The Philosophy of Mathematics and the Independent 'Other'

A thesis submitted for the degree of
Doctor of Philosophy at The University of
Queensland in November 2004

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Statement of Originality

This work is original.

Statement of Contribution to Jointly Published Works

No part of this work has formed a contribution to a jointly published work, nor have I contributed previously to a jointly published work.

Statement of Contribution by Others

This work is entirely my own, except where indicated in text.

Signed:

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Acknowledgments

Graham Priest for patience and good advice, Marshall Roberts for technical help and constant support, and Dominic Hyde for useful comments and maintaining morale.

Abstract

This thesis explores some of the fundamental notions motivating mathematical realism, in particular the idea of a separate, independent mathematical realm. This notion, which at its core is the concept of an 'other', forms the work's primary focus. Various versions of mathematical realism are explored, including structuralism, naturalism and a transcendental approach – all of which attempt to incorporate or engage with an 'other' mathematical realm. Arguing that none of these approaches retains the original intuitive notion of a truly separate 'other', an alternative account is offered, drawing inspiration from Husserl's phenomenology and Derrida's analysis of metaphysics as theology. The account proposes that, in order to retain the concept of an 'other', the independent mathematical realm must be uncircumscribable by any account, and literally beyond comprehension. But, in order to preserve mathematical realism as a philosophy, and in accordance with its fundamental optimism (that the mathematics we access somehow is the mathematical realm and other than us), mathematics must also be within our grasp and able to be fully seen as it is in and of itself. So mathematics must be fully within and fully outside of our grasp (or consciousness). It is argued that this paradox is unavoidable and irreducible, but not necessarily a drawback for realism.

TABLE OF CONTENTS

Introduction

This thesis is about mathematical realism. 'Mathematical realism' will for now serve simply as a cover term for the host of versions of the idea that mathematics is somehow real (or perhaps, given the vagaries of words like 'real', that mathematics is not 'unreal'), rather than a construct of our making, or a product of our imagination.

Just as most of us comprehend and live in the physical world apparently believing (or at least acting and thinking as though) it is not unreal, so most mathematicians and, I would venture, all mathematical realists, comprehend and interact with the mathematical world.

The idea that the mathematical realm is no more unreal than the physical realm is an attractive one – inviting the belief that mathematics is another reality or part of a broader reality; as interesting, as inviting and as separate an entity as we could hope to encounter. Among other things, this idea presents mathematics as a still largely uncharted realm of infinite possibility – a place to explore and discover; something other – the encountering of which is potentially able to teach us about more than ourselves; something to learn.

But can anything like this idea – which at this stage is just the loose, intuitive concept of an intriguing, entirely separate 'other' – be translated into a set of principles and incorporated into a coherent philosophy? The same question can be asked of quite a number of features of (versions of) mathematical realism, such as the idea that mathematical reality is unique, or fixed, the idea that its objects are abstract, and so on.

In fact, it turns out that most of the features of mathematical realism (perhaps of any philosophy) – including its attractive qualities, or apparent strengths – are notoriously hard to pin down. There are good reasons for this. In the first place, articulating the precise nature of each quality is not easy. The concept 'real' is itself a good example. Where one person can use the term 'real' to indicate a reality strictly separate from ourselves, others can understand the same term as representing a *belief* in some sort of external realm, as a linguistic tool whose function does not depend on the existence of anything whatsoever, or even as a sort of shorthand for an internal reality, entirely the construct of our imagination. Similar ambiguities exist for terms like 'independent', 'metaphysical' and 'abstract'. Even the term 'other' is not without its subtleties, as any theologian will testify.

Secondly, there is the problem of deciding just what counts as a quality or a strength, and what counts as a weakness of mathematical realism. For example, one realist might see uniqueness as a vital aspect of the realist's mathematical realm - believing perhaps that any relativity with regard to mathematical reality itself threatens the idea of, and our drive to discover 'the truth'. On the other hand, another mathematical realist might dismiss the notion of a fixed universe altogether, favouring instead the notion that no single model or structure can fully represent everything we would want to call a mathematical truth. But regardless of which feature of mathematical realism is highlighted, all either stem from, or have some kind of important interplay with, the concept that mathematical reality is something other than ourselves – something *else*.

For this reason, the concept of an 'other' is considered in what follows to be one of the most important aspects of mathematical realism and, as such, its overt incorporation is counted as a desideratum for mathematical realist accounts in general.

The claim that all features of mathematical realism stem from or interact in an important way with the notion of an 'other' can be illustrated by many scenarios, including the following. Whether or not the mathematical realm is seen as unique can depend on whether or not that realm is seen as a reality whose existence is separate from our own. If our existence does not influence the existence of that realm, to ask just what *it* is seems a legitimate question, and so the belief that there is a single correct description of that reality naturally attaches to, indeed perhaps relies upon, the idea that that reality is independent of, or other than ourselves. Other instances reasoned along similar lines are easily imagined.

