZFC) Let $\mathcal{M}'$ be the extension of a partially iterative possibility model $\mathcal{M}$ whose stages are the height of $\mathcal{M}$ and "x occurs at β" is interpreted as "$x \in M\mathcal{M}_\beta$". Then, $[\![\phi]\!]_{\mathcal{M}} = [\![\phi]\!]_{\mathcal{M}'}$ (for $\phi \in \mathcal{L}_\in$).

It is also relatively easy to see how we might formulate Plenitude and Priority in a second-order object language if we ultimately wanted to axiomatise the iterative conception of possible sets.

5 Possibilities, possible worlds, and forcing extensions

How does this all relate to more familiar presentations of forcing? In particular, how does the forcing possibility model relate to so-called forcing *extensions*: that is, models of the form $M[G]$? In this section, I answer this question. In brief: forcing extensions are just possible worlds for a possibility model.

Possibilities are like parts of possible worlds. It is therefore natural to think of a possible world as a set of possibilities: namely, the possibilities that it has as parts. At a minimum, this set of possibilities should have two features. First, it should be rich enough to witness every proposition or its negation. A possible world is a complete specification of a way the world could be and so the possibilities that compose it should witness this fact. Second, it should be modest enough not to witness both some proposition and its negation. A possible world is a way the world could be, and the world could not be inconsistent, so the possibilities that compose it should witness this fact too.

Definition 4 Say that $X \subseteq \mathbb{P}$ is a *possible world* (for $\mathbb{P}$) when it is *complete*—for every proposition Y, X either contains a possibility in Y or a possibility in $\overline{Y}$—and *consistent*—there is no proposition Y such that X contains both a possibility from Y and a possibility from $\overline{Y}$ (equivalently: any two of its possibilities are compossible).

Unfortunately, for the most interesting class of possibility spaces—indeed, for the possibility spaces we almost always use in forcing—there are, provably, no possible worlds!

Theorem 10 (ZFC) Suppose that below any possibility in $\mathbb{P}$ there are two incompossible possibilities. Then, there are no possible worlds for $\mathbb{P}$.[26]

[26] To see why these are the most interesting cases, suppose we have a possibility p that cannot be so extended. Then, by working through the clauses for truth at a possibility, it is easy to see that p will be complete: it will either make ϕ or $\neg\phi$ true for each ϕ. It will, in other words, act exactly like a classical model, like a possible world in a Kripke model. Not only that: it will act like the universe of sets V. At any such possibility, every possible set will be identical to some boring possible set. As a consequence, p will make a sentence true just in case it is true in V! If anything interesting is going to happen in the associate forcing possibility model, therefore, it ain't going to happen at p.

Nevertheless, although there aren't any possible worlds, there *could*—in some important senses of "could"—be such worlds. Let me explain.

We construct the forcing possibility model using the axioms of ZFC. So, any classical model of ZFC will contain what it thinks is a forcing possibility models. Now consider a classical countable transitive model M of ZFC and let $\mathbb{P}$ be a partial order in M.[27] Then M will think there is a forcing possibility model $\mathcal{M}$. And as above, it will think that there are no possible worlds for $\mathbb{P}$. And we, from the outside, will think there are no possible worlds for $\mathbb{P}$ either. Nevertheless, we can prove that because there are only countably many propositions in M, there will be a set of possibilities that is consistent and complete, not simpliciter, but for the propositions in M.

Definition 5 Let M be a classical model. Then, say that $X \subseteq \mathbb{P}$ is a *possible world for* M (and $\mathbb{P}$) when it is *complete for* M—for every proposition $Y \in M$, X either contains a possibility in Y or a possibility in $\overline{Y}$—and *consistent*—there is no proposition Y such that X contains both a possibility from Y and a possibility from $\overline{Y}$ (equivalently: any two of its possibilities are compossible).

When M is a countable transitive model, there is, outside of M, a possible world for M.

Theorem 11 (ZFC) Suppose M is a countable transitive model and $\mathbb{P}$ is a partial order in M. Then, there is a possible world for $\mathbb{P}$. Indeed, for each $p \in \mathbb{P}$, there is a possible world for $\mathbb{P}$ containing p.

Now, what happens to the forcing possibility model $\mathcal{M}$ in M if we treat the possibilities in some such world as actual? It turns out that we get a classical model, a possible world in the Kripkean sense. More precisely, say that a proposition $X \in M$ is *true at* W just in case W contains a possibility p which implies X—that is, just in case $\exists p \in W(p \vDash X)$ (equivalently: $\exists p \in W(p \in X)$). Let W be a possible world for M. Then, we can define a classical model $\mathcal{M}_W$ from $\mathcal{M}$ in the obvious way. We let the domain of $\mathcal{M}_W$ be exactly the same as the domain of $\mathcal{M}$, and we stipulate that:

- $\mathcal{M}_W \vDash x \in y$ iff $[\![x \in y]\!]$ is true at W
- $\mathcal{M}_W \vDash x = y$ iff $[\![x = y]\!]$ is true at W

A simple induction on the complexity of ϕ then establishes that ϕ is true in $\mathcal{M}_W$ iff it is true at W. Formally:

$$\mathcal{M}_W \vDash \phi \leftrightarrow [\![\phi]\!] \text{ is true at } W$$

By an obvious extension of our terminology, we can say that $\mathcal{M}_W$ is a possible world for $\mathcal{M}$. It is then easy to see that necessary truth in $\mathcal{M}$ is simply truth in all possible worlds for $\mathcal{M}$! Formally, where W ranges over possible worlds for M, we can prove:

$$[\![\phi]\!] \equiv \mathbb{P} \quad \Leftrightarrow \quad \forall W(\mathcal{M}_W \vDash \phi)$$

[27] Here and throughout the following discussion, I will implicitly appeal to the fact that various notions are absolute for transitive models. For example, because $\mathbb{P}$ is a partial order in M, it will be a partial order in V.

