Supreme Deity, to know all things, to have created all things, to be immanent in all things. Further, he declares that it is by a knowledge of him that salvation is attained.

"They verily who worship Me with devotion, they are in Me, and I also in them."

"He who knoweth Me, unborn, beginningless, the great Lord of the world, he among mortals is without delusion, liberated from all sin."

"Having pervaded this whole universe with a portion of Myself, I exist."

"He who seeth Me everywhere, and seeth everything in Me, of him will I never lose hold, and he will never lose hold of Me."

"By Me all this world is spread out, the embodiment of the unmanifested."

Such are a few of the sayings of Krishna, taken at random from this wonderful discourse.

Then, again, Dr Fraser and other mythologists have shown us that most of the other characteristic doctrines and traditions of the Christian religion bear a remarkable resemblance, to say no more, to the customs and traditions of primitive people all over the world. The notion of the sacrifice of a divine human person, "himself to himself," at about the vernal equinox, for the redemption of the sins of the people was evidently no new one. And so one might take one doctrine after another and show that it is not peculiar to the Christian religion. This religion is not *unique* either in its ethical teachings, in its doctrinal teachings, or in the mystical experiences of its devotees.

I am afraid I have left no space to speak of the disturbing effects on the lay mind of the Higher Criticism. Great numbers of the laity, no doubt, are waiting to see in what form the Bible will emerge from the melting pot purged of its dross.

When our pulpits are occupied by men whose reading has been wide and catholic: when their minds are no longer invincibly biassed by their education; when their career and their very livelihood no longer depend upon the acceptance of certain antiquated beliefs, then, and not till then, shall we find our churches better attended by the more reflecting and better informed members of the laity.

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THE AXIOM OF INFINITY.

(*Hibbert Journal*, April 1904.)

Professor Keyser's very interesting article on "The Axiom of Infinity" contains a contention of capital importance for the theory of infinity. The view advocated by those who, like myself, believe all pure mathematics to be a mere prolongation of symbolic logic, is, that there are no new axioms at all in the later parts of mathematics, including among these both ordinary arithmetic and the arithmetic of infinite numbers. Professor Keyser maintains, on the contrary, that a special axiom is covertly invoked in all attempted demonstrations of the existence of the infinite. I believe that, in so thinking, he has been misled by the brevity, and perhaps obscurity, with which writers on this subject have usually stated their arguments. I am myself, as yet, obnoxious to the same charge; for the strict and detailed proof, with all the apparatus of logical rigour, is too long to be given incidentally, and was therefore reserved by me for vol. ii. of my *Principles of Mathematics.* It is possible, however, with a little care, so to set forth the outline of the proof as to make it appear that, whether "exquisite" or not, it is certainly not "round."[1]

I presuppose, in setting forth this argument, the definition of number, and the proof that, with the suggested definition, every class has some perfectly well-defined number of terms. These matters I have discussed at length in Part II. of the above-mentioned work; and so far as appears, Professor Keyser has no fault to find in regard to them.

The first step is to demonstrate that there is such a number as 0. The number of things fulfilling any condition which nothing fulfils is defined to be 0; and it may be shown that there are such conditions. For example, nothing is a proposition which is both true and false. Consequently, the number of things which are propositions that are both true and false is 0. Thus there is such a number as 0.

We next define the number 1 as follows. The number of terms in a class is 1 if there is a term in the class such that, when that term is taken away, the number of terms remaining is 0. That classes having one member exist is not hard to prove; for example, the class of things identical with the number 0 consists of the number 0 alone, and has only one member.

We proceed in like manner to the number 2, and we prove that the class consisting of the numbers 0 and 1 has two members, from which it follows that the number 2 exists.

It is in the next stage of the argument that, if I am not mistaken, Professor Keyser has been misled by an undue brevity. He appears to think that, at this point, the advocates of infinity are content with a vague "and so on"—a sort of *etcetera* which is intended to cover a multitude of sins. But *etceteras*, common as they are in ordinary mathematics, where they are represented by rows of little dots, are not tolerated by the stricter symbolic logicians. I shall try to show how it is that the argument proceeds without them.

We first prove the principle of mathematical induction[2]—a principle

[1] See *Hibbert Journal, loc. cit.*, pp. 549–550.

[2] I omit the proofs of propositions here assumed. Some of these proofs will be found in § 4 of my article in the *Revue de Mathématiques*, vol. vii.; others in Mr A. N. Whitehead's article in the *American Journal of Mathematics*, vol. xxiv.

which, in this domain, does work for us such as could hardly be expected but from an *etcetera*. This principle states that any property possessed by the number 0, and possessed by $n+1$ when it is possessed by n, is possessed by all finite numbers. By means of this principle, we prove that, if n be any finite number, the number of numbers from 0 to n, both inclusive, is $n+1$. Consequently, if n exists, so does $n+1$. Hence, since 0 exists, it follows by mathematical induction that all finite numbers exist. We prove also that, if m and n be two finite numbers other than 0, $m+n$ is not identical with either m or n. It follows that, if n be any finite number, n is not the number of finite numbers, for the number of numbers from 0 to n is $n+1$, and $n+1$ is different from n. Thus no finite number is the number of finite numbers; and therefore, since the definition of cardinal numbers[1] allows no doubt as to the existence of a number which is the number of finite numbers, it follows that this number is infinite. Hence, from the abstract principles of logic alone, the existence of infinite numbers is rigidly demonstrated.

The above is the strict proof appropriate to pure mathematics, since the entities with which it deals are exclusively those belonging to the domain of pure mathematics. Other proofs, such as the one from the fact that the idea of a thing is different from the thing, are not appropriate to pure mathematics, since they do, as Professor Keyser points out, assume premisses not mathematically demonstrable. But such proofs are not on that account circular or otherwise fallacious. Accepting the five postulates enumerated by Professor Keyser on p. 549 as assumed by Dedekind, I deny wholly that any one of the five *presupposes* the actual infinite. It is true that they together *imply* the actual infinite; it is indeed their purpose to do so. But it is too common, in philosophising, to confound implications with presuppositions. At this rate, all deduction would be circular. The contention advanced by Professor Keyser is essentially the following: If the conclusion (the existence of the infinite) were untrue, one of the premisses would be untrue; consequently the premisses beg the conclusion, and the argument is circular. But in all correct deductions, if the conclusion is false, so is at least one of the premisses. The falsehood of the premisses presupposes the falsehood of the conclusion; but it by no means follows that the truth of the premisses presupposes the truth of the conclusion. The root of the error seems to be that, where a deduction is very easily drawn, it comes to be viewed as actually part of the premisses; and thus very elementary arguments acquire the appearance, quite falsely, of *petitiones principii*.

Another point which calls for criticism is the psychological form of Professor Keyser's statement of the axiom of infinity. He states this axiom (p. 551) as follows: "Conception and logical inference alike presuppose absolute certainty that an act which the mind finds itself capable of performing is intrinsically performable endlessly." This statement is rendered vague by the word *intrinsically*; but I sincerely hope there is no

[1] See my *Principles of Mathematics*, chap. xi.

such presupposition in inference, since it is a most certain empirical fact that the mind is not capable of endlessly repeating the same act. Even apart from the fact that man is mortal, he is doomed to intervals of sleep; when he is drunk, he cannot perform mental acts which he can perform when he is sober, and so on. I am aware, of course, that such accidents are intended to be eliminated by the word *intrinsically*; but when they are, as they must be, explicitly and in terms eliminated, we get an axiom so complicated, and so plainly full of empirical elements, that it would require extraordinary boldness to present it as underlying all logic. The only escape would be to say that "the mind" is to be taken to mean God's mind. But few will maintain nowadays that the existence of God is a necessary premiss for all logic.[1]

The truth is that, throughout logic and mathematics, the existence of the human or any other mind is totally irrelevant; mental processes are studied by means of logic, but the subject-matter of logic does not presuppose mental processes, and would be equally true if there were no mental processes. It is true that, in that case, we should not know logic; but our knowledge must not be confounded with the truths which we know, and in the case of logic, although our knowledge of course involves mental processes, that which we know does not involve them. Logic will never acquire its proper place among the sciences until it is recognised that a truth and the knowledge of it are as distinct as an apple and the eating of it.

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[1] See my *Philosophy of Leibniz* (Cambridge, 1900), chap. xv., especially § 111.