Given the centrality of the concept of an 'other' for mathematical realism, the central aim of this work is to flesh out and examine just what it takes to articulate the notion of an entirely separate reality in a philosophy of mathematics. Note that the fleshing out of an 'entirely other' reality is opposed to the fleshing out of some derivative of the same intuitive notion. That is, the following is not an attempt to accommodate various versions or degrees of independence, or 'otherness'. It is an attempt to better understand and explicate the most extreme concept conceivable (and, perhaps, inconceivable) of an 'other'. This work, therefore, is about a particular realism - one which posits a reality whose existence is as separate from our own as a person could desire. The most extreme position is taken for two reasons. The first is simplicity. The task of pinning down the intuitive notion of an 'other', it seems to me, is best accomplished by examining this notion at its most extreme, where the various components and implications of such a notion are as stark as they get.

The second reason is another of the primary motivations for this work. The realism I want to explore is this particular realism at its most extreme. The question I am seeking to answer is how a philosophy of mathematics can incorporate the (apparently) cut-and-dried realist beliefs underpinning the practice of mathematics and, analogously, the realist beliefs underpinning our interaction with the physical world. In particular, how a philosophy of mathematics can incorporate the belief or presumption that the reality that mathematics deals with is a reality entirely separate from ourselves.

That is, the primary task of this work is not to present the most acceptable realist model, nor to justify realism in general, but rather to ask what it takes for a particular (cut-and-dried, black-and-white, or 'ideal') realist model to work, or even just to be expressed philosophically. This question does, I believe, have important implications for other realist models as well, especially if they derive from, or are informed by this 'ideal'. I hope also to show that the account I put forward – an account incorporating the ideal or 'strong' versions of various beliefs that other models also incorporate – ultimately has no more problems, or is no less acceptable than those other models. If this is indeed the case, then there needs to be some further argument weighing the possible advantages of the ideal against derivative or relative versions of the same realist beliefs and ideas.

Put negatively, the main problem I try to avoid here is the problem of not incorporating or addressing the notion of a fully independent other in a realist philosophy of mathematics. It is in accordance with this aim that I identify this *as* a problem for the other accounts I address. The first few chapters aim to show that less than the idea of independence is addressed in the accounts I discuss – which simply means that these accounts have not avoided my main problem. The rest of the work is an attempt to show that the problems that are encountered by my own account are no greater than those surrounding the question or problem of our relationship to the *physical* world.

Some other desiderata for my realist philosophy of mathematics are likewise motivated by the aim to articulate commonsense or everyday background realist beliefs regarding everyday objects. These include: (1) that the mathematical reality is analogous with the physical world's reality; (2) that our understanding of mathematical reality depends on that reality; (3) that mathematical reality existed before we came to know it; (4) that mathematical reality is 'metaphysical'; (5) that mathematical reality is 'non-arbitrary'; (6) that mathematical reality is unique; and (7) that meaning is objective, and so determinate. But, as all of these notions depend on the notion of a reality independent from ourselves, the first item on the list remains the most important. The concluding chapters argue that in order to incorporate an 'other' reality, a considerable degree of mystery must be written into or accommodated by a philosophy of mathematics; akin in many ways to the mystery inherent to many theologies. Whether or not this conclusion weighs in favour of or against the mathematical realist's agenda is left as a matter of interpretation.

One could see such a conclusion as a vindication of the view that mathematical realism is ultimately no more comprehensible, workable, nor logical than a simple leap of faith. On the other hand, one might view the overt incorporation of a 'leap of faith' as a long overdue acceptance that faith is an unavoidable part of any philosophy whatsoever, particularly a realist philosophy – and hence as the beginning of a more concerted effort to understand a crucial component of the philosopher's work that has hitherto been pushed out of sight. I hope this work ultimately encourages the latter rather than the former interpretation, but whether or not it does is not as important in the agenda as is the achievement of a more modest aim: that of simply introducing the option, for the mathematical realist, of a favourable interpretation of an otherwise apparently unfavourable conclusion. Such an option introduces the possibility of further cross-pollination between philosophy of mathematics and other studies, such as theology, that deal overtly with belief systems and faith. Taken in concert with the more traditional bounds of analytic philosophy, the new boundaries offer new possibilities for the philosophy of mathematics and consequently for our understanding of our relationship with the objects and the realm of our mathematical knowledge.

A Note on Structure

The program, to acknowledge and articulate the beliefs underpinning the sort of mathematical realism inspired by, or attempting in some way to incorporate, the idea of a completely independent mathematical reality, falls into two sections. The first section begins to outline, in broad strokes, the theories presented in more detail in the second section. So, while some seemingly obscure or specific terms (e.g. 'filled intention', 'embodied reality' and 'independent' versus 'other' reality) are used in the first section without full definitions or explanations, these are revisited and clarified in the final few chapters.

The first section looks at various versions of mathematical realism – all of which can be seen as taking some kind of positive approach to the idea that mathematical reality is independent. This is done in order to more specifically identify some of the problems or difficulties encountered when attempting to incorporate the notion of an 'other' in mathematical realism. There are, of course, a great many more philosophies of mathematics that are relevant in some way to this work. Notably absent are Richard Tieszen's Husserlian approach, Hartry Field's anti-realism, and Crispin Wright's realism.[1] Rather than discuss each relevant philosophy of mathematics, though, I have taken the few I believe to be in some way representative of a particular approach. Also, the philosophies most relevant to this work are self proclaimed realist philosophies (with the exception of Folina's), or more overtly realist than those listed above. Further, of the host of realist philosophies, I discuss only the few that need to be discussed in order to gain an overall view of the sort of problems I hope to identify in the expression of and approach to mathematical realism in general.