More generally, we can show that truth at a possibility is simply truth in all possible worlds in which the possibility obtains. Formally, we can show that:

$$p \vDash [\![\phi]\!] \quad \Leftrightarrow \quad \forall W \ni p(\mathcal{M}_W \vDash \phi)$$

Now the G that you see in standard presentations of forcing is called a *generic filter* for M and $\mathbb{P}$. And although the definitions are different, generic filters *are* possible worlds. The $M[G]$ that you see in standard presentations of forcing is called a *generic extension* of M. And, again, although the definitions are different, generic extensions are simply the transitive collapses of possible worlds. In particular, $M[G]$ is simply the transitive collapse of $\mathcal{M}_G$.

We can think of *width potentialism* as the view that although there may not be possible worlds, there can—in some primitive sense of "can"—be possible worlds. Here's one way to make that precise. For a universe of sets $\mathcal{U}$ satisfying ZFC and partial order $\mathbb{P} \in \mathcal{U}$, as above we say that a subset X of $\mathbb{P}$ is a *possible world* (for $\mathbb{P}$ and $\mathcal{U}$) when it is *complete* for $\mathcal{U}$—for any proposition $Y \in \mathcal{U}$, X either contains a possibility in Y or a possibility in $\overline{Y}$—and *consistent*—there is no proposition $Y \in \mathcal{U}$ such that X contains a possibility in Y and a possibility in $\overline{Y}$. Let W be a possible world for $\mathbb{P}$ and $\mathcal{U}$ and let $\mathcal{M}$ be the forcing possibility model in $\mathcal{U}$ for $\mathbb{P}$. As before, we say that a proposition $X \in \mathcal{U}$ is *true at* W just in case W contains a possibility p which implies X—that is, just in case $\exists p \in W(\{p\} \vDash X)$. We then use W to define the corresponding classical model $\mathcal{M}_W$. The central width potentialist claim can then be formulated precisely as follows.

> Let $\mathcal{U}$ be a universe, $\mathbb{P}$ a partial order in $\mathcal{U}$, and $\mathcal{M}$ the forcing possibility model in $\mathcal{U}$ for $\mathbb{P}$. Then there could be a universe $\mathcal{U}'$ which is isomorphic (modulo identity) to $\mathcal{M}_W$ for some possible world W for $\mathbb{P}$ and $\mathcal{U}$.

By working with the right partial orders, we can use this claim to obtain universes making all sorts of target claims true or false. For example, the central width potentialist claim implies that as long as there is at least one universe, there could be a universe making $\neg\mathsf{CH}$ true. Moreover, given the association of sets in $\mathcal{U}$ with the corresponding boring possible sets in the forcing possibility model, we can show that for any sets $\vec{x} \in \mathcal{U}$, $[\![\phi(\vec{s_x})]\!]$ is true in W just in case $\phi(\vec{x})$ is true in $\mathcal{U}'$ and thus that $\mathcal{U} \subseteq \mathcal{U}'$.[28] It then follows that there are also universes making all sorts of statements about the elements of $\mathcal{U}$ true. For example, let x be the powerset of ω in $\mathcal{U}$. Then the central width potentialist claim implies that there is a universe containing x and ω in which some subset of ω is not in x. So, no universe contains absolutely all subsets of ω. In general, for any infinite set x in $\mathcal{U}$ and any set y in $\mathcal{U}$, there is a universe containing x and y in which some subset of x is not in y. So, no universe contains absolutely all subsets of any infinite set x. Similarly, given any set x in $\mathcal{U}$, the central width potentialist claim implies that there is a universe $\mathcal{U}' \supseteq \mathcal{U}$ in which x is countable. So, every set in any universe is countable in some extended universe.[29]

[28] To prove this, we let i be the composition of the function from sets in $\mathcal{U}$ to the corresponding boring sets in the forcing model with the isomorphism from $\mathcal{M}_W$ to $\mathcal{U}'$. A simple induction then establishes that i is the identity function on $\mathcal{U}$.

[29] Given natural assumptions, this is actually equivalent to the central width potentialist claim. The reason is that when the powerset of $\mathbb{P}$ in $\mathcal{U}$ is countable in some $\mathcal{U}' \supseteq \mathcal{U}$, we can explicitly define a possible world for $\mathbb{P}$ and $\mathcal{U}$ in $\mathcal{U}'$. We can then use that world to explicitly define a transitive subcollection $\mathcal{C} \subseteq \mathcal{U}'$ which is isomorphic to $\mathcal{M}_W$ (modulo identity). So if we assume that such collections also count as universes, then we have a witness to the central width potentialist claim.

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