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FIRST- AND SECOND-ORDER LOGIC OF MASS TERMS

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ABSTRACT. Provided here is an account, both syntactic and semantic, of first-order and monadic second-order quantification theory for domains that may be non-atomic. Although the rules of inference largely parallel those of classical logic, there are important differences in connection with the identification of argument places and the significance of the identity relation.

KEY WORDS: identity, infinite divisibility, non-atomic domains of quantification, quantities

The logic of mass terms is a generalisation of standard predicate logic. It allows for domains of quantification which have parts, but do not consist of individuals. The rules of inference are largely those of normal predicate logic. The main point of divergence concerns the identification of argument places (reflexivisation). As there may be no individuals, the idea that distinct occurrences of the same variable always refer to the same individual cannot be applied in specifying the semantics.

The first-order system is developed syntactically in Section 1, the second-order system in Section 2. Formal semantics for the logic of mass terms¹ are arrived at indirectly by first translating the statements of the logic of mass terms into a standard first-order calculus, whose domain of quantification is the totality of quantities, i.e. the totality of parts of the domain of mass quantification.

Soundness and completeness results for the first-order logic of mass terms are obtained in Section 3, for second-order logic in Section 4.

1. FIRST-ORDER LOGIC OF MASS TERMS

Informal Grammar and Semantics

Mass Terms and Quantity Terms

The basic statements of L_1 are quantified statements. They have the form of 'All snow is white' and are written

$$(\forall p \varepsilon \mu) Fp.$$



μ is a mass term, F a predicate letter, p a quantificational variable. Mass terms are treated as referring expressions. There is a significant parallelism between mass terms and the more familiar kind of general term, represented by count nouns like ‘horse(s)’, ‘number(s)’, etc. I will refer to this more familiar type of general term as *general count terms*. Unlike singular terms, it is not an individual object that a mass term like ‘milk’ or a count noun like ‘horse(s)’ refers to. Rather, ‘horse(s)’ refers to all (and only) horses and ‘milk’ refers to all (and only) milk.

Still, there are individuals in the semantic neighbourhood of mass terms, namely quantities. E.g., there is the quantity of all milk and there is the quantity of milk in a particular glass. Quantities are related to mass terms in the same way as classes, e.g., the class of all horses or the class of horses in a particular stable, are related to general count terms. From a general count term G one obtains the singular term ‘the class of G s’ and from a class term C the general term ‘members of C ’. Analogously, if μ is a mass term ‘the quantity of (all) μ ’ refers to an individual quantity, and if κ is a singular term referring to a quantity, then ‘of κ ’ can in many cases be used as a mass term referring to all of κ . (‘All of the quantity κ is . . .’, ‘Some of the quantity κ is . . .’.)

The semantic difference between mass terms and quantity terms can also be described in this way: quantity terms refer to quantities collectively, mass terms distributively, just as class terms refer to classes collectively, general count terms distributively. For to refer to a class distributively is to refer to its members, i.e. each and every one of its members.

Having emphasised the semantic difference between mass terms and quantity terms, I now proceed to ignore this distinction in the formal language. It will be left open whether a Greek letter like “ μ ” is to be read as a mass term or as a quantity term. The significance of whole sentences is not affected by the ambiguity, as long as “ \forall ”, “ \exists ”, “ ε ” and relative clauses are systematically re-interpreted. E.g.,

$$(\forall p \varepsilon \mu) \dots p \dots$$

can be read indifferently as “All / milk . . .” or as “All of / the quantity of milk . . .”. In some contexts one way of resolving the systematic ambiguity may be more natural than the other.

For any general count term G there is a related predicate ‘is a G ’ which is true of everything that G refers to and of nothing else. Similarly, for every mass term μ there is a related predicate ‘is μ ’ that is true of all that μ refers to and only of what μ refers to; e.g., ‘is water’ is true of all water and only of water. In the formal language the expression ‘ $p \varepsilon \mu$ ’ represents the predicate ‘is μ ’, derived from the mass term μ .

One further aspect of the parallelism between mass terms and general count terms is the existence of a device that restricts the reference of the term to what satisfies a certain predicate. Complex mass terms like ‘milk that was exported to the EU in 1998’ consist of a mass term plus relative clause. The expression ‘ $\{\mu p \mid A\}$ ’ is intended to refer to all μ that is A .

A predicate letter F is interpreted as a characteristic that may apply here and there in the domain.

$$(\forall p \varepsilon \mu) Fp$$

counts as true under an interpretation, if the characteristic associated with F obtains everywhere in the extension of μ ; or, in other words, if all of the quantity that μ refers to has the characteristic associated with F , i.e. all μ is F .

This account of universal quantification differs from the familiar account for a sentence like ‘All canaries are yellow’ (‘ $(\forall x \varepsilon G) Fx$ ’) only in the way in which the general term, i.e. either mass term or general count term, refers. The sense of the generality conveyed by the quantifier appears to be the same whether the range of quantification is the extension of a mass term or of a general count term; the form of words – ‘all of the extension of . . . has the characteristic F ’ – can serve in either case. Therefore it is appropriate to use the same symbol ‘ \forall ’ in both cases.

Quantificational variables are used to tie quantifiers to argument places in complex predicates. While they do not really have a role in the basic statements of L_1 , they are required, since the language allows also for relations and for multiple quantification (‘Some wood is denser than all water’). Predicates may also be truth-functionally complex. In short, in all these respects L_1 is like a first-order language with restricted quantifiers.

Reflexivisation

It would seem therefore that the logic of mass terms does not differ from standard first-order logic. This is not the case, though. The differences emerge in connection with reflexivisation, the identification of argument places. I write $p \left\{ \begin{smallmatrix} q \\ r \end{smallmatrix} \right. Rqr$ to indicate the predicate obtained by identifying the argument places of the relation R . In general, writing $p \left\{ \begin{smallmatrix} q \\ r \end{smallmatrix} \right. A$ amounts to reflexivising the open sentence A by identifying the argument places indicated by the free variables q and r , respectively; the resulting single argument place is now indicated by p .

Among the possible interpretations that are intended to be covered are domains of quantification, such as spaces and intervals of time, that do not have minimal, or atomic, parts. An example will help clarify how reflexivisation works when the domain is thus infinitely divisible. Where there are atomic parts, however, these parts can play the role of individuals

of classical logic. $(\forall p \varepsilon \mu) p \{r^q\} Rqr$ then signifies that every atomic part of μ is R -related to itself. This interpretation is clearly not feasible when the domain is infinitely divisible so that there are no atomic parts. For example, take as the domain of quantification a line λ , i.e. a continuous 1-dimensional space, not understood as a collection of points but as infinitely divisible. Let S be the relation *is less than 1 mm distant from*. Then

$$(\forall p \varepsilon \lambda) p \{r^q\} Sqr$$

is true because it is true everywhere in λ that there is less than 1 mm distant from there itself. In this case we can find intervals κ that are sufficiently small for it to be true that

$$(\forall q \varepsilon \kappa)(\forall r \varepsilon \kappa) Sqr$$

i.e. everywhere in κ is less than 1 mm distant from anywhere in κ , and the line λ can be completely covered with these intervals.

On the other hand, let L be the relation *is to the left of* and R its converse *is to the right of*. Then

$$(\forall p \varepsilon \lambda) p \{r^q\} Lqr$$

and

$$(\exists p \varepsilon \lambda) p \{r^q\} Lqr$$

are false since it is not true anywhere in λ that there is to the left of there itself. The example illustrates that for

$$(\exists p \varepsilon \mu) p \{r^q\} Rqr$$

to be the case there has to be at least one part κ of μ for which it is true that

$$(\forall q \varepsilon \kappa)(\forall r \varepsilon \kappa) Rqr.$$

Hence if

$$(\forall q \varepsilon \kappa)(\forall r \varepsilon \kappa) Rqr$$

is false whatever part κ of μ is considered, then

$$(\exists p \varepsilon \mu) p \{r^q\} Rqr$$

is false.

The usual method of reflexivisation, namely to replace the occurrences of q and of r by p , is not sufficiently scope sensitive. The reflexivisation device does not commute with negation. To illustrate, consider

again the line λ . We have seen that $(\exists p \varepsilon \lambda) p \{^q_r Lqr$ is false. Therefore $(\forall p \varepsilon \lambda) \sim p \{^q_r Lqr$ is true. On the other hand, a fortiori, $(\forall p \varepsilon \lambda) p \{^q_r Lqr$ and $(\forall p \varepsilon \lambda) p \{^q_r Rqr$ are false. But since λ is infinitely divisible, the converse of L, namely R is the same as the negation of L. Hence $(\forall p \varepsilon \lambda) p \{^q_r \sim Lqr$ is false, while $(\forall p \varepsilon \lambda) \sim p \{^q_r Lqr$ is true.

The Language L_1

Symbols

1. Denumerably many first-order variables: p_1, p_2, \dots ;
2. Denumerably many atomic mass terms: κ, κ', \dots , and Δ ;
3. Countably many predicate letters;
4. Countably many 2-place relation letters;
5. $\sim \& \supset \vee \equiv \forall \exists | \{ () \}$.

Formation Rules

Formulae

- (L1.i) An expression consisting of a predicate letter followed by a first-order variable is a formula;
- (L1.ii) An expression consisting of a relation letter followed by two, not necessarily distinct, first-order variables is a formula;
- (L1.iii) If p is a first-order variable and μ a mass term, then $p \varepsilon \mu$ is a formula;
- (L1. \sim) If A is a formula, then $\sim A$ is a formula;
- (L1.&) If A and B are formulae then $(A \& B)$ is a formula;
- (L1.R) If A is a formula and p is not free in A , unless p is identical with q or r , then $p \{^q_r A$ is a formula;
- (L1. \forall) If A is a formula, μ a mass term, and p a first-order variable, then $(\forall p \varepsilon \mu) A$ is a formula.

Mass Terms

- (L1.iv) Atomic mass terms are mass terms;
- (L1.C) If μ is a mass term and A a formula with no free first-order variable besides p , then $\{\mu p | A\}$ is a (complex) mass term.

The expressions $(A \vee B)$, $(A \supset B)$ and $(A \equiv B)$ are defined in the usual way; $(\exists p \varepsilon \mu) A$ is defined as $\sim(\forall p \varepsilon \mu)\sim A$. The notion of a free occurrence of a variable is also defined in the usual way, with the additional stipulation that the free variables in $p \{^q_r A$ are p plus any free variables in A other than q and r . A *closed formula* or *statement* is a formula without free occurrences of variables.

A variable q is *free for* p in a formula A if and only if the variable p has no free occurrences in A which lie within the scope of a quantifier $(\forall q \varepsilon \mu)$, or of a reflexivisation operator $p_i \left\{ \begin{smallmatrix} p_j \\ p_k \end{smallmatrix} \right.$, where q is p_j or p_k .

Provability in L_1

The following clauses are counterparts of rules of natural deduction, with Γ and Θ sets of formulae of L_1 . They are like the usual clauses except where reflexivisation is involved. And since (*Refl A*) merely records a syntactic convention, justifications are needed only for (*Refl I*) and (*Refl E*).

(*Refl I*) allows us to infer that all of a quantity is A -related to itself from the premise that all of the quantity is A -related to all of the quantity. The inference is valid in first-order logic and it is intuitively also valid in the logic of mass terms. For example, suppose a certain volume V of space is small enough for it to be true that everywhere in V is less than 1 m distant from anywhere in V . It then follows that everywhere in V is less than 1 m distant from there itself.

(*Refl E*) is less transparent. The premise here is that for every non-null subquantity κ of μ it is not true that all of κ is A -related to all of κ . The conclusion that can be drawn according to (*Refl E*) is that it is nowhere true in μ that there is A -related to there itself. For illustration recall the line λ , a continuous 1-dimensional space, and let A be the relation *to the left of*. Since λ is continuous and the mass quantifier does not range over points, we recognise as true the statement that nowhere in λ is to the left of there itself. (*Refl E*) allows us to infer this statement from the fact that however small an interval on λ one chooses, it is false that all of the interval is to the left of all of the interval.

Inference Rules

(*Reiteration*) $\Gamma \vdash A$ for any member A of Γ

(*Thinning*) If $\Gamma \vdash A$, then $\Gamma, \Theta \vdash A$

(*Cut*) If $\Gamma \vdash A$ and $\Theta, A \vdash B$, then $\Gamma, \Theta \vdash B$

(\sim I) If $\Gamma, A \vdash B$ and $\Gamma, A \vdash \sim B$, then $\Gamma \vdash \sim A$

(*DN*) If $\Gamma \vdash \sim \sim A$, then $\Gamma \vdash A$

($\&$ I) If $\Gamma \vdash A$ and $\Gamma \vdash B$, then $\Gamma \vdash A \& B$

($\&$ E) If $\Gamma \vdash A \& B$, then $\Gamma \vdash A$ and $\Gamma \vdash B$

(*Refl I*) If $\Gamma \vdash (\forall q \varepsilon \mu)(\forall r \varepsilon \mu) A$, then $\Gamma \vdash (\forall p \varepsilon \mu) p \left\{ \begin{smallmatrix} q \\ r \end{smallmatrix} \right. A$, provided that if p is different from q and from r then p is not free in A .

(*Refl E*) If $\Gamma, (\exists p \varepsilon \kappa) p \varepsilon \kappa, (\forall p \varepsilon \kappa) p \varepsilon \mu \vdash \sim(\forall q \varepsilon \kappa)(\forall r \varepsilon \kappa) A$, then $\Gamma \vdash (\forall p \varepsilon \mu) \sim p \left\{ \begin{smallmatrix} q \\ r \end{smallmatrix} \right. A$, provided that κ does not occur in Γ, μ , or A .