The second section of this work is my own attempt to deal with the problem of incorporating an independent reality into a mathematical realist account.

[1] I refer here primarily to Wright [1987], Tieszen [1989] and Field [2001]. All are relevant to this thesis, but not especially needed in order to establish its main points.

Citations

Author/date references are provided in the first instance of citing an external source. If subsequent references to the same work are made immediately following this first citation (i.e. before any other work is cited), only author and/or page number are provided.

Square brackets within a quotation indicate my additions unless otherwise stated. Double square brackets within a quotation are part of the quoted text.

Part 1

Chapters one through four all deal with versions of mathematical realism that seem to include a significant degree of independence in their account of mathematical reality.

Chapter One

In chapter one, I look at Shapiro's structuralism, and argue that the sort of reality that Shapiro ascribes to structures is ultimately too vague to enable them – as the reality posited – to escape some of the problems more commonly associated with anti-realist accounts. In particular, I argue that it is not clear just how independent Shapiro's structures are from the epistemic routes by which they are accessed. That is, I argue that along with not escaping anti-realist problems, in the end Shapiro's structuralism is no less problematic than the traditional platonism upon which he seeks to improve. This is important since at least part of his program is an attempt to overcome what he sees as the more serious problems with traditional platonism. I try to show that his ante rem independence for structures does not set his account far enough apart from anti-realist accounts or eradicate their pitfalls.[2] That is, the independence of the

[2] According to Shapiro [1997], the term "ante rem" applied to universals is the notion that "universals exist prior to and independent of any items that may instantiate them" (p. 84). For Shapiro "[t]he important distinction is between *ante rem* realism and the others ... [his question] is whether, and in what sense, structures exist independently of the systems that exemplify them. Is it reasonable to speak of the natural-number structure, the real-number structure, or the set-theoretic hierarchy on the off chance that there are no systems that exemplify these structures?" (p. 85). In fact it is the centrality of this question that motivates my later examination of the notion of dependence. If indeed the structures and the systems that exemplify them are two different things, what is the relationship between these two things? Without some other

reality he proposes (structures) is not strong enough, or not obviously so, to set that reality apart from the mechanisms by which we perceive it. Nor, I argue, are these structures (i.e. those arrived at via the mechanisms of our perception) simply identifiable with fully independent structures. And they cannot be, at least not without this identity being an ad hoc or somehow unnatural stipulation.

To support these claims I argue that Shapiro does not distinguish, nor does his account have the tools to distinguish, between the object of our perception (accessed via pattern recognition and abstraction) and the independent objects he proposes (the ante rem structure that existed prior to our perception – or to our act of perception). Ultimately, Shapiro simply stipulates the independence of his structures and this stipulation is opposed to one of the main aims of this work – the incorporation or addressing of independence in a philosophy of mathematics as a fundamental principle. I argue that Shapiro's independence is not fundamental, or at least that its role is ambiguous and therefore his stipulation is still more ad hoc than fundamental.

I argue that in order to incorporate the idea of independence in a 'non ad hoc' fashion the retention of the Caesar problem is required. That is, the Caesar problem *remaining* a problem is a necessary pre-condition for the incorporation into a philosophy of mathematics of the idea of an 'other' mathematical reality. The retention of this problem *as* a problem amounts to the belief that the problem has a determinate solution, regardless of whether or not we can discover it (which in turn amounts to the preservation of the idea that there is an independent other with its own determinate nature, and so on!). In other words, without this open-ended statement of belief, realism loses the possibility of proposing a fully independent reality.

Finally, I argue that two notions central to Shapiro's account undermine each other. The first is that we discover structures. The second is that structures exist ante rem. These ideas are undermined in Shapiro's account simply because each is defined and defended in terms of the other. This leads to the problem of

distinguishing feature, can the difference between the two be maintained? I answer this question negatively, and go on to suggest the idea that the systems depend on the structures as one such feature.

establishing the difference between an inherently circular (but nonetheless useful and insightful) foundation for mathematics such as that offered by Shapiro's ante rem structures, and the more metaphysical appeal to an external reality proposed in my own account. Just what this difference amounts to is a major focus of the following chapter.

Chapter Two

Chapter two discusses the difference between an external and an internal justification for mathematics, and argues that for the realism I am interested in exploring only an external justification counts. The case for an internal or set-theoretic justification is presented through Maddy's naturalistic interpretation of Gödel's realism, while the case for an external justification is presented through my own interpretation of the same.

This chapter explores the importance for mathematical realism of philosophical or metaphysical issues and asks just what, for my mathematical realist, would constitute a ground for mathematics. In other words, it asks just what could count as a solution to the problem of justifying mathematics and what could not, once even the most basic tenets of the mathematical realism under discussion are granted.

