Possible Patterns

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Abstract

"There are no gaps in logical space", writes Lewis (1986), giving voice to sentiment shared by many philosophers. But different natural ways of trying to make this sentiment precise turn out to conflict with one another. One is a *pattern* idea: "Any pattern of instantiation is metaphysically possible". Another is a *cut and paste* idea: "For any objects in any worlds, there exists a world that contains any number of duplicates of all of those objects." Jumping off from discussions from Forrest and Armstrong (1984) and Nolan (1996), we use resources from model theory to show the inconsistency of certain packages of combinatorial principles and the consistency of others.

1 Two Combinatorial Ideas

A famous Humean slogan has it that there are no necessary connections between distinct existences. ("There is no object, which implies the existence of any other if we consider these objects in themselves," Hume 1978, 1.III.vi.) Many contemporary philosophers have endorsed this kind of "combinatorialist" idea: "there should be no arbitrary limits to what's possible" (Sider 2009), there are "no gaps in logical space" (Lewis 1986, 87), "there are no brute necessities" (Dorr 2008; see also e.g. Kleinschmidt 2015). This picture might be motivated epistemologically: brute necessities would make trouble for any tight connection between what's conceivable and what's metaphysically possible. Or it might be motivated metaphysically: arbitrary-looking constraints on metaphysical possibility, whether arising from primitive essences, powers, laws, or the the necessary will of God, seem occult. We won't be evaluating these motivations here: rather, we'll be examining logical limits on what such a view could consistently say.

Here's one way of trying to articulate the Humean idea:

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> Any pattern of instantiation of any fundamental properties and relations is metaphysically possible (Wang 2013, 538; Saucedo 2011; see also Armstrong 1989, 49).

This "pattern" idea is still not completely clear. If "any pattern" means "any actually instantiated realized pattern", then this says no more than the truism that what is actual is possible; and if it means "any pattern which is metaphysically possible to instantiate, it says no more than the tautology that what is possible is possible.[1] We'll be exploring ways of spelling out this "pattern" idea. But in response to similar difficulties, David Lewis concluded that similar principles "cannot be salvaged as principles of plenitude"[2] and "we need a new way to say … that there are possibilities enough, and no gaps in logical space" (1986, 87). So Lewis proposed a second, mereological way of articulating the combinatorial idea—the "cut and paste" idea.

> Possibility is governed by a combinatorial principle. We can take apart the distinct elements of a possibility and rearrange them. We can remove some of them altogether. We can reduplicate some or all of them. We can replace an element of one possibility with an element of another. When we do, since there is no necessary connection between distinct existences, the result will itself be a possibility (2009, 209; see also 1986, sec. 1.8; Nolan 1996; for critical discussion see Wilson 2015).

(In Lewis's earlier work (1986) he mainly applies this combinatorial idea to spatiotemporal parts of possible worlds. But in this 2009 presentation, he also tries to capture some of the pattern idea within the cut and paste idea—rearranging, as he puts it, "not only spatiotemporal parts, but also abstract parts—specifically, the fundamental properties".)

It turns out that there are serious problems for straightforwardly unifying these two different combinatorial ideas: natural ways of spelling out the "pattern" idea and the "cut and paste" idea are inconsistent with one another. Peter Forrest and David Armstrong (1984) attempted to show this (though this isn't exactly how they put it); Daniel Nolan (1996) showed that their argument was dialectically ineffective, and the idea has since been neglected. But different arguments do successfully reveal conflict between the two combinatorial ideas. We'll show the inconsistency of a certain combinatorialist package; we'll go on to also show how to devise consistent alternative packages based on the "pattern" idea. Our main technical tools for this

[1] Wang and Saucedo both recognize these difficulties. They each consider ways of articulating the slogan in terms of the logical consistency of certain sentences; we'll come back to this idea in section 5. Note also that Wang puts the slogan forward as a target, not as her own view: see footnote 9.

[2] He attributes the point to Peter van Inwagen, specifically concerning the slogan: "absolutely every way that a world could possibly be is a way that some world is."

project come from model theory: we'll deploy the mathematical theory of *relational structures* to regiment the intuitive notion of patterns of instantiation.

We should note that these results don't rely on Lewis's brand of modal realism, nor indeed on any particular commitments about the nature of possibilia. Neither do they rely (as some arguments have) on the idea that possible worlds or possible objects form a set. Furthermore, these consistency and inconsistency results have philosophical ramifications for many views other than a full-blown Humean picture of metaphysical possibility. They apply even if just certain special aspects of reality are freely recombinable—a single relation, perhaps. They also apply to other modalities besides metaphysical possibility. For instance, parallel issues arise for views that say certain qualitative patterns of instantiation are *epistemically* possible. (Recall that Hume's original target was *a priori* necessary connections.) So our results should not just be of interest to the metaphysician, but also to the epistemologist.

2 Patterns of Properties

A well-known argument due to Forrest and Armstrong (1984) reveals that a certain kind of combinatorialism is inconsistent. We'll present a variant of this argument. The variant is close to the original in spirit, but while the original version only targeted Lewis's specific modal realist conception of possible worlds, our version abstracts away from those commitments: it doesn't depend on any particular view of the nature of worlds. We've also taken the opportunity to put things in terms that are continuous with arguments and ideas we present later in this paper.[3]

The Forrest–Armstrong argument targets the combination of two principles. The first principle is a version of the "pattern" idea: given a possible world W,

[3] Throughout this paper we freely appeal to the truth at every possible world of ZFCU—standard set theory adapted to a setting with urelements. (We do not assume Urelement Set—that there is a set of non-sets.) We also make the simplifying assumptions that set-membership is rigid, and that it is not contingent what pure sets there are.

We make some additional assumptions about the logic of possible worlds. First, we assume that truth-at-a-world is closed under logical consequence:

Closure If ψ is a logical consequence of $\phi_1, \phi_2 \ldots$, and at W, ϕ_1, and at W, ϕ_2, and …, then at W, ψ.

But we will qualify this principle later in this section: see footnotes 7 and 13. We also assume for simplicity that it is not contingent what worlds there are, nor what is true at them: that is, for any world W at which ϕ, at every world V, W is a world at which ϕ. (Given natural background assumptions about the connection between modal operators and possible worlds, this is tantamount to assuming the modal logic S5.)

> There will then be some property, F-ness … which each electron in W may or may not have, and may or may not have independently of whether the other electrons in W have it. For each sub-set of the N electrons it will be possible that precisely the electrons in that sub-set have the property F-ness" (1984, 165).

To make it more transparent that "electron" is a placeholder in the argument, we'll instead talk about "marbles": these are supposed to be some kind of concrete particulars that are candidates for instantiating recombinable properties. We'll make this "pattern" principle precise as follows:

Property At any world, if D is the set of marbles and X is a subset of D, then at some world: the marbles are precisely the elements of D, and the F marbles are precisely the elements of X.

(This principle is in fact a little stronger than Forrest and Armstrong's, since it requires the "pattern world" to be one in which there are no other marbles besides the D-marbles. Nothing important turns on this, but it will simplify some connections later on.)

The second principle is a version of Lewis's "cut and paste" idea:

> [G]iven any number of possible worlds, W_1, W_2 …, there exists a possible world, having wholly distinct [i.e., non-overlapping] parts, such that one of these parts is an internally exactly resembling duplicate of W_1 […], another a duplicate of W_2, and so on (1984, 164).

Their argument is presented in terms of duplication (specifically targeting Lewis's early formulations of recombination), but we'll put this a bit more abstractly.[4] If Ω is a set of possible worlds, then where Forrest and Armstrong say that W^+ has distinct duplicates of the worlds in Ω as parts, we'll instead say "W^+ **disjointly embeds** Ω." In due course we'll give a definition of disjoint embedding, but for the moment all that is important is this fact about it:

Enough Marbles If W^+ disjointly embeds Ω, and at each world W in Ω there is at least one marble, then at W^+ there are at least as many marbles as elements of Ω.

So our version of the principle says:

[4] This bypasses some concerns about the part-whole structure of possible worlds, and whether the recombinable properties are intrinsic.

Paste For any set of worlds Ω there is a world W^+ that disjointly embeds Ω.

Note that when Forrest and Armstrong say "any number of possible worlds" they mean it literally: their "paste" principle only applies to worlds that have a cardinal number. With orthodoxy, we suppose this requires that they are not too numerous to form a set, and we've made this explicit in our formulation of Paste, by saying "for any *set* of worlds."[5]

The argument against Property and Paste relies on several background assumptions. Forrest and Armstrong apply their "paste" principle to the plurality of all possible worlds, which requires:

World Set There is a set of all worlds.

Similarly, when they say "the N electrons" they are implicitly assuming that the electrons also have a cardinal number, and thus they assume:

Marble Set At any world, there is a set of all marbles.[6]

There are powerful arguments against these set-theoretic assumptions, and accordingly we will relax them in section 4. But since that introduces extra technical complications, we will assume World Set and Marble Set for now.

The argument uses two further auxiliary premises.

Possible Marble At some world, there is at least one marble.
Possibilities For any sets X and Y, if at W the F marbles are precisely the elements of X, and at W the F marbles are precisely the elements of Y, then $X = Y$.

[5]Forrest and Armstrong alternatively consider applying the argument to some broader notion of an "aggregate" of all worlds. It's not clear what they have in mind, but they might be gesturing at a version of the argument using plural quantification: we take up this idea in section 4.

[6]Assuming that (at every world) marbles and worlds are non-sets, Marble Set and World Set are consequences of the necessity of the more familiar Urelement Set axiom, which says that there is a set of all non-sets.

Of course, some hold that possible worlds *are* sets—for example, sets of propositions. In this case World Set would not follow from Urelement Set.

There are challenges to Urelement Set that don't apply to Marble Set. For example, suppose that for each marble m and any distinct sets A and B, the fusion of m and A is a distinct non-set from the fusion of m and B. Then there will, after all, be as many non-sets as sets, violating Urelement Set (see Uzquiano 2006). But this argument makes no trouble for Marble Set on its own.

(Equivalently, for any distinct sets X and Y, if at W the F marbles are the elements of X and at W' the F marbles are the elements of Y, then W and W' are distinct worlds.) This final premise is about the uniqueness of what is true at a possible world. We'll return to this in a moment.[7]

Here is the main Forrest–Armstrong result, in our setting:

> Given Possible Marble, Marble Set, World Set, and Possibilities, it follows that Property and Paste are not both true.

Let Ω be the set of all worlds at which there is at least one marble. (This set exists, given World Set and the axiom of Separation.) Then by Paste, there is some world W^+ that disjointly embeds Ω, and so at W^+ there are at least as many marbles as there are worlds in Ω. By Marble Set these marbles form a set D, and by Possible Marble D is non-empty. Then Property tells us that at W^+, for each subset X of D there is some world in Ω at which the F marbles are precisely the elements of X; and by Possibilities these worlds are numerically distinct from one another. Thus, at W^+, there are at least as many worlds in Ω as subsets of D. So there are at least as many members of D as subsets of D. But this contradicts a standard result of set theory:

Cantor's Theorem There are strictly more subsets of D than members of D.

In short, at the world W^+ we would have to have

$$D \geq \Omega \geq 2^D > D$$

which is impossible. QED.

The Forrest–Armstrong argument establishes an interesting result, but it is not one that has dialectical force against their original target, namely David Lewis's system: for Lewis rejects some of the assumptions on which the argument relies (Nolan 1996).[8] At a crucial point in the argument we infer that there are as many distinct

[7] In fact, we can derive Possibilities from our background principles, using the necessity of identity. Suppose that at at W, X contains the F marbles, and at W, Y contains the F marbles. Then by the unqualified version of Closure (footnote 3), together with the fact that the axiom of Extensionality is true at W, it follows that at W, $X = Y$.