- (*Refl A*) $p \left\{ \begin{smallmatrix} q \\ r \end{smallmatrix} \right. Rqr \dashv\vdash Rpp$
- ($\forall I$) *If $\Gamma, q \varepsilon \mu \vdash A(q/p)$, then $\Gamma \vdash (\forall p \varepsilon \mu)A$, provided that*
- (a) *q is free for p in A*
 - (b) *$A(q/p)$ results from A by replacing every free occurrence of p by q*
 - (c) *there are no free occurrences of q in $(\forall p \varepsilon \mu) A$*
 - (d) *there are no free occurrences of q in Γ*
- ($\forall E$) *If $\Gamma \vdash (\forall p \varepsilon \mu) A$, then $\Gamma, q \varepsilon \mu \vdash A(q/p)$, provided that*
- (a) *q is free for p in A*
 - (b) *$A(q/p)$ results from A by replacing every free occurrence of p by q*
 - (c) *there are no free occurrences of q in $(\forall \mu p) A$*
- (The rationale of condition (c) is to avoid applying reflexivisation simultaneously with quantifier elimination.)
- (ΔI) $\vdash p \varepsilon \Delta$
- (*Compl I*) *If $\Gamma \vdash p \varepsilon \mu$ and $\Gamma \vdash A$, then $\Gamma \vdash p \varepsilon \{\mu q \mid A(q/p)\}$ provided that there is no free variable in A besides p , and q is free for p in A*
- (*Compl E*) *If $\Gamma \vdash p \varepsilon \{\mu q \mid A(q/p)\}$, then $\Gamma \vdash p \varepsilon \mu$ and $\Gamma \vdash A$. provided that there is no free variable in A besides p , and q is free for p in A*

Derived Rules

- ($\supset I$) *If $\Gamma, A \vdash B$, then $\Gamma \vdash A \supset B$*
- ($\supset E$) *If $\Gamma \vdash A \supset B$ and $\Gamma \vdash A$, then $\Gamma \vdash B$*
- ($\exists I$) *If $\Gamma \vdash q \varepsilon \mu$ and $\Gamma \vdash A(q/p)$, then $\Gamma \vdash (\exists p \varepsilon \mu)A$, provided that*
- (a) *q is free for p in A*
 - (b) *$A(q/p)$ results from A by replacing every free occurrence of p by q*
 - (c) *there are no free occurrences of q in $(\exists p \varepsilon \mu)A$*
- ($\exists E$) *If $\Gamma \vdash (\exists p \varepsilon \mu)A$ and $\Gamma, q \varepsilon \mu, A(q/p) \vdash C$, then $\Gamma \vdash C$, provided that*
- (a) *q is free for p in A*
 - (b) *$A(q/p)$ results from A by replacing every free occurrence of p by q*
 - (c) *there are no free occurrences of q in $(\exists \mu p)A$*
 - (d) *there are no free occurrences of q in Γ .*

Consequences of the Derivation Rules

LEMMA 1.1. (a) $(\exists p \varepsilon \mu)p \varepsilon \mu, (\forall p \varepsilon \mu) A \vdash (\exists p \varepsilon \mu) A$
 (b) $(\exists p \varepsilon \mu)p \varepsilon \mu \vdash A \equiv (\forall \mu p) A$, if p is not free in A .

LEMMA 1.2. (a) $\vdash (\forall p \varepsilon \mu)p \in \mu$
 (b) $(\forall p \varepsilon \mu) \sim(p \varepsilon \mu') \dashv\vdash (\forall p \varepsilon \mu') \sim(p \varepsilon \mu)$
 (c) $(\forall p \varepsilon \mu)p \varepsilon \mu', (\forall p \varepsilon \mu')A \vdash (\forall p \varepsilon \mu)A$
 (d) $(\forall p \varepsilon \mu)p \varepsilon \mu', (\forall p \varepsilon \mu')p \varepsilon \mu'' \vdash (\forall p \varepsilon \mu)p \varepsilon \mu''$
 (e) $(\forall p \varepsilon \mu)A, (\forall p \varepsilon \mu') \sim A \vdash (\forall p \varepsilon \mu) \sim(p \varepsilon \mu')$
 (f) $(\forall p \varepsilon \mu)p \varepsilon \mu'', (\forall p \varepsilon \mu') \sim(p \varepsilon \mu'') \vdash (\forall p \varepsilon \mu) \sim(p \varepsilon \mu')$

LEMMA 1.3. *Suppose q is free for p in A . Then*

- (a) $\vdash (\forall p \varepsilon \{\mu q \mid A(q/p)\})A$
 (b) $\vdash (\forall p \varepsilon \{\mu q \mid A(q/p)\})p \varepsilon \mu$
 (c) $(\exists p \varepsilon \mu)A \dashv\vdash (\exists p \varepsilon \{\mu q \mid A(q/p)\})p \varepsilon \{\mu q \mid A(q/p)\}$

Proof by (Compl I), (Compl E), (\exists I), and (\exists E). □

LEMMA 1.4. *Suppose q is free for p in A . Then*

- (a) $(\forall p \varepsilon \{\mu q \mid A(q/p)\})B \dashv\vdash (\forall p \varepsilon \mu)(A \supset B)$
 (b) $(\exists p \varepsilon \{\mu q \mid A(q/p)\})B \dashv\vdash (\exists p \varepsilon \mu)(A \& B)$

If all μ is μ' , i.e. if $(\forall p \varepsilon \mu)p \varepsilon \mu'$, μ is a part of μ' . If μ and μ' are parts of one another they are the same quantity. It is useful to introduce the following abbreviations.

DEFINITION 1.1. (a) $\mu \subseteq \mu'$ for: $(\forall p \varepsilon \mu)p \varepsilon \mu'$
 (b) $\mu = \mu'$ for: $(\forall p \varepsilon \mu)p \varepsilon \mu' \& (\forall p \varepsilon \mu')p \varepsilon \mu$

LEMMA 1.5. (a) $\vdash \mu \subseteq \Delta$
 (b) $\vdash \mu = \{\Delta q \mid q \varepsilon \mu\}$
 (c) $\vdash \{\Delta q \mid q \varepsilon \mu \& A\} = \{\mu q \mid A\}$

Proof by (Δ I) and Lemma 1.3. □

LEMMA 1.6. *If $\Gamma, (\exists p \varepsilon \kappa)p \varepsilon \kappa, (\forall p \varepsilon \kappa)p \varepsilon \mu \vdash \sim(\forall p \varepsilon \kappa)A$, then $\Gamma \vdash (\forall p \varepsilon \mu)\sim A$, provided that κ does not occur in Γ, μ , or A .*

Proof. Assume $\Gamma, (\exists p \varepsilon \kappa)p \varepsilon \kappa, (\forall p \varepsilon \kappa)p \varepsilon \mu \vdash \sim(\forall p \varepsilon \kappa)A$ and that κ does not occur in Γ, μ , or A . Then

$$\Gamma, (\exists p \varepsilon \{\mu q \mid A(q/p)\})p \varepsilon \{\mu q \mid A(q/p)\}, \\ (\forall p \varepsilon \{\mu q \mid A(q/p)\})p \varepsilon \mu \vdash \sim(\forall p \varepsilon \{\mu q \mid A(q/p)\})A.$$

By Lemma 1.3(a) and Lemma 1.3(b)

$$\Gamma \vdash \sim(\exists p \varepsilon \{\mu q \mid A(q/p)\})p \varepsilon \{\mu q \mid A(q/p)\}$$

and by Lemma 1.3(c)

$$\Gamma \vdash (\forall p \varepsilon \mu) \sim A. \quad \square$$

LEMMA 1.7. *If*

$$\begin{aligned} \Gamma, (\exists p_{i_1} \varepsilon \kappa_{j_1})p_{i_1} \varepsilon \kappa_{j_1}, \dots, (\exists p_{i_n} \varepsilon \kappa_{j_n})p_{i_n} \varepsilon \kappa_{j_n}, \\ (\forall p_{i_1} \varepsilon \kappa_{j_1})p_{i_1} \varepsilon \mu_1, \dots, (\forall p_{i_n} \varepsilon \kappa_{j_n})p_{i_n} \varepsilon \mu_n \\ \vdash \sim(\forall p_{i_1} \varepsilon \kappa_{j_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{j_n})A \end{aligned}$$

and $\kappa_{j_1}, \dots, \kappa_{j_n}$ are distinct atomic mass terms not occurring in Γ , μ_1, \dots, μ_n, μ , or A , then

$$\Gamma \vdash (\forall p_{i_1} \varepsilon \mu_1) \dots (\forall p_{i_n} \varepsilon \mu_n) \sim A.$$

Proof by repeated use of Lemma 1.6. □

LEMMA 1.8. *If*

$$\begin{aligned} \Gamma, (\exists p_{i_1} \varepsilon \kappa_{j_1})p_{i_1} \varepsilon \kappa_{j_1}, \dots, (\exists p_{i_n} \varepsilon \kappa_{j_n})p_{i_n} \varepsilon \kappa_{j_n}, \\ (\exists p_i \varepsilon \kappa)p_i \varepsilon \kappa, (\forall p_{i_1} \varepsilon \kappa_{j_1})p_{i_1} \varepsilon \mu_1, \dots, \\ (\forall p_{i_n} \varepsilon \kappa_{j_n})p_{i_n} \varepsilon \mu_n, (\forall p_i \varepsilon \kappa)p_i \varepsilon \mu \\ \vdash \sim(\forall p_{i_1} \varepsilon \kappa_{j_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{j_n})(\forall p_j \varepsilon \kappa)(\forall p_k \varepsilon \kappa)A \end{aligned}$$

and $\kappa_{j_1}, \dots, \kappa_{j_n}$ are distinct atomic mass terms not occurring in Γ , μ_1, \dots, μ_n , or A , then

$$\Gamma \vdash (\forall p_{i_1} \varepsilon \mu_1) \dots (\forall p_{i_n} \varepsilon \mu_n)(\forall p_i \varepsilon \kappa) \sim p_i \left\{ \begin{matrix} P_j \\ P_k \end{matrix} \right. A.$$

Proof by (Refl E) and Lemma 1.7. □

The discussion of formal semantics will be postponed until second-order quantification theory has been described. It will turn out that for every statement of the first-order logic of mass terms there is an equivalent statement of the second-order logic of mass terms of a certain kind (basically, without first-order quantifiers). The details of the formal semantics then naturally flow from this equivalence.

2. MONADIC SECOND-ORDER LOGIC OF MASS TERMS

The second-order logic of mass terms is of interest for two reasons. Firstly, there does not seem to be much use for first-order theories by themselves. Interesting theories, such as topology, require second-order logic, at least the monadic part of second-order logic. Moreover, monadic second-order logic of mass terms is frequently appealed to in the philosophical literature under the title of *mereology*. It appears to be of some interest therefore to identify the proper place of mereology in the space of logical theories.

The second reason for investigating the second-order logic of mass terms is to arrive at semantics for the first-order logic of mass terms. It can be shown that first-order statements can in a sense be reduced to second-order statements. For every first-order statement there exists an equivalent second-order statement that is pure in the sense of containing first-order quantifiers only attached to atomic predicates. This means that the semantics for first-order logic can be derived from second-order semantics, the latter being straightforwardly classical.

Only the monadic part of second-order logic will be developed. The basic second-order statements have the form

$$(\Pi \alpha \subseteq \mu) A \quad \text{and} \quad (\Sigma \alpha \subseteq \mu) A.$$

The variables α, β etc. belong to the same category as mass terms or as quantity terms, depending on what μ is taken to be. This means that the systematic ambiguity explicated in Section 1 is carried over into the second-order language L_2 . Evidently, the ambiguity does not affect the truth conditions of statements.

The intended meaning of $(\Pi \alpha \subseteq \mu) A$ (of $(\Sigma \alpha \subseteq \mu) A$) is that A is true of every (of some) *non-null* sub-quantity of the quantity μ . It is undoubtedly easier to convey this intention in the language of quantity terms rather than mass terms. For if in ordinary language first-order mass quantification is rare, second-order mass quantification is rarer. The examples below may well look contrived. First existential quantification: The word ‘some’, attached to a mass term, appears at times to indicate first-order quantification, at other times second-order quantification. In

Some sand is wet

‘some’ has the role of the first-order ‘ \exists ’ and the sentence is symbolised

$$(\exists p \varepsilon \text{ sand}) \text{wet } p.$$

But the sentence

Some sand is, all of it, wet

which has the same content, has to be construed differently. The word ‘all’ must indicate first-order quantification and ‘it’ must be a pronoun standing in for a mass term. For the sentence to be true,

$$(\forall p \varepsilon \mu) \text{ wet } p$$

has to be true, where μ is some sand. If we take α to be a variable in the grammatical category of mass terms, we can write

$$\text{Some sand } \alpha: (\forall p \varepsilon \alpha) \text{ wet } p$$

and here ‘some’ must be a second-order and not a first-order quantifier. Hence

$$(\Sigma \alpha \subseteq \text{sand})(\forall p \varepsilon \alpha) \text{ wet } p.^2$$

The Language L₂

L₂ is an extension of L₁ which provides for second-order variables and for second-order quantifiers which bind those variables.

Symbols

1. Denumerably many second-order variables: $\alpha_1, \alpha_2, \dots$;
2. the second-order quantifiers Π and Σ , the latter a defined symbol.

Formation Rules

The following rules (L2.Π) and (L2.v) are added to the formation rules (L1.i)–(L1.C) of L₁.

(L2.Π) If A is a formula, μ a mass term, and α a second-order variable, then $(\Pi \alpha \subseteq \mu) A$ is a formula.

(L2.v) Second-order variables are mass terms.

$(\Sigma \alpha \subseteq \mu) A$ is introduced as an abbreviation of $\sim(\Pi \alpha \subseteq \mu) \sim A$. A free occurrence of a second-order variable is one not bound by a second-order quantifier. A formula of L₂ counts as closed (as a statement) if it does not contain free occurrences of first-order or second-order variables. A *closed* mass term is one that does not contain free occurrences of second-order variables.³

Pure Statements

The main result to be proved in this section is that any statement of L₂ is equivalent in L₂ to what is to be called a *pure statement* of L₂.