The case for external justification begins with the observation that Gödel's work can be interpreted as both in favour of, and against, its necessity. This is possible, I believe, because of an ambiguity in Gödel's use of the term 'meaning'. It seems that for Gödel, 'meaning' refers to different things in different contexts. That is, in some contexts it seems that Gödel uses the term to refer to an independent mathematical reality, but in others he appears to use the term to refer to our *understanding* of mathematical reality. I suggest that this is because, to Gödel, the two possible referents are, ideally at least, identical.

Which of the two referents is favoured will depend on the interpreter's own bias: Maddy reads Gödel as privileging internal meaning over external. In other words, she reads Gödel as locating meaning on the level of our understanding of mathematical formalisms rather than on any independent abstract plane. I read Gödel as privileging external meaning. Regardless of issues of interpretation, though, I argue that there is for the realist ultimately no way of avoiding the external.

Maddy argues that philosophy is metaphysical. I agree, but further argue that, because of the nature of mathematical realism and philosophy, more than just what fits easily under the rubric 'philosophy' is metaphysical. 'Mathematical realism' itself and 'justification' are both examples of 'metaphysics' once just about any degree of realism is granted.

According to this stance, I interpret Gödel's own understanding of these terms as metaphysical. His realism and his interest in Husserl motivates and provides some evidence for the validity of such an interpretation. This expansion of the term 'metaphysical' is defended and expressed with greater precision through the ideas of Derrida and Husserl and through ideas more or less directly inspired by their work. Also defended by these means is the idea that Maddy's separation of philosophy from concepts like 'mathematical realism' and 'justification' is itself in need of further justification.

Phenomenological concepts and terminology are then brought to bear on Gödel's mathematical realism, which results in the claim that Gödel's sought-after justification for mathematics can be aligned with Husserl's filled intention. That the two concepts so readily fit together is, I believe, no coincidence: the Husserlian framework that Gödel appeals to, and Gödel's realism itself, are in many ways paradigmatic of mathematical realism in general and the idea of an independent 'other' is a predominant occupation for them both (albeit from two separate angles).

Metaphysics is defined as that which prioritises presence over presentation, or as that which has 'the sign' dependent on 'the signified', or, in mathematical realist terms: that which has the formalism dependent on the independent mathematical reality it represents (insofar as the formalism is interpreted as representing, as opposed to creating reality).

Interpreting Gödel's as an overtly philosophical realism, though, involves more than simply arguing that his discourses can be read as 'philosophic'. I give an interpretation of Gödel's realism whereby the possibility of meaning itself depends on the existence of an independently existing mathematical realm, and meaning is bound to that realm, rather than to the realm of our understanding. Using this framework to further explore Gödel's ideas helps to establish the importance of an external ground for mathematical realism, and to highlight its potential benefits.

Chapter Three

Chapter three begins with the supposition that one of the benefits of the realist's idea of an external ground for mathematics is in the ability of a ground, via some sort of reference to that ground, to provide a sort of 'first cause' or reason for the existence of mathematics itself (or, mathematics as we understand it – a.k.a. mathematical formalisms).

Initially dubbed 'non-arbitrariness', this benefit or trait of realism is more fully articulated via the framework Gödel provides in his [1995a] wherein the idea that realist accounts are a 'representation of a system of truths' is taken to be one of their 'essential strengths'. This idea, interpreted according to the schema Gödel outlines, itself entails that the mathematical reality posited (in those accounts possessing this 'essential strength') is, at least to some degree, independent – or other than – what we comprehend. And this independence in turn entails a specific problem for these accounts: the problem of justifying the claim that the fundamental axioms – indeed of any part of mathematics we wish to call true and correct – embodies this independent reality.

Thus, the provision of a ground and the problems presented by the provision of a ground (otherwise known as the problem of justification – this problem is more fully articulated in the final chapters), taken together provide an interpretation of 'non-arbitrariness' in realist accounts.

Arbitrariness, on the other hand, is defined as a corresponding 'groundlessness' and perceived lack of any problem of justification. That is – arbitrariness is the perceived absence of the need to justify mathematical claims by connecting mathematics as we know it with any 'other' sort of reality, independent of that which we comprehend.

The trait of arbitrariness is then compared with that of 'piecemealness', defined according to Zalta and Linsky [1995]. This comparison is designed to show that while the traditional platonist and my own realist accounts may suffer from piecemealness, they do not suffer from arbitrariness. Which of course leads to the question: which is worse? I argue that the trait of non-arbitrariness at the very least mitigates the problems presented by the trait of piecemealness – and so conclude that it is worse to be arbitrary than piecemeal.

Non-arbitrariness not only ensures the presence of the problem of justification; it also has ramifications for a realist theory of meaning, the presentation of which is begun in the previous chapter and elaborated in more detail in the second section. To show this, I argue that the difference between a meaningful and a meaningless mathematical formalism, for the realist, is a corresponding provision or lack of a ground, and that it is in the ground itself that meaning 'resides' (rather than in the formalisms, or in 'mathematics itself').

As a result, none of the means that we possess of accessing or describing a mathematical object can, taken alone, account for a meaningful, correct, mathematical formalism. That is, when both meaning and reality are taken to be fully independent of us, *only* the identification of formalism with an outside ground can provide us with such a thing as a meaningful mathematical formalism. (Of course, some other relationship between us and what is independent of us might also provide us with a meaningful mathematical formalism, but I proceed on the assumption that all other relationships must surely provide a more tenuous link than identification).