[8] Nolan puts the point like this: "[T]hey talk as if there is trans-world identity of electrons. [This is bad] because Lewis does not think that there is any such thing, and they are supposed to be discussing a problem for *Lewis'* theory" (1996, 243, original emphasis). Strictly speaking, trans-world identity isn't exactly what's at issue, but rather the *haecceitistic* principle that isomorphic possibilities, which merely differ regarding which particular marbles are F, are witnessed by distinct possible worlds. But the best-known way of rejecting this haecceitistic principle is Lewis's, which goes by way of rejecting trans-world identity. (It's a bit surprising that Lewis himself did not seem to notice this problem with the argument: for his response see Lewis 1986, sec. 2.2.)

possible worlds as sets of marbles, since for each set of marbles X there is some world at which just the X-marbles are F (let's say "red"). Take the case of two singleton sets $\{\text{Mary}\}$ and $\{\text{Marvin}\}$. Then in particular, the argument requires that, given that Mary and Marvin are distinct, the world at which just Mary is red is distinct from the world at which just Marvin is red. But Lewis is a *counterpart theorist*, and a counterpart theorist can resist this. For it to be true at W that just Mary is red, it suffices that Mary have some *counterpart* which is the only red marble in W. The same goes for Marvin. Crucially, Lewis allows that the very same marble can be a counterpart for *both* Mary and Marvin—and thus the very same world can do double-duty for both possibilities (see Lewis 1986, 232ff.). So, for the counterpart theorist, Property does not imply that there are as many distinct possible worlds as there are sets of marbles. In particular, given this counterpart-theoretic gloss on truth-at-a-world, the backround premise Possibilities fails: it can be true at W_1 that Mary is red, and also true at W_2 that Mary is not red, and yet it does not follow that W_1 and W_2 are distinct.

A more general lesson is that the Forrest–Armstrong argument is only effective against the *haecceitist* combinatorialist: someone who accepts both Property—which says there are possible worlds witnessing property-distributions that differ merely with respect to what individual marbles are like—as well as Possibilities—which guarantees the distinctness of these possible worlds. It would be nice, then, to try to rehabilitate a version of the argument with more general application, by only appealing to *qualitatively* distinct possible worlds.

Moreover, the slogan we began with—

> Any pattern of instantiation of any fundamental properties and relations is metaphysically possible

—is much more plausible when read as a principle concerning *qualitative* patterns, rather than as a constraint on *de re* possibilities. The de re reading is not just haecceitistic, but in fact radically anti-essentialist. For example, suppose that "marbles" include both photons and electrons, and suppose that each electron is essentially an electron, and each photon is essentially a non-electron. Then reading "is F" as "is an electron" would make Property false—but this essentialist picture is still compatible with a qualitative gloss on the combinatorialist slogan.[9]

[9] Wang (2013, 539–40) raises a related objection to combinatorialism: if every pattern of instantiation of fundamental properties and relations is metaphysically possible, and *is located at* is fundamental, then something which is not a region could have something located at it. Qualitative pattern principles do not have this *de re* modal consequence.

Wang also raises other objections which apply equally well to qualitative principles: the best contenders we currently have for fundamental properties and relations don't look especially well suited

From here on, we'll be focusing on principles that concern the possibility of certain sorts qualitative patterns, rather than haecceitistically loaded combinatorial principles. (That isn't to say we are assuming *anti-haecceitism*, nor are we assuming Lewisian doctrines associated with counterpart theory. For the purposes of this paper we aim to stay neutral on such questions.) In particular, rather than Forrest and Armstrong's Property, which is about distinct *de re* possibilities for the F-ness of particular marbles, we can explore the prospects for a principle about different qualitative patterns of F-ness over the marbles. Here's a natural principle to try (Nolan 1996, 243):

Property Pattern At any world, if D is the set of marbles and X is a subset of D, then at some world: the number of marbles is the same as the cardinality of D, and the number of F marbles is the cardinality of X.

Daniel Nolan points out that, unlike Forrest and Armstrong's Property principle, Property Pattern does not lead to any contradiction with Paste. We can show this with a simple model (1996, 243–44). There is a countable infinity of worlds, each of which contains a countable infinity of marbles. For each natural number n there is a world containing just n red marbles, and another world containing just n non-red marbles. There is also one world containing infinitely many red marbles and infinitely many non-red marbles. (To see that this respects Paste we need only recall the standard fact that the disjoint union of countably many countable sets is countable.)

The combinatorial idea of Property Pattern can also be generalized to patterns of arbitrarily many properties. Doing this precisely takes a bit of work—our next task. In what follows, in order to talk about different sorts of patterns, we will appeal to the standard model-theoretic notion of a *structure*.[10] Suppose P is a set of monadic properties.[11] (We'll generalize this in the next section.) A P-**structure** is a pair of

to free recombination. For instance, determinate properties like *having 1kg mass* and *having 2kg mass* are incompatible with one another, and distance relations obey geometric constraints. In this paper our focus is on the logical limits of recombination theses: we do not really rest anything on whether recombinable properties and relations are *fundamental*, or which ones they might be.

[10]For reference see e.g. Hodges (1997). Philosophers are most likely to be familiar with structures in the context of the semantics for first-order logic (e.g. Sider 2010, sec. 4.2). In that setting, the *signature* of a structure—our set of properties P—is usually left implicit, and relationships between different structures are not emphasized. Another minor difference is that in that setting structures with empty domains are not usually allowed.

[11]Model theorists are usually neutral about what sort of things are the elements of a signature—they might, for instance, be symbols or numbers. But nothing in the standard formalism prevents us from using properties for this purpose, and we'll find this choice convenient.

a set D, called the **domain**, and a function that takes each property F in P to some subset of D, called the **extension** of F in that structure. One of the overarching ideas of this paper is that model theory—the mathematical theory of structures—is a valuable tool for modal theorizing.[12]

A **marble structure** is any structure whose domain contains every marble and nothing else. For example, suppose the properties in P are just redness and circularity, and there are just two marbles Mary and Marvin. Then there are sixteen marble structures (with signature P). For instance, one structure S_1 has these extensions:

$$S_1 \qquad \begin{array}{rcc} \text{circular} & \mapsto & \varnothing \\ \text{red} & \mapsto & \{\text{Mary}\} \end{array}$$

Another has these:

$$S_2 \qquad \begin{array}{rcc} \text{circular} & \mapsto & \{\text{Marvin}\} \\ \text{red} & \mapsto & \{\text{Mary}, \text{Marvin}\} \end{array}$$

Among these sixteen abstract marble structures, one is special: the one that assigns redness just to marbles which are really red, and that assigns circularity to those marbles which are really circular. Call this the *real* marble structure. In general:

> If S is a marble P-structure, S is **real** iff, for each property F in P, the S-extension of F includes just the F marbles.

At any world there is exactly one real marble P-structure (given Marble Set), though generally there are many abstract marble structures.

Structures provide a way of precisifying the idea of a "pattern of properties", and articulating claims about their possibility. What we still need to spell out is what it is for a possible world to *realize* a certain pattern.

Remember, realizing a pattern shouldn't require that any particular marbles instantiate these properties: the pattern principle we are articulating is not haecceitistic. For example, for the pattern represented by S_1 to be metaphysically possible requires that at some world there are two things, neither of which is circular and just one of which is red. For S_2, we require a world at which there are two red things, just one of which is circular. What is it for two structures with different domains to represent the same qualitative pattern? The standard answer is *isomorphism*. Structures are isomorphic iff there is a one-to-one correspondence between their domains

[12] The idea that combinatorial principles can be articulated in terms of structures has been suggested from time to time, but not worked out at the level of detail required for the results we will be investigating. (E.g., Bricker 1991, 608; Hazen 2004, 332.)

which respects the extension of each property in P. (See appendix A for more official statements of standard definitions.)

But also, for a world W to realize the pattern represented by S_1 shouldn't even require that Mary and Marvin *exist* at W. If they don't, then it's plausible that the structure S_1 also does not exist at W. In general, we want to make "cross-world" structural comparisons, while allowing that particular marbles and marble-structures may only exist contingently. Our strategy is to appeal to an intermediary structure whose existence is not contingent. For this purpose we will make the natural assumption that the size and structure of the universe of *pure sets* does not vary from world to world.[13] This assumption is not unassailable; but if it fails, we take this to mainly make trouble for *expressing* the idea of cross-world isomorphism. Presumably this is an idea that one would like to make sense of somehow or other.[14]

Let a **pure structure** be a structure whose domain is a pure set. Given that the universe of pure sets is fixed, the existence, size, and isomorphism facts for pure structures are also fixed. Putting these ideas together, we can now say what it is for a possible world to realize a certain pattern of properties.

> A world W **realizes** a P-structure S iff, for some pure structure S' which is isomorphic to S, at W, the real marble P-structure is isomorphic to S'.

Using this definition, we can do what we set out to do, making precise the generalization of Property Pattern to arbitrarily many properties. Let P be any set of properties.

***P*-Pattern** At each world, for any marble P-structure S, some world realizes S.

[13] In addition to assuming that the existence, elements, and identities of pure sets do not vary from world to world, we also make another more technical assumption. Counterpart theorists typically reject Closure (see footnotes 3 and 7). Perhaps at W, Marvin is red (thanks to one of Marvin's counterparts), and at W, Marvin is green (thanks to another counterpart), but it's not the case that at W, Marvin is red *and* green. But even the counterpart theorist should accept Closure for the special case of *qualitative* statements, which make no reference to any particular individuals: in that case, counterparts are inert. More generally, we also will assume Closure for the special case of statements which make reference to no particular objects other than pure sets, possible worlds, or properties and relations. The counterpart theorist may still wish to resist this assumption: perhaps even abstracta bear non-trivial counterpart relations. But exploring this view would raise extra technical complications; we leave it to others.

[14] An alternative kind of non-contingent structure is available to *Necessitists*, who hold that (necessarily) everything exists necessarily (e.g. Williamson 2002). Necessitists can make sense of cross-world isomorphism without any need for special pure structures. For related discussion see Fritz (2013); Fritz and Goodman (2016).

Note that, using our definition of "realizes" and our background assumptions about pure sets, we can derive a useful principle which is analogous to Possibilities, without the tendentious de re commitments: each possible world tells a story which is unique *up to isomorphism*.[15]

Structural Possibilities At any world, for any world W and P-structures S_1 and S_2, if W realizes S_1 and W realizes S_2, then S_1 and S_2 are isomorphic.

Forrest and Armstrong's "paste" principle was about duplication and parts. We've noted that for our purposes it's more perspicuous to discuss a different, more abstract relation between worlds: *disjoint embedding*. An embedding is, intuitively, an isomorphism between one structure and *part* of another. This is a one-to-one (but not necessarily onto) function from the domain of one structure to the domain of another, which respects the extension of each P-property. If there is an embedding from S_1 to S_2 then we say S_2 **embeds** S_1. A structure S^+ **disjointly embeds** a family of structures iff there are embeddings of each of them into S^+, such that the ranges of the embeddings of different structures in the family have no elements in common. (Again, see appendix A for more careful statements.)

We can also naturally extend these notions from structures to worlds.

> A world W **embeds** a world V (with respect to P) iff W realizes some P-structure S and V realizes some P-structure T such that S embeds T.
>
> A world W^+ **disjointly embeds** a set of worlds Ω (with respect to P) iff W^+ realizes some P-structure S^+, each Ω-world W realizes some P-structure S_W, and S^+ disjointly embeds the family of structures S_W for $W \in \Omega$.

(Earlier we appealed to *Enough Marbles*: if W^+ disjointly embeds Ω, and at each Ω-world there is at least one marble, then at W^+ there are at least as many marbles as Ω-worlds. This can now be derived from the definition of disjoint embedding.)

Now we can generalize Paste to arbitrary sets of properties:

***P*-Paste** For any set of worlds Ω, there is a world W^+ that disjointly embeds Ω (with respect to P).

[15] Suppose that W realizes each of S_1 and S_2. That is, there are pure structures T_1 and T_2 isomorphic to S_1 and S_2, respectively, such that at W there is a real marble structure is isomorphic to T_1, and also at W there is a real marble structure is isomorphic to T_2. We also know that at W, there is at most one real marble structure. Since these truths-at-W make reference to no objects other than pure structures, by our qualified version of Closure (see footnote 13), it follows that at W, T_1 and T_2 are isomorphic to one another. Since isomorphism facts for pure structures do not vary from world to world, T_1 is isomorphic to T_2, and thus S_1 is isomorphic to S_2.

With these definitions, we can state a possibility result that generalizes Nolan's observation. It turns on the existence of a certain sort of "universal" structure:

Theorem 1 Let P be any set of monadic properties. There is a P-structure U which disjointly embeds isomorphic copies of every P-structure no larger than U.

(To be explicit, the size of a structure is the cardinality of its domain.) Here's the idea of the proof of Theorem 1. Each element of a structure has a certain profile of properties—a certain subset of P which includes just the properties that apply to that element. We can characterize a P-structure (up to isomorphism) just by specifying how many elements it has with each different profile of properties. Call this specification—a function from subsets of P to cardinal numbers—the structure's *global profile*. We can find a suitable infinite cardinal κ so that there are only κ different global profiles for structures no bigger than κ. Then we can glue together one representative structure for each of these κ different global profiles in one big structure, whose size is $\kappa \times \kappa = \kappa$. See appendix A for further details.

We can use Theorem 1 to give a model for P-Pattern and P-Paste (along with the other background assumptions). The idea is that there is a world W^+ that realizes the "universal" κ-sized P-structure given by Theorem 1. The set of all possible worlds includes one representative from each isomorphism type of structure with at most κ elements—satisfying P-Pattern. The structures realized by any set of worlds in this model can be disjointly embedded in the universal structure realized by W^+—satisfying P-Paste.

Thus:

> If P is a set of monadic properties, then P-Pattern and P-Paste are jointly consistent (together with World Set, Marble Set, and Possible Marble).[16]

The Forrest–Armstrong result relied on the fact that there strictly more subsets of D than elements of D. Theorem 1 shows that this does not carry over from particular sets to qualitative patterns of properties. That is, there exist sets D for which there are no more isomorphism-types of P-structures on D than elements of D.

[16] By "consistent" we mean having a (Kripke) model. A model assigns an arbitrary extension to each property in P in each world in the model: each P-property plays the role of a primitive predicate. Of course, some of these extensions may not represent realistic possibilities. For example, even if P includes both *red* and *colored*, there is no guarantee that the extension of *red* is a subset of the extension of *colored*. Likewise, even if P includes *self-identical*, there are models that leave some individuals out of its extension in some worlds. Many absurd theses count as consistent in this formal sense. This caveat applies to all of our consistency claims.

It's an important feature of this model that the grand world W^+ has even more individuals than there are properties in the set P. Indeed, the possibility result is blocked if we add a further principle:

P-Plenitude At each world, there are at least as many properties in P as there are marbles.

Then we can argue (for any set of monadic properties P):

> Given World Set, Marble Set, and Possible Marble, it follows that P-Plenitude, P-Pattern, and P-Paste are not all true.

We can use similar reasoning to the Forrest–Armstrong argument: let $\mathbf{\Omega}$ be the set of all worlds containing marbles, and let W^+ disjointly embed $\mathbf{\Omega}$. Possible Marble and P-Pattern ensure that, for each P-profile, $\mathbf{\Omega}$ includes some world at which some marble has that profile. So at W^+ there must be at least one representative marble with each P-profile, and thus at W there are at least as many marbles as there are P-profiles. Since there are strictly more P-profiles than properties in P, this contradicts P-Plenitude.

$$D \geq 2^P > P$$

Note that the assumption of *disjoint* embeddings is dispensable for this argument. If W^+ merely embeds each world in $\mathbf{\Omega}$, with no regard for disjointness, this still ensures that W^+ includes at least one marble with each P-profile. So (given the background assumptions) P-Plenitude and P-Pattern also conflict with this weaker principle, which drops the disjointness condition:

Overlapping P-Paste For any set of worlds $\mathbf{\Omega}$, there is a world W that embeds each $\mathbf{\Omega}$-world.

But P-Plenitude is not obviously a well-motivated constraint. For instance, if P is a set only containing fundamental qualitative monadic properties, then there might well not be that many of them.[17]

Here are the main observations so far. Forrest and Armstrong's argument raised a problem for a *haecceitistic* package of recombination principles. These haecceitistic

[17] The distinction between Overlapping P-Paste and P-Paste is analogous to the distinction Nolan (1996, 241–42) draws between the "Principle of Recombination" and the "Stronger Principle of Recombination".

principles are alien to Lewis's worldview—and in any case, it's more natural to understand combinatorialist slogans about "patterns of fundamental properties and relations" as concerning *qualitative* patterns. Moreover, we've seen that qualitative patterns of monadic properties don't lead to the kind of cardinality explosion that drives the Forrest–Armstrong argument. So far, this style of impossibility result doesn't make serious trouble for qualitative combinatorialist views.

3 Patterns of Relations

There are no obvious logical difficulties for qualitative recombination of monadic properties. But when we extend these combinatorial ideas to relations, things look quite different.

We'll now let the signature P contain not just monadic properties, but relations of any finite adicity. (We'll count properties as "monadic relations".) Call P **monadic** iff it contains only monadic properties; otherwise, if P contains at least one relation of adicity at least two, P is **polyadic**. In this more general case, a P-**structure** is a pair of a domain together with a function that takes each n-adic relation F in P to some set of ordered n-tuples of elements of D (the **extension** of F). Extending the notions of isomorphism and embedding to polyadic signatures is routine; explicit definitions are provided in appendix A for reference. With these more general definitions in place, the statements of the principles in section 2—specifically, P-Pattern, P-Paste, and Overlapping P-Paste—make sense for for a polyadic signature P without any further modification. But it turns out that these generalized principles stand in importantly different relationships. In fact, whereas the Forrest–Armstrong-style argument involving monadic *properties* didn't end up presenting any problem for qualitative pattern and paste principles, the same kind of argument using *relations* raises serious trouble.[18]

The argument turns on the following mathematical fact, which is a kind of generalization of Cantor's Theorem.

Theorem 2 Let P be a polyadic signature. For any set D there are strictly more

[18]Kit Fine pointed out that in this respect recombination principles are closely analogous to *abstraction principles* of the sort that play a role in neo-Fregean philosophy of mathematics. "Monadic" abstraction principles, such as Hume's Principle for cardinal numbers, are consistent, while similar "dyadic" abstraction principles, such as the analogous principle for ordinal numbers, are inconsistent. Furthermore, the proofs of both of these facts are closely related to ours. (See Boolos 1998, 184 and 222.)

non-isomorphic P-structures with domain D than elements of D. In short,

$$\mathrm{Iso}_P D > D$$

This points to a crucial contrast between the monadic and polyadic cases: as we have seen, Theorem 1 implies that this statement is false if we replace "polyadic" with "monadic". We'll sketch two different proofs, using different ideas. (The two proofs also lead to interestingly different strengthenings of Theorem 2.) Details are again provided in appendix A.

For a warm-up, suppose D is countably infinite, consider a single dyadic relation *zapping*, and consider a structure S where zapping has the structure of the ordering of natural numbers (so a "zero" marble zaps everything in D, including itself, another marble "one" zaps everything in D except zero, and so on). This order structure is **rigid**, in the sense from model theory: there is only one isomorphism from S to S—namely, the identity function, which takes each element of D to itself.[19]

Next, suppose that in addition to the relation of zapping, P also contains one monadic property, *redness*. How many ways are there to distribute redness over the order structure given by S? That is, how many different ways are there of extending the {zapping}-structure S to a {zapping, redness}-structure? One way assigns redness to just the first thing, another assigns redness to just the odd-numbered marbles, and so on. In general, for each set X of marbles in S, there is a corresponding P-structure S_X which has the same zapping structure as S, and which has X as the extension of redness. Furthermore, each of these structures is qualitatively distinct.[20] So there are at least as many isomorphically distinct P-structures as subsets of D—and thus, by Cantor's Theorem, strictly more such structures than elements of D.

We can get from here to Theorem 2 in two steps. The first step is to notice that the extra monadic property of redness wasn't really needed. Instead of varying structures according to which elements are red, we can do the same thing with harmless modifications of the zap-ordering itself. In the original ordering based on the natural numbers, every element zaps itself. For each set X, we can come up with a modified ordering S_X where, instead of coloring the individuals in X red, we switch off their self-zapping. This modified ordering still naturally matches marbles

[19] In other words, a rigid structure is one that has no non-trivial automorphisms, where an *automorphism* is an isomorphism from a structure to itself, and the *trivial* automorphism is the identity function. This use of "rigid" is unrelated to the modal meaning common among philosophers.

[20] This is because any isomorphism f from S_X to S_Y has to be, in particular, an isomorphism of the underlying zap-structure, which means that f must be the identity function, and so, since f preserves the extension of redness, $X = Y$.

up with natural numbers. (The marble at position n is the one which zaps everything except for the marbles in positions before n, and perhaps itself.) In particular, S_X is still *rigid*, which is what we needed for the argument.

The second step is to note that the assumption that there are only countably many marbles was inessential. In fact, a standard fact of set theory—the **Well-Ordering Principle** (which is equivalent to the axiom of Choice)—says that *any* set can be ordered in a way which has the same rigidity property as the natural numbers ordering.[21]

The second proof of Theorem 2 uses a different fundamental theorem from set theory, concerning ordinals (for this formulation see e.g. Clark 2016):

Burali-Forti's Theorem For any cardinal κ, there are strictly more than κ isomorphically distinct well-ordered sets with at most κ elements.

For example, let κ be countable infinity, and consider the dyadic relation of zapping. There are many ways of putting non-isomorphic well-ordering zap-structures on countable sets. It could be an ω-sequence (structured just like the natural numbers), or an ω-sequence with an extra element at the tail, or two copies of an ω-sequence end to end, or an ω-sequence of end-to-end ω-sequences, and so on. What we learn from Burali-Forti is that in fact there are *uncountably* many qualitatively different order structures; and furthermore this generalizes from the countable case to any size.

Theorem 2 has this important consequence for recombination.

> Let P be polyadic. Given World Set, Marble Set, and Possible Marble, then P-Pattern and P-Paste are not both true.

This follows from Theorem 2 the same way that the Forrest–Armstrong result followed from Cantor's Theorem. If Ω is the set of worlds containing marbles, and W^+ disjointly embeds Ω, then at W^+ the set of marbles D is at least as large as Ω, which (by P-Pattern and Structural Possibilities) has at least as many elements as there are non-isomorphic structures with domain D, which by Theorem 2 is strictly larger than D. In short:

$$D \geq \Omega \geq \mathrm{Iso}_P D > D$$

[21] As it turns out, the principle we really need here—that every set is the domain of some rigid relational structure—is strictly weaker than Choice. Like Choice, though, it is independent of ZF set theory. See Hamkins and Palumbo (2012).

Contradiction, QED.[22]

So there's an important disanalogy between recombination for relations and recombination for monadic properties. Any set allows more qualitative patterns of a dyadic relation than it has members; but the same isn't true for qualitative patterns of monadic properties. So while qualitative recombination principles may remain well-behaved for monadic properties, analogous principles for relations lead to combinatorial explosion.

It's striking that the requirement of *disjoint* embeddings in P-Paste plays an essential role in this impossibility result (unlike the P-Plenitude result in section 2). In particular, the argument that at W^+ there are at least as many marbles as Ω-worlds (i.e., $D \geq \Omega$) requires that the "paste" world has at least one distinct marble representing each pattern. If the embedded structures are allowed to overlap, then this is not guaranteed. In fact, we can show that if P-Paste is weakened to Overlapping P-Paste, combinatorial explosion is once again averted. This is due to another striking mathematical fact.

Theorem 3 If P is finite, then there is a countable P-structure that embeds every countable P-structure.

Since the proof of this result is more involved, we defer it to appendix A. The proof uses a construction from model theory called a *Fraïssé Limit*.[23] In the special case of a single dyadic relation, this "universal" structure is called the **random graph**, on account of one of its striking properties (Erdős and Rényi 1963; Rado 1964). Let D be a countable set. Suppose an angel visits each ordered pair of elements (d_1, d_2) in D and flips a fair coin: if it comes up heads, d_1 zaps d_2, and otherwise not. Once every pair has been visited, we have a certain zapping-structure with domain D. It turns out that with probability one, the resulting structure is isomorphic to the random graph. (This property can also be used to provide an alternative existence proof for the structure in question—using the fact that a set of measure one must be non-empty!)

[22] Note that the proof of Theorem 2 using Burali-Forti's Theorem shows something stronger: a weaker principle than P-Pattern also conflicts with P-Paste. Namely:

Order Pattern At any world, for any marble structure S which is a well-ordering, some world realizes S.

The proof of Theorem 2 using Cantor's Theorem also shows something stronger: not only is $\mathrm{Iso}_P D > D$, but in fact $\mathrm{Iso}_P D \geq 2^D$.

[23] Thanks to Alex Kruckman on `http://math.stackexchange.com` for pointing us in the right direction.

Using Theorem 3, we can argue:

> If P is finite, then P-Pattern and Overlapping P-Paste are consistent (together with World Set, Marble Structure, and Possible Marble).

Once again, we can provide a model by letting the worlds include one representative from each isomorphism type of countable P-structures—thus satisfying P-Pattern. These include a world W^+ that realizes the "universal" structure from Theorem 3, which embeds every world—thus satisfying Overlapping P-Paste.

In fact, a stronger version of Theorem 3 also holds, which generalizes beyond the finite case. (See appendix A.)

Theorem 4 If κ is an inaccessible cardinal and $P < \kappa$, there is P-structure of size κ that embeds every P-structure of size at most κ.

So, for an arbitrary set of relations P, given the existence of a sufficiently large inaccessible cardinal, P-Pattern can be maintained consistently with Overlapping P-Paste.

4 Pluralities of Worlds

So far our arguments have relied heavily on two set-theoretic background assumptions: World Set and Marble Set. Nolan (1996) and Sider (2009) have thought that these were the key culprits that make trouble for recombination—and it's true that rejecting them does evade one kind of combinatorial argument (see also Pruss 2001). But as we'll see in this section, the main impossibility result of section 3 does not essentially turn on these set-theoretic assumptions.

Let's begin by examining Daniel Nolan's independent argument against the combination of World Set and Marble Set. His basic observation (adapted to our terminology) is that World Set and Marble Set are jointly inconsistent with this principle:

Sizes For any cardinal κ, at some world there are at least κ marbles.

Here's a version of the argument. Suppose Marble Set: so for each world W, there is a certain cardinal which at W is the number of marbles. Call any such cardinal a "world cardinal". Suppose World Set: then there is a set of all world cardinals.[24]

[24] By the axiom of Replacement.

For any set of cardinals, there is a cardinal κ strictly greater than each of them.[25] Then no cardinal greater than or equal to κ is itself a world cardinal—contradicting Sizes, QED.[26]

Why might the combinatorialist accept Sizes? As Nolan presents things, Sizes follows from a combinatorial principle that Lewis entertained (without quite endorsing): "for any objects in any worlds, there exists a world that contains *any number* of duplicates of all of those objects" (Nolan 1996, 239, our emphasis; paraphrasing Lewis 1986, 89). This leaves off Lewis's caveat: "size and shape permitting". What Nolan recommends is that we don't impose the caveat, and instead drop the combination of World Set and Marble Set.

(There are two ways this could go. One way would be to keep Marble Set, but say that for each cardinal, there is a world with that many marbles—and thus too many worlds to form a set. The other way would be to say that there is at least one vast world containing too many marbles to form a set—which all by itself would suffice to make Sizes true.)

An alternative way of motivating Sizes uses our ideology of structures, rather than the Lewisian ideology of duplication. Sizes follows from this principle:

Copy For any cardinal number κ and for any world W, there is a world W^+ that disjointly embeds at least κ isomorphic copies of the structure realized by W.

Just as with Sizes, the Copy principle is fine as long as we don't try to maintain both Marble Set and World Set.

[25] This follows from a combination of the axiom of Unions—getting us an upper bound for the cardinals—and Power Set—getting us a cardinal strictly greater than that upper bound.

[26] This version of the argument has the advantage of not relying on either Lewisian modal realism or Williamsonian Necessitism. (compare Uzquiano 2015.)

Nolan's version of this argument was a bit simpler, but depends on more contentious background assumptions. Suppose that there is a set of *all possible marbles*. Then this set has a certain cardinality, and there is another cardinal κ which is greater yet. Then Sizes says there is some world in which there are at least κ marbles—and so there are at least κ possible marbles, contradiction, QED. Note that this version of the argument relies on this inference:

If at some world W, there are at least κ marbles, then there are at least κ possible marbles.

This inference is unproblematic in the Lewisian framework in which Nolan is working. It is similarly unproblematic in the Necessitist framework defended by Timothy Williamson, according to which, if there could have been at least κ marbles, then there are at least κ things which could have been marbles (see Sider 2009). But the relevant inference is rejected by Contingentists, who hold that it is contingent which things are anything at all. The version of the argument we presented doesn't rely on this contentious inference.

(It does still rely on a restricted and less contentious Necessitist assumption: namely, that it is not contingent what *cardinals* there are.)

So giving up one or both of these set assumptions—World Set and Marble Set—is a perfectly good way of escaping Nolan's problem of Sizes. But the problem of recombinable relations we presented in the previous section is not like that: as we'll now show, it does not rely in any essential way on either of these set-theoretic assumptions. We can restate a generalized version of the argument using plural quantification.[27]

Our formulations of *P*-Paste and *P*-Pattern both involved structures whose domains and extensions were sets. But it's natural to extend the underlying idea to versions which—in the absence of World Set and Marble Set—are stronger. Consider a dyadic relation of *zapping*. A standard structure for zapping specifies a set of ordered pairs as its extension. If there is no set of all marbles, then there are ways of distributing zapping over pairs of marbles where those pairs are too numerous to form a set—and thus they don't comprise a zapping-extension in any structure, in the standard sense we presented in section 2. But we can still naturally extend the idea of a zapping-pattern to this case, by replacing singular quantification over structures with plural quantification over ordered pairs. And we can show that, in a naturally extended sense, it's still true that there are strictly more isomorphically distinct *patterns of pairs* of marbles than there are marbles.

The rest of this section will provide more detail about how this works. What we need to do is translate standard structure-theoretic talk into plural language. This requires some care, but it is essentially straightforward.

(In what follows, we'll use the plural quantificational expression "there are zero or more", rather than the alternative "there are one or more". Nothing essentially turns on this choice, but it makes certain results easier to state. Apparent singular quantification over "pluralities" is just a heuristic shorthand for more serious plural talk.)

A structure, as we defined it before—or to be more explicit in what follows, a **set structure**—was defined as a pair of a certain set—the domain—and an extension function—a function from relations to sets of *n*-tuples. Instead of quantifying over these set-theoretic objects, we can instead quantify plurally over the things *in* the domain and the tuples in the extensions, directly. Where before we said "there is a structure *S* such that …", instead we can say "there are some things, the *S-domain-things*, and there are some tuples, the *S-extension-tuples*, such that …". For a general signature, we can think of a "plural structure" as an *indexed family* of pluralities: a plurality for the domain, and a plurality of *n*-tuples for each *n*-place relation.

[27] We use standard plural logic (with full impredicative comprehension). (See e.g. Linnebo 2014.) We also assume that pluralities of abstracta are fixed, in this sense: if each of the *X*'s is a pure set or a possible world, then *x* is one of the *X*'s iff at every world *x* is one of the *X*'s.

While sets of sets are unproblematic, orthodox plural logic does not provide us with the resources to straightforwardly quantify over "pluralities of pluralities". But there is a coding trick that lets us get around this obstacle in certain cases.[28] Suppose that for each i among the I's, there are certain things, the X_i's: then we can think of this as a *family of pluralities* indexed by the I's. (This is also called a *class-valued function* from the I's.) We can encode an indexed family of pluralities like this using a plurality of pairs. In the case at hand, for each n-place relation F (in P) we want to represent a "plural extension" for F, which consists of certain n-tuples, the S_F's. We can do this using a plurality of ordered pairs, the S's, such that each of the S's is an ordered pair of some n-place relation F, and some n-tuple. Then the S_F's are those n-tuples $(d_1, \ldots, d_n)$ such that $(F, (d_1, \ldots, d_n))$ is one of the S's.

We also want to represent a domain: we can do this by picking some canonical object Dom which is not a relation (for example, the word "domain" or the number 0), and include among the S's some ordered pairs whose first element is Dom. Then the S-domain consists of those things d such that (Dom, d) is one of the S's.

In general, we'll say the X's **code a family of pluralities indexed by the** I's iff the X's are ordered pairs each of which has one of the I's as its first element. Then for any i among the I's, we can let the X_i's be those things x such that (i, x) is one of the X's. So, in particular, the S's **code a plural P-structure** iff they code a family of pluralities indexed by the relations in P together with Dom, where for each n-adic relation F in P, each of the S_F's is an n-tuple of things among the S_{Dom}'s. The S_{Dom}'s are the S-**domain**, and the S_F's are the S-**extension** of F.

In what follows, it will sometimes be convenient to speak singularly, saying "there is a plural P-structure S such that …". But it's important to bear in mind that, like talk of "pluralities" or "families", this is intended to be cashed out plurally, not as singular quantification over any kind of *object* which is itself a plural structure.

We can similarly extend other structure-theoretic notions to the plural case, such as isomorphism and disjoint embedding. For instance, if S and S' are plural structures, we'll say the X's **code an isomorphism** from S to S' iff each of the X's is an ordered pair (d, d') where d is in the S-domain and d' is in the S'-domain, and these pairs satisfy suitable conditions. The details are straightforward, but tedious, so we'll relegate them to appendix B.

In order to compare structures across worlds, it will again be helpful to appeal to "fixed" structures. Once again, we will deploy a fixed universe of pure sets for this purpose. Let a **pure plural structure** be a plural structure each of whose domain-

[28] This coding trick, from Paul Bernays, takes advantage of the Curry correspondence between $I \to 2^X$ and $2^{I \times X}$. (See Uzquiano 2015, 9)

things is a pure set. Then we can define realization as before. At any world, the **real marble plural structure** S has as its domain all of the marbles, and for each relation F in P, the S-extension of F consists of just the n-tuples of marbles that stand in F. Then if W is a world, and the S's code a plural P-structure:

> W **realizes** S iff, for some pure plural P-structure S' which is isomorphic to S, at W, the real marble plural structure is isomorphic to S'.

Now we can state our generalized pattern principle:

Plural Pattern At any world, for any S's that code a plural structure whose domain consists of the marbles, some world realizes S.

We can also straightforwardly extend the definition of disjoint embedding (with respect to a signature P) to pluralities of worlds. (Again, this is in appendix B.) As in the set case, the definition has the following important consequence. Let a **marble-world** be a world at which there is at least one marble.

Enough Marbles If each of the W's is a marble-world, and W^+ disjointly embeds the W's, then at W^+ there are at least as many marbles as the W's.

(As is standard, plural cardinal comparisons can be spelled out in terms of pluralities of pairs: at W^+ there are some ordered pairs that code a one-to-one function from the marbles to the W-worlds.)

Now we can state a plural generalization of our Paste principle:

Disjoint Plural Paste For any worlds, the W's, there is some world W^+ such that W^+ disjointly embeds the W's.

The key point is that these plurally generalized recombination principles face exactly the same difficulty as the set-theoretic versions. We can now adapt our impossibility result from section 3 to show:

> Global Choice, Possible Marble, Plural Pattern and Disjoint Plural Paste are not all true.