DEFINITION 2.1. A *pure statement* is a closed formula of L_2 which

- (a) does not contain complex mass terms;
- (b) is such that within the scope of a first-order quantifier ($\forall p \varepsilon \mu$) there are no connectives or operators except, possibly, other first-order quantifiers.

Provability in L_2

To the clauses in Section 1 we add second-order quantifier introduction and elimination clauses.

- (Π) If $\Gamma, (\exists p \varepsilon \kappa)p \varepsilon \kappa, (\forall p \varepsilon \kappa)p \varepsilon \mu \vdash A(\kappa/\alpha)$,
then $\Gamma \vdash (\Pi \alpha \subseteq \mu) A$, provided that
- (a) $A(\kappa/\alpha)$ results from A by replacing every free occurrence of α by κ ;
 - (b) κ does not occur in $(\Pi \alpha \subseteq \mu) A$;
 - (c) κ does not occur in Γ .
- (ΠE) If $\Gamma \vdash (\Pi \alpha \subseteq \mu) A$,
then $\Gamma, (\exists p \varepsilon \mu')p \varepsilon \mu', (\forall p \varepsilon \mu')p \varepsilon \mu \vdash A(\mu'/\alpha)$, provided that
- (a) μ and μ' are closed mass terms;
 - (b) $A(\mu'/\alpha)$ results from A by replacing every free occurrence of α by μ' .

Derived Rules for the Second-Order Existential Quantifier

- (ΣI) If $\Gamma \vdash (\forall p \varepsilon \mu')p \varepsilon \mu, \Gamma \vdash (\exists p \varepsilon \mu')p \varepsilon \mu'$, and $\Gamma \vdash A(\mu'/\alpha)$,
then $\Gamma \vdash (\Sigma \alpha \subseteq \mu) A$, provided that
- (a) μ and μ' are closed mass terms;
 - (b) $A(\mu'/\alpha)$ results from A by replacing every free occurrence of α by μ' .
- (ΣE) If $\Gamma \vdash (\Sigma \alpha \subseteq \mu) A$ and $\Gamma, (\exists p \varepsilon \kappa)p \varepsilon \kappa, (\forall p \varepsilon \kappa)p \varepsilon \mu, A(\kappa/\alpha) \vdash C$,
then $\Gamma \vdash C$, provided that
- (a) $A(\kappa/\alpha)$ results from A by replacing every free occurrence of α by κ ;
 - (b) κ does not occur in $(\Sigma \alpha \subseteq \mu) A$;
 - (c) κ does not occur in Γ .

Comprehension Principle

Characteristic of higher-order logic are comprehension principles, which bring complex predicates into the range of second-order quantifiers. Given the present understanding of the second-order quantifiers, the appropriate principle for the logic of mass terms is

$$(Compr) \vdash (\exists p \varepsilon \mu)A \supset (\Sigma \alpha \subseteq \mu)(\forall p \varepsilon \mu)(p \varepsilon \alpha \equiv A)$$

If A has no more than one free variable then the appropriate instance of this schema is a theorem of the second-order logic of mass terms because of the presence of complex mass terms in the languages L_1 and L_2 . By (*Compl I*) and (*Compl E*) one obtains

$$\vdash (\forall p \varepsilon \mu)(p \varepsilon \{\mu q \mid A(q/p)\} \equiv A)$$

The instance of the Comprehension Principle then follows by (ΣI).

The next theorem lists important consequences of the Comprehension Principle.

- THEOREM 2.1. (a) $(\exists p \varepsilon \mu)A \dashv\vdash (\Sigma \alpha \subseteq \mu)(\forall p \varepsilon \alpha)A$
 (b) $(\forall p \varepsilon \mu)A \dashv\vdash (\Pi \alpha \subseteq \mu)(\exists p \varepsilon \alpha)A$
 (c) $(\exists p \varepsilon \mu)A \dashv\vdash (\Sigma \alpha \subseteq \mu)(\Pi \beta \subseteq \alpha)(\exists p \varepsilon \beta)A$
 (d) $(\forall p \varepsilon \mu)A \dashv\vdash (\Pi \alpha \subseteq \mu)(\Sigma \beta \subseteq \alpha)(\forall p \varepsilon \beta)A$

By Theorem 1.2 and the second-order quantifier rules one obtains the following theorem.

- THEOREM 2.2. (a) $(\forall p \varepsilon \mu)A \dashv\vdash (\Pi \alpha \subseteq \mu)(\forall p \varepsilon \alpha)A$
 (b) $(\exists p \varepsilon \mu)A \dashv\vdash (\Sigma \alpha \subseteq \mu)(\exists p \varepsilon \alpha)A$
 (c) $(\Pi \alpha \subseteq \mu)(\Sigma \beta \subseteq \alpha)A \dashv\vdash (\Pi \alpha \subseteq \mu)(\Sigma \beta \subseteq \alpha)(\Pi \gamma \subseteq \beta)(\Sigma \delta \subseteq \gamma)A(\delta/\beta)$
 (d) $(\Pi \alpha \subseteq \Delta)(\exists q \varepsilon \alpha)(\exists r \varepsilon \alpha)A \dashv\vdash (\Pi \alpha \subseteq \Delta)(\Sigma \beta \subseteq \alpha)(\Pi \gamma \subseteq \beta)(\exists q \varepsilon \gamma)(\exists r \varepsilon \gamma)A$.

Second-order quantification also helps to clarify the significance of reflexivisation.

- THEOREM 2.3. (a) $(\exists p \varepsilon \mu)p \left\{ \begin{smallmatrix} q \\ r \end{smallmatrix} \right. A \vdash (\Sigma \alpha \subseteq \mu)(\forall q \varepsilon \alpha)(\forall r \varepsilon \alpha)A$
 (b) $(\Sigma \alpha \subseteq \mu)(\forall q \varepsilon \alpha)(\forall r \varepsilon \alpha)A \vdash (\exists p \varepsilon \mu)p \left\{ \begin{smallmatrix} q \\ r \end{smallmatrix} \right. A$
Proof. (a) By (*Refl E*) and second-order quantifier rules.
 (b) By (*Refl I*) and Theorem 2.1. □

THEOREM 2.4 (Substitution of co-referential mass terms). *If* $\Gamma \vdash \mu = \mu'$, *then* $\Gamma \vdash A(\mu'//\mu) \equiv A$.

First-Order, Second-Order Equivalences

LEMMA 2.5. *Suppose q is free for p in A . Then*

- (a) $(\Pi \alpha \subseteq \{\mu p \mid A\})B \dashv\vdash (\Pi \alpha \subseteq \mu)((\forall p \varepsilon \alpha)A \supset B)$
- (b) $(\Sigma \alpha \subseteq \{\mu p \mid A\})B \dashv\vdash (\Sigma \alpha \subseteq \mu)((\forall p \varepsilon \alpha)A \& B)$

THEOREM 2.6. *Let B and C be formulae of L_2 , not containing complex mass terms. Then*

- (a) $(\forall p \varepsilon \mu) \sim B \dashv\vdash (\Pi \alpha \subseteq \mu) \sim (\forall p \varepsilon \alpha) B$
- (b) $(\forall p \varepsilon \mu)(B \& C) \dashv\vdash (\forall p \varepsilon \mu) B \& (\forall p \varepsilon \mu) C$
- (c) $(\forall p \varepsilon \mu)(B \supset C) \dashv\vdash (\Pi \alpha \subseteq \mu)((\forall p \varepsilon \alpha) B \supset (\forall p \varepsilon \alpha) C)$
- (d) $(\forall p \varepsilon \mu) p \left\{ \begin{smallmatrix} q \\ r \end{smallmatrix} \right. B \dashv\vdash (\Pi \alpha \subseteq \mu)(\Sigma \beta \subseteq \alpha)(\forall q \varepsilon \beta)(\forall r \varepsilon \beta) B$
- (e) $(\forall p \varepsilon \mu)(\forall q \varepsilon \mu') B \dashv\vdash (\forall q \varepsilon \mu')(\forall p \varepsilon \mu) B$
- (f) $(\forall p \varepsilon \mu)(\Pi \alpha \subseteq \mu') B \dashv\vdash (\Pi \alpha \subseteq \mu')(\forall p \varepsilon \mu) B$, *provided α is not free in μ .*

Proof. (a) By Theorem 2.1(b).

(c) By Theorem 2.2(a), and by Theorem 2.1(b) and Lemma 1.1.

(d) By Theorem 2.1(b) and Theorem 2.3. \square

The equivalences of Theorem 2.6 imply the following generalisations from one universal mass quantifier to a sequence of such quantifiers.

THEOREM 2.7. *Let B and C be formulae of L_2 , not containing complex mass terms. Then*

- (a) $(\forall p_{i_1} \varepsilon \mu_{j_1}) \dots (\forall p_{i_n} \varepsilon \mu_{j_n}) \sim B \dashv\vdash$
 $(\Pi \alpha_{k_1} \subseteq \mu_{j_1}) \dots (\Pi \alpha_{k_n} \subseteq \mu_{j_n}) \sim (\forall p_{i_1} \varepsilon \alpha_{k_1}) \dots (\forall p_{i_n} \varepsilon \alpha_{k_n}) B$,
where $\alpha_{k_1}, \dots, \alpha_{k_n}$ do not occur in B .
- (b) $(\forall p_{i_1} \varepsilon \mu_{j_1}) \dots (\forall p_{i_n} \varepsilon \mu_{j_n})(B \& C) \dashv\vdash$
 $(\forall p_{i_1} \varepsilon \mu_{j_1}) \dots (\forall p_{i_n} \varepsilon \mu_{j_n}) B \& (\forall p_{i_1} \varepsilon \mu_{j_1}) \dots (\forall p_{i_n} \varepsilon \mu_{j_n}) C$.
- (c) $(\forall p_{i_1} \varepsilon \mu_{j_1}) \dots (\forall p_{i_n} \varepsilon \mu_{j_n})(B \supset C) \dashv\vdash (\Pi \alpha_{k_1} \subseteq \mu_{j_1}) \dots$
 $(\Pi \alpha_{k_n} \subseteq \mu_{j_n})((\forall p_{i_1} \varepsilon \alpha_{k_1}) \dots (\forall p_{i_n} \varepsilon \alpha_{k_n}) B$
 $\supset (\forall p_{i_1} \varepsilon \alpha_{k_1}) \dots (\forall p_{i_n} \varepsilon \alpha_{k_n}) C)$,
where $\alpha_{k_1}, \dots, \alpha_{k_n}$ do not occur in B or C .
- (d) $(\forall p_{i_1} \varepsilon \mu_{j_1}) \dots (\forall p_{i_n} \varepsilon \mu_{j_n}) p_{i_1} \left\{ \begin{smallmatrix} p_j \\ p_k \end{smallmatrix} \right. B \dashv\vdash (\Pi \alpha_{k_1} \subseteq \mu_{j_1}) \dots$
 $(\Pi \alpha_{k_n} \subseteq \mu_{j_n})(\Sigma \alpha_{l_1} \subseteq \alpha_{k_1}) \dots (\Sigma \alpha_{l_n} \subseteq \alpha_{k_n})$
 $(\forall p_j \varepsilon \alpha_{l_1})(\forall p_k \varepsilon \alpha_{l_1})(\forall p_{i_2} \varepsilon \alpha_{l_2}) \dots (\forall p_{i_n} \varepsilon \alpha_{l_n}) B$,
where $\alpha_{k_1}, \dots, \alpha_{k_n}$ and $\alpha_{l_1}, \dots, \alpha_{l_n}$ do not occur in B .
- (e) $(\forall p_{i_1} \varepsilon \mu_{j_1}) \dots (\forall p_{i_n} \varepsilon \mu_{j_n})(\forall q \varepsilon \mu') B \dashv\vdash$
 $(\forall q \varepsilon \mu')(\forall p_{i_1} \varepsilon \mu_{j_1}) \dots (\forall p_{i_n} \varepsilon \mu_{j_n}) B$.

- (f) $(\forall p_{i_1} \varepsilon \mu_{j_1}) \dots (\forall p_{i_n} \varepsilon \mu_{j_n})(\Pi \alpha \subseteq \mu)B \dashv\vdash$
 $(\Pi \alpha \subseteq \mu)(\forall p_{i_1} \varepsilon \mu_{j_1}) \dots (\forall p_{i_n} \varepsilon \mu_{j_n})B,$
provided α is not free in $\mu_{j_1}, \dots, \mu_{j_n}$.

Identity

As in standard predicate logic, the second-order quantifiers permit definition of the first-order identity relation.

DEFINITION 2.2. $p = q$ for: $(\Pi \alpha \subseteq \Delta)(p \varepsilon \alpha \equiv q \varepsilon \alpha)$.