Enlarging upon the work of the previous chapter, the nature of this identification is presented, in this chapter, as analogous to Husserl's 'filled intention'. This particular analogy is employed in order to capture the idea that the independent ground is necessarily *present* (and existent), before mathematics can be meaningful (at least for the realist I am proposing).

One problem with this analogy is its implication that the independent reality (outside our comprehension) and the mathematical formalism (within our comprehension) are precisely one and the same, or interchangeable. But a realist agreeing with this implication would look no different than a constructivist claiming, say, that what we comprehend determines what *is*.

In an attempt to discover just what needs to be retained in order for the realist to avoid the foregoing scenario, the traits 'arbitrariness' and 'non-arbitrariness' are further examined. Defining the latter as 'relevance', I argue that arbitrariness is associated with self-reference, and relevance is associated with reference to an 'other'. I then argue that, at least for the realist, meaningful mathematics depends not only on the existence of an other, but also to reference to this other. Whereas for the constructivist or, more generally, for those accounts that privilege what we comprehend over what is outside our comprehension, reality depends on the comprehended formalism (what *is* is just whatever it is that we comprehend).

Therefore, the dependence on there being an 'other' and the privileging of this other over internal concerns (or over that which we comprehend) is crucial to the realist. It is this factor or desideratum that has to be maintained against – or kept in tension with – the identification of mathematical formalisms with independent mathematical reality, i.e. of what is internal (to humanity) with what is external (always other than humanity).

The remainder of the chapter is taken up with a comparison between two kinds of principled platonism – Zalta and Linsky's, and my own. The latter is the initial attempt to formulate the problem of incorporating an independent reality in a philosophy of mathematics. This amounts to the problem of how to include both the possibility of identifying what we comprehend with this 'other', and the principle that this 'other' remains always different from what we comprehend. This problem is examined and developed in greater depth in the final three chapters.

Chapter Four

This chapter looks at one framework in which the problems surrounding realism and anti-realism in mathematics are commonly understood, and compares this framework with my own (as it has thus far developed throughout the foregoing chapters).

The former framework is utilised in Folina's work to set up and identify the problems and the strengths associated with realism and constructivism. It revolves around the gap (or lack of a gap, in the case of constructivism) between what the facts are and what we take them to be.

There is an implication in this division that meaning is associated with 'what we take the facts to be', and that the main problem for the realist is to overcome the gap between a meaningful mathematical statement or formalism, and the independent mathematical reality 'meant' by that formalism.

The Gödel-inspired schema set up in previous chapters is presented in this chapter as an alternative framework in which to express the strengths and weaknesses of realist and constructive accounts.

Meaning, on the Gödel-inspired framework, is associated with the independent realm, rather than with what we understand. As a result, the main problem for realism is the justification of the presumption that mathematical formalisms are indeed meaningful. The realist (according to this framework) presumes that mathematics is justified. The 'problem of justification' referred to earlier, then, is best understood as the recognition or demonstration of this presumption, not its establishment.

The main problem presented by Folina's initial framework is addressed well by her own Poincaré-inspired account. The problems I see as still remaining for her account are comparable to those remaining for my own – with the exception that her account is less able than my own to accommodate the notion of an independent other. This is not something she specifically sets out to do, nor is her account supposed to be realist (although it is supposed to incorporate realist strengths).

I show, minimally, that her account is not more fruitful than my own, nor more effective in answering the questions raised by the initial framework around which it is formed. That is, as the final few chapters all help to show, Folina's (and indeed almost any) account remains as problematic as the traditional realism outlined in its initial framework. And since Folina's initial set-up embodies the common if not predominant terms of reference for accounts and solutions offered across the spectrum of realist and anti-realist concerns, it is a point worth noting.

From a certain angle, weighing Folina's Poincaréan account against my own ultimately means choosing between two different mysteries – the mystery of an independent 'other' and the mystery of the Poincaréan intuitions. This is the case, more or less, for each account I address. The mysteries inherent in these accounts often do more epistemological work than the mystery of an independent other. But just so long as that work involves an attempt to maintain, incorporate, even simply gesture toward, or indicate that 'other', I believe it will always encounter in the final analysis a conundrum as deep as the one I propose. This alone is a reason to retain an overt, fundamental expression of the presence of just such an other, since you might as well gain all of its benefits if you are going to have to carry all of its problems.

There is, though, another angle from which to view Folina's account. Hers is a transcendental account and, as such, can be interpreted as answering the problem of access in a way that leaves room for the notion of a transcendent reality that produces or is at least partially responsible for the way we are (or for the pre-conditions of a finite thinking being). This means it goes somewhat further toward an inclusion of a fully independent/mysterious 'other' than most middle road or realist accounts. A transcendental approach can be thought of as taking us up to the border of 'the other', but then leaving us there. On the other hand, a phenomenological approach – one general heading under which my own account falls – can be thought of more positively, as an illustration of what occurs or might occur if indeed there were a relationship between us and it – or, in other words, if indeed we could see/access what is not us.

