Indefinite Divisibility

Jeffrey Sanford Russell

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Abstract

Some hold that the lesson of Russell's paradox and its relatives is that mathematical reality does not form a "definite totality" but rather is "indefinitely extensible". There can always be more sets than there ever are. I argue that certain contact puzzles are analogous to Russell's paradox this way: they similarly motivate a vision of *physical* reality as iteratively generated. In this picture, the divisions of the continuum into smaller parts are "potential" rather than "actual". Besides the intrinsic interest of this metaphysical picture, it has important consequences for the debate over absolute generality. It is often thought that "indefinite extensibility" arguments at best make trouble for mathematical platonists; but the contact arguments show that nominalists face the same kind of difficulty, if they recognize even the metaphysical possibility of the picture I sketch.

> … endure not yet
> A breach, but an expansion.
>
> John Donne,
> "A Valediction: Forbidding Mourning"

1 Extensibility and Divisibility

Sets are supposed to be plenitudinous. It's tempting to express this plenitude like this:

Sets. For any things, there is some set whose members are just those things.

But this is inconsistent. *The sets* are some things, so

There is some set whose members are just the sets.

This article descends from an earlier paper I presented at the 2009 Arché/CSMN graduate conference; this version doesn't share much with the original besides its title. Thanks to Andrew Bacon, Einar Bohn, John Hawthorne, Hartry Field, Kit Fine, Shieva Kleinschmidt, Colin Marshall, Ted Sider, Gabriel Uzquiano, and participants at the California Metaphysics Workshop on the Relation Between Logic and Metaphysics, for comments along the way.

Since no set is its own member,

> There is some set which is distinct from every set.

Contradiction, QED. (Of course, Russell's version bypasses the assumption that no set is a self-member, by considering *the non-self-members* rather than *the sets*.)

There are two main strategies for resolving the inconsistency.[1] The first strategy—which has become mathematical orthodoxy—is to somehow restrict Sets: for instance, it might only hold when the quantifier "for any things" is restricted to things that are not too numerous, or things that all live below some level in the iterative hierarchy of sets. This restriction distinguishes *set-forming* pluralities from the rest: for example, the natural numbers are set-forming; the sets are not. But while we may have grown used to this standard solution, it's unsettling. What can it be that prevents certain things from forming a set? Indeed, we can make perfect sense of a collection—a *set*$^+$ or *proper class*—of all sets. But then it looks like the sets$^+$ are what set theorists were really interested in—they are more deserving of the name "set" than the restricted species that we were calling "sets". We still can't form a set$^+$ of all sets$^+$—but nothing stops us from forming a set^{++} of all sets$^+$—and on it goes. Wherever we put the boundary between the set-forming and otherwise, it looks like an arbitrary restriction. We seem to have given up on the ambitions of set theory as a *general* theory of collections.

The alternative strategy takes these ever-more-inclusive collections seriously: the universe of sets does not form a "definite totality", but rather is *indefinitely extensible* (Dummett 1993, 441). As with the orthodox strategy, Sets is amended so that the domain of "for any things" is less inclusive than that of "there is a set"—or to put it the other way around, the second quantifier is *more* inclusive than the first.

> For any things, there is some$^+$ set whose elements are just those things.

But (on pain of arbitrariness) we can't say the quantifier "some$^+$" is absolutely general, either: we can also say

> For any$^+$ things, there is some^{++} set whose elements are just those things.

And so on. However many things you quantify over, there is always a more inclusive quantifier, "some^{+++++}" and so on, that you might use instead. Thus the proponent of indefinite extensibility rejects *absolute generality*. Very roughly, we can never quantify over absolutely everything there is—for there is always more yet.

[1] There are other options, such as revising classical logic, which I will ignore. Note I'm also setting aside the view of Yablo (2004) that plural comprehension is restricted such that there aren't any such things as *the sets*.

There are serious difficulties here. David Lewis charges the view with incoherence. "Maybe [the defender of indefinite extensibility] replies that some mystical censor stops us from quantifying over absolutely everything without restriction. Lo, he violates his own stricture in the very act of proclaiming it!"(1991, 68). How can one explain the limits of generality without violating those very limits? In order to articulate the view, for present purposes I'll follow a proposal developed by Kit Fine (2006), Oystein Linnebo (2010), and James Studd (2013). It is tempting to describe the formation of sets in temporal or modal terms—the hierarchy is indefinitely *extensible*, rather than indefinitely *extended.* The proposal is that we should take this seriously as describing the *potential* for further sets, involving a notion analogous to modality or tense. It's not clear what this sort of "potential" comes to. (Surely it isn't metaphysical possibility, or literal ordering in time. Fine calls it "postulational possibility", and suggests that it may be primitive.[2]) But even if we don't perfectly understand this idea, let's take it seriously and try to use it.

I'll follow Studd's suggestion that we should describe the potential for ever-expanding ontologies using formal analogues of *tense operators*.[3] For instance, the revised principle of plenitude for sets says

Potential Sets. Always, for any things, there *will be* some set whose elements are just those things.

Again, we don't intend this to literally describe events in time. We are describing some kind of structure of iterative generation. To get a rough picture (and to find models for the purpose of consistency proofs) we can translate these tense operators into quantification over *stages* (or "ontologies"). But we won't take stage-quantification seriously as a reduction of the operators: indeed, there will be too many stages to really quantify over all at once. The key logical point is that these tense-like operators are "prophylactic" (using Meghan Sullivan's term, 2011, 11): they protect us from ontological contamination. There *will* be sets not among the sets there *are* (in this specialized sense of "will").

The plenitude of sets provides one motivation for indefinite extensibility, and thus one line of attack on stable absolute generality—but not the only one. Other things

[2] "I doubt that one can provide an account of [postulational modalities] in essentially different terms—and in this respect, of course, they may be no different from some of the other modalities" (Fine 2006, 33).

[3] In particular, I'll say "always" for Studd's box, "will" for Studd's forward-looking diamond, "will always" for Studd's forward-looking box, and similarly for "was" and "was always". The reason time talk is nicer than possibility talk is that it is bidirectional, which helps us express useful things about the structure of the "stages" of being. (For example, we can express the fact that stages are well-ordered using the schema "If it is not the case that φ and it will be the case that φ, then it will be the case that (φ and it was never the case that φ).") But as it happens I won't do anything that really turns on choosing tense over modality as our template.

offer parallel challenges: cardinal numbers, ordinals, properties, propositions. Each of these entities recommend plenitude principles of their own (such as: any *well-ordered* things have an ordinal) which turn out in their own ways to raise inconsistencies. These inconsistencies can similarly be resolved by invoking some sort of (similarly difficult to explicate) *potential* to extend the universe with yet more entities.

One feature that sets, numbers, properties, and propositions have in common is that they are abstract: *nominalists* reject them all. Someone who thinks that strictly speaking there really aren't any sets or numbers or propositions—and thus rejects their plenitude outright—will not see these indefinite extensibility arguments as any threat to absolute generality. (The nominalist might see this rather as a pretty good *modus tollens*, showing that the platonist's conception is not even coherent.) Thus it's supposed that only the platonist absolutist is vulnerable to this line of attack (see Hellman 2006).

I'll argue that the challenge of indefinite extensibility is not a special feature of abstract reality. Certain conceptions of *physical* objects motivate the view that even the universe of concreta can be indefinitely extensible.[4] In particular, there are puzzles about the structure of continuous physical objects which play a role analogous to the paradoxes of set theory. Natural plenitude principles turn out to be inconsistent when understood straightforwardly. Again, mathematical orthodoxy rejects these principles and replaces them with something less ambitious; but again, the original conception can be rendered consistent using the indefinite extensibilist's tools—and thus rejecting any stable absolute generality.

The case of physical objects is a bit more delicate than the case of sets, because it's contingent whether there really *are* any continuous physical objects of the sort I'll describe. Regions of space-time or fields are reasonable candidates—but perhaps these are not really continuous, at least in the demanding sense I'll put forward. But the mere *possibility* of such things is enough to raise trouble. The dialectic is structurally parallel to Ted Sider's challenge (1993) to Peter van Inwagen's mereological theory (1991). Van Inwagen claims that everything is either an atom or a living thing; Sider points out that insofar as we should think this thesis true at all, we should think it a

[4]The general idea arises from non-standard analysis, and is suggested by Michael Potter:

> For the proposal now under consideration is that we should conceive of the continuum as indefinitely *divisible* in much the same way as the hierarchy is indefinitely extensible, and it seems inevitable that if this idea is thought through it will eventually lead us to abandon the idea that the continuum is a *set* of points at all (2004, 146).

Daniel Nolan also alludes to this:

> If the mathematical realm can be said to be "indefinite" in size, should we say the same about the non-mathematical realm? (2004, 319)

My technical approach starts from John Conway's proper-class sized continuum of "surreal numbers" (Conway 2000 [1976]). More on this later.

necessary truth; moreover, it is incompatible with (non-living) "atomless gunk". Since atomless gunk is a metaphysical possibility, van Inwagen's thesis is at best contingently true, and thus false. Similarly, the proponents of stable absolute generality—"absolutists"—don't mean for us to believe that *it just so happens* that there are objects exhausting absolutely all there is (or will be). If this thesis is true at all, it is necessarily true. The kind of thing we'll consider is incompatible with absolutism—so its mere possibility is incompatible with the necessity of absolutism.

Of course, there are pressing worries about whether indefinite extensibility is even coherent. I'm following a "modalist" strategy for making sense of the view; but many questions remain unanswered, and I think it's inconclusive whether the strategy is successful. Should it fail, and the idea of indefinite extensibility itself turn out to be incoherent, then of course the kinds of things I'll describe aren't possible after all. I'm open to the thought that it will turn out that way. But if it does, the point is equally devastating for the indefinite extensibility of set theory. My main point is that insofar as the plenitude of the mathematical universe challenges absolute generality, so does the possible plenitude of the physical universe. The parity between these two cases still stands even if both sorts of plenitude turn out to be incoherent.

Like in Sider's argument against van Inwagen, the possibility we'll envision involves parts. A physical object might be *indefinitely divisible*: its parts do not form a "definite totality", but rather whatever parts it may have, there will be still more parts not among them. Mere "atomless gunk" of the sort Sider appealed to is not divisible enough for our purposes. We can perfectly well describe an ordinary atomless object by quantifying over a fixed domain of continuum-many parts. To approach the limits of stable absolutely general quantification we must look for something more divisible yet.

Daniel Nolan raises the possibility of what he calls "hypergunk": an object that has arbitrarily large sets of parts (2004). He defends the view that hypergunk is metaphysically possible: it is logically consistent; its existence does not appear to be analytically false or a Kripke-Putnam kind of *a posteriori* impossibility; it can be clearly described in fairly natural terms; and "it may also be a reasonably natural way of spelling out a natural conception of unlimited divisibility" (2004, 307). He points out that some, such as C.S. Peirce, have apparently taken the space of the actual world to be something like this—and if this is a mistake, it seems to be empirical.[5]

Hypergunk does not meet our needs, either. The trouble is that the definition of hypergunk is nominalistically unacceptable, since it is framed in terms of the cardi-

[5] Hazen (2004) provides a proof that hypergunk is consistent, relative to set theory plus a large cardinal axiom. But Hazen also argues that hypergunk is not a genuine possibility, on the grounds that there is necessarily a set of all non-sets. Hazen's argument provides another reason (apart from nominalist scruples) to be interested in direct characterizations of very large continua that aren't parasitic on set theory: perhaps the nature of *sets* guarantees they outrun Nolan's hypergunk, understood set-theoretically.

nalities of sets. Perhaps the definition can be reframed to avoid any appeal to sets, but I don't know how that would go; I won't pursue it. But even though it is encumbered with set-theoretic baggage, there is a close connection between Nolan's hypergunk and some of the possibilities I'll describe. In these possibilities, the parts of continuous objects keep going on and on in the same way as the ordinals—and so, in the presence of set theory, it follows that these objects are in fact hypergunk. But that isn't the path I'll take: my puzzles arise without appeal to sets or ordinals.

(A technicality: I do use plural quantification in my formulations. Nowadays many nominalists are happy to speak plurally, in accord with the teaching of Boolos (1984). But some nominalists still think that's fishy: maybe it amounts to "set theory in sheep's clothing", to use Quine's famous phrase. I don't think anything very important turns on the choice for our purposes: rather than plurals, I think we can say what we need to say using schematic generalizations, which even the austerest nominalists deign to use. But doing things that way raises extra technical complications, which I want to set aside.[6])

Instead of either gunk or hypergunk, we'll look at certain contact puzzles about the structure of continuous physical objects. Nothing which satisfies plausible-seeming contact principles can have all of its parts "exist at once": rather than being *divided* into arbitrarily small parts, we can only make sense of such things as indefinitely *divisible*. If it is really possible for there to be such indefinitely divisible things, then generality absolutism is not necessarily true—and so it is actually false. And aside from worries about absolute generality, these contact puzzles and the picture that emerges from them are interesting in their own right. They motivate non-standard continua that present striking alternative possibilities for the structure of spatio-temporal extension.

2 Things Fall Apart

Here's a simple idea. Some objects are scattered. A cloud of particles consists of many small parts, none of which are in contact with one another. A bikini consists of two separate parts. In contrast, there may be objects that are not scattered but rather continuous. Since the word "continuous" nowadays has a technical meaning, let me introduce some new terms.

> Some things are **scattered** iff no two of them are in contact with one another.
>
> A thing is **weakly cohesive** iff it is not a sum of two scattered parts.

[6] For example, Studd's first-order schematic formulations of modalized set theory require extra complications to ensure that we only consider *rigid* predicates as instances; plurals build in that rigidity automatically.

> A thing is **strongly cohesive** iff it is not a sum of two or more scattered parts.

Strong cohesion is a plausible gloss on continuity—though as we'll see, a problematic one.[7]

(I'll say more about the technical background for this notion of contact later. A point worth noting here: the mereologist's "part" is used in a somewhat extended sense, so things count as parts of themselves. Similarly, our notion of "contact" is somewhat extended, so that things that share a common part count as being in especially intimate contact. I'll also use "*x* touches *y*" as a synonym for "*x* and *y* are in contact".)[8]

You might have thought that a continuous object—such as a region of space—was not divisible into scattered parts. The standard mathematical treatment rejects this thought. According to mathematical orthodoxy, the continuum is made entirely of unextended points. Furthermore, these points are *dense* in the sense that between any two points there is a third—and thus, by applying this principle repeatedly, there are infinitely many points between any two. So points are never in contact with one another, and nothing composed of two or more points is strongly cohesive: the continuum of points is disintegrated.

The claim that extended things are composed of unextended points is perplexing in many ways—such as the ancient paradoxes of measure and continuous change. A rival historical tradition rejects this claim, and maintains that all genuine parts of space (and spatially extended objects) are extended: points, lines, and surfaces are mere abstractions or idealizations.[9] Here's one way of stating the view that everything

[7] The etymology of "continuous" is "held together". Aristotle writes: "continuity belongs to things that naturally in virtue of their mutual contact form a unity" (2008, V.3, pp. 126–7). See Zimmerman (1996b), p. 10: "But if there is a gap—however small—between *every* pair of discrete extended parts of a sphere, then the sphere is nothing but gaps through and through!"

[8] Taking contact as our basic notion is not the mathematically orthodox approach to topology, which instead takes the notion of an open set of points as basic. We want to proceed in a way that is neutral as to the existence of points—or sets. For development of this idea see Clarke (1981); Zimmerman (1996b); Roeper (1997), 255ff.; Russell (2008), 253ff.

[9] For example, Kant writes:

> The property of magnitude by which no part of them is the smallest (no part is simple) is called their continuity. Space and time are *quanta continua*, because no part of them can be given except as enclosed between boundaries (points and instants), thus only in such a way that this part is again a space or a time. Space therefore consists only of spaces, time of times. Points and instants are only boundaries, i.e. mere places of their limitation; … and from mere places … neither space nor time can be composed (Kant 1999 [1781], 292).

For historical discussion see Zimmerman (1996a), as well as the references in the preceding footnote.

is extended. (For simplicity, we restrict our quantifiers in what follows to the parts of continuous physical objects.)

Interiors. Everything has an **interior part**: a part of *x* which is not in contact with anything that does not overlap *x*.

(Overlap means sharing a common part. Given standard assumptions, Interiors is equivalent to principle that there are no mere boundaries; see Russell (2008), p. 254.)

But rejecting unextended parts cannot save continuity, in the sense of strong cohesion. Let Big be an arbitrary thing. We can find scattered parts that together compose Big by going through all of Big's parts one by one. Say Small is a part of Big. If Small doesn't overlap anything we picked earlier, then Small has an interior part, Smaller, which is not in contact with anything we picked earlier. In that case, add such an interior part to our collection. Otherwise, just move on to the next part and keep going. When we're done, we'll have picked out a scattered collection of Big's parts (all of the various Smallers) such that every part of Big overlaps at least one of them. From this it follows that Big is the sum of these scattered parts.[10]

To make this argument precise requires a technical assumption: **Global Choice**, which generalizes the standard Axiom of Choice from *sets* to arbitrary *pluralities*. The nicest version of the proof I know uses the plural form of an elegant equivalent statement of Choice, the Teichmüller-Tukey Lemma. Let "the *X*'s fit" stand in for an arbitrary formula with one plural variable. We say "fitting has **finite character**" iff

For any *X*'s, the *X*'s fit iff any finitely many *Y*'s among the *X*'s fit.

The following principle follows from Global Choice:[11]

[10] This argument uses a similar idea to the Cantor-Forrest-Arntzenius measure argument discussed in Russell (2008). (See also Forrest 1996; Forrest 2007; Arntzenius 2008.) In fact, though I won't take this up here, I believe that the indefinite extensibility approach also offers a promising alternative response to the measure argument.

[11] Here's a sketch of how Global Tukey's Lemma can be derived from Global Choice. First, another technical point: I don't know how to even *state* Global Choice properly without appealing to something like plural quantification over ordered pairs. (The appendix to Lewis 1991 gives a way of reconstructing pairs from purely mereological resources, if there are enough things to go around; see also the discussion of options in Hawthorne and Uzquiano 2011, 60–61.) Let's not fuss over that too much, and assume any two things have a unique pair. Given this, we can also simulate quantification over "functions" from pluralities to individuals, using plural quantification over suitable pairs of individuals. Global Choice says that there are pairs which encode a global choice function, taking each (non-empty) plurality to one of its members.

If there is no maximal fitting plurality, then for any fitting *X*'s there is some further thing, not among the *X*'s, that fits with the *X*'s. Using Global Choice, we can then code a "function" that takes each fitting plurality to some such thing $g(X)$. Then we can use g to recursively construct ever-longer well-ordered

Global Tukey's Lemma. If fitting has finite character, then there is a maximal fitting plurality: that is, some (zero or more) things fit together and do not fit with anything else.

We now prove:

Theorem 1. Given classical mereology and Global Choice, Interiors implies that no composite thing is strongly cohesive.

Proof. Let Big be any composite thing; we'll show that it is a sum of at least two scattered parts. First, "the X's are scattered proper parts of Big" has finite character: indeed, for some things to satisfy this property, it suffices that any *two* of them are scattered proper parts of Big. Then by Global Tukey's Lemma there is a maximal plurality of scattered proper parts of Big: call them the **Scattered Parts**. The maximality condition says that each proper part of Big is in contact with one of the Scattered Parts. It then follows that Big is a sum of the Scattered Parts. Suppose otherwise, so Big has some part Small which doesn't overlap any Scattered Part. Then by Interiors, Small has a part Smaller, which is a part of Big not in contact with any Scattered Part. This nearly finishes the proof; we just need to rule out the edge case where Smaller is not a *proper* part of Big—that is, "Smaller" is Big itself. In that case no Scattered Part overlaps Big, which can only hold if there are zero Scattered Parts. Since Big is composite, it has a proper part, which would then be in contact with no Scattered Part, contradicting their maximality. (And of course there are at least two Scattered Parts: otherwise Big would be a sum of fewer than two proper parts, which is impossible in classical mereology.) □

So, whether things have unextended point parts, or are entirely made of extended parts, they still break down into scattered parts.[12]

This contact puzzle is related to another. Let's return to the orthodox continuum of points. Take any single point. It is not in contact with any other single point—but

sequences of individuals, starting from an empty sequence and using the recurrence relation that each thing in the sequence is given by applying g to all the things before it in the sequence. (These sequences can also be represented by pluralities of pairs.) We can carry out this construction using the idea of the proof of the Recursion Theorem (or Zorn's Lemma, or the Well-Ordering Theorem)—by considering all the various well-ordered sequences of different lengths that obey the recurrence relation, and showing that we can join them up into a maximal sequence of that sort. I won't go into the details here. Then the things that appear in the maximal sequence must fit together—and applying g to them would provide an even longer sequence obeying the recurrence relation. That's a contradiction. So there must be a maximal fitting plurality after all.

[12]These alternatives are not exhaustive—for instance, the "no zero" conception of space defended by Arntzenius (2008) countenances things with neither point parts nor interior parts. But I don't expect that to help with this problem.

it *is* in contact with the sum of *all* other points—there is nothing left between it and them. Thus the orthodox continuum violates this principle:

Distributive Contact. If x touches the sum of the Ys, then x touches some Y.

But while we may have grown used to this odd topological consequence, it's perplexing. While there may be emergent unities which are greater than their parts, a *mere sum* isn't supposed to be like that. A mereological sum is often taken—in some elusive sense—just to *be* its parts.[13] The failure of Distributive Contact is a knock against that idea. In that case there are important topological facts which irreducibly concern the whole. *It* is in contact with things *they* are separated from.

The *finite* version of Distributive Contact is a standard axiom of mereotopology (Roeper 1997, 255, principle A5).

Finitely Distributive Contact. If x touches the sum of y_1 and y_2, then x touches y_1 or x touches y_2.

But what justification might there be for this principle that does not extend to infinite sums? Here's one: the orthodox continuum obeys the finite principle but not the infinite principle, and the orthodox continuum is the best we have. A good argument—but one which is weakened if we discover coherent alternatives.

John Hawthorne (2000) argues against Distributive Contact. (He simply calls it "The Contact Principle".) His argument also assumes the mathematically orthodox structure of the continuum, but in a different, instructive way. (My version here is a variant of his.) Consider two cube blocks in contact with one another, side by side. Since we are in the orthodox continuum of points, let's suppose that one of them is *closed*, including the boundary plane between them, and the other is *open*, excluding it—so there is no overlap, and no intervening empty space between them. Accordingly, call the blocks Open and Closed. Now consider certain parts of Open. The first part consists of all of Open except its half closest to Closed. The second part is all of Open except the third of it closest to Closed. Next, all of Open except the closest quarter. Then all but the closest fifth. And so on. Call these the *Zeno Parts*. Open is the sum of the Zeno Parts: every part of Open sticks out some finite distance from its boundary with Closed, and thus overlaps some Zeno Part—and indeed infinitely many of them. But again, each of the Zeno Parts—say the nth—is still separated from Closed by the remaining $1/n$th of Open. So Closed is in contact with the sum of the Zeno Parts, but not with any particular Zeno Part.

[13] "The fusion is nothing over and above the cats that compose it. It just is them. They just are it. Take them together or take them separately, the cats are the same portion of Reality either way." (Lewis 1991, 81–82; for further discussion see e.g. Hawley 2014). Compare Peter Forrest's (2007) discussion of similar principles about size and overlap.

This difficulty doesn't essentially turn on point parts, or on the difference between open and closed regions. In fact, the same problem arises if we reject those peculiarities and instead maintain that everything has an interior part. Even then, Distributive Contact implies that everything breaks into two separated pieces—an even worse conclusion than the failure of strong cohesion that we considered earlier.

Theorem 2. Given classical mereology, Distributive Contact and Interiors imply that nothing composite is weakly cohesive.

Proof. Let Small be a proper part of Big. Then let Separate be the sum of all the parts of Big that do not touch Small. (There must be at least one such part, since classical mereology ensures that Big has a part disjoint from Small, and this in turn has an interior part.) By Distributive Contact, Small does not touch Separate. Any part of Big which is disjoint from Small has (by Interiors) a part which does not touch Small, which is thus a part of Separate. So every part of Big overlaps either Small or Separate; that is, Big is a sum of Small and Separate, which are not in contact. So Big is not weakly cohesive. □

(We can argue in the same way for a somewhat stronger result, weakening the premise Interiors a bit. Call x **regular** iff: for any two parts of x which are not in contact, there is some further part of x which is in contact with neither of them. The idea is that if two parts of a continuum are separated, then there ought to be something *separating* them, which is strictly between them.[14] Distributive Contact implies, using classical mereology, by the same argument just given, that nothing composite is regular. Theorem 2 follows from this, because Interiors implies that anything weakly cohesive is regular.)

(Note also the close connection between Theorems 1 and 2. In fact, Distributive Contact implies that anything weakly cohesive is also strongly cohesive. If x is the sum of *more* than two scattered parts, then it is also the sum of any one of those parts together with the sum of the rest of them; Distributive Contact implies that these are also not in contact with one another. So assuming Global Choice would also let us derive Theorem 2 as a consequence of Theorem 1. The proof given here has the advantage of not relying on that technical assumption.)

[14] Descartes held something like this, applying it as an argument against the vacuum:

> For, when there is nothing between two bodies, they must necessarily touch each other; and it is manifestly contradictory for them to be apart, or for there to be a distance between them, and yet for this distance to be nothing: because all distance is a mode of extension, and therefore cannot exist without an extended substance (Descartes 1982 [1644], pt. 2, principle 8; quoted in Maudlin 1993).

Is Distributive Contact just a mistake, then? I think that conclusion would be hasty. Return to the case of the Zeno Parts. In the orthodox continuum, it's clear that the sum of these is the complete block Open. This is because a standard continuous line obeys this version of

The Archimedean Principle. If I is any interval part of an interval J, then there is some natural number n such that I is at least $1/n$ times the length of J.

In other words, the standard continuum has no infinitesimal parts. This is what ensures that every part of Open sticks out into at least one of the Zeno Parts.

But this principle may be false, or at least contingent. Non-standard analysis gives us models in which it fails, where there are distances smaller than every ordinary positive fraction. (The Compactness theorem guarantees the existence of such models in which the entire first-order theory of the standard continuum still holds, as Robinson 1996 [1966] exploited.) Could a non-Archimedean continuum avoid the problem, and vindicate Distributive Contact?

Not directly. If Open has infinitesimal parts in addition to the Zeno Parts, we can repeat the argument using those: consider the parts comprising all but an infinitesimal part of Open adjacent to Closed, for each infinitesimal. Then Open will again be composed of these parts, but Closed will touch none of them. To avoid the result that a block can be composed of infinitesimal Zeno-ish parts, we'll need to carve out of the block a remainder even smaller than any of the infinitesimals.

Rather than despair, this suggests a way of escape: there is simply no limit to the smallness of the divisions we can make. When we divide something into Zeno-ish parts, there are always even tinier parts that still remain. The continuum is not merely non-Archimedean, but *indefinitely divisible*.

To get a clearer view of this, consider a continuous line stretched out from left to right. It has intervals as parts. (I'll speak about the left and right *endpoints* of these intervals, as a convenient way of discussing their ordering relationships; but as before, don't think of endpoints as genuine parts of the line. We could instead do things with more complicated ordering relations, like "I is leftwise-left-of J" rather than "the left endpoint of I is left of the left endpoint of J".)

In the standard picture, the continuum is *dense*: each interval is divisible into smaller interval parts. Within any interval I we can find a smaller interval J whose endpoints lie strictly between the endpoints of I. And indeed, for any finitely many *nested* intervals, there is a further interval which is smaller than each of them. This picture partly vindicates Anaxagoras's hypothesis: "Nor is there a least of what is small, but there is always a smaller…" (Burnet 1920, 126; quoted in Sider 1993). Thinking of intervals as the units of smallness, for each one or finitely many of them there is a smaller.

But this is only a partial vindication. For in the standard continuum there is, in a sense, a least of what is small. There are *infinite* sequences of nested intervals which converge on a single point—and thus there is still in this picture an *ideal* "least of what is small" at the limit of a descending chain of parts. (Indeed, these chains of nested intervals reveal an underlying atomism to the continuum we are describing: such chains are a standard way of reconstructing points from pointless topologies; see e.g. Roeper 1997.)

What prevents there from being a further part at the bottom of such an infinite descending chain of intervals? Nothing at all. Indeed, non-standard analysis gives us a model of the continuum which embeds the standard continuum, but which additionally has infinitesimal intervals smaller than each nested chain of standard intervals. So the standard continuum does not divide things as finely as it could.

What if instead of merely a *dense* continuum, there was an *indefinitely* dense continuum, such that any descending chain of nested intervals could be extended with a yet smaller interval? This would be a more complete vindication of Anaxagoras's picture of "no least of what is small"—ever more infinitesimal smallness without end. (Ockham seems to have held a view along these lines, though it's a bit obscure: "I say that if the world would have been from eternity, and God would have in every hour made one division in one continuum, there would not be as yet a complete division." *Quotlibeta Septem*, book I, question 9, quoted in Birch 1936, 503.)

On the face of it, this picture is inconsistent. Consider *all* the intervals which share certain leftmost endpoint. Their right endpoints are linearly ordered, so these are a nested chain of intervals. But no interval can be smaller than all of them: it would have to share the same left edge as they do, and thus be one of those intervals originally surveyed—and it can't be smaller than itself.[15]

But perhaps the problem comes in when we try to survey *all* intervals at once—just as there are problems with surveying all *sets* at once. Perhaps the continuum isn't arbitrarily *divided*, but rather arbitrarily *divisible*. Whatever parts a continuous thing may have, there *will be* smaller parts not yet among them. Can this perspective hold things together, allowing there to be things which aren't entirely composed of scattered

[15] I'll spell out this argument in a bit more detail. Say the Lefties are all the intervals that share a left endpoint with a certain interval I. Suppose J is a proper part of every Lefty. Then since J is part of I, the left endpoint of J must not be further left than I's left endpoint. If J's left endpoint is further right than I's, then there is a Lefty left of J, and J cannot be a part of it. So J is itself a Lefty, and thus not a proper part of every Lefty.

parts, and preventing contact with the whole in the absence of contact with any part?

3 Putting Things Back Together

Yes.

First let's articulate some mereological principles more explicitly. Here's one way of axiomatizing classical mereology.

Core Mereology. Part is reflexive, transitive, and antisymmetric.

Supplementation. If everything that overlaps x overlaps y, then x is part of y.

(Or equivalently: if x is *not* part of y, then some part of y does not overlap x.) And finally:

Sums. (i) x is a sum of the X's iff: every X is part of x, and x is part of anything that has every X as a part.

(ii) Any (one or more) things have a sum.

Note that I'm not taking "sum" to be defined in the usual way, but rather as another primitive to be constrained by principles—because as we go we'll be investigating several rival understandings of sums. Note also that the principle offered here in Sums (i) is not the standard definition. Here I'm using the notion of a *least upper bound*—the smallest thing that has each X as a part. In contrast, let's adopt this standard definition:

> x is a **fusion** of the X's iff for every y, y overlaps x iff y overlaps some X.

One consequence of classical mereology (in particular, Supplementation) is that sums are fusions. In other words, this principle holds:

Distributive Overlap. x overlaps the sum of the X's iff x overlaps some X.

Besides mereology, let's note some standard principles about contact (following Roeper 1997, 255).

Core Topology. Contact is reflexive, symmetric, and monotonic (the last meaning that if x touches a part of y, then x touches y).

(Remember that we are counting overlap as a case of contact, and of course anything shares a part with itself.) In addition to these standard assumptions, we'd like to make some that are less standard.

Distributive Contact. If x touches the sum of the Ys, then x touches some Y.

(As I mentioned earlier, normally only the finite version of this is assumed.)

Interiors. Everything has an interior part.

Cohesion. Something is composite and weakly cohesive.

Recall that Distributive Contact (with mereology) implies that anything weakly cohesive is also strongly cohesive. So I won't always bother to distinguish the two notions.

In the previous section, we showed (by several routes) that these principles taken all together are inconsistent. But the resources of indefinite extensibility give us a way of salvaging them. Several of these principles—in particular, Sums, Supplementation, and Interiors—posit the existence of a certain thing. But perhaps the correct way of stating these principles is not to posit "actual" existence, but merely "potential" existence. Remember, this is how indefinite extensibility tackles Sets: for any things, there *will* be a set. So what if there merely *will* be a sum, a remainder, or an interior part? In fact, any of these three revisions will work on its own, giving rise to three different pictures of indefinitely divisible continua. Let's briefly examine each of them. (Further details and proofs are presented in an appendix.)

The first (and simplest) idea is to replace Interiors with

Potential Interiors. Anything *will* have an interior part.

This principle is consistent with classical mereology together with the other topological principles stated above—including Distributive Contact and Cohesion. There is a simple model. In one dimension, we can think of the continuum as beginning as a single undivided unity; at the next stage it is divided into two adjacent parts; at the next stage, each part is divided again; and so on. There are only countably many stages in this picture, and at each stage there are only finitely many parts. The continuum has a *potential* infinity of parts, but never an *actual* infinity—a broadly Aristotelian picture. Let's call this the **Finitist Model**.

In this picture, there are no such things as the Zeno Parts—not all at once. There *will* eventually be each of the Zeno Parts, but it will never be the case that there are more than finitely many of them. The block is divisible into arbitrarily many parts, but never *infinitely* many. This version is relatively small and tame; the other two are vaster.

Here's a second idea. We can replace Supplementation with the principle

Potential Supplementation. If it *will always* be the case that everything that overlaps x overlaps y, then x is part of y.

Equivalently: if x is not part of y, then y *will* have a part disjoint from x.

As an example, consider again a collection of Zeno Parts of a solid block. What we can say is that these fail to compose the block, but rather compose a proper part of the block, the Zeno Sum. There isn't yet anything that makes up the difference between the Zeno Sum and the original block—but there *will* be. When there is, we can consider a still larger collection of Zeno-ish parts, but these again will fail to compose the entire block. Thus we can maintain the principle that whatever touches the whole touches some part.

This picture gives up a bit of classical mereology. What we have instead is **Heyting mereology**: this stands formally to intuitionistic propositional logic in the same way that classical mereology stands formally to classical propositional logic.[16] In particular, it should be noted that in this sort of mereology *least upper bounds* and *fusions* can come apart—and least upper bounds are really a more plausible candidate for the ordinary notion of a sum (see Russell 2008, 265–266, discussing a view of Peter Forrest's).

To find a model for these principles (and thus prove they are really consistent) we can turn to a very rich kind of non-standard analysis: the "surreal numbers" discovered by John Conway (2000 [1976]; see also Knuth 1974; Ehrlich 2012). This is a vast continuum—in fact, too vast even to form a set, in orthodox set theory. The basic idea is a generalization of Dedekind's "cut" method for constructing the standard continuum of real numbers from the rational numbers. Dedekind identified the number π with the set of all rational numbers less than π. Similarly, we can consider the set of all the *real* numbers less than π as fixing a number infinitesimally less than π—a number between π and all the standard real numbers less than it. And we can go on iterating this, building up larger and larger, denser and denser non-standard continua.

For each ordinal α, we can construct a linearly ordered set of **points at stage** $\boldsymbol{\alpha}$, which we'll think of as ordered from left to right. Each later stage extends each of the earlier ones by adding more points. We begin with the empty set. For each successor ordinal $\alpha + 1$, we can construct the new $(\alpha + 1)$-points as cuts: leftward-closed sets of α-points. At each limit ordinal, we simply gather together all of the previous stages into a single ordered set. To reach *all* of the surreal numbers, we accumulate together all of the points that occur at any stage of this construction, for arbitrary ordinals. But for our purpose—which is to come up with standard set-theoretic models for a consistency proof—it's better to just stop at some suitably large bound, rather than really going on indefinitely.

[16] In particular, while classical mereology says that composition has the structure of a complete Boolean algebra, with the bottom element dropped out, Heyting mereology says that composition has the structure of a complete *Heyting* algebra, with the bottom element dropped out. See the appendix for details.

This construction gives us ever vaster ontologies of points—but what we're looking for are growing ontologies of extended objects. We can accomplish this by representing continuous objects with certain sets of α-points—in particular, unions of closed intervals. We represent parts by subsets, and we represent contact by sharing at least a point. (Points do not themselves represent objects in this ontology; so contact is distinct from overlap, which requires sharing at least an interval.) It's easy to check some of the main important properties of this model, such as that contact is distributive. (This follows from the fact that the intersection of a set A with a union of sets B_i is the same as the union of the intersections of A with each B_i.)

(The Finitist Model in fact can be constructed using these same tools, only considering the finite stages.)

The third picture is a variant that uses a different notion of a sum—not as a fusion or a least upper bound, but as a *stable* least upper bound. We can keep each of the original principles except Sums, which we replace with this:

Potential Sums. (i) x is a sum of the X's iff each X is part of x, and it *will always* be the case that x is part of anything that has each X as a part.

(ii) Any things *will* have a sum.

Note that the definition of a least upper bound is "extrinsic"—it requires there to be no *lesser* upper bound. If the ontology can expand, then in principle a smaller upper bound could come into being. (Something similar goes for fusions: it could be that some new thing comes into being that overlaps x without overlapping any X.) If this can happen, the notion of a sum is more naturally understood as involving a stronger constraint: not just being *temporarily* the smallest thing containing each X, but being *permanently* thus. This requires sums to be stable. The trade-off is that stable sums aren't guaranteed to immediately exist. Like sets, this view has it that mereological sums of concrete things are iteratively generated.

In this picture, the Zeno Parts do not compose the whole block, and neither do they compose something less than the whole block—rather, they don't compose anything at all. They *will* compose something less than the whole block; and when they do, there will also be further infinitesimal parts of the block that make up the difference. At that stage, these infinitesimal parts fail to compose anything—but again, they *will* compose something, and when they do there will be yet smaller infinitesimal parts to make up the difference. And so on. Continuous objects are ever divisible into longer and longer Zeno sequences, and never thereby exhaustively divided.

Of course, a proof of consistency isn't the same as a proof of *possibility*. But what we've accomplished here clears away a significant objection to the possibility of strongly cohesive objects and the Distributive Contact principle. As I see it, the main challenge still facing the intelligibility of these kinds of continua is to clarify the—admittedly obscure—notion of "potential". This is still a difficult problem—one shared with those who defend an indefinitely extensible *mathematical* realm.

4 Further Morals

These are intriguing pictures of the structure of the physical world. If they are possible, then they show that indefinite extensibility can be motivated independently of platonism about abstract objects. I'll point out a few further lessons. The possibility of indefinite multitudes of conreta closes off certain strategies for making sense of the rejection of absolute generality.

First, indefinite divisibility puts pressure against accounts that enlist "limited", inextensible predicates to play roles that extensible predicates cannot. For instance, Geoffrey Hellman takes a "sortalist" line, suggesting that even though there is something defective about "everything" and "every ordinal", nonetheless there is no obstacle to our using "every *donkey*" and the like (Hellman 2006, 90ff.). This saves the apparent datum that "There are no talking donkeys" is perfectly intelligible: it says "Every donkey is non-talking". He goes on to suggest that a "flat-out" denial of existence, like "There are no ghosts", should be interpreted as a denial of ghosthood within some sufficiently broad, but still limited category: for instance, it might mean "Every space-time-occupant is a non-ghost".

The possibility of indefinite divisibility, though, would show that there aren't nearly as many limited predicates to go around as one might think. "Space-time occupant" won't do, for instance. Neither will Hellman's other suggestion, "cause", since there's no reason to think the parts of strongly cohesive physical objects couldn't enter into causal relations. The problem runs deeper if we consider more exotic elaborations of the general picture. For instance, a donkey might have been a continuous extended object; and it might even have turned out that among such a donkey's parts were donkey-*homunculi* (*asinunculi*?)—and so on indefinitely. If that could happen, then "donkey" isn't a limited predicate either—and there won't be many limited predicates left for implementing Hellman's strategy.

A different thought that is put under pressure is that the inconstancy of what there is arises from something special about the nature of abstract objects. Sets and numbers and properties seem flimsier than tables and chairs. Perhaps they are "constructed" or "stipulated" or "postulated". Kit Fine's position seems to be along these lines: indefinite extensibility is explained in terms of "postulational possibility", which in turn he explains in terms of reinterpretation. "The possibility that there are more sets, for example, depends upon a reinterpretation in what it is for there to be a set" (2006, 33). But even if this is plausible for sets—I find the idea difficult to get the hang of—it is radical to say that physical objects could be thus postulated into being, by merely reinterpreting our quantificational expressions. (We could stipulate that "There is a talking donkey" expresses a truth—for instance, the truth that there is a philosopher—but this would just be changing the subject. I don't think it's relevantly a way there might *be* a talking donkey.) Indeed, Fine recognizes, "It is plausibly part of the meaning of "donkey" that donkeys cannot be introduced into the domain through postulation" (using this fact to explain how we can sensibly and categorically say, "There are no talking donkeys") (2006, 41–42). If even donkeys can be divided indefinitely, then Fine's account will not extend to this case.

Or perhaps instead we should follow this radical idea where it leads. If parts are (in some elusive sense) merely potential, perhaps we really should think of them as importantly like abstract objects. Perhaps physical objects are divided merely "in thought", rather than "in reality"—the part is in some sense merely an abstraction from the whole—and thus physical objects are in some sense susceptible to being stipulated or postulated into being.[17] These strike me as wild and obscure ideas, though some may like the sound of them. I leave them to others to develop.

A Consistency Proofs

In this appendix I'll sketch consistency proofs for the three models described in Section 3, using familiar structures from modal and tense logic.

In what follows, let a **mereotopology** be a set (the **domain**) with two binary relations: **part** and **contact**. I won't build any general structural features of these relations into the label "mereotopology", and will just state further constraints explicitly in what follows. A **subspace** of a mereotopology M is a mereotopology whose domain is a subset of the domain of M, and has the part and contact relations which restrict the corresponding relations in M to the smaller domain.

As usual, we say x **overlaps** y iff x and y have a common part. If M' is a subspace of M, we say M' **respects overlap** iff whenever x and y are in the domain of M' and overlap in M, then x and y overlap in M' as well. (Note that the converse automatically follows from the definition of subspace: any things that overlap in the subspace M' must also overlap in M.)

[17] Leibniz held a view like this of space and time, but not of matter.

> But space, like time, is not something substantial, but ideal, and consists in possibilities, or an order of coexistents that is in some way possible. And thus there are no divisions in it but such as are made by the mind, and the part is posterior to the whole. In real things, on the contrary, units are prior to the multitude, and multitudes exist only through units. (*Die Philosophischen Schriften von G.W. Leibniz* II. 278, as quoted in Russell 2013 [1900], 245)

A **stage model** consists of an ordered set of **stages**, and for each stage α a mereotopology M_α: the **ontology at stage** α. We require stages to be well-ordered and unbounded, and if $\alpha < \beta$, then we require the ontology M_α to be an overlap-respecting subspace of M_β.

It's straightforward to specify interpretations of a language that involves (i) predicates for part and contact (ii) singular and plural quantification, and (iii) "forward" and "backward" looking tense operators. Given a stage-model, we interpret a formula φ in a stage model with respect to a stage α, an assignment f of an individual (in the domain of any stage) to each individual variable, and an assignment g of a set of individuals (in the domain of any stage) to each plural variable. Since the clauses of such interpretations are standard and familiar, I won't spell them all out explicitly. A few examples should suffice.

- "There are some X's such that φ"" is true at (α, f, g) iff φ is true at (α, f, g') for some assignment g' that agrees with g except perhaps at X, and for which $g'(X)$ is a subset of the domain of α.
- "x is an X" is true at (α, f, g) iff $f(x)$ is an element of $g(X)$.
- "It will be the case that φ" is true at (α, f, g) iff φ is true at (β, f, g) for some stage β after α.

Say that a stage-model **satisifes** φ iff φ is true at every stage α, with respect to any "proper" singular and plural assignments, in the sense that their values are in the domain of α.

Note a few important principles that are satisfied by every stage model (in addition to standard tense-logical principles: see Studd 2013 for details).

Expanding Domains. Everything will always be something.

Stable Pluralities. x is an X iff it will always be the case that x is an X.

Stable Part. x is part of y iff it will always be the case that x is part of y.

Stable Overlap. x overlaps y iff it will always be the case that x overlaps y.

Stable Contact. x touches y iff if will always be the case that x touches y.

Next we'll construct a simple version of Conway's surreal numbers: a nested sequence of linearly ordered sets S_α for each ordinal α. (I'll call elements of S_α "α-points", and speak of them as ordered from left to right.) For each successor ordinal, each $(\alpha + 1)$-point is either an α-point, or else an α-*cut*: a subset $X \subseteq S_\alpha$ which is leftward-closed, in the sense that if $x \in X$ and y is left of x then $y \in X$. For any $(\alpha + 1)$-points x and y, we say x is left of y iff one of the following cases holds:

i. x and y are both α-points and x is left of y in S_α;
ii. x is an α-point and y is an α-cut, and $x \in y$;
iii. x is is a cut and y is an α-point, and $y \notin x$;
iv. x and y are both cuts, and $x \subset y$.

It's straightforward to check that if S_α is linearly ordered, then so is $S_{\alpha+1}$. For each limit ordinal, S_λ is the union $\bigcup_{\alpha<\lambda} S_\alpha$, with its inherited ordering.

Here is the most important feature of these nested ordered sets that we'll use in what follows—a powerful density property.

Lemma 1. For any ordinal α and any subsets $L, R \subseteq S_\alpha$, if each member of L is left of each member of R, then there is some $x \in S_{\alpha+1}$ which is to the right of each element of L, and to the left of each element in R. More briefly: if $L < R$ in S_α, then for some x in $S_{\alpha+1}$, $L < x < R$.

Proof. Let x be the leftward closure of L: the set of all α-points which are either in L or to the left of some point in L. This is clearly leftward-closed, and thus an $(\alpha+1)$-point. It's straightforward to check from the definitions that $L < x < R$. □

If x is left of y in S_α, then the closed interval $[x, y]$ in S_α is the set $\{z \in S_\alpha \mid x \le z \le y\}$. We say x is in the *interior* of an interval I iff $x \in I$ and x is not one of the endpoints. Here's a basic fact about closed intervals.

Lemma 2. Suppose I and J are closed intervals, a is a point in I, and a is in the interior of J. Then the intersection $I \cap J$ is a closed interval.

Proof. Say $I = [x_1, x_2]$ and $J = [y_1, y_2]$. We know that $x_1 < x_2$ and $y_1 < y_2$, and also $x_1 \le a < y_2$ and $y_1 < a \le x_2$. So $\max(x_1, y_1) < \min(x_2, y_2)$. These are the endpoints of the closed interval which is $I \cap J$. □

In what follows, let λ be a limit ordinal. For each $\alpha < \lambda$, let the **α-intervals** be the closed intervals in S_λ with endpoints in S_α. Let $H_\lambda(\alpha)$ be the set of all (non-empty) unions of α-intervals. (I'll sometimes drop the subscript λ.) This is a mereotopology, where a set counts as part of another iff it is a subset, and two sets count as in contact iff they have a non-empty intersection. We'll now check some important properties of $H_\lambda(\alpha)$.

Classical mereology comprises the following principles:

Core Mereology. Part is reflexive, transitive, and antisymmetric.

Sums. (i) x is a sum of the X's iff: every X is part of x, and x is part of anything that has every X as a part. (That is, x is the least upper bound of the X's.)

(ii) Any (one or more) things have a sum.

Supplementation. If everything that overlaps x overlaps y, then x is part of y.

Heyting mereology includes Core Mereology and Sums, but replaces Supplementation with this (which is also a consequence of classical mereology):

Distributive Part. Any part of the sum of the X's is a sum of parts of X's

(Less tersely: if x is part of the sum of the X's, then there are Y's, each of which is part of some X, such that x is the sum of the Y's.)

Lemma 3. $H_\lambda(\alpha)$ satisfies the principles of Heyting mereology.

Proof. Core Mereology and Sums are sufficiently obvious. Note in particular that the sum of elements of $H(\alpha)$ is their union. This just leaves Distributive Part. Suppose $A \subseteq B = \bigcup_i B_i$ in $H(\alpha)$. We need to show that there are C_j, each of which is part of some B_i, such that $A = \bigcup_j C_j$. It will suffice to show that for for each point a in A, there is some α-interval that contains a and which is part of A and some B_i. There is some α-interval I such that $a \in I \subseteq A$. I contains at least one α-point besides a; let's say (without loss of generality) that it's to the left of a. Then let L be the set of all α-points left of a: by Lemma 1, there is some $(\alpha + 1)$-point b between L and a—and thus b is in the interior of I. Since I is part of A, which is part of B, b must also be in some B_i, and thus b is in some α-interval $J \subseteq B_i$. J contains b, which is left of a, and (since J is an α-interval) J also contains at least one α-point to the right of b, and thus not to the left of a; so a is also in J. Then by Lemma 2, $I \cap J$ is a closed interval that contains a and is a common part of A and B_i. □

Though $H_\lambda(\alpha)$ doesn't always satisfy Supplementation, we do have the following closely related fact.

Lemma 4. If A is not part of B in $H(\alpha)$, then there is some C in $H(\alpha + 1)$ which is part of B and does not overlap A.

Proof. Suppose that that A and B are elements of $H(\alpha)$, and A is not a subset of B, so there is some $b \in B - A$. Then Lemma 1 (applied to the points in A which are to the left of b, $\{b\}$, and the points in A which are to the right of b) guarantees that there is some $(\alpha + 1)$-interval I which contains b in its interior, and excludes A. Since $b \in B$, there is also some closed α-interval J such that $b \in J \subseteq B$. By Lemma 2, $I \cap J$ is an $(\alpha + 1)$-interval that is part of A, and has no common part with A. □

Lemma 5. $H_\lambda(\alpha)$ satisfies these topological principles:

> **Core Topology.** Contact is reflexive, symmetric, and monotonic.
>
> **Distributive Contact.** Anything that touches the sum of the X's also touches some X.

Proof. Trivial from the definitions and basic set theory. □

Lemma 6. Any α-interval in $H_\lambda(\alpha)$ is cohesive. Thus $H_\lambda(\alpha)$ satisfies

> **Cohesion.** Something is composite and cohesive.

Proof. Let I be an α-interval, and suppose $I = A \cup B$ for some A and B in $H(\alpha)$. Then (WLOG) suppose I's left endpoint is in A. Let $B^- = B \cap S_\alpha$, and let A^- be the set of points in $A \cap S_\alpha$ which are to the left of each point in B^-. Then by Lemma 1 there is an $(\alpha + 1)$-point x between A^- and B^-; this point must also be in I. It can't be that $x \in B$ (since every point in B lies in a closed interval whose left endpoint is in B^-, and thus $x < B$). So x must be in A, and thus some α-interval J is part of A and contains x. Let y be J's right endpoint; y cannot be in A^-, since $A^- < x \leq y$. Since y is not in A^-, there is some point $b \in B^-$ such that $b \leq y$. Since $x < B$, $x < b \leq y$, and thus since x and y are both in J, b is also in J, which is part of A. So $b \in A \cap B$, and thus A and B are in contact. □

Lemma 7. If A is in $H(\alpha)$, then there is some B in $H(\alpha + 2)$ which is an interior part of A. (That is, B is a part of A, and any C in contact with B overlaps A.) Thus if α is a limit ordinal, then $H_\lambda(\alpha)$ satisfies

> **Interiors.** Every x has an interior part.

Proof. Let A be an element of $H(\alpha)$, and let I be an α-interval which is part of A. By two applications of Lemma 1, there is a pair of points in $H(\alpha + 2)$ which are between the endpoints of I, and thus a closed interval J, such that every point in J is in the interior of I. It follows that J is an interior part: if $C \cap J$ contains a point x, then x is in some interval K which is part of C, and then by Lemma 2, $I \cap K$ is an interval which is a common part of A and C.

In the case where α is a limit ordinal, we can apply this reasoning to the stage $\beta < \alpha$ where the endpoints of I first appear; then there is an interior part J in stage $\beta + 2 < \alpha$, and thus J is in $H(\alpha)$ as well. □

Let the **Principles** be Core Mereology, Sums, Supplementation, Distributive Part, Core Topology, Distributive Contact, Interiors, and Cohesion.

Theorem 3 (The Finitist Model). There is a stage model that satisfies all of the Principles except Interiors, and instead satisfies

> **Potential Interiors.** Everything will have an interior part.

Proof. Let stages be finite ordinals, and for each stage $\alpha < \omega$, let the ontology M_α be $H_\omega(\alpha)$. Lemmas 3, 5, 6, and 7 show that this model satisfies the principles of Heyting mereology, Core Topology, Distributive Contact, Cohesion, and Potential Interiors. This just leaves Supplementation. This follows from the fact that for $\alpha < \omega$, there are only finitely many points in each stage S_α, which is clear from the construction.

If $b \in B - A$, then there are α-points $x < b < y$ which are nearest to b. These have the property that for any α-point z, if $z < y$ then $z \leq b$, and if $x < z$ then $b \leq z$. Then the interval $[x, y]$ has no common interval part with A. If it did, A would contain some α-interval $[l, r]$ which includes a point a such that $x < a < y$. Then since $l \leq a < y$, $l \leq b$, and likewise since $x < a \leq r$, $b \leq r$. So b would be in $[l, r]$, and thus in A, contradicting our assumption. So $[x, y]$ is a part of B which does not overlap A. □

Theorem 4 (The Intuitionist Model). There is a stage model that satisfies all of the Principles except Supplementation, and instead satisfies

> **Potential Supplementation.** If it will always be the case that everything that overlaps x overlaps y, then x is part of y

Proof. Let κ be a regular uncountable ordinal. Let stages be *limit* ordinals prior to κ. (These are well-ordered and have no last element, since if λ were the greatest limit ordinal less than κ, then the ordinals $\lambda + n$ for finite n would be countable and cofinal with κ.) For each stage λ, let $M_\lambda = H_\kappa(\lambda)$. We can then apply Lemmas 3, 4, 5, 6, and 7. □

Theorem 5 (The Stable Sum Model). There is a stage model that satisfies all of the Principles except Sums, and instead satisfies

> **Potential Sums.**
>
> (i) x is a sum of the X's iff every X is part of x, and it will always be the case that x is part of anything that has each X as a part. (That is, x is a *stable* least upper bound of the X's.)
>
> (ii) Any (one or more) things will have a stable sum.

Proof. Let the stages be as in Theorem 4. For each stage λ, let the ontology M_λ be the union $\bigcup_{\alpha<\lambda} H_\kappa(\alpha)$ (with the usual part and contact relations).

In this model, the only thing that counts as a *sum* of some A_i in M_α in the sense of Potential Sums is again their union. The union may not itself be an element of M_α (since the component intervals of the A_i's may not all be in any $H(\beta)$ for $\beta < \alpha$). Still, it certainly is in the ontology of the stage immediately after α (since each component interval of any A_i must be in $H(\alpha)$). And clearly there is nothing in any later stage that has each A_i as a part without also having their union as a part. Again, the rest of the Principles follow from straightforward application of the lemmas. □

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