The identity relation has the usual higher-order properties of symmetry and transitivity,

$$p = q \vdash q = p$$

and

$$p = q, q = r \vdash p = r.$$

Reflexivity, however, does not necessarily obtain. The usual proof of reflexivity from symmetry and transitivity or directly from the definition is not available, since it involves identification of argument places. In fact, the identity relation is reflexive in a domain Δ , i.e.

$$(\forall p \varepsilon \Delta)p = p,$$

only if that domain is atomic. If Δ is non-atomic (infinitely divisible), then

$$(\forall p \varepsilon \Delta)\sim(p = p).$$

In order to prove these assertions it is necessary to express divisibility, atomicity, and non-atomicity in terms of the formal concepts at hand. Quantity μ is divisible if, for some part α of the domain Δ , some of μ is in α and some is not. Hence

$$(A1) \mu \text{ is divisible iff } (\exists q \varepsilon \mu)(\exists r \varepsilon \mu)(\Sigma \alpha \subseteq \Delta)\sim(q \varepsilon \alpha \equiv r \varepsilon \alpha)$$

$$(A2) \mu \text{ is indivisible iff } (\forall q \varepsilon \mu)(\forall r \varepsilon \mu)(\Pi \alpha \subseteq \Delta)(q \varepsilon \alpha \equiv r \varepsilon \alpha)$$

And Δ is infinitely divisible if each one of its parts is divisible, while Δ is atomic if every part of Δ has itself an indivisible part.

$$(A3) \Delta \text{ is non-atomic (infinitely divisible) iff}$$

$$(\Pi \beta \subseteq \Delta)(\exists q \varepsilon \beta)(\exists r \varepsilon \beta)(\Sigma \alpha \subseteq \Delta)\sim(q \varepsilon \alpha \equiv r \varepsilon \alpha)$$

(A4) Δ is atomic iff

$$(\Pi \beta \subseteq \Delta)(\Sigma \gamma \subseteq \beta)(\forall q \varepsilon \gamma)(\forall r \varepsilon \gamma)(\Pi \alpha \subseteq \Delta) \\ (q \varepsilon \alpha \equiv r \varepsilon \alpha)$$

On the other hand,

$$(\forall p \varepsilon \Delta) \sim (p = p),$$

i.e.

$$(\forall p \varepsilon \Delta) \sim p \left\{ \begin{array}{l} q \\ r \end{array} \right. (\Pi \alpha \subseteq \Delta)(q \varepsilon \alpha \equiv r \varepsilon \alpha)$$

is, by virtue of Theorem 2.6, equivalent to

$$(\Pi \beta \subseteq \Delta) \sim (\Pi \gamma \subseteq \beta)(\Sigma \delta \subseteq \gamma)(\forall q \varepsilon \delta)(\forall r \varepsilon \delta) \\ (\Pi \alpha \subseteq \Delta)(q \varepsilon \alpha \equiv r \varepsilon \alpha)$$

and hence to

$$(\Pi \beta \subseteq \Delta)(\Sigma \gamma \subseteq \beta)(\Pi \delta \subseteq \gamma)(\exists q \varepsilon \delta)(\exists r \varepsilon \delta) \\ (\Sigma \alpha \subseteq \Delta) \sim (q \varepsilon \alpha \equiv r \varepsilon \alpha),$$

which by Theorem 2.2(d) is equivalent to

$$(\Pi \beta \subseteq \Delta)(\exists q \varepsilon \beta)(\exists r \varepsilon \beta)(\Sigma \alpha \subseteq \Delta) \sim (q \varepsilon \alpha \equiv r \varepsilon \alpha),$$

i.e. as claimed, to the infinite divisibility of Δ .

Similarly,

$$(\forall p \varepsilon \Delta) p = p$$

is equivalent by definition to

$$(\forall p \varepsilon \Delta) p \left\{ \begin{array}{l} q \\ r \end{array} \right. (\Pi \alpha \subseteq \Delta)(q \varepsilon \alpha \equiv r \varepsilon \alpha)$$

and hence by Theorem 2.6 to

$$(\Pi \beta \subseteq \Delta)(\Sigma \gamma \subseteq \beta)(\forall q \varepsilon \gamma)(\forall r \varepsilon \gamma)(\Pi \alpha \subseteq \Delta) \\ (q \varepsilon \alpha \equiv r \varepsilon \alpha),$$

i.e. as claimed, to the atomicity of Δ .

In sum,

$$(\forall p \varepsilon \Delta) \sim (p = p)$$

characterises the domain as non-atomic, and

$$(\forall p \varepsilon \Delta)(p = p)$$

as atomic.⁴

Representation by Pure Statements of L_2

It is now possible to prove that every statement of L_2 is equivalent in L_2 to a pure statement of L_2 . As explained before, a *pure statement of L_2* is a closed formula of L_2 which

- (a) does not contain complex mass terms
- (b) is such that within the scope of a first-order quantifier ($\forall p \varepsilon \mu$) there are no connectives or operators except, possibly, other universal first-order quantifiers.

Complex mass terms can be eliminated from first-order quantifiers in L_1 itself with the help of Lemma 1.4 and from second-order quantifiers by Lemma 2.5.

A statement A of L_2 that is free of complex mass terms can be transformed into a pure statement A^P of L_2 with the help of the equivalences assembled in Theorem 2.6.

And the equivalences of Theorem 2.7 allow us to transform any formula A of L_2 which does not contain complex mass terms into an equivalent pure formula A^P . In order to facilitate the recursive transformation we add to L_2 denumerably many further atomic mass terms π_1, π_2, \dots , the resulting language being called L_2^+ . If A is an open formula of L_2 , A^P is in general a formula of L_2^+ ; but if A is a closed formula of L_2 , then A^P does not contain any of the additional mass terms π_i and is therefore a statement, a pure statement, of L_2 .

DEFINITION 2.3. Let A be a formula of L_2 without complex mass terms. Then A^P is recursively defined as follows.

- (P.i) If A is Fp_i , then A^P is $(\forall p_i \varepsilon \pi_i)Fp_i$
- (P.ii) If A is $Rp_i p_j$, then A^P is $(\forall p_i \varepsilon \pi_i)(\forall p_j \varepsilon \pi_j)Rp_i p_j$
- (P.iii) If A is $p_i \varepsilon \mu$, then A^P is $(\forall p_i \varepsilon \pi_i)p_i \varepsilon \mu$
- (P.~) If A is $\sim B$ and p_{i_1}, \dots, p_{i_n} are the variables free in A , then A^P is $(\prod \alpha_{j_1} \subseteq \pi_{i_1}) \dots (\prod \alpha_{j_n} \subseteq \pi_{i_n}) \sim B^P(\alpha_{j_1}/\pi_{i_1}, \dots, \alpha_{j_n}/\pi_{i_n})$, where $\alpha_{j_1}, \dots, \alpha_{j_n}$ do not occur in B^P
- (P.&) If A is $B \& C$, then A^P is $B^P \& C^P$
- (P.⊃) If A is $B \supset C$ and p_{i_1}, \dots, p_{i_n} are the variables free in A , then A^P is $(\prod \alpha_{j_1} \subseteq \pi_{i_1}) \dots (\prod \alpha_{j_n} \subseteq \pi_{i_n}) (B^P(\alpha_{j_1}/\pi_{i_1}, \dots, \alpha_{j_n}/\pi_{i_n}) \supset C^P(\alpha_{j_1}/\pi_{i_1}, \dots, \alpha_{j_n}/\pi_{i_n}))$, where $\alpha_{j_1}, \dots, \alpha_{j_n}$ do not occur in B^P or C^P
- (P.Refl) If A is $p_{i_1} \left\{ \begin{matrix} P_j B \\ p_k \end{matrix} \right.$ and p_{i_1}, \dots, p_{i_n} are the variables free in A , then A^P is $(\prod \alpha_{j_1} \subseteq \pi_{i_1}) \dots (\prod \alpha_{j_n} \subseteq \pi_{i_n}) (\sum \alpha_{k_1} \subseteq \alpha_{j_1}) \dots (\sum \alpha_{k_n} \subseteq \alpha_{j_n}) B^P(\alpha_{k_1}/\pi_{j_1}, \alpha_{k_1}/\pi_{k_1}, \alpha_{k_2}/\pi_{i_2}, \dots, \alpha_{k_n}/\pi_{i_n})$ where $\alpha_{j_1}, \dots, \alpha_{j_n}$ and $\alpha_{k_1}, \dots, \alpha_{k_n}$ do not occur in B^P

- (P.∀) If A is $(\forall p_i \varepsilon \mu)B$, then A^P is $B^P(\mu/\pi_i)$
(P.Π) If A is $(\Pi \alpha_I \subseteq \mu)B$, then A^P is $(\Pi \alpha_I \subseteq \mu)B^P$

Theorem 2.7 guarantees the equivalence of the quantificational closure of A with A^P .

THEOREM 2.8. *Let A be any formula of L_2 without complex mass terms and let p_{i_1}, \dots, p_{i_n} be the variables that are free in A . Then $(\forall p_{i_1} \varepsilon \pi_{i_1}) \dots (\forall p_{i_k} \varepsilon \pi_{i_k})A \dashv\vdash A^P$.*

In particular, given that by Lemmas 1.4 and 2.5 any statement of L_2 is equivalent in L_2 to a statement of L_2 without complex mass terms we have the following result.

THEOREM 2.9. *Let A be any statement of L_2 . Then there exists a pure statement A^P of L_2 which is equivalent to A .*

In a pure statement of L_2 first-order quantifiers occur only with atomic formulae, i.e. in statements of the types $(\forall p \varepsilon \mu)Fp$, $(\forall p \varepsilon \mu)(\forall q \varepsilon \mu')Rpq$, and $(\forall p \varepsilon \mu)p \varepsilon \mu'$. $(\forall p \varepsilon \mu)Fp$ can therefore be regarded as a second-level predicate applied to μ , $(\forall p \varepsilon \mu)(\forall q \varepsilon \mu')Rpq$ and $(\forall p \varepsilon \mu)p \varepsilon \mu'$ as second-level relations applied to μ and μ' . With this understanding any trace of first-order quantification has vanished from pure statements of L_2 . This feature of pure statements will be exploited when it comes to devising semantics for L_1 and L_2 .

EXAMPLE. The L_1 statement

$$(\forall p \varepsilon \Delta)[\sim Fp \supset p \begin{matrix} q \\ r \end{matrix} Rqr]$$

is equivalent to the pure L_2 statement

$$(\Pi \alpha \subseteq \Delta)[(\Pi \beta \subseteq \alpha)\sim(\forall p \varepsilon \beta)Fp \supset (\Pi \beta \subseteq \alpha)(\Sigma \gamma \subseteq \beta)(\forall q \varepsilon \gamma)(\forall r \varepsilon \gamma)Rqr]$$

Second-Order Logic of Mass Terms as Mereology

The terms μ, ρ, \dots , including second-order variables, are being used in a systematically ambiguous way. They can be interpreted either as belonging to the grammatical category of mass terms and as referring distributively to quantities, or as belonging to the category of singular terms and referring collectively to quantities. If one focuses on the second interpretation, the

second-order quantifiers ($\Pi \alpha \subseteq \mu$) can be understood as restricted first-order quantifiers of a certain kind. It is this understanding of the quantifiers which will lead to formal semantics for the first-order logic of mass terms.

It is intuitively clear that the part-of relation is a fundamental relation for quantities. Indeed the part-of relation has played a role in the informal semantics for the logic of mass terms and Definition 1.1 introduces the notation $\mu \subseteq \mu'$ for μ is a part of μ' . It is hardly surprising then to realise that the second-order logic of mass terms is the logic of the part-of relation, i.e. is mereology. By Lemma 1.5(a) every quantity is a part of the domain Δ (i.e. the quantity to which, on any particular interpretation, the term Δ refers). And according to the rules (PII) and (PIE) in Section 1 the quantifier ($\Pi \alpha \subseteq \Delta$) ranges over all subquantities of Δ . The term

$$\{\Delta p \mid \sim(p \varepsilon \Delta)\}$$

does not designate a quantity, as there is no empty quantity. However, if the element to which the term refers is added as null-element, the mereological structure becomes a Boolean algebra with the Boolean operations defined in the following natural way.

DEFINITION 2.4.

- | | |
|--|----------------------|
| (a) $-\mu = \{\Delta p \mid \sim(p \varepsilon \mu)\}$ | <i>Complement</i> |
| (b) $\mu \wedge \mu' = \{\Delta p \mid p \varepsilon \mu \ \& \ p \varepsilon \mu'\}$ | <i>Meet</i> |
| (c) $\mu \vee \mu' = \{\Delta p \mid p \varepsilon \mu \vee p \varepsilon \mu'\}$ | <i>Join</i> |
| (d) $\Lambda\{\alpha \mid A\} = \{\Delta p \mid (\Pi \alpha \subseteq \Delta)(A \supset p \varepsilon \alpha)\}$ | <i>Infinite Meet</i> |
| (e) $\forall\{\alpha \mid A\} = \{\Delta p \mid (\Sigma \alpha \subseteq \Delta)(A \ \& \ p \varepsilon \alpha)\}$ | <i>Infinite Join</i> |
| (f) $1 = \Delta$ | <i>Unit Element</i> |
| (g) $0 = \{\Delta p \mid \sim(p \varepsilon \Delta)\}$ | <i>Null Element</i> |

With the help of the rules for complex mass terms the identities characteristic of Boolean algebras can then be proved. The result is not at all surprising. There exists a complete parallelism with the elementary theory of classes encapsulated in ordinary second-order logic. And as I have pointed out before, classes are related to count nouns in the same way as quantities are related to mass terms.

In any interpretation of L_2 , classically construed, the elements of the domain over which the second-order quantifiers range form a Boolean algebra. The Boolean algebra is generally not complete. But it is of interest to note the following infinite joins which must exist in every interpretation.

LEMMA 2.10. $\vdash \{\mu q \mid A(q/p)\} = \forall\{\alpha \mid \alpha \subseteq \mu \ \& \ (\forall p \varepsilon \alpha)A\} = \mu \wedge \forall\{\alpha \mid (\forall p \varepsilon \alpha)A\}$

Pure Statements of L_2 *Predicates and Relations*

On the interpretation that takes μ, ρ, \dots to be singular terms for quantities the language L_2 has two types of first-order quantifiers, namely $(\forall p \varepsilon \mu)$ and $(\Pi \alpha \subseteq \mu)$. If we now concentrate on the pure statements of L_2 , the ordinary first-order mass-quantifier can be eliminated. For in pure statements of L_2 quantifiers of the form $(\forall p \varepsilon \mu)$ occur only in formulae of the forms $(\forall p \varepsilon \mu)Fp$, $(\forall p \varepsilon \mu)(\forall p \varepsilon \rho)Rpq$ and $(\forall p \varepsilon \mu)p \varepsilon \rho$. $(\forall p \varepsilon \mu)Fp$ attributes a certain property to the quantity μ , namely that all of it is F . And $(\forall p \varepsilon \mu)(\forall p \varepsilon \rho)Rpq$ asserts that μ and ρ stand in a certain relation, namely that all of μ is R -related to all of ρ , while $(\forall p \varepsilon \mu)p \varepsilon \rho$ asserts the relation $\mu \subseteq \rho$.

So, corresponding to any first-order predicate F defined on the domain Δ there is a predicate \mathbf{F} defined for the sub-quantities of Δ as specified in the following definition. There is a similar correlation between relations defined on Δ and relations among the sub-quantities of Δ .

DEFINITION 2.5.

- (a) $\mathbf{F}\alpha$ for: $(\forall p \varepsilon \alpha)Fp$
- (b) $\mathbf{R}\alpha\beta$ for: $(\forall p \varepsilon \alpha)(\forall p \varepsilon \beta)Rpq$

Treating $(\forall p \varepsilon \mu)Fp$ and $(\forall p \varepsilon \mu)(\forall p \varepsilon \rho)Rpq$ as atomic statements $\mathbf{F}\mu$ and $\mathbf{R}\mu\rho$ of the first-order logic of quantities, all trace of first-order mass quantification has disappeared. So, given the result that for any statement of L_2 there exists an equivalent pure statement of L_2 (Theorem 2.9), the meaning of every statement of L_2 can be explained by invoking the familiar semantics of classical first-order logic. The details will be given in the next two sections.

EXAMPLE. The L_1 statement

$$(\forall p_1 \varepsilon \Delta)[\sim Fp_1 \supset p_1 \{ \begin{smallmatrix} p_2 \\ p_3 \end{smallmatrix} R p_2 p_3]$$

is equivalent to the pure L_2 statement

$$(\Pi \alpha \subseteq \Delta)[(\Pi \beta \subseteq \alpha) \sim \mathbf{F}\beta \supset (\Pi \beta \subseteq \alpha)(\Sigma \gamma \subseteq \beta) \mathbf{R}\gamma\gamma]$$

Predicates \mathbf{F} and relations \mathbf{R} of the kind introduced in Definition 2.5 satisfy the following conditions, known as distributive and cumulative reference conditions. Lemma 1.2 and Theorem 2.1 yield proof of these conditions.

LEMMA 2.11. (a) $F\alpha \ \& \ \beta \subseteq \alpha \vdash F\beta$
 (b) $(\Pi \beta \subseteq \alpha)(\Sigma \gamma \subseteq \beta)F\gamma \vdash F\alpha$

LEMMA 2.12. (a) $(R\alpha\beta \ \& \ \alpha' \subseteq \alpha \ \& \ \beta' \subseteq \beta) \vdash R\alpha'\beta'$
 (b) $(\Pi \alpha' \subseteq \alpha)(\Pi \beta' \subseteq \beta)(\Sigma \alpha'' \subseteq \alpha')(\Sigma \beta'' \subseteq \beta')R\alpha''\beta'' \vdash R\alpha\beta$

3. FORMAL SEMANTICS FOR THE LOGIC OF MASS TERMS

Model-theoretic semantics for the logic of mass terms can be formulated on the basis of the systematic translatability of statements of L_1 and L_2 into pure statements of L_2 . The latter can be construed as statements of a first-order language with restricted quantifiers whose truth conditions can be formulated without difficulty. Thereby one arrives at formal semantics for the logic of mass terms. The present section is devoted to first-order semantics; second-order semantics will be dealt with in the next one.

Interpretations

Since the domain of any interpretation constitutes a Boolean algebra, as was shown in Section 2, formal interpretations are based on such algebras. An interpretation $J = \langle A, V \rangle$ consists of a Boolean algebra A and a semantic function V . Frequently only the non-null members of A are considered. These are then referred to as the *positive* members of A . V associates with every atomic mass term κ an element of A , and, in particular, with the term Δ the unit element 1 of A . Further, V associates with every predicate letter F a set $V(F)$ of elements of A , which meets these 2 conditions, reflecting Lemma 2.11.

- (1) If $\alpha \in V(F)$ and $\beta \subseteq \alpha$, then $\beta \in V(F)$
- (2) If, for every positive $\beta \subseteq \alpha$, there exists a positive $\gamma \subseteq \beta$ with $\gamma \in V(F)$, then $\alpha \in V(F)$

And V associates with every relation letter R a set of pairs of elements of A which meets these 2 conditions (cf. Lemma 2.12).

- (3) If $\langle \alpha, \beta \rangle \in V(R)$, $\alpha' \subseteq \alpha$, and $\beta' \subseteq \beta$, then $\langle \alpha', \beta' \rangle \in V(R)$
- (4) If, for every positive $\alpha' \subseteq \alpha$ and positive $\beta' \subseteq \beta$, there exist positive $\alpha'' \subseteq \alpha'$ and $\beta'' \subseteq \beta'$ with $\langle \alpha'', \beta'' \rangle \in V(R)$, then $\langle \alpha, \beta \rangle \in V(R)$

By a *positive sequence* I shall mean an infinite sequence $\sigma = \langle \sigma_1, \sigma_2, \dots \rangle$ whose terms are positive elements of A . $\tau \subseteq \sigma$ is to mean that $\tau_i \subseteq \sigma_i$ for $i = 1, 2, \dots$. The positive sequence $S_{j,k}^i \sigma$ is like σ except that its j -th and k -th elements both equal σ_i . The positive sequence $S_i^j \sigma$ is like σ

except that its i -th element equals σ_j . And the positive sequence $S_i^\rho \sigma$ is like σ except that its i -th element equals ρ . Truth in an interpretation will be defined via reference and satisfaction by positive sequences. I write ' $J \models_\sigma A$ ' for ' σ satisfies formula A in J ' and ' $J(\mu)$ ' for 'the reference of μ in J '.

Satisfaction

The recursive satisfaction clauses are motivated by the clauses of Definition 2.3.

- (J.i) If A is Fp_i , then $J \models_\sigma A$ iff $\sigma_i \in V(F)$
- (J.ii) If A is $Rp_i p_j$, $i \neq j$, then $J \models_\sigma A$ iff $\langle \sigma_i, \sigma_j \rangle \in V(R)$
- (J.ii') If A is $Rp_i p_i$, then $J \models_\sigma A$ iff for every positive sequence $\sigma' \subseteq \sigma$ there exists a positive sequence $\sigma'' \subseteq \sigma'$ such that $\langle \sigma_i'', \sigma_i'' \rangle \in V(R)$
- (J.iii) If A is $p_i \varepsilon \mu$, then $J \models_\sigma A$ iff $\sigma_i \subseteq J(\mu)$
- (J.~) If A is $\sim B$, then $J \models_\sigma A$ iff, for every positive sequence $\sigma' \subseteq \sigma$, $J \not\models_{\sigma'} B$
- (J.&) If A is $B \& C$, then $J \models_\sigma A$ iff $J \models_\sigma B$ and $J \models_\sigma C$
- (J.refl) If A is $p_i \left\{ \begin{smallmatrix} P^j \\ P^k \end{smallmatrix} B \right.$, then $J \models_\sigma A$ iff for every positive sequence $\sigma' \subseteq \sigma$ there exists a positive sequence $\sigma'' \subseteq \sigma'$ such that $J \models_{S_{j,k}^{\sigma''}} B$
- (J.∀) If A is $(\forall p_i \varepsilon \mu)B$, then $J \models_\sigma A$ iff $J \models_{S_i^\delta \sigma} B$, where $\delta = J(\mu)$

Reference

The clause for complex mass terms reflects Lemma 2.10.

- (J.atom) If μ is an atomic mass term κ , then $J(\mu) = V(\kappa)$
- (J.compl) If μ is $\{\rho p_i \mid A\}$, then $J(\mu) = J(\rho) \wedge V\{\alpha \mid J \models_{S_i^\rho \langle 1,1,\dots \rangle} A\}$

Infinite Joins

Note that (J.compl) presupposes that certain infinite unions of elements of the Boolean algebra A exist. Explicitly:

- (J.V) All infinite joins $V\{\rho \mid J \models_{S_i^\rho \langle 1,1,\dots \rangle} A\}$ exist, where A has no free variables besides p_i and ρ ranges over the elements of A .

This means that the semantics do not amount to a recursive definition of satisfaction and reference given arbitrary choices of A and V . Rather, the satisfaction and reference clauses form part of the characterisation of interpretations. The condition that certain infinite joins exist indirectly constrains A and V .

Truth

A statement A is true for an interpretation $J = \langle A, V \rangle$, $J \models A$, iff $J \models_{\langle 1,1,\dots \rangle} A$.

Validity

A statement A is valid, $\models A$, iff $J \models A$ for every interpretation J .

Logical Consequence

$\Gamma \models A$ iff in every interpretation $J = \langle A, V \rangle$, for every positive sequence σ , if $J \models_{\sigma} B$ for every $B \in \Gamma$, then $J \models_{\sigma} A$.

Lemmas about Positive Infinite Sequences

With the formal definition of semantic consequence in place, the soundness and completeness of the system of inference rules for the first-order logic of mass terms introduced in Section 1 can be established. First, a number of lemmas which will be required.

LEMMA 3.1. (a) If $J \models_{\sigma} A$, then, for every positive $\sigma' \subseteq \sigma$, $J \models_{\sigma'} A$;
 (b) If, for every positive $\sigma' \subseteq \sigma$, there exists a positive $\sigma'' \subseteq \sigma'$ with $J \models_{\sigma''} A$, then $J \models_{\sigma} A$.

Proof by induction on the complexity of A , using the satisfaction clauses. \square

LEMMA 3.2. Let A be a statement in which p_i does not occur free. Then

$$J \models_{\sigma} A \quad \text{if and only if} \quad J \models_{S_i^{\delta} \sigma} A,$$

where δ is any element of the Boolean algebra A .

LEMMA 3.3. Let A be a formula; p_j a variable which is different from p_i , is free for p_i in A , and does not occur free in A ; let $A(p_j/p_i)$ be the result of substituting p_j for all free occurrences of p_i in A . Then $J \models_{\sigma} A(p_j/p_i)$ iff $J \models_{S_i^j \sigma} A$.

Soundness

THEOREM 3.4 (Soundness). Let A and the members of Γ be formulae of L_1 . Then $\Gamma \models A$ if $\Gamma \vdash A$.

The *proof* is lengthy and uneventful. I present here just the parts dealing with the quantification rules and reflexivisation.

(\forall I) Assume that $\Gamma, p_j \varepsilon \mu \models A(p_j/p_i)$ and that the applicability conditions of (\forall I) are met. Suppose $J \models_{\sigma} \Gamma$. Then, if $J \models_{\sigma} p_j \varepsilon \mu$, i.e. by

(J.iii) $\sigma_j \subseteq \delta = J(\mu)$, then $J \models_{\sigma} A(p_j/p_i)$, i.e. $J \models_{S_i^j \sigma} A$ by Lemma 3.3. Since p_j does not occur in Γ , $J \models_{S_i^j \sigma} A$, i.e. $J \models_{\sigma} (\forall p_i \varepsilon \mu)A$ by (J.V). So $\Gamma \models (\forall p_i \varepsilon \mu)A$.

($\forall E$) Assume $\Gamma \models (\forall p_i \varepsilon \mu)A$ and suppose that $J \models_{\sigma} \Gamma$ and $J \models_{\sigma} p_j \varepsilon \mu$, i.e. $\sigma_j \subseteq J(\mu)$. Then $J \models_{\sigma} (\forall p_i \varepsilon \mu)A$, i.e. $J \models_{S_i^j \sigma} A$, where δ is $J(\mu)$. $\sigma_j \subseteq \delta$ and so $S_i^j \sigma \subseteq S_i^j \delta$. Hence $J \models_{S_i^j \sigma} A$ by Lemma 3.1 and $J \models_{\sigma} A(p_j/p_i)$ by Lemma 3.3. So $\Gamma, p_j \varepsilon \mu \models A(p_j/p_i)$.

(*Refl I*) Suppose $J \models_{\sigma} (\forall p_j \varepsilon \mu)(\forall p_k \varepsilon \mu)A$, i.e. by (J.V) $J \models_{S_j^j S_k^j \sigma} A$, where $\delta = J(\mu)$; hence $J \models_{S_j^j S_k^j \delta} A$, since p_i is not free in A unless $i = j$ or $i = k$. Then $J \models_{\sigma} (\forall p_i \varepsilon \mu)p_i \{ \begin{smallmatrix} P_j \\ P_k \end{smallmatrix} A$, i.e. by (J.V) $J \models_{S_i^j \sigma} p_i \{ \begin{smallmatrix} P_j \\ P_k \end{smallmatrix} A$, i.e. by (J.Refl) for every $\sigma' \subseteq S_i^j \sigma$ there exists a $\sigma'' \subseteq \sigma'$ such that $J \models_{S_{j,k}^i \sigma''} A$. For suppose $\sigma' \subseteq S_i^j \sigma$ and let σ'' be σ' . Then $S_{j,k}^i \sigma'' \subseteq S_{j,k}^i S_i^j \sigma$. But $S_{j,k}^i S_i^j \sigma = S_j^j S_k^j S_i^j \sigma$. So $S_{j,k}^i \sigma'' \subseteq S_j^j S_k^j S_i^j \sigma$. Hence $J \models_{S_{j,k}^i \sigma''} A$ by Lemma 3.1.

(*Refl E*) Suppose

$$\Gamma, (\exists p_i \varepsilon \kappa)p_i \varepsilon \kappa, (\forall p_i \varepsilon \kappa)p_i \varepsilon \mu \models \sim(\forall p_j \varepsilon \kappa)(\forall p_k \varepsilon \kappa)A,$$

where κ does not occur in Γ , μ , or A . Consider an arbitrary interpretation $J = \langle A, V \rangle$ and suppose that $J \models_{\sigma} \Gamma$. Assume for reductio that $J \not\models_{\sigma} (\forall p_i \varepsilon \mu) \sim p_i \{ \begin{smallmatrix} P_j \\ P_k \end{smallmatrix} A$. Then by (J.V) $J \not\models_{S_i^j \sigma} \sim p_i \{ \begin{smallmatrix} P_j \\ P_k \end{smallmatrix} A$, where δ is $J(\mu)$. Hence by (J. \sim) there exists a positive sequence $\sigma' \subseteq S_i^j \sigma$ such that $J \models_{\sigma'} p_i \{ \begin{smallmatrix} P_j \\ P_k \end{smallmatrix} A$. So by (J.Refl) there exists a positive sequence $\sigma' \subseteq S_i^j \sigma$ such that, for every positive sequence $\sigma'' \subseteq \sigma'$, there is a positive sequence $\tau \subseteq \sigma''$ with $J \models_{S_{j,k}^i \tau} A$. Hence by the transitivity of \subseteq there exists a positive sequence $\tau \subseteq S_i^j \sigma$ such that $J \models_{S_{j,k}^i \tau} A$.

Now consider a second interpretation $J^* = \langle A, V^* \rangle$, where J^* differs from J only in that $J^*(\kappa) = V^*(\kappa) = \rho = \tau_i$, τ_i being the i -th element in the sequence τ . Since κ does not occur in Γ , $J^* \models \Gamma$. Since $\tau \subseteq S_i^j \sigma$, $\rho = \tau_i \subseteq \delta = J(\mu)$ and so $J^* \models_{S_i^j \sigma} p_i \varepsilon \mu$ by (J.iii) and hence $J^* \models_{\sigma} (\forall p_i \varepsilon \kappa)p_i \varepsilon \mu$ by (J.V). Since τ is a positive sequence, $V^*(\kappa)$ is positive and so $J^* \models_{\sigma} (\exists p_i \varepsilon \kappa)p_i \varepsilon \kappa$. By the supposition it follows that $J^* \models_{\sigma} \sim(\forall p_j \varepsilon \kappa)(\forall p_k \varepsilon \kappa)A$, i.e., for every positive sequence $\sigma' \subseteq \sigma$, $J^* \not\models_{\sigma'} (\forall p_j \varepsilon \kappa)(\forall p_k \varepsilon \kappa)A$. Hence, for every positive sequence $\sigma' \subseteq \sigma$, $J^* \not\models_{S_j^j S_k^j \sigma'} A$; and since κ does not occur in A , for every positive sequence $\sigma' \subseteq \sigma$, $J \not\models_{S_j^j S_k^j \sigma'} A$. Let the sequence ν be identical with τ , except that $\nu_i = \sigma_i$. Since $\tau \subseteq S_i^j \sigma$, $\nu \subseteq \sigma$ and so $J \not\models_{S_j^j S_k^j \nu} A$. As p_i is not free in A , unless $i = j$ or $i = k$, $J \not\models_{S_j^j S_k^j \nu} A$ by Lemma 3.2.

But $S_i^\rho v = \tau$ and $S_j^\rho S_k^\rho S_i^\rho v = S_j^\rho S_k^\rho \tau = S_{j,k}^i \tau$, which means that $J \not\models_{S_{j,k}^i \tau} A$, q.e.a. So the assumption that $J \models_\sigma (\forall p_i \mu) \sim p_i \left\{ \begin{smallmatrix} P_j \\ P_k \end{smallmatrix} A \right.$ has to be rejected. Consequently, $J \models_\sigma (\forall p_i \varepsilon \mu) \sim p_i \left\{ \begin{smallmatrix} P_j \\ P_k \end{smallmatrix} A \right.$, and therefore $\Gamma \models (\forall p_i \varepsilon \mu) \sim p_i \left\{ \begin{smallmatrix} P_j \\ P_k \end{smallmatrix} A \right.$. \square

Completeness

The completeness proof involves an adaptation of the well-known methods used in Henkin's proof. A consistent set Θ_0 of statements can be enlarged to a maximally consistent set Θ of a certain kind, and from this an interpretation can be constructed under which all statements in Θ_0 are true. First, the definition of the required maximally consistent set and the interpretation based on it.

DEFINITION 3.1. Let L^+ be an extension of L_1 . Then Θ is a *maximally consistent set of statements* (closed formulae) of L^+ *providing examples* if and only if the following four conditions are met:

- (i) For any statement A , exactly one of $A, \sim A$ is a member of Θ ;
- (ii) If $\Gamma \subseteq \Theta$ and $\Gamma \vdash A$, then $A \in \Theta$;
- (iii) If a statement of the form

$$\sim(\forall p_{i_1} \varepsilon \mu_1) \dots (\forall p_{i_n} \varepsilon \mu_n) A$$

is a member of Θ , there exist atomic mass terms $\kappa_{j_1}, \dots, \kappa_{j_n}$ such that the statements

$$\begin{aligned} &(\exists p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \kappa_{j_1}, \dots, (\exists p_{i_n} \varepsilon \kappa_{j_n}) p_{i_n} \varepsilon \kappa_{j_n}, \\ &(\forall p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \mu_1, \dots, (\forall p_{i_n} \varepsilon \kappa_{j_n}) p_{i_n} \varepsilon \mu_n \end{aligned}$$

and

$$(\forall p_{i_1} \varepsilon \kappa_{j_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{j_n}) \sim A$$

are all members of Θ ;

- (iv) If a statement of the form

$$\sim(\forall p_{i_1} \varepsilon \mu_1) \dots (\forall p_{i_n} \varepsilon \mu_n) (\forall p_i \varepsilon \mu) \sim p_i \left\{ \begin{smallmatrix} P_j \\ P_k \end{smallmatrix} A \right.$$

is a member of Θ , there exist atomic mass terms $\kappa_{j_1}, \dots, \kappa_{j_n}$, and κ such that the statements

$$\begin{aligned} &(\exists p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \kappa_{j_1}, \dots, (\exists p_{i_n} \varepsilon \kappa_{j_n}) p_{i_n} \varepsilon \kappa_{j_n}, \\ &(\exists p_i \varepsilon \kappa) p_i \varepsilon \kappa, \\ &(\forall p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \mu_1, \dots, (\forall p_{i_n} \varepsilon \kappa_{j_n}) p_{i_n} \varepsilon \mu_n, \\ &(\forall p_i \varepsilon \kappa) p_i \varepsilon \mu \end{aligned}$$

and

$$(\forall p_{i_1} \varepsilon \kappa_{j_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{j_n}) (\forall p_j \varepsilon \kappa) (\forall p_k \varepsilon \kappa) A$$

are all members of Θ .

If Θ is so defined, then $A \in \Theta$ if and only if $\Theta \vdash A$.

The Interpretation J^Θ

We construct an interpretation $J^\Theta = \langle A^\Theta, V^\Theta \rangle$ for which all the members of a maximally consistent set Θ which provides examples are true. The elements of the Boolean algebra A^Θ are the equivalence classes of mass terms under the equivalence relation \approx^Θ , defined as follows.

DEFINITION 3.2. If μ and μ' are closed mass terms, then $\mu \approx^\Theta \mu'$ for: $\Theta \vdash \mu = \mu'$.

Let $[\mu]_{\approx^\Theta}$ be the equivalence class of μ for the equivalence relation \approx^Θ . The Boolean operations \neg , \wedge and \vee , and the ordering relation \subseteq on A^Θ are then defined as follows (compare Definition 2.4).

DEFINITION 3.3. Let $[\mu]_{\approx^\Theta}$ and $[\mu']_{\approx^\Theta}$ be the equivalence classes of closed mass terms μ and μ' under \approx^Θ . Then

- (a) $\neg[\mu]_{\approx^\Theta} = [\{\Delta p \mid \sim(p \varepsilon \mu)\}]_{\approx^\Theta}$
- (b) $[\mu]_{\approx^\Theta} \wedge [\mu']_{\approx^\Theta} = [\{\Delta p \mid p \varepsilon \mu \ \& \ p \varepsilon \mu'\}]_{\approx^\Theta} = [\{\mu p \mid p \varepsilon \mu'\}]_{\approx^\Theta}$
- (c) $[\mu]_{\approx^\Theta} \vee [\mu']_{\approx^\Theta} = [\{\Delta p \mid p \varepsilon \mu \ \vee \ p \varepsilon \mu'\}]_{\approx^\Theta}$
- (d) $[\mu]_{\approx^\Theta} \subseteq [\mu']_{\approx^\Theta}$ iff $\Theta \vdash (\forall p \varepsilon \mu) p \varepsilon \mu'$

The unit element 1^Θ of the Boolean algebra is $[\Delta]_{\approx^\Theta}$, the null element 0^Θ is $\neg[\Delta]_{\approx^\Theta}$. Hence $\mu \in 1^\Theta$ if and only if $\Theta \vdash (\forall p \varepsilon \Delta) p \varepsilon \mu$ and $\mu \in 0^\Theta$ if and only if $\Theta \vdash (\forall p \varepsilon \Delta) \sim(p \varepsilon \mu)$ or, equivalently, if and only if $\sim(\exists p \varepsilon \mu) p \varepsilon \mu$.

The interpretation J^Θ is completed by stipulating that, for every atomic mass term κ , $V^\Theta(\kappa) = [\kappa]_{\approx^\Theta}$; for every predicate letter F , $[\mu]_{\approx^\Theta} \in V^\Theta(F)$ iff $\Theta \vdash (\forall p \varepsilon \mu) Fp$; and, for every relation letter R , $\langle [\mu]_{\approx^\Theta}, [\mu']_{\approx^\Theta} \rangle \in V^\Theta(R)$ iff the $\Theta \vdash (\forall p \varepsilon \mu) (\forall q \varepsilon \mu') Rpq$.

Having specified the interpretation J^Θ , it needs to be proved that $J^\Theta \models A$ iff $\Theta \vdash A$. A sequence of elements of A^Θ is a sequence of equivalence classes of closed mass terms. It will be convenient to designate a sequence $\langle \sigma_1, \sigma_2, \dots, \sigma_i, \dots \rangle$ of closed mass terms by σ and the sequence $\langle [\sigma_1]_{\approx^\Theta}, [\sigma_2]_{\approx^\Theta}, \dots, [\sigma_i]_{\approx^\Theta}, \dots \rangle$ of their equivalence classes by $[\sigma]_{\approx^\Theta}$. A Θ -positive sequence $[\sigma]_{\approx^\Theta}$ is such that $[\sigma_i]_{\approx^\Theta} \neq 0^\Theta$, i.e. $\Theta \vdash (\exists p \varepsilon \sigma_i) p \varepsilon \sigma_i$, for $i = 1, 2, \dots$; the sequence σ itself will also be called

Θ -positive. $[\sigma']_{\approx\Theta} \subseteq [\sigma]_{\approx\Theta}$ when $\Theta \vdash (\forall p \varepsilon \sigma'_i) p \varepsilon \sigma_i$ for $i = 1, 2, \dots$; in this case I also write $\sigma' \subseteq \sigma$. Finally, I write $J^\Theta \models_\sigma A$ instead of $J^\Theta \models_{[\sigma]_{\approx\Theta}} A$.

DEFINITION 3.4. Let L^+ be an extension of L_1 . Let Θ be a maximally consistent set of statements of L^+ providing examples, σ a Θ -positive sequence of mass terms, and p_{i_1}, \dots, p_{i_n} the free variables in A . Then

$$\Theta \vdash_\sigma A$$

is short for

$$\Theta \vdash (\forall p_{i_1} \varepsilon \sigma_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma_{i_n}) A.$$

LEMMA 3.5. Let L^+ be an extension of L_1 . Let Θ be a maximally consistent set of statements providing examples. Let σ be a positive sequence of closed mass terms of L_1 and let p_{i_1}, \dots, p_{i_n} be the variables free in A . Then

- (a) If $\Theta \vdash_\sigma A$, then for every Θ -positive sequence of mass terms $\sigma' \subseteq \sigma$, $\Theta \vdash_{\sigma'} A$;
- (b) If, for every Θ -positive sequence of closed mass terms $\sigma' \subseteq \sigma$, there exists a Θ -positive sequence of closed mass terms $\sigma'' \subseteq \sigma'$ with $\Theta \vdash_{\sigma''} A$, then $\Theta \vdash_\sigma A$.

Proof. (a) Assume $\Theta \vdash (\forall p_{i_1} \varepsilon \sigma_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma_{i_n}) A$. Let $\sigma' \subseteq \sigma$, i.e. $\Theta \vdash (\forall p \varepsilon \sigma'_i) p \varepsilon \sigma_i$, for $i = 1, 2, \dots$. Then $\Theta \vdash (\forall p_{i_1} \varepsilon \sigma'_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma'_{i_n}) A$ by Lemma 1.2 and the maximal consistency of Θ .

(b) Assume that, for every Θ -positive sequence of closed mass terms $\sigma' \subseteq \sigma$, there exists a Θ -positive sequence of closed mass terms $\sigma'' \subseteq \sigma'$ with $\Theta \vdash (\forall p_{i_1} \varepsilon \sigma''_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma''_{i_n}) A$, and suppose that not $\Theta \vdash (\forall p_{i_1} \varepsilon \sigma_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma_{i_n}) A$. Then $\Theta \vdash \sim (\forall p_{i_1} \varepsilon \sigma_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma_{i_n}) A$, since Θ is maximally consistent. Since Θ provides examples (clause (iii) of Definition 3.1), there are atomic mass terms $\kappa_{j_1}, \dots, \kappa_{j_n}$ such that the following hold.

$$\begin{aligned} \Theta \vdash (\exists p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \kappa_{j_1}; \dots; \Theta \vdash (\exists p_{i_n} \varepsilon \kappa_{j_n}) p_{i_n} \varepsilon \kappa_{j_n} \\ \Theta \vdash (\forall p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \sigma_{i_1}; \dots; \Theta \vdash (\forall p_{i_n} \varepsilon \kappa_{j_n}) p_{i_n} \varepsilon \sigma_{i_n} \end{aligned}$$

and

$$(1) \quad \Theta \vdash (\forall p_{i_1} \varepsilon \kappa_{j_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{j_n}) \sim A$$

Consider the sequence σ' which is identical with σ except that $\sigma'_{i_1} = \kappa_{j_1}, \dots, \sigma'_{i_n} = \kappa_{j_n}$. Then σ' is a Θ -positive sequence of closed mass terms,

$\sigma' \subseteq \sigma$, and there exists, as assumed, a Θ -positive sequence of closed mass terms $\sigma'' \subseteq \sigma'$ so that the following hold:

$$(2) \quad \Theta \vdash (\forall p_{i_1} \varepsilon \sigma_{i_1}'') \dots (\forall p_{i_n} \varepsilon \sigma_{i_n}'') A$$

and

$$\Theta \vdash (\forall p_{i_1} \varepsilon \sigma_{i_1}'') p_{i_1} \varepsilon \kappa_{j_1}, \dots, \Theta \vdash (\forall p_{i_n} \varepsilon \sigma_{i_n}'') p_{i_n} \varepsilon \kappa_{j_n}.$$

But from (1) by (a)

$$(3) \quad \Theta \vdash (\forall p_{i_1} \varepsilon \sigma_{i_1}'') \dots (\forall p_{i_n} \varepsilon \sigma_{i_n}'') \sim A.$$

Since σ'' is a Θ -positive sequence of closed mass terms, (2) and (3) are inconsistent, and therefore Θ is inconsistent, contrary to what was assumed. So, rejecting the supposition, we infer that

$$\Theta \vdash (\forall p_{i_1} \varepsilon \sigma_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma_{i_n}) A. \quad \square$$

LEMMA 3.6. *Let L^+ be an extension of L_1 . Let Θ be a maximally consistent set of statements of L^+ providing examples. Let σ be a Θ -positive sequence of closed mass terms of L^+ and let there be no free variables in A other than p . Then*

$$\mathbb{V}\{\rho\}_{\approx^\Theta} \mid \Theta \vdash (\forall p \varepsilon \rho) A = \{[\Delta q \mid A(q/p)]\}_{\approx^\Theta}.$$

Proof. (a) $\Theta \vdash (\forall \{\Delta q \mid A(q/p)\} p) A$ by Lemma 1.3.

(b) If $\Theta \vdash (\forall p \varepsilon \rho) A$, then $\Theta \vdash (\forall p \varepsilon \rho) p \varepsilon \{\Delta q \mid A(q/p)\}$ by (ΔI) and (Compl I) . \square

THEOREM 3.7. *Let L^+ be an extension of L_1 . Let Θ be a maximally consistent set of statements of L^+ providing examples. Let $J^\Theta = \langle A^\Theta, V^\Theta \rangle$ be the interpretation based on Θ , let σ be a Θ -positive sequence of closed mass terms, and let A be a formula, μ a mass term of L^+ . Then*

- (a) $J^\Theta \models_\sigma A$ iff $\Theta \vdash_\sigma A$;
- (b) $J^\Theta(\mu) = [\mu]_{\approx^\Theta}$.

The *proof*, by simultaneous induction on the complexity of A and of μ , is omitted.

Consequently,

THEOREM 3.8. *Let L^+ be an extension of L_1 . A statement A is true for the interpretation J^Θ based on a maximally complete set Θ of statements of L^+ which provides examples if and only if $\Theta \vdash A$.*

Construction of a Maximally Consistent Set Θ

It remains to be shown that any consistent set of statements of L_1 can be expanded to a maximally consistent set providing examples. By adding to the language L_1 denumerably many atomic mass terms one obtains a language L^+ , whose atomic mass terms are $\kappa_1, \kappa_2, \dots$. Let C_1, C_2, \dots, C_l be an enumeration of all pairs $\langle \langle \kappa_{l_1}, \dots, \kappa_{l_n} \rangle, A \rangle$, where A is a formula of L^+ and the sequence $\langle \kappa_{l_1}, \dots, \kappa_{l_n} \rangle$ of atomic mass terms of L^+ has as many elements as there are free variables in A .

Let $\Gamma = \Gamma_0$ be a consistent set of statements of L_1 . For each l from 1 on, Γ_l is defined as follows, depending on the pair $C_l = \langle \langle \kappa_{l_1}, \dots, \kappa_{l_n} \rangle, A_l \rangle$:

Case 1. A_l is of the form $\sim p_i \left\{ \begin{smallmatrix} P_j \\ p_k \end{smallmatrix} B \right.$. Let $p_{i_1}, \dots, p_{i_{n-1}}$ be the free variables in A_l other than p_i .

Case 1.1. $\Gamma_{l-1} \cup \{(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_{n-1}} \varepsilon \kappa_{l_{n-1}}) (\forall p_i \varepsilon \kappa_{l_n}) \sim p_i \left\{ \begin{smallmatrix} P_j \\ p_k \end{smallmatrix} B \right\}$ is consistent. Then

$$\begin{aligned} \Gamma_l &= \Gamma_{l-1} \\ &\cup \{(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_{n-1}} \varepsilon \kappa_{l_{n-1}}) (\forall p_i \varepsilon \kappa_{l_n}) \sim p_i \left\{ \begin{smallmatrix} P_j \\ p_k \end{smallmatrix} B \right\} \end{aligned}$$

Case 1.2. $\Gamma_{l-1} \cup \{(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_{n-1}} \varepsilon \kappa_{l_{n-1}}) (\forall p_i \varepsilon \kappa_{l_n}) \sim p_i \left\{ \begin{smallmatrix} P_j \\ p_k \end{smallmatrix} B \right\}$ is inconsistent. Then

$$\begin{aligned} \Gamma_l &= \Gamma_{l-1} \\ &\cup \{ \sim (\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_{n-1}} \varepsilon \kappa_{l_{n-1}}) (\forall p_i \varepsilon \kappa_{l_n}) \sim p_i \left\{ \begin{smallmatrix} P_j \\ p_k \end{smallmatrix} B \right\} \\ &\cup \{ (\exists p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \kappa_{j_1}, \dots, (\exists p_{i_{n-1}} \varepsilon \kappa_{j_{n-1}}) p_{i_{n-1}} \varepsilon \kappa_{j_{n-1}}, \\ &\quad (\exists p_i \varepsilon \kappa) p_i \varepsilon \kappa \} \\ &\cup \{ (\forall p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \kappa_{l_1}, \dots, (\forall p_{i_{n-1}} \varepsilon \kappa_{j_{n-1}}) p_{i_{n-1}} \varepsilon \kappa_{l_{n-1}}, \\ &\quad (\forall p_i \varepsilon \kappa) p_i \varepsilon \kappa_{l_n} \} \\ &\cup \{ (\forall p_{i_1} \varepsilon \kappa_{j_1}) \dots (\forall p_{i_{n-1}} \varepsilon \kappa_{j_{n-1}}) (\forall p_j \varepsilon \kappa) \\ &\quad (\forall p_k \varepsilon \kappa) B \}, \end{aligned}$$

where $\kappa_{j_1}, \dots, \kappa_{j_{n-1}}$ and κ are atomic mass terms of L^+ which do not occur in any statement in Γ_{l-1} .

Case 2. A_l is not of the form $\sim p_i \left\{ \begin{smallmatrix} P_j \\ p_k \end{smallmatrix} B \right.$. Let p_{i_1}, \dots, p_{i_n} be the free variables in A_l .

Case 2.1. $\Gamma_{l-1} \cup \{(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n}) A_l\}$ is consistent. Then

$$\Gamma_l = \Gamma_{l-1} \cup \{(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n}) A_l\}$$

Case 2.2. $\Gamma_{l-1} \cup \{(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n}) A_l\}$ is inconsistent. Then

$$\begin{aligned} \Gamma_l = & \Gamma_{l-1} \\ & \cup \{ \sim (\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n}) A_l \} \\ & \cup \{ (\exists p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \kappa_{j_1}, \dots, (\exists p_{i_n} \varepsilon \kappa_{j_n}) p_{i_n} \varepsilon \kappa_{j_n} \} \\ & \cup \{ (\forall p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \kappa_{l_1}, \dots, (\forall p_{i_n} \varepsilon \kappa_{j_n}) p_{i_n} \varepsilon \kappa_{l_n} \} \\ & \cup \{ (\forall p_{i_1} \varepsilon \kappa_{j_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{j_n}) \sim A_l \}, \end{aligned}$$

where $\kappa_{j_1}, \dots, \kappa_{j_n}$ are atomic mass terms of L^+ which do not occur in any statement in Γ_{l-1} . Finally, define Θ as $\bigcup_{l=0}^{\infty} \Gamma_l$.

I omit the proof that Θ , so defined, is consistent.

THEOREM 3.9. *Suppose Γ is a consistent set of statements of L_1 . Then there exists a maximally consistent set Θ of statements of an extension L^+ of L_1 which provides examples, such that $\Gamma \subseteq \Theta$.*

The Completeness Theorem

THEOREM 3.10 (Completeness). *Let A and the members of Γ be statements of L_1 . Then $\Gamma \vdash A$ if $\Gamma \models A$.*

Proof. Suppose not $\Gamma \vdash A$. Then $\Gamma \cup \{\sim A\}$ is consistent. By Theorem 3.9 $\Gamma \cup \{\sim A\} \subseteq \Theta$, where Θ is a maximally consistent set providing examples. By Theorem 3.8 the members of Γ as well as $\sim A$ are true on the interpretation J^Θ . Hence A is not true on J^Θ and $\Gamma \not\models A$. \square

4. SOUNDNESS AND COMPLETENESS OF MONADIC SECOND-ORDER LOGIC OF MASS TERMS

To the formal semantics for the logic of mass terms of Section 4 has to be added a satisfaction clause for the second-order quantifier. For that the notion of a (κ, μ) -variant is needed. Note that while in L_2 a mass term μ may contain occurrences of second-order variables, only closed mass terms have a reference $J(\mu)$ in an interpretation J . All mass terms considered in this section, other than second-order variables themselves, are closed mass terms. Therefore none of the formulae to be dealt with in this section contains free occurrences of second-order variables.

DEFINITION 4.1. Let J be the interpretation $\langle A, V \rangle$, μ a closed mass term and κ an atomic mass term; and let V^* be like V except that $V^*(\kappa)$ is

an element of A with $0 \neq V^*(\kappa) \subseteq J(\mu)$. Then $J^* = \langle A, V^* \rangle$ is a (κ, μ) -variant of J . (Since κ is atomic, $J^*(\kappa) = V^*(\kappa)$. If $J(\mu) = 0$, J has no (κ, μ) -variant.)

With this notion at hand, the satisfaction clause for the second-order quantifier is

(J.Π) $J \models_{\sigma} (\Pi \alpha \subseteq \mu)B$ iff $J^* \models_{\sigma} B(\kappa/\alpha)$ for every (κ, μ) -variant J^* of J , where κ does not occur in $(\Pi \alpha \subseteq \mu)B$.

On the syntactic side we have the two inference rules for the second-order quantifier which were introduced in Section 2.

(ΠI) *If $\Gamma, (\exists p \varepsilon \kappa)p \varepsilon \kappa, (\forall p \varepsilon \kappa)p \varepsilon \mu \vdash A(\kappa/\alpha)$, then $\Gamma \vdash (\Pi \alpha \subseteq \mu)A$, provided that*

- (a) $A(\kappa/\alpha)$ results from A by replacing every free occurrence of α by κ ;
- (b) κ does not occur in $(\Pi \alpha \subseteq \mu)A$;
- (c) κ does not occur in Γ .

(ΠE) *If $\Gamma \vdash (\Pi \alpha \subseteq \mu)A$, then $\Gamma, (\exists p \varepsilon \mu')p \varepsilon \mu', (\forall p \varepsilon \mu')p \varepsilon \mu \vdash A(\mu'/\alpha)$, provided that*

- (a) μ and μ' are closed mass terms;
- (b) $A(\mu'/\alpha)$ results from A by replacing every free occurrence of α by μ' .

The soundness and completeness of the enlarged system of inference rules relative to the semantics for second-order logic can be established quite easily by extending the proofs in the previous section.

Soundness

It needs to be shown that the two rules above are validity preserving.

(ΠI) Assume that

$$\Gamma, (\exists p \varepsilon \kappa)p \varepsilon \kappa, (\forall p \varepsilon \kappa)p \varepsilon \mu \models A(\kappa/\alpha)$$

and that the provisos (a) to (c) are met. Now suppose that $J \models_{\sigma} \Gamma$. Let $J^* = \langle A, V^* \rangle$ be an arbitrary interpretation which differs from J at most in that $0 \neq V^*(\kappa) \subseteq J(\mu)$. Then $J^* \models_{\sigma} \Gamma$, since κ does not occur in Γ ; $J^* \models_{\sigma} (\forall p \varepsilon \kappa)p \varepsilon \mu$ by (J.∀) and (J.iii), since $J^*(\kappa) = V^*(\kappa) \subseteq J(\mu)$; and $J^* \models_{\sigma} (\exists p \varepsilon \kappa)p \varepsilon \kappa$, since $J^*(\kappa) \neq 0$. So, $J^* \models_{\sigma} A(\kappa/\alpha)$. Hence $J^{\#} \models_{\sigma} A$ for any interpretation $J^{\#} = \langle A, V^{\#} \rangle$ which differs from J at most

in that $0 \neq V^\#(\kappa) \subseteq J(\mu)$, i.e. for any (κ, μ) -variant $J^\#$ of J . Hence $J \models_\sigma (\Pi \alpha \subseteq \mu)A$ by (J. Π), and so $\Gamma \models (\Pi \alpha \subseteq \mu)A$.

(ΠE) Assume that $\Gamma \models (\Pi \alpha \subseteq \mu)A$ and that the proviso of (ΠE) is met. Suppose that $J \models_\sigma \Gamma$, $J \models_\sigma (\forall p \varepsilon \mu')p \varepsilon \mu$, i.e. $J(\mu') \subseteq J(\mu)$, and $J \models_\sigma (\exists p \varepsilon \mu')p \varepsilon \mu'$, i.e. $J(\mu') \neq 0$. Then $J \models_\sigma (\Pi \alpha \subseteq \mu)A$, hence $J^* \models_\sigma A(\kappa/\alpha)$, where J^* is the (κ, μ) -variant $\langle A, V^* \rangle$ of J with $V^*(\kappa) = J(\mu')$. Therefore $J \models_\sigma A(\mu'/\alpha)$, and so

$$\Gamma, (\exists p \varepsilon \mu')p \varepsilon \mu', (\forall p \varepsilon \mu')p \varepsilon \mu \models A(\mu'/\alpha).$$

THEOREM 4.1 (Soundness). *Let A and the members of Γ be formulae of L_2 . Then $\Gamma \models A$ if $\Gamma \vdash A$.*

Completeness

The Maximally Consistent Set Θ

The second-order completeness proof is just a modification of the completeness proof of Section 3. To begin with, Definition 3.1 of a *maximally consistent set of statements providing examples* receives an additional clause.

DEFINITION 4.2. Let L^+ be an extension of L_2 . Θ is a *maximally consistent set of statements of L^+ providing examples* if and only if in addition to (i)–(iv) of Definition 3.1 the following condition is met:

(v) If a statement of the form

$$\sim(\forall p_{i_1} \varepsilon \mu_1) \dots (\forall p_{i_n} \varepsilon \mu_n)(\Pi \alpha \subseteq \mu)B,$$

is a member of Θ , there exist atomic mass terms $\kappa_{j_1}, \dots, \kappa_{j_n}$, and κ such that the statements

$$(\exists p_{i_1} \varepsilon \kappa_{j_1})p_{i_1} \varepsilon \kappa_{j_1}, \dots, (\exists p_{i_n} \varepsilon \kappa_{j_n})p_{i_n} \varepsilon \kappa_{j_n},$$

$$(\exists p \varepsilon \kappa)p \varepsilon \kappa,$$

$$(\forall p_{i_1} \varepsilon \kappa_{j_1})p_{i_1} \varepsilon \mu_1, \dots, (\forall p_{i_n} \varepsilon \kappa_{j_n})p_{i_n} \varepsilon \mu_n,$$

$$(\forall p \varepsilon \kappa)p \varepsilon \mu;$$

and

$$(\forall p_{i_1} \varepsilon \kappa_{j_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{j_n}) \sim B(\kappa/\alpha)$$

are all members of Θ .

With L^+ , A^Θ , V^Θ and J^Θ defined as before, we are able to prove the following theorem, which extends Theorem 3.7.

THEOREM 4.2. *Let L^+ be an extension of L_2 . Let Θ be a maximally consistent set of statements of L^+ providing examples. Let $J^\Theta = \langle A^\Theta, V^\Theta \rangle$ be the interpretation based on Θ , let σ be a Θ -positive sequence of closed mass terms, and let A be a formula, μ a closed mass term of L^+ . Then*

- (a) $J^\Theta \models_\sigma A$ iff $\Theta \vdash_\sigma A$;
- (b) $J^\Theta(\mu) = [\mu]_{\approx^\Theta}$.

Proof. To the inductive proof of Theorem 3.7 we need to add consideration of the case in which A is $(\Pi \alpha \subseteq \mu)B$, with p_{i_1}, \dots, p_{i_n} the free variables in B .

- (a) Suppose $\Theta \vdash_\sigma (\Pi \alpha \subseteq \mu)B$, i.e.

$$\Theta \vdash (\forall p_{i_1} \varepsilon \sigma_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma_{i_n})(\Pi \alpha \subseteq \mu)B.$$

Let μ' be any closed mass term with $\Theta \vdash (\exists p \varepsilon \mu')p \varepsilon \mu'$ and $\Theta \vdash (\forall p \varepsilon \mu')p \varepsilon \mu$.

Then by (ΠE) and $(\forall E)$, $(\forall I)$

$$\Theta \vdash (\forall p_{i_1} \varepsilon \sigma_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma_{i_n})B(\mu'/\alpha),$$

i.e. $\Theta \vdash_\sigma B(\mu'/\alpha)$. Therefore by the inductive hypothesis $J^\Theta \models_\sigma B(\mu'/\alpha)$. Let $J^{\Theta*}$ be the interpretation $\langle A^\Theta, V^{\Theta*} \rangle$, where $V^{\Theta*}$ is like V^Θ , except that $V^{\Theta*}(\kappa)$ is $[\mu']_{\approx^\Theta}$. Since $[\mu']_{\approx^\Theta} \subseteq [\mu]_{\approx^\Theta} = J^\Theta(\mu)$ by the inductive hypothesis, $J^{\Theta*}$ is a (κ, μ) -variant of J^Θ . So, $J^{\Theta*} \models_\sigma B(\kappa/\alpha)$ for every (κ, μ) -variant $J^{\Theta*}$ of J^Θ . Hence by $(J.\Pi)$

$$J^\Theta \models_\sigma (\Pi \alpha \subseteq \mu)B.$$

- (b) Suppose not $\Theta \vdash_\sigma (\Pi \alpha \subseteq \mu)B$, i.e.

$$\text{not } \Theta \vdash (\forall p_{i_1} \varepsilon \sigma_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma_{i_n})(\Pi \alpha \subseteq \mu)B.$$

Then, given that Θ is maximally consistent,

$$\Theta \vdash \sim(\forall p_{i_1} \varepsilon \sigma_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma_{i_n})(\Pi \alpha \subseteq \mu)B.$$

Since Θ provides examples (clause (v)) there exists a Θ -positive sequence $\sigma' \subseteq \sigma$ and an atomic mass term κ such that

$$\begin{aligned} \Theta \vdash (\exists p \varepsilon \kappa)p \varepsilon \kappa, \\ \Theta \vdash (\forall p \varepsilon \kappa)p \varepsilon \mu, \end{aligned}$$

and

$$\Theta \vdash (\forall p_{i_1} \varepsilon \sigma'_{i_1}) \dots (\forall p_{i_n} \varepsilon \sigma'_{i_n})\sim B(\kappa/\alpha),$$

i.e. $\Theta \vdash_{\sigma'} \sim B(\kappa/\alpha)$. Hence not $\Theta \vdash_{\sigma'} B(\kappa/\alpha)$, since Θ is maximally consistent. And by the inductive hypothesis $J^\Theta \not\vdash_{\sigma'} B(\kappa/\alpha)$. Since $V^\Theta(\kappa) \neq 0^\Theta$ and $V^\Theta(\kappa) \subseteq J^\Theta(\mu) = [\mu]_{\approx^\Theta}$ (again by the inductive hypothesis), J^Θ itself is a (κ, μ) -variant of J^Θ . Hence by (J. Π) $J^\Theta \not\vdash_{\sigma'} (\Pi \kappa \subseteq \mu)B$. And so $J^\Theta \not\vdash_\sigma (\Pi \kappa \subseteq \mu)B$ by Lemma 3.2. \square

The Construction of Θ

In order to ensure that Θ meets condition (v), the following needs to be added to the description of the construction of the successive sets Γ_l in Section 3.

Case 3. A_l is of the form $(\Pi \alpha \subseteq \mu)B$. Let p_{i_1}, \dots, p_{i_n} be the free variables in B .

Case 3.1. $\Gamma_{l-1} \cup \{(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n})(\Pi \alpha \subseteq \mu)B\}$ is consistent. Then

$$\Gamma_l = \Gamma_{l-1} \cup \{(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n})(\Pi \alpha \subseteq \mu)B\}$$

Case 3.2. $\Gamma_{l-1} \cup \{(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n})(\Pi \alpha \subseteq \mu)B\}$ is inconsistent. Then

$$\begin{aligned} \Gamma_l = & \Gamma_{l-1} \\ & \cup \{\sim(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n})(\Pi \alpha \subseteq \mu)B\} \\ & \cup \{(\exists p_{i_1} \varepsilon \kappa_{j_1})p_{i_1} \varepsilon \kappa_{j_1}, \dots, (\exists p_{i_n} \varepsilon \kappa_{j_n})p_{i_n} \varepsilon \kappa_{j_n}, \\ & \quad (\exists p \varepsilon \kappa)p \varepsilon \kappa\} \\ & \cup \{(\forall p_{i_1} \varepsilon \kappa_{j_1})p_{i_1} \varepsilon \kappa_{l_1}, \dots, (\forall p_{i_n} \varepsilon \kappa_{j_n})p_{i_n} \varepsilon \kappa_{l_n}, \\ & \quad (\forall p \varepsilon \kappa)p \varepsilon \mu\} \\ & \cup \{(\forall p_{i_1} \varepsilon \kappa_{j_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{j_n})\sim B(\kappa/\alpha)\}, \end{aligned}$$

where $\kappa_{j_1}, \dots, \kappa_{j_n}$ and κ are atomic mass terms of L^+ which do not occur in any statement in Γ_{l-1} .

The Consistency of Θ

Finally, the proof that the set of statements Θ obtained by the construction is consistent needs to be supplemented by considering Case 3.2, in which

$$\Gamma_{l-1} \cup \{(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n})(\Pi \alpha \subseteq \mu)B\}$$

is inconsistent, which means that

$$\Gamma_{l-1} \vdash \sim(\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n})(\Pi \alpha \subseteq \mu)B.$$

The assumption that Γ_{l-1} is consistent, but Γ_l inconsistent must be shown to be untenable. By the inconsistency of Γ_l

$$\begin{aligned} &\Gamma_{l-1}, (\exists p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \kappa_{j_1}, \dots, (\exists p_{i_n} \varepsilon \kappa_{j_n}) p_{i_n} \varepsilon \kappa_{j_n}, \\ &(\exists p \varepsilon \kappa) p \varepsilon \kappa, (\forall p_{i_1} \varepsilon \kappa_{j_1}) p_{i_1} \varepsilon \kappa_{l_1}, \dots, \\ &(\forall p_{i_n} \varepsilon \kappa_{j_n}) p_{i_n} \varepsilon \kappa_{l_n}, (\forall p \varepsilon \kappa) p \varepsilon \kappa \\ &\vdash \sim(\forall p_{i_1} \varepsilon \kappa_{j_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{j_n}) \sim B(\kappa/\alpha) \end{aligned}$$

But then, by Lemma 1.7,

$$\begin{aligned} &\Gamma_{l-1}, (\exists p \varepsilon \kappa) p \varepsilon \kappa, (\forall p \varepsilon \kappa) p \varepsilon \mu \\ &\vdash (\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n}) B(\kappa/\alpha) \end{aligned}$$

and by (Π) and ($\forall E$), ($\forall I$)

$$\Gamma_{l-1} \vdash (\forall p_{i_1} \varepsilon \kappa_{l_1}) \dots (\forall p_{i_n} \varepsilon \kappa_{l_n}) (\Pi \alpha \subseteq \mu) B,$$

which means that, contrary to assumption, Γ_{l-1} is inconsistent. So,

THEOREM 4.3. *Suppose Γ is a consistent set of statements of L_2 . Then there exists a maximally consistent set Θ of statements of an extension L^+ of L_2 which provides examples, such that $\Gamma \subseteq \Theta$.*

The Completeness Theorem

By familiar reasoning it can now be inferred that L_2 is complete.

THEOREM 4.4 (Completeness). *Let A and the members of Γ be statements of L_2 . Then $\Gamma \vdash A$, if $\Gamma \models A$.*

Discussion

The completeness result appears to be in conflict with the well-known incompleteness of second-order logic. The discrepancy is easily explained. The familiar incompleteness result presupposes the ‘standard’ interpretation of second-order logic: the second-order quantifiers range over all properties on the domain of individuals or, equivalently, all subsets of the domain. The domain of the interpretation not only is the range of the first-order quantifier, it also determines the range of the second-order quantifier. When the domain is infinite, the range of the second-order quantifiers goes well beyond the totality of properties expressible (the totality of subsets characterisable) in L_1 .

If the ‘non-standard’ interpretation of second-order logic is adopted, the range of the second-order quantifiers is not completely determined by the

domain, the range has to be specified separately, it need not consist of all properties on the domain (all subsets of the domain), but it must include all those properties (subsets) which can be characterised in L_1 . If validity is defined in terms of non-standard interpretations, then second-order logic is complete.

At first glance, the interpretation of second-order quantification offered here corresponds to the ‘standard’ interpretation: the quantifiers range over all sub-quantities of the domain Δ , i.e. every non-null element of the Boolean algebra A . However, if we think of the second-order quantifiers as ranging over *properties* a different picture emerges. A property, we have seen, is a set ϕ of quantities which meets 2 conditions, namely

- (1) If $\alpha \in \phi$ and $\beta \subseteq \alpha$, then $\beta \in \phi$
- (2) If, for every positive $\beta \subseteq \alpha$, there exists a positive $\gamma \subseteq \beta$ with $\gamma \in \phi$, then $\alpha \in \phi$

While for every property F expressible in L_1 there is a quantity of all of the domain that is F , thanks to constraint (J1.x), there is no general requirement that for every property there is a quantity comprising all of the domain that has the property. For the Boolean algebra A need not be complete. This means that properties and quantities are not in complete correspondence and the present reading of the second-order quantifiers amounts to a ‘non-standard’ interpretation; for that reason the monadic second-order logic for mass terms presented here is complete.

In order to formulate the counterpart for mass terms of the *standard interpretation*, one would have to require that the Boolean algebra over whose elements the second-order quantifiers range be complete, i.e. its domain be the completion of A .

NOTES

¹ Semantics for the first-order logic of mass terms were first presented in Roesper, P., ‘Semantics for Mass Terms with Quantifiers’, *Noûs* **17** (1983), 251–265, and Roesper, P., ‘Generalisation of First-Order Logic to Nonatomic Domains’, *J. Symbolic Logic* **50** (1985), 815–838.

² Again, second-order universal quantification cannot easily be rendered in ordinary language.

$$(\exists \text{ sand}, m)(\forall m p) \text{ wet } p$$

might be translated as

Take some sand, no matter which: all of it is wet.

³ Mass terms never contain free occurrences of first-order variables.

⁴ Note that $(\forall p \varepsilon \Delta)(\forall q \varepsilon \Delta)\sim(p=q)$ and $(\forall p \varepsilon \Delta)(\exists q \varepsilon \Delta)p=q$ are equivalent to $(\forall p \varepsilon \Delta)\sim(p=p)$ and $(\forall p \varepsilon \Delta)p=p$, respectively, and can therefore also serve to characterise non-atomic and atomic domains, respectively.

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