Part 2

My own account is an attempt to show that there are relationships that hold between three things: the word (the recognisable physical mark or symbol), the word's sense (what the word means – this is also called 'the formalism') and the independent other (the independent object – also called the independent mathematical object, or reality). I offer a model that draws together these three factors (over two levels), and leaves room for the 'other', without the pitfalls of representation and with the full expression of the desiderata listed. For a graphical representation of a more detailed version of this model, see the diagram on page 136 of this text.

Chapter Five

I begin by outlining an approach to the classical problems that realism faces which, initially at least, sets those problems aside in favour of a thorough examination of the similarities between physical and abstract objects. Particularly I focus on the notion that our understanding of mathematical formalisms depends on the presence of the relevant mathematical objects themselves, and our reception of that presence. I contend that although this idea is generally less contentious when applied to physical objects than to abstract objects, it is equally applicable to both.

I defend this contention through ideas inspired by Husserl's phenomenology. The way I understand it, phenomenology's starting point is with the idea that there is an 'other'. Indeed its main focus is the complex and intricate nature of the relationship between that other and ourselves. Specifically, Husserl studied the way in which the 'real world' of ordinary material things is given to consciousness as existing. I utilise this idea and couple it with the more Fregean (or analytic) optimism that indeed such objects do exist. I then propose that real (or successful) understanding depends on (at least) two things: (1) the assumption of an object whose existence is independent of our understanding and (2) the accuracy of that assumption. I argue that phenomenology opens up the possibility of studying the existence assumption itself, and so can provide a means by which to show the second of the two requirements above. My account, then, incorporates an assumption that there are independent objects and studies this assumption via phenomenology in order to illustrate the nature of the assumption itself, the reasonableness of that assumption and, finally, to express just what such an assumption would look like if indeed it were accurate.

In other words, there is on one level a relationship between what we understand and the assumed existing object. But exploring the nature of the independent object's existence further yields another relationship on another level between the assumed existing object and the independent, actual existing object. This is a deeper expression of the three factors between which I seek to draw identities: 'the word' (our understanding), 'the word's sense' (the formalism/what we understand) and 'the independent other' (which is and is not the 'other' itself). I propose that successful understanding takes place when our understanding of a thing is ultimately identical to the thing itself, just because the thing itself is identical to the assumed existing 'thing itself' that we understand.

The phenomenological framework wherein these identities are studied in more detail offers yet another name for 'the thing we understand' – the noemata (a.k.a. 'the formalism'). 'The word'/'our understanding' is known as noesis. My proposal consists of the notion that the first relationship above depends on the second. That is, (in phenomenological terms) the relationship between noemata and noesis depends (for its accuracy, but also even for its existence) on there being a prior relationship between noemata and the independent object. As I have suggested, this notion (of the dependence of 'real' understanding on the relevant object's real existence) is, perhaps, more in accordance with Frege than with Husserl. But the theory of meaning I go on to express is distinctly un-Fregean. Specifically, I contend that it is impossible to have sense without reference (so long as 'sense' and 'reference' are able to be aligned with 'formalism' and 'independent object').

Chapter Six

Chapter six outlines a theory of meaning according to which meaning is inextricable from the realm of independent objects. The theory is that the existence of an independent object (or realm) is a necessary pre-condition for the meaningfulness of anything we say or even think about that realm. The definition of inextricability demonstrates how meaning can be understood as objective, by showing how it depends on the independent reality itself being present. The definition also explicates the productive nature of that independent reality. For instance, a scenario is provided in order to demonstrate how an independent reality can produce meaning. The notion that an independent reality produces mathematics derives from the idea that mathematical meaning in particular cannot exist without it.

The theory, in general terms, is as follows. 'Natural' or 'real' meaning is inextricable from independent reality. The notion of inextricability encompasses the notion that there is an independent reality which produces mathematical formalisms. More precisely, mathematical formalisms cannot exist without a prior presence, or an origin – an original productive act. The formalism in turn produces our understanding (it gives us something *to* understand). When we understand correctly, then what we understand *is* what the formalism means, and what the formalism means *is* the relevant independent object or reality.

I also try to show what 'meaningless' mathematics would look like according to my theory – that is, the real or natural meaning of a mathematical formalism is contrasted with its 'shallow' or workable meaning. This accommodates the notion that a mathematical formalism's objective (real or natural) meaning need not be recognised as such. Importantly, this allows that widely accepted mathematical meanings of mathematical formalisms do 'get it right', regardless of any theories of meaning. That is, my own expression of the presumption that mathematics is correct most of the time holds that mathematical formalisms do indeed have objective meaning, since their objects do in fact exist independently, and are themselves what is meant.

So, what we understand, what we take as the meaning of a mathematical formalism, what we believe is the relevant object and what in fact is the object – the independent 'other' embodied by the formalism – all can be the same thing and most of the time they are. But, so long as the idea of an 'other' is to be preserved at all, and so long as the notion that it is this 'other' upon which real meaning depends, then our understanding, the meaning and the independent object, must remain separable. The final chapter deals with this apparent paradox. The strategy there is twofold. First, I attempt to make some sense of the paradox and second, I argue that regardless of whether or not it makes any sense, the paradox itself is just as much an integral part of the realist's philosophy as the idea of an independent reality. It cannot be overcome without the loss of the realist notion of an independent reality. In short, in order to accommodate the idea of an 'other' in a mathematical realist account, one must accommodate this paradox.