The main idea of the argument is the same as before. Consider a single dyadic relation of *zapping*. Given Disjoint Plural Paste, there is a world W^+ that disjointly embeds all of the marble-worlds—those worlds at which there is at least one marble.

By Enough Marbles, at W^+ there are at least as many marbles as marble-worlds. Possible Marble ensures that there is at least one marble-world, and so at W^+ there is at least one marble. Plural Pattern says that at W^+, for any pairs of marbles, some world realizes that zapping pattern—and in particular, this is a marble-world. So there is a distinct marble-world for each isomorphically distinct way of choosing pairs of marbles.[29] Thus at W^+ there are at least as many *marbles* as *patterns of pairs of marbles*. But as we'll show, this is impossible.

What remains to be shown is that there are strictly more patterns of pairs of marbles than there are marbles. To show this, we can use a plural generalization of Cantor's Theorem (Bernays 1942). This theorem formalizes the intuitive idea that there are strictly more pluralities of marbles than there are marbles. We can state this using the same trick for coding *indexed families of pluralities*. The idea is that no family of pluralities indexed by D's can include *every* plurality of D's.

Cantor–Bernays Theorem Let the X's code a family of pluralities indexed by the D's. Then there are (zero or more) D's which are not the X_d's for any d.

Bernays' proof is an easy application of the usual Cantor–Russell trick: let the R's be the (zero or more) things d such that d is not among the X_d's. Suppose for reductio that for some d, the X_d's are the R's. Then it follows that d is not among the R's. That is, d is among the X_d's, and so by construction d *is* one of the R's, which is a contradiction.

We can extend Bernays' result about pluralities to an analogous result about plural structures. This formalizes the intuitive thought that there are more isomorphism-types for a relation on marbles than there are marbles.

Theorem 5 Let P be a polyadic signature. Let the S's code a family of plural P-structures indexed by the D's. (That is, for each d among the D's, the S_d's code some plural structure.) If the D's can be well-ordered, then some plural structure on the D's is not isomorphic to S_d for any d.

The proof is a straightforward "pluralization" of the proof of Theorem 2 we gave in section 3. For details, see appendix B.[30]

[29] This step relies on the plural analogue of Structural Possibilities, which can be shown in the same way as the set version from our background assumptions about the fixity of pluralities of pure sets.

[30] Here is a technicality. (Thanks to Daniel Nolan for very helpful discussion.) To derive our impossiblity result from Theorem 5, we need the further claim that, in any possible world, the marbles can be well-ordered. While the fact that any *set* can be well-ordered is equivalent to the set-theoretic

5 Unrestricted Patterns

So far we've been exploring the difficulties that arise for the combination of two different combinatorial ideas. First, the "pattern" idea,

> Any pattern of instantiation of any fundamental properties and relations is metaphysically possible.

Second, the "cut and paste" idea,

> For any objects in any worlds, there exists a world that contains any number of duplicates of all of those objects.

Let's now consider the prospects for the "pattern" idea taken on its own. This turns out to be very powerful.

Let P be some arbitrary polyadic signature: some set of properties and relations, including at least one relation which is not a monadic property. In this section "structures" are to be understood as structures with signature P.

We have considered two ways of spelling out the "pattern" slogan. First:

Marble Set Pattern At any world, for any (set) structure S whose domain is the set of marbles, some world realizes S.

(In section 2 we called this "P-Pattern".) The second way (which in section 4 we called Plural Pattern) generalizes from set structures, which are limited in size to what can be contained in a single set, to a pluralized version that is not so limited.

axiom of Choice, it turns out that the plural generalizations of these principles—*Global Well-Ordering* and *Global Choice*—are not equivalent: in fact, there are models of Global Choice without Global Well-Ordering (Howard, Rubin, and Rubin 1978).

Still, Global Choice does imply that any plurality of *pure sets* can be well-ordered (see e.g. Linnebo 2010, 161–62). It follows that, if there are no more X's than pure sets, then the X's can be well-ordered. Furthermore, recall that we defined "realizes" in terms of isomorphism with a pure plural structure. It follows from this definition that any plural structure with a domain outnumbering the pure sets is unrealizable. So in fact, Plural Pattern implies that there are no more marbles than pure sets. (See our discussion of Limitation of Size in section 5).) Given this, Global Choice implies that the marbles can be well-ordered.

More generally, Howard, Rubin, and Rubin (1978) show that many different formulations of Choice-like principles whose set-theoretic formulations are equivalent can subtly come apart in the context of plural logic. Fortunately for us, these subtleties shouldn't matter so long as there are no more marbles than pure sets: any standard plural formulation of Choice should do as far as Theorem 5 is concerned.

Marble Plural Pattern At any world, for any plural structure S whose domain consists of all marbles, some world realizes S.

(Recall that apparently singular quantification over plural structures, like "plurality"-talk, is cloaked plural quantification: the variable S here is really a plural variable.)

Each of these Pattern principles has consequences that go beyond those of Lewis's "cut and paste" duplication principle. The duplication principle does not guarantee that, if there could be a red marble and a square marble, then there could be a red square marble; nor does it guarantee that if there could be a zapping pair of marbles, then there could be marbles that don't zap one another, or that any marble could zap itself (see also Wilson 2015, 148). By contrast, Marble Set Pattern and Marble Plural Pattern each imply all of these conditionals (assuming the signature P includes redness, squareness, and zapping).

That said, the duplication principle has consequences that the Pattern principles by themselves do not secure: if there could be one marble, then there could be three, or infinitely many, or indeed κ-many for any cardinal κ. That is, if there could be at least one marble, then this principle we discussed in section 4 follows:

Sizes For any cardinal κ, at some world there are at least κ marbles.

But neither Marble Set Pattern nor Marble Plural Pattern implies Sizes. In fact, for any non-zero cardinal κ, both Marble Pattern principles are consistent with there being at most κ marbles at any world. Unlike "cut and paste", these Pattern principles (on their own) don't give us any way of getting larger worlds from smaller ones. This suggests that they don't entirely do justice to the picture that motivated the combinatorial slogan about "any pattern of instantiation".

We can do better by slightly modifying the Pattern principles. Notice that in the statement of these principles, marbles are really playing two distinct roles. One role is as possible "realizers" of structures. The other role is as "generators" of structures. The pattern principle roughly says: any abstract structure could be concretely realized. The abstract structures are generated using a domain of objects, and a set of properties and relations. But there is no obstacle to using, say, *numbers* to generate an abstract structure that can be concretely realized by, say, *people*. For example, an abstract structure for the *loving* relation with a domain of numbers just amounts to a set of ordered pairs of numbers. Notice that the existence of such structures has nothing to do with whether numbers are capable of love. For some *people* to realize this structure just requires that the pattern of loving among them be isomorphic to those pairs of numbers.

The key point is that even if we are using marbles as "realizers", this doesn't preclude us from using different things as "generators". So we might choose generators that exist in great multitudes—like numbers. If every structure generated by such a multitude can be realized by marbles, then in particular, there can be a multitude of marbles. This motivates strengthening Marble Set Pattern as follows:

Unrestricted Set Pattern At any world, for every (set) structure S, some world realizes S.

(Compare Bricker's principle (P1) 1991, 612.) This formulation simply drops the words "whose domain is the set of marbles" from Marble Set Pattern. The principle says that any structure with *any* set-sized domain will do. Note that Sizes immediately follows from this.

Recall that in section 2 we defined "realizes" in terms of *pure* structures (to allow cross-world comparisons), and we are assuming that it is not contingent what pure sets there are. This means we can make two simplifications to Unrestricted Set Pattern without losing any power: we can restrict the structure-quantifier to *pure* structures, and we can drop the words "at any world". Given our background assumptions, this version is equivalent:

Pure Set Pattern For every pure set structure S, some world realizes S.

There are as many distinct isomorphism types of pure set structures as pure sets.[31] Thus Pure Set Pattern implies that there are at least as many worlds as pure sets, and thus the worlds are themselves too plentiful to form a set (by the axiom of Replacement). So Unrestricted Set Pattern is inconsistent with the principle World Set.

Even so, Unrestricted Set Pattern is consistent taken on its own. It is also consistent together with the principle Marble Set (which, recall, says that at each world there is a set of all marbles). One way this could be is if for each pure set structure S there is a possible world W_S that realizes S, and these are all of the possible worlds. This would clearly satisfy Pure Set Pattern, and thus Unrestricted Set Pattern.

In previous sections we've considered four different "paste"-style principles. (In section 2 we used the names "P-Paste" and "Overlapping P-Paste" for the first two.)

[31] Indeed, there is an isomorphically distinct structure for each ordinal, and there are as many ordinals as sets.

Disjoint Set Paste For any set of worlds Ω, some world realizes a set structure that disjointly embeds set structures realized by each world in Ω.

Overlapping Set Paste For any set of worlds Ω, some world realizes a set structure that embeds a set structure realized by each world in Ω.

Disjoint Plural Paste For any worlds, the *W*'s, some world realizes a plural structure that disjointly embeds a family of plural structures realized by the *W*'s.

Overlapping Plural Paste For any worlds, the *W*'s, some world realizes a plural structure that embeds a plural structure realized by each *W*-world.

What is the upshot of Unrestricted Set Pattern for these various principles?

First, suppose Marble Set is true. Unrestricted Set Pattern and Marble Set together imply that both Plural Paste principles are false: no single set of marbles is big enough to embed every set-sized marble pattern. But Unrestricted Set Pattern and Marble Set also jointly imply that both Set Paste principles are *true*: each *set* of worlds can be disjointly embedded in some world. (This is because any set of set-structures has a *disjoint sum*: see appendix A.)

Things are a bit messier if Marble Set is false. In that case, some world doesn't realize any set structure at all—it has too many marbles for that. So both Set Paste principles come out trivially false. Also, without Marble Set, Unrestricted Set Pattern does not imply either of the Plural Paste principles, nor their negations.

Let's now consider the stronger *plural* version of this unrestricted pattern principle:

Unrestricted Plural Pattern (UPP) At any world, for any plural structure *S*, some world realizes *S*.

Again, because we defined "realizes" in terms of non-contingent pure plural structures, UPP implies that every plural structure is isomorphic to some *pure* plural structure. In particular, UPP implies

Limitation of Size At any world, there are no more things than pure sets.

Against our set-theoretic background, this principle is equivalent to (the necessitation of) an influential proposal from Von Neumann: some things form a set iff they are not in one-to-one correspondence with everything.[32] (For discussion of

[32] Note also that Limitation of Size implies Global Choice and Global Well-Ordering. (Limitation of Size puts everything in one-to-one correspondence with the ordinals; we can use this correspondence to define a global choice function.) In the presence of the Urelement Set axiom (that there is a set of all non-sets) the converse implication from Global Choice to Limitation of Size holds as well (Linnebo 2010, 151–52 and 161–162). But since we are not assuming Urelement Set, in our context Limitation of Size is in fact a stronger claim than Global Choice.

this principle's merits, see Hawthorne and Uzquiano 2011, sec. 6.3.)

In fact, UPP is equivalent to the conjunction of Limitation of Size with a restricted pattern principle (given our background assumption that it's not contingent what pure sets there are):

Pure Plural Pattern For any pure plural structure *S*, some world realizes *S*.

We noted that Unrestricted Set Pattern is inconsistent with World Set. Unrestricted Plural Pattern has more radical consequences yet for worlds: indeed, on one natural way of understanding it, the principle is inconsistent. Theorem 5 tells us that there are strictly more isomorphically distinct plural structures than *things*. But if *worlds* are things, then since UPP requires that there are as many worlds as isomorphism types, this would imply that there are strictly more worlds than worlds, which cannot be. The issue here arises from the fact that UPP lets us use any sort of thing as generators—and so in particular, if worlds are things, then they can generate patterns themselves.

There are two natural ways to respond to this. One is to back off from the fully unrestricted principle, and stick to a restricted principle that doesn't allow worlds as pattern-generators: for example, Pure Plural Pattern is a natural fall-back principle. As we noted, if Limitation of Size holds, then Pure Plural Pattern is just as strong as Unrestricted Plural Pattern. But (putting this another way) if *worlds* are things, then in fact Pure Plural Pattern implies that Limitation of Size is false: there are strictly more worlds than pure sets.

The second response is to understand quantification over worlds as a *façon de parler*—just as we have done with quantification over *pluralities* or *plural structures* or *families* or *isomorphism types*. The idea is that there aren't any such *objects* as worlds; but rather, this is a convenient shorthand for some other more perspicuous idiom. If our goal is just to restate UPP, then this could be the idiom of familiar modal operators ("boxes and diamonds"):

> Necessarily, for any plural structure *S*, possibly *S* is realized.

If worlds aren't things, then they can't be used as generators for structures, and so collapse is averted.

Paraphrases using modal operators won't work for every use of world-quantifiers in this paper—in particular, plural quantification over worlds and cardinal comparisons pose special challenges. For a more general solution, one might invoke some higher-order idiom, such as quantification in sentence or operator position (see Fine 1977, 137ff.). For consistency, we'll understand the principles we discuss to be officially expressed in a *sorted* language that distinguishes world-quantifiers from first-order object-quantifiers.[33]

Similar issues may arise not just for worlds, but also for other plenitudinous domains, such as propositions, properties, events, facts, etc. (We'll return to this shortly.)

Like Unrestricted Set Pattern, Unrestricted Plural Pattern implies Sizes: for each cardinal, some world has at least that many marbles. But the plural principle also generates even larger patterns yet, whose domain-things are more numerous than *any* cardinal. For instance, since there are plural patterns whose domains include *all* the pure sets, UPP implies that there could be as many marbles as there are pure sets. Thus UPP is inconsistent with Marble Set.[34]

Even though UPP is inconsistent with World Set, and also inconsistent with Marble Set, it's still consistent taken on its own (given the caveat about world-quantifiers). One way this could be is if there is one possible world realizing each pure plural structure, and no other possible worlds. This guarantees Pure Plural Pattern. If furthermore at each world there are no objects besides marbles and sets, then since at each world there are no more marbles than pure sets, Limitation of Size follows.[35] Finally, as we noted, Limitation of Size and Pure Plural Pattern together imply Unrestricted Plural Pattern.

Now let's examine how this plural-structure-based way of articulating recombination interacts with the "cut and paste" idea. It follows directly from the Cantorian argument we presented in section 4 that Unrestricted Plural Pattern is inconsistent with Disjoint Plural Paste. But all three of the other Paste principles we've considered—Overlapping Plural Paste, Disjoint Set Paste, and Overlapping Set Paste—not only are consistent with UPP, but in fact follow from UPP.

Disjoint Set Paste and Overlapping Set Paste each follow from this stronger Paste principle:

[33] We'll also need plural quantifiers for both individual and world types, as well as a sort of quantifier for "cross-categorial ordered pairs", where one element is of world-type and the other is of individual-type—or at least some surrogate for these quantifiers, such as even-higher-order relational quantification. We'll suppress these technical details to keep things readable.

[34] This depends on our background assumption that there couldn't be more pure sets than there are.

Note also that if we assume that (at every world) no marble is a set, then UPP is inconsistent with the necessity of the Urelement Set axiom. This may also put further pressure on Limitation of Size: for instance, if at each world distinct pluralities of marbles have distinct fusions, then there could be strictly more fusions of marbles than pure sets (see Hawthorne and Uzquiano 2011).

[35] This relies on Global Choice (see Uzquiano 2015, proposition 2 in the appendix).

Disjoint Paste for Not Very Many Worlds For any worlds the W's, if the W's are not more numerous than the things, there is some world W^+ that disjointly embeds the W's.

By "**Many**" we will mean as numerous as the things, and by "**Very Many**" we will mean even *more* numerous than the things. (Remember, "worlds" are not things themselves—and indeed, taken all together they are more numerous than the things. Limitation of Size says that there are Many pure sets—but not *Very* Many.) The basic reason why Disjoint Paste for Not Very Many Worlds follows from Unrestricted Plural Pattern is that—putting things a bit roughly—any not-Very-Many pluralities of things have a disjoint union, which is another plurality of things. Thus any not-Very-Many pure plural structures can be disjointly embedded in another pure plural structure (their "disjoint sum"), which suffices for Disjoint Set Paste—given Limitation of Size, since in that case every world realizes some pure plural structure.

The derivation of Overlapping Plural Paste is less obvious: this relies on a plural generalization of Theorem 4 based on Fraïssé's construction. This generalization shows the following remarkable fact (see appendix B):

Theorem 6 Given Global Choice, there is a pure plural structure that embeds every pure plural structure.

Again, since Limitation of Size implies that every world realizes a pure plural structure, this suffices for Overlapping Paste. The plural structure guaranteed by Theorem 6—we'll call it the **universal plural structure**—is a kind of mathematical pluriverse: an abstract universe that, in a sense, includes every abstract universe. If we lived in a world that realized this structure, then something very much like Lewisian modal realism would be true.[36]

Note also that the universal plural structure contains many copies of itself—in fact, as many copies as things.[37] So we also have two strong duplication-style principles that follow from Unrestricted Plural Pattern with Limitation of Size. Let a **plural part** of a world be a plural-substructure of the plural structure realized by that world, and say that a world embeds a plural structure S iff it realizes a structure that

[36]Except Lewis holds that the concrete universes are *isolated*, in the sense that no fundamental relations—or at least no "spatio-temporally analogous" fundamental relations—link non-world-mates (1986, 75–78). In contrast, the universal plural structure is not divisible into relationally isolated substructures.

[37]The basic reason for this is that "Many times Many equals Many": we can divide up a plurality of Many things into Many disjoint subpluralities of Many things. Then we can paint the universal plural structure onto each of these subpluralities.

embeds *S*. (Again quantification over "plural parts" is really shorthand for a plural quantification.)

Overlapping Plural Copy For any plural parts of any worlds, some world embeds Many isomorphic copies of each of them.

Disjoint Plural Copy for Not Very Many Parts For any not-Very-Many plural parts of any worlds, some world disjointly embeds Many isomorphic copies of each of them.

These principles are very similar in spirit to Lewis's duplication principle—but these are not extra postulates, but rather consequences of Unrestricted Plural Pattern. And unlike Lewis's version, there is no pressure to tack on any extra caveats like "size and shape permitting" to this package.

Unrestricted Plural Pattern looks to us like a promising articulation of the combinatorialist idea—"there are no gaps in logical space". But we should note that this version wasn't available to Lewis: it's integral to his vision that worlds are genuine concrete individual objects, and as we noted earlier, this conception of possible worlds is incompatible with UPP—since then worlds themselves would be generators of patterns. Putting this point another way, we have a vindication of Forrest and Armstrong's original idea, understood broadly: a combinatorial argument against Lewis's theory of possible worlds.

Unrestricted Plural Pattern also makes trouble for other metaphysical views besides Lewis's. Notice first that the argument against Lewisian modal realism doesn't essentially rely on construing worlds as *concrete*: it also applies to any view according to which worlds are particular sets, or *sui generis* abstract objects (whether these are structured "states of affairs" or unstructured simples), as long as they are something. More generally, UPP conflicts with any view according to which there are *at least as many* objects as marble-worlds. For example, you might think that for each world *W* there is a certain necessarily existing state of affairs which obtains just at *W*. Any view like this is incompatible with UPP.

Here's another example. Some philosophers, having become convinced that statues and lumps of clay can be distinct while entirely coinciding, go on to embrace "bazillion-thingism": in addition to familiar objects like statues and lumps of clay, there are many less familiar coincidents. Some are more modally fragile—like Tate-Museum-statues that are destroyed by transport—and some are more modally robust—like clay-aggregates that can survive utter dispersal (Bennett 2004, 356; see also Yablo 1987; Hawthorne 2006; Leslie 2011). One ambitious version of bazillion-thingism says that each marble *M* coincides with a distinct thing for each way of choosing either a marble or nothing from every non-actual possible world. In that case there are even more objects coinciding with M than there are marble-worlds. So this kind of plenitudinous ontology is also at odds with Unrestricted Plural Pattern. Of course, our point here is just to point out the tension between Unrestricted Plural Pattern and this kind of plenitude—we take no stand on which way it should be resolved.

We've been examining combinatorial principles based on the model-theoretic ideas of structures and isomorphisms. It's worth taking a moment to compare this to a different approach. Raul Saucedo (2011, 242–3) makes the following proposal for explicating the idea that every pattern of certain properties and relations is metaphysically possible:

> Suppose that L is a first-order language with standard logical vocabulary (the truth-functional connectives, first-order variables and quantifiers, and the identity symbol), whose non-logical vocabulary consists of only a stock of predicates. Let's assume that every n-place predicate of L expresses exactly one n-place property or relation, and that every n-place property or relation is expressed by exactly one n-place predicate of L. Then we may formulate recombination principles for properties and relations as follows:
>
> Any such-and-such sentence of L is true at some metaphysically possible world.

Saucedo discusses several ways of filling in "such-and-such"; here is one:

> Any sentence of L that has a model is true at some metaphysically possible world. (p. 245)

(This is a bit stronger than his favored version of the principle, which adds in some extra qualifications. Some of these can be handled simply by restricting which properties and relations are expressed by predicates in L.)

As Saucedo acknowledges (his footnote 14), an ordinary first-order language is expressively limited. So this recombination principle is accordingly weaker than one might wish. For example, standard metalogical results about first-order logic show that Saucedo's principle is compatible with there being no world where a certain binary relation expressed by a predicate in L has the structure of an ω-sequence. Likewise, it is compatible with every property expressed by a predicate in L having just countably many instances at every world. So this principle does not guarantee the metaphysical possibility of many perfectly good infinite patterns.

One might try to overcome these limitations (as Saucedo also suggests) by switching to a more expressive language (see also Dorr and Hawthorne 2013, 14ff.). In fact, in order to get a principle as strong as Unrestricted Set Pattern, one would have to go all the way up to a language which includes Many sentences—no ordinary set-sized language will do. To get a principle with the strength of Unrestricted Plural Pattern we have to go further yet—for instance, by considering the Very Many pluralities of sentences in such a large language.[38]

But these complications are avoidable. These are efforts to find *languages* which are expressive enough to characterize every structure. But why not simply talk about structures themselves directly, as we have done, without this detour through transfinite syntax? The structures were in the background of the sentential approach anyway, since *having a model* (that is, being true in some structure) was our test all along for which sentences are *logically consistent*, and thus, according to the sentential recombination principle, true at some world. Furthermore, as we hope we've demonstrated throughout this paper, examining structures directly can provide us with illuminating insights into the space of possible patterns.

A Set Structures

In this appendix we'll present proofs of three key model-theoretic facts we used in this paper. (1) For any set of monadic properties, there is a structure that disjointly embeds every structure of its size. (2) For polyadic relations, there is *no* non-empty structure that disjointly embeds every structure of its size. (3) In either case, there *is* a structure that embeds each structure of its size, if we don't require disjointness.

We begin with some standard definitions, for reference.

Definition 1 (a) A **signature** is a set P each element of which has some **adicity** (which is some positive natural number). As in the main text, we'll call the elements of P **relations**. A signature P is **monadic** iff each of its members has adicity one. Otherwise P is **polyadic**.

(b) A (set) P-**structure** is a pair of a set D, the **domain**, and a function that takes each n-place relation F in P to a subset of D^n, the **extension** of F.

(c) An **element** of a structure S is an element of its domain. The **size** of S (written $|S|$) is the number of its elements.

(d) Let S_1 and S_2 be P-structures with domains D_1 and D_2, respectively. An **embedding** of S_1 in S_2 is a one-to-one function $f : D_1 \to D_2$ such that,

[38] Unrestricted Plural Pattern is thus plausibly a counterexample to Dorr and Hawthorne's conjecture in Dorr and Hawthorne (2013 footnote 23).

for each n-place F in P, and for each n-tuple of elements $(d_1, \ldots, d_n)$ in D, $(d_1, \ldots, d_n)$ is in the S_1-extension of F iff $(fd_1, \ldots, fd_n)$ is in the S_2-extension of F. The structure S_2 **embeds** S_1 iff there is an embedding of S_1 in S_2.

(e) An **isomorphism** is an embedding which is also an onto function. S_1 and S_2 are **isomorphic** ($S_1 \cong S_2$) iff there is an isomorphism from S_1 to S_2.

(f) S_1 is a **substructure** of S_2 (written $S_1 \subseteq S_2$) iff the domain of S_1 is a subset of the domain of S_2, and the function the takes each element of S_1 to itself is an embedding.

Note that an embedding of S_1 in S_2 is an isomorphism between S_1 and some substructure of S_2. If S_2 embeds S_1 then clearly $|S_2| \geq |S_1|$.

Definition 2 S^+ **disjointly embeds** a family of structures S_i indexed by I iff for each $i \in I$ there is an embedding $f_i : S_i \to S^+$ such that for $i \neq j$, the ranges of f_i and f_j have no elements in common.

Some of our arguments will use the *sum* of some structures S_i, which "glues together" a family of structures without overlap. This is the minimal structure that disjointly embeds the family S_i. The domain of the sum-structure has as its domain a disjoint union of the original domains, and its extension for each relation F is the corresponding disjoint union of the S_i-extensions for F.

Definition 3 Let S_i be a family of P-structures, for $i \in I$. The **disjoint sum** $\coprod_{i\in I} S_i$ is the P-structure whose domain consists of all ordered pairs (i, d) for $i \in I$ and d in S_i, and whose extension for each n-place F in P is the set of all n-tuples $((i, d_1), \ldots, (i, d_n))$ for $i \in I$ and $(d_1, \ldots, d_n)$ in the S_i-extension of F.

The following facts about sums are clear from the definitions.

Lemma 1 A structure S^+ disjointly embeds the family of structures S_i iff S^+ embeds their disjoint sum $\coprod_i S_i$.

Lemma 2 Let $S^+ = \coprod_{i\in I} S_i$ and let $\lambda = |I|$. If S_i is non-empty for each $i \in I$, then $|S^+| \geq \lambda$. If $|S_i| \leq \kappa$ for each $i \in I$, then $|S^+| \leq \kappa \times \lambda$.

Definition 4 (a) A P-structure U is **weakly universal** iff U embeds each structure which is no larger than U.

(b) A P-structure U is **strongly universal** iff U disjointly embeds a representative of each isomorphism type of structure no larger than U.

Clearly any strongly universal structure is also weakly universal.

Theorem 1 If P is a monadic signature, there exists a strongly universal P-structure.

Proof. When P consists of just monadic properties, we can fully describe a P-structure by specifying how many elements it has with each profile of P-properties. (In the monadic case, we don't have to keep track of any connections between different elements.) If S is a P-structure, then for each d in S, let the **individual profile** of d be the set of $F \in P$ such that d is in the extension of F. Let the **global profile** for S be the function that takes each subset $Q \subseteq P$ to the number of elements in S which have individual profile Q: this is some cardinal which is at most $|S|$. Two P-structures are isomorphic iff they have the same global profile.

We can choose a cardinal κ such that there are no more than κ different global profiles corresponding to structures of size at most κ. In particular, let $\pi = |P|$, and let κ be an infinite cardinal such that

$$\kappa^{2^\pi} = \kappa$$

(If π is finite, this equation holds for any infinite κ. More generally, it holds for any inaccessible $\kappa > \pi$. It also holds for $\kappa = 2^\mu$ for any infinite $\mu \geq 2^\pi$.) Let Φ be the set of all functions from subsets of P to cardinals which are at most κ. Then Φ clearly has at most $\kappa^{2^\pi} = \kappa$ elements.

For each function $f \in \Phi$, we can choose a representative structure S_f whose global profile is f. So every structure of size at most κ is isomorphic to S_f for some $f \in \Phi$. Finally, let U be the sum $\coprod_{f \in \Phi} S_f$. Since Φ has cardinality κ and $|S_f| \leq \kappa$ for each $f \in \Phi$, by Lemma 2, $|U| \leq \kappa \times \kappa = \kappa$. (In fact, $|U| = \kappa$.) So U disjointly embeds a representative of each isomorphism type of structure no larger than U, which means that U is strongly universal. □

Now suppose P is a polyadic signature. Let $\kappa > 0$, and let S_i for $i \in I$ be a family of structures including one representative of each isomorphism type of structure with size κ. Let $\lambda = |I|$. In section 3 we proved Theorem 2: $\lambda > \kappa$. Moreover, if U is a structure that disjointly embeds the family S_i, by Lemma 1 and Lemma 2,

$$|U| \geq \left|\coprod_i S_i\right| \geq \lambda > \kappa$$

Corollary 1 If P is a polyadic signature, there is no non-empty strongly universal P-structure.

Next we'll prove the main "possibility" result from section 3: there are structures that embed *every* structure that is not too big. First, recall the following definitions.

Definition 5 (a) κ is **regular** iff any union of strictly fewer than κ sets, each of which has strictly fewer than κ elements, has strictly fewer than κ elements.

(b) κ is **inaccessible** iff κ is regular and for any $\lambda < \kappa$, we also have $2^\lambda < \kappa$.

Theorem 4 Let P be any signature. If κ is an inaccessible cardinal strictly larger than P, then there is a weakly universal P-structure of size κ.

This fact follows from a theorem by Roland Fraïssé. Since the existing presentations of the proof we've been able to find are either insufficiently general for our purposes (for the countable case see Hodges 1997, 158–64) or else use forbiddingly high-powered technical machinery (e.g. Caramello 2008, and references therein), we'll sketch a proof here. This proof sketch will also help us generalize to the plural case.

It's worth noting that the theorem does not rely on any large cardinal axioms; but if large enough inaccessibles do not exist, then it is vacuous. To use Theorem 4 to deduce that there exist weakly universal P-structures for each signature P requires the additional premise that every cardinal is exceeded by some inaccessible. This is independent of ZFCU.

In what follows, let P be any signature, and let $\kappa > |P|$ be an inaccessible cardinal. A **small structure** is a P-structure with size strictly less than κ.

The proof of Theorem 4 turns on a certain *homogeneity* property. Our strategy will be to prove two main lemmas:

1. If U is homogeneous, then U embeds every structure of size at most κ.
2. There exists a homogeneous structure of size κ.

Very roughly, a homogeneous structure looks basically the same everywhere. More precisely, if U is homogeneous, then any small substructure of U can be extended however we like to bigger small structures.[39]

Definition 6 If $A \subseteq B$ and $f : A \to U$ and $g : B \to U$ are embeddings, then g **extends** f iff $g(d) = f(d)$ for every d in A (figure 1).

$$\begin{tikzcd} A \arrow[r, "\subseteq"] \arrow[dr, "f"'] & B \arrow[d, "g"] \\ & U \end{tikzcd}$$

Figure 1: Extending an embedding

Definition 7 A structure U is **homogeneous** iff for any small structures $A \subseteq B$, and any embedding $f : A \to U$, there is some embedding $g : B \to U$ that extends f.

Here's the idea of the first step. Suppose U is homogeneous, and let A be any structure of size at most κ. We can build A up as the *limit* of an infinite expanding chain of small substructures $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$. Start with the trivial embedding of the empty structure in U. Then we can use the homogeneity property to extend this embedding to A_0, and then extend it further to A_1, and so on. This expanding chain of embeddings also has a limit, and this is an embedding of A in U.

Let's make this idea more precise.

Definition 8 Let α be an ordinal. An **α-chain** is a sequence of structures A_i for each ordinal $i < \alpha$, such that $A_i \subseteq A_j$ whenever $i \leq j < \alpha$.

Lemma 3 For any α-chain of structures A_i, there is a unique **limit structure** A^+ such that, for any structure B, $A^+ \subseteq B$ iff $A_i \subseteq B$ for every $i < \alpha$. The limit of a chain of structures is denoted $\lim_{i<\alpha} A_i$.

(In fact, the limit is the structure whose domain is the union of the domains of each structure A_i, and for each $F \in P$ the extension of F is the union of the extensions of F in each structure A_i.)

We note without proof the following facts about limits. (These rely on our background assumption that κ is regular.)

[39] Our term "homogeneous" corresponds to what what Hodges calls "weak homogeneity". In contrast, in Hodges' terminology, an *ultrahomogeneous* structure S is one such that any isomorphism of small substructures of S can be extended to an automorphism of S. The short label "homogeneous" is often used for this property instead—which is not too bad, because for structures no larger than κ, the properties turn out to be equivalent (see Hodges 1997, Lemma 6.1.4 for the countable case).

Lemma 4 Let A^+ be the limit of an α-chain of small structures A_i.

(a) $|A^+| \leq \kappa$. If $\alpha < \kappa$, then A^+ is small.

(b) If B is a small structure and $f : B \to A^+$ is an embedding, then for some $i < \kappa$, f also embeds B in A_i.

(c) For any structure A such that $|A| \leq \kappa$, A is the limit of some κ-chain of small structures.

Definition 9 For an α-chain of structures A_i, a **chain of embeddings** of A_i in B is a sequence of embeddings $f_i : A_i \to B$ for $i < \alpha$, such that f_j extends f_i whenever $i \leq j < \alpha$.

Lemma 5 For any chain of embeddings $f_i : A_i \to B$, there is a **limit embedding** $g : \lim_i A_i \to B$ that extends each embedding f_i (figure 2).

$$\begin{tikzcd}
A_0 \arrow[r, "\subseteq"] \arrow[drrr, "f_0"'] & A_1 \arrow[r, "\subseteq"] \arrow[drr, "f_1"] & A_2 \arrow[r, "\subseteq"] \arrow[dr, "f_2"] & \cdots & \lim_i A_i \arrow[dl, "g"] \\
 & & & B &
\end{tikzcd}$$

Figure 2: A limit of a chain of embeddings

Now we have the tools to prove the first step of the main theorem.

Lemma 6 If U is homogeneous, then U embeds every structure of size at most κ.

Proof. Suppose A is a structure of size at most κ. Then A is a limit of some κ-chain of small structures A_i. We can use homogeneity to inductively construct a corresponding chain of embeddings $f_i : A_i \to U$. Let $i < \kappa$, and suppose for the inductive hypothesis that we already have a chain of embeddings $f_j : A_j \to U$ for $j < i$. Let $B = \lim_{j<i} A_j$, and let g be the limit of the chain of embeddings f_j. Then B is a substructure of A_i, and g is an embedding of B in U. By homogeneity, we can extend g to an embedding $f_i : A_i \to U$. In this way we can inductively construct a κ-chain of embeddings of the structures A_i in U. Then the limit of this chain is an embedding of the limit structure A in U. □

The second step—the construction of a homogeneous structure—relies on an additional structure-theoretic operation, which is called **amalgamation**. This is a kind of "paste with overlap". Suppose that we have two structures B and C with a "common part", in the sense that some substructure of B is isomorphic to a substructure of C. Then we can glue B and C together in a way that respects their common part.

Lemma 7 ***(The Amalgamation Property)*** Suppose A, B, and C are small structures, $A \subseteq B$, and $f : A \to C$ is an embedding. Then there is a small structure D such that $C \subseteq D$, and there is an embedding $g : B \to D$ that extends f (figure 3).

$$\begin{tikzcd} A \arrow[r, "\subseteq"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, "\subseteq"] & D \end{tikzcd}$$

Figure 3: Amalgamation

Note that in the special case where A is the empty structure, the Amalgamation Property just says that B and C can both be embedded in some structure D—so the disjoint sum of B and C is a special case of amalgamation.[40]

We can use these ideas to build up a homogeneous structure in stages. First, we can list all the small structures in a length-κ sequence $B_0, B_1, \ldots$ (by Lemma 8 below). We can use this to recursively define a chain of small structures. Start with the empty structure. Given a structure S, we can consider all of the possible ways of extending any of its small substructures to any of the first i structures in the B-sequence. Then we can extend S to a larger (but still small) structure S^+ that extends all of these substructures in all of these ways (Lemma 9). Go on this way through the whole B-sequence, take the limit of the resulting κ-chain, and we're done.

Lemma 8 Suppose P is a signature with π elements and $\kappa > \pi$ is inaccessible. Then there are κ isomorphism types of P-structures with strictly fewer than κ elements.

Lemma 9 Let S be a small structure, and let Φ be a set of fewer than κ triples (A, B, f) consisting of small structures $A \subseteq B$ and an embedding $f : A \to S$. Then there is a small structure S^+ such that for each triple $(A, B, f) \in \Phi$ there is an extension of f to an embedding $g : B \to S^+$.

Proof Sketch. We can inductively define a chain of structures S_i. We'll start with the initial structure S. Then we'll repeatedly apply the Amalgamation Property: taking each triple (A, B, f) one by one, we can extend the embedding of A in S_i to

[40]In the jargon from category theory, disjoint sums, limits, and amalgamation are all *colimits*. Thus many of the facts in this appendix can be subsumed under the fact that, for any relational signature P, the category of P-structures with embeddings has all small colimits.

an embedding of the larger structure B in S_{i+1}. The final structure S^+ is the limit of this chain. □

Lemma 10 There exists a homogeneous structure of size κ.

Proof Sketch. By Lemma 8, there is a length-κ sequence of small structures B_i, such that every small structure is isomorphic to B_i for some $i < \kappa$.

We'll recursively define a chain of small structures U_i. For any ordinal $i < \kappa$, first let $S = \lim_{j<i} U_i$. Let Φ be the set of all triples (A, B_j, f) consisting of a substructure $A \subseteq B_j$ for some $j < i$, and an embedding $f : A \to S$. It can be checked that $|\Phi| < \kappa$, so we can apply Lemma 9: there is a small structure $U_i \supseteq S$ that appropriately extends every embedding in Φ.

Now take the limit: let $U = \lim_{i<\kappa} U_i$. By Lemma 4, $|U| \leq \kappa$. (It's also clear that $|U| \geq \kappa$, so U has exactly κ elements.) The last thing to check is that U is homogeneous. This follows from the fact that, for any small structures $A \subseteq B$ and any embedding $f : A \to U$, the triple (A, B, f) showed up somewhere in the construction (up to isomorphism). □

B Plural Structures

In this appendix we spell out some of the definitions used in section 4 and section 5, and prove two key facts. The first is the main impossibility result, Theorem 5, which says that, for a signature including relations, there are more isomorphically distinct plural structures on a certain domain than things in that domain. The second is the main possibility result, Theorem 6, which says that there is a "universal" pure plural structure which embeds every pure plural structure. (Note that while the "possibility" theorem obviously relies on set theory, the "impossibility" theorem does not, beyond some basic things about ordered tuples.)

We'll use capital letters for plural variables, and for concision we'll use the notation $x \in X$ to mean "x is one of the X's", and the notation $\{x \mid \varphi(x)\}$ for the plural term "those things x such that $\varphi(x)$"—that is, for plural comprehension. Plural identity $X = Y$ means that each X is a Y and each Y is an X.

Recall that our plural quantifiers are to be understood as saying "there are zero or more things" rather than "there are one or more things". Nothing essentially turns on this, but it makes various things easier to state. (Otherwise we would need to add many caveats of the form "or else there is no x such that $\varphi(x)$.")

Definition 10 The X's **code a family of pluralities** indexed by the I's iff each of the X's is an ordered pair (i, x) for some $i \in I$. For each $i \in I$, let $X_i = \{x \mid (i, x) \in X\}$.

Definition 11 The S's **code a plural P-structure** iff the S's code a family of pluralities indexed by the elements of $P \cup \{\mathsf{Dom}\}$ (where Dom is the label for the domain, some object which is not an element of P), and for each relation F in P, the S_F's are ordered n-tuples of the S_{Dom}'s. The S-**domain** is S_{Dom}, and the S-**extension** of F is S_F.

("The S-domain" and "the S-extension of F" are really plural terms.)

Definition 12 (a) The X's **code a plural function** from the A's to the B's iff the X's are ordered pairs such that, for each a among the A's, there is exactly one b such that (a, b) is among the X's. Let $X(a)$ denote this unique b.

(b) The X's are **one-to-one** iff for any b among the B's there is at most one a among the A's such that $X(a) = b$.

(c) The X's are **onto** iff for any b among the B's there is at least one a among the A's such that $X(a) = b$.

Definition 13 Suppose the S_1's and the S_2's each code plural structures.

(a) The X's **code a plural embedding** of S_1 in S_2 iff the X's code a plural function from the domain of S_1 to the domain of S_2, the X's are one-to-one, and for any n-tuple $(d_1, \dots, d_n)$ of things in the S_1-domain, $(d_1, \dots, d_n)$ is in the S_1-extension of F iff $(X(d_1), \dots, X(d_n))$ is in the S_2-extension of F.

(b) The X's **code a plural isomorphism** from S_1 to S_2 iff the X's code an embedding of S_1 in S_2 and are onto.

We can also define disjoint embedding as before, using the same trick for representing indexed families of pluralities—this time, to represent an indexed family of *plural structures*.

Definition 14 Consider a family of pluralities S_i, where for each $i \in I$, the S_i's code a plural P-structure. Let the S^+'s code a plural structure. Then S^+ **disjointly embeds** the family of plural structures S_i iff there are some X's which code a family of pluralities indexed by the I's, such that (i) for each i among the I's, the X_i's code an embedding from S_i to S^+, and (ii) for any distinct $i \in I$ and $j \in I$, there are no d and d' such that $X_i(d) = X_j(d')$.

Definition 15 Suppose the O's are ordered pairs of D's. The O's **well-order** the D's iff, for any X's among the D's, there is a unique $x \in X$ such that $(x, y) \in O$ for every $y \in X$. (In other words, any D's have a unique O-least element.)

Note that well-orderings are reflexive: if the O's well-order the D's, then $(d, d) \in O$ for each $d \in D$. (This follows from the definition by considering the singleton plurality $\{d\}$.)

Theorem 5 (See section 4.) Let P be a polyadic signature. Let the S's code a family of plural P-structures indexed by the D's. (That is, for each d among the D's, the S_d's code some plural structure.) If the D's can be well-ordered, then some plural structure on the D's is not isomorphic to S_d for any d.

It suffices to handle the case where P includes a single dyadic relation, say *zapping*. (Increasing the adicity of the relation or adding further relations to the signature can only increase the number of isomorphically distinct structures.) If the Z's are ordered pairs of D's, let (D, Z) be the plural structure that assigns the Z's as the extension of zapping.

Our proof of Theorem 5 relies on the following fact about well-orderings (which we won't prove here):

Lemma 11 ***(Rigidity Lemma)*** Suppose the O's well-order the D's. If the Z's code an isomorphism from (D, O) to (D, O), then $Z(d) = d$ for every $d \in D$.

The proof also relies on the counting fact we discussed in section 4:

Cantor–Bernays Theorem Let the X's code a family of pluralities indexed by the D's. Then there are (zero or more) D's which are not the X_d's for any d.

Proof of Theorem 5. First we'll show that there is an isomorphically distinct plural structure on the D's for each plurality of D's; then we can apply the Cantor–Bernays Theorem.

By hypothesis, there are O's that well-order the D's. For any X's among the D's, let the O_X's be the O-pairs excluding just the identity pairs (x, x) for $x \in X$.

$$O_X = \{(x, y) \in O \mid x \neq y \text{ or } x \notin X\}$$

For any X's and Y's among the D's, if the Z's code an isomorphism from (D, O_X) to (D, O_Y), it follows that the Z's also code an isomorphism from (D, O) to (D, O). Thus, by the Rigidity Lemma, $Z(d) = d$ for every $d \in D$'s—and so it follows that X's just are the Y's. In short, if $(D, O_X) \cong (D, O_Y)$ then $X = Y$.

Now suppose the S's code a family of plural structures indexed by D. Then we can construct a corresponding family of pluralities. If there are any Y's such that $S_d \cong (D, O_Y)$, let $X_d = Y$. (We just showed that there is at most one such plurality. If there are no such Y's, we can let the X_d's be something arbitrary, such as all the D's.) By the Cantor–Bernays Theorem, there are some Y's among the D's such that, for every $d \in D$, $X_d \neq Y$. Thus (D, O_Y) is a plural structure which is not isomorphic to S_d for any $d \in D$. □

That completes the impossibility result. Next we'll outline a proof of the main possibility result.

Theorem 6 Given Global Choice, there exists a pure plural structure that embeds every pure plural structure.

We can prove this by "pluralizing" the proof of Theorem 4. First, we can straightforwardly generalize the idea of a limit of a chain of structures to the case where we have a very long "chain" of structures indexed by *all* of the ordinals.

Definition 16
(a) The X's **code an absolute sequence** iff the X's code a plural function from the ordinals: that is, for each ordinal i, there is exactly one pair (i, x) among the X's.
(b) The A's **code an absolute chain of structures** iff the A's code an absolute sequence such that, for each ordinal i, A_i is a (set) structure, and A_i is a substructure of A_j for any ordinals $i \leq j$.
(c) The F's **code an absolute chain of embeddings** iff for each ordinal i, F_i is a (set) embedding and F_j extends F_i for any ordinals $i \leq j$.

In general, an absolute chain of set structures may have no *set* structure as a limit—the structures may eventually exceed the size of any particular set. But even an absolute chain will still have a *plural* structure as a limit, which is given by taking all together everything which is part of the domain of any structure A_i.

Lemma 12
(a) For any absolute chain of pure set structures A_i, there is a unique pure plural structure A^+, the **limit** of the chain $\lim_i A_i$, such that for any plural structure B, A^+ is a (plural) substructure of B iff A_i is a substructure of B for every ordinal i.

(b) For any absolute chain of (set) embeddings $f_i : A_i \to B$, there is a unique plural embedding of $\lim_i A_i$ in B that extends f_i for every ordinal i.

Furthermore, the "absolute infinity" which is the length of the ordinals has the same relevant features as inaccessible cardinals that make the construction of a universal structure work. We can run things essentially as before, understanding "small" to mean "set-sized". For example:

Lemma 13 (a) Given Global Choice, every pure plural structure is a limit of some absolute chain of set structures.

(b) If A is the limit of an absolute chain of set structures A_i, then any set-substructure of A is also a substructure of A_i for some ordinal i.

Lemma 14 Suppose U is a *homogeneous* pure plural structure, in the sense that for any set structures $A \subseteq B$, any embedding $f : A \to U$ can be extended to an embedding of B in U. Then U embeds every pure plural structure.

Proof Sketch. Suppose A is some pure plural structure. By Lemma 13, A is a limit of an absolute chain of set structures A_i. Then as in Lemma 6, we can construct an absolute chain of embeddings $f_i : A_i \to U$. The limit is a plural embedding of A in U. □

Then we can construct a homogeneous pure plural structure by exactly the same method as Lemma 10, as the limit of an absolute chain of set structures. Each element of the chain is guaranteed to permit extensions of a certain *set* of (A, B, f) triples. As in the set case, the limit of this absolute chain satisfies the homogeneity property for *every* such triple. This completes the proof of Theorem 6.

Corollary 2 Given Limitation of Size, there exists a plural structure that embeds every plural structure.

Proof. Limitation of Size guarantees that every plural structure is isomorphic to some pure plural structure, and thus is embeddable in the universal plural structure of Theorem 6. (Since Global Choice follows from Limitation of Size, we don't need to include this as an additional hypothesis.) □

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