Chapter Seven

The final chapter focuses on the notion of independence and on presenting a realist philosophy of mathematics able to express and encompass that notion. I show that any realism that at the outset includes the idea of an independent reality that we can and do know, must at some stage encounter the paradox above. More specifically, I give an account holding that there is an independent realm whose reality is at least as robust as the physical world's, and holding that mathematics is mostly correct, and that correct mathematics is the result of our being able to access this realm. I then argue that such an account is posing a problem of justification (the problem of justifying the claim to 'know' mathematics) and that such an account will at least encounter, if not find itself committed to, the following: that the mathematical reality that we do see or comprehend can itself be said to be identical with independent mathematical reality just in case the independent reality is not the thing we see or comprehend.

I attempt to describe this paradox in more detail in order to show that it is, if not completely within our grasp, comprehensible to some degree. I believe that some of Husserl's ideas enable just such a description. That is, phenomenology provides a single conceptual framework wherein both ends of the paradox can be expressed. This does not mean that the paradox is reconciled within that framework, but the fact that one account can naturally and comprehensibly deal with both of its ends is important, and goes some way to providing a way to accept that the paradox has a definite role in any discussion of mathematical realism.

Simply and informally put, I take Husserl as saying something like, 'we cannot know what's outside of our comprehension, therefore we cannot know what's independent or 'other' than us ... but here's how we *do*.' Or, in other words, phenomenology provides a picture of just how things would look if we could know the fundamentally unknowable and that picture correlates well with our real experience of the world and its phenomena.

I argue that the paradox and Husserl's accommodation of it provide the best description of exactly what it is that the realist is committed to by holding that we can 'get it right' (that mathematics is correct and true) and that the reality posited is independent. And so long as the realist wants to argue that mathematical reality is always something different from us and our constructs, but that this fact does not rule out the possibility of our being able to see or access that reality, then he needs some way of accommodating the paradox above.

I accept the paradox but, at the same time, I argue the following: (1) that the paradox is at least as good an explanation of our relationship to mathematical objects as any posited in a philosophy of mathematics so far; (2) that the explanation the paradox provides is tenable or workable; and (3) that accepting the paradox improves our understanding of that relationship. This last point just says that the paradox, though contradictory, is not so far-fetched that we cannot imagine a scenario where both its claims are true. Indeed that scenario is the proposed (phenomenological) 'solution' I put forward.

I argue for (1) using Derrida's arguments to the effect that philosophy in general is no further ahead on the problems posed than my own particular stance. That is, Derrida's arguments show that all philosophies, so long as they share certain features – in particular, the idea of a 'ground' – can get no further in describing our relationship with mathematical reality than an open acceptance of the paradox above.

Addressing (2), I argue that phenomenology, as an account of our relationship to pure phenomena, accommodates the mysteriousness of an 'other' while at the same time tells a story as to how our perception or understanding can access this mystery. That is, phenomenology embraces the paradox, refines it, and makes it 'work' as a solution to the problem posed. The solution is not a resolution. In the end, the answer I put forward to the problem of justification is that it must remain unresolved. The claims that generate the problem need to remain in tension with one another because they cannot be reconciled without being undermined. The 'solution', then, is to describe what we and the world would be like if the paradox accurately encapsulated our real experience with regard to the independent 'other' that is mathematical reality.

Point (3) is not directly defended. Whether or not my analysis of the paradox takes us further than simply ignoring it or attempting to obliterate it, is left up to the reader to judge.

1. Independence and Justification

1.1. Introduction

In his introduction to Shapiro [1997], Stewart Shapiro outlines a number of desiderata for his work. These begin with the statement that mathematical assertions ought to be taken literally, that is "at face value" (p. 3). Shapiro says that his first desideratum suggests realism in mathematics for two reasons. The first of these reasons is that mathematical assertions tend to talk about mathematical objects as though they exist. The second is that scientific assertions tend to talk about scientific objects as though they exist, and scientific language is effectively inseparable from mathematical language. Specifically since model theoretic semantics applies to ordinary and scientific language, this same model theoretic semantics ought to apply to mathematical language.

This means that the initial desideratum boils down for Shapiro to two separate desiderata. The first – realism in ontology – arises from the fact that model theoretic semantics has the singular terms of its language denoting objects and the variables ranging over a domain of discourse. Taking mathematics at face value then, means taking mathematical objects to exist. The second separate desideratum – realism in truth-value – arises from the fact that the model theoretic framework attributes each well-formed, meaningful sentence with a determinate and non-vacuous truth-value, either true or false. Thus the primary requirement that Shapiro places on his own account is "to develop an epistemology for mathematics while maintaining the ontological and semantic commitments [above]" (p. 4). He seeks to fulfil this requirement both negatively and positively.

1.2. The Equivalence Solution

On the negative side he argues against what he sees as the best of the anti-realist programs – those that seek to explain most of mathematics as it stands. His primary claim against these programs is that they involve at least as much of the epistemic and semantic problems involved in realism:

> In a sense the problems are equivalent - for example, a common manoeuvre today is to introduce a 'primitive' such as a modal operator, in order to reduce ontology. The

> proposal is to trade ontology for ideology. However in the context at hand – mathematics – the ideology introduces epistemic problems quite in line with the problems with realism. The epistemic difficulties with realism are generated by the richness of mathematics itself (p. 5).

In turn, I argue in what follows that the difficulties generated by this argument are greater than the difficulties generated by a more traditional platonism. The burden on the latter is to show how we can know anything about an abstract, eternal, acausal realm of real, existing mathematical objects. The burden on the former, however, is to show how the burden associated with the existence posited for mathematical objects in realism is any greater, or indeed in the final analysis, any different, than that which I will argue ought to be associated with the non-existence posited for mathematical objects in anti-realism. What I will show, in the following, is that denying the existence of mathematical objects requires as least as much philosophical justification as asserting it.

1.3. The Caesar Problem

Another of Shapiro's main (negative) arguments is his argument that the 'Caesar problem' is not a real problem for mathematical realism. In short, the Caesar problem poses the question: precisely what are the real or original objects of mathematics and how can these be identified? Specifically, how can numbers be distinguished from any other kinds of object, such as the object that is Julius Caesar? Since there is no 'in principle' way of deciding between the various candidates, it is tempting to conclude with Benacerraf (see Benacerraf [1965]) that the singular terms of mathematical language do not denote objects at all, at least nothing akin to ordinary or scientific objects. Shapiro ([1997]) argues that the acceptability of this conclusion "depends on what it is to *be an object* ... on what sort of questions can be legitimately asked about objects and what sort of questions have determinate answers waiting to be discovered" (p. 5).

Shapiro's argument says that there is no real fact determining the matter of whether the places of a structure and other objects, or whether the places of two different structures, are identifiable. According to Shapiro, the only questions of identity with a determinate answer are those asked within the context of the single structure. That is, there is no determinate answer to the question 'is the 2 of the natural numbers identifiable with the 2 of the real numbers?' but there is a determinate answer to the question 'is the 2 of the natural numbers identifiable with the 4 of the natural numbers?' So whether or not the places of the natural number structure exist as objects in the universe, or existed as objects before we identified them as such, is for Shapiro a non-question.

This has the consequence that our identification or 'discovery' of mathematical structures may or may not be a discovery or identification of objects that existed previous to whatever procedure enabled the discovery or identification itself. I argue that this means that the structures of Shapiro's structuralism are at worst entirely new objects whose existence depends on the various epistemic routes by which they are encountered, and at best, are unable to be identified with any unique realm of real, pre-existing mathematical objects. I also argue that this means that the notion (desirable for the realist) that dependence of mathematical truth, proof, coherence, etc. depends on mathematical reality cannot properly be incorporated into Shapiro's account.

It should be noted, though, that Shapiro himself does not seem to see this as a problem. Rather, he aligns his program with Putnam's internal realism wherein "the notions of object and existence are not treated as sacrosanct, as having just one possible use" (p. 128, quoting Putnam).

1.4. The Response

In sum, in this chapter, I will be looking at Shapiro's structuralism as a version of mathematical realism. I will argue that, although Shapiro's account does address the problem of how we come to know what we know when we do mathematics, it does so at the expense of a clear, specific account of the sort of reality that mathematics is about. In particular, it remains unclear just what structures are and how such things can be said to occupy a place in the realm of independent, objective reality.

Specifically, I will argue that the sort of independence that Shapiro attributes to his structures – i.e. the 'ante rem' existence of structures – does, in the end, little explanatory or practical work. That is, the 'ante rem' existence Shapiro attributes to structures is too vague or too weak a notion to serve the realist aims he seems to want it to serve. I suggest that the problems with Shapiro's 'ante rem' structures derive in part from his rejection of a single fixed mathematical universe and in part from the lack of support his account offers for the strong notion of independence that he initially appears to favour.

In particular, the notion of 'ante rem' existence does not settle traditional realist problems, such as the various offshoots of Benacerraf's problem. As we saw above, these most notably include problems of reference and correctness: for instance, the problem of showing that mathematical names and formalisms correctly correspond to or 'pick out' their genuine counterparts in the independently existing mathematical realm, if these counterparts are indeed independent of their instances.

I also argue that Shapiro's 'ante rem' existence for structures does not, in the end, defend the idea commonly held by realist mathematicians, that there is a determinate yes or no answer to mathematical questions and unproven mathematical theories.

1.5. Strong Ante Rem Independence

Structuralism is often summed up with the phrase 'mathematics is the science of structures'. According to the structuralist (Shapiro [1997]), what we study when we do mathematics are structures: "the abstract form(s) of ... system(s) ... [where a system is] a collection of objects with certain relations" (pp. 73-74).

Shapiro reworks the structuralist slogan 'that mathematical objects are places in structures'. He points out that there is an intuitive difference between an object and the place or space it occupies. There are, then, two separate items to consider – places in structures, and objects that occupy those places.

In Shapiro's account, an object is an 'office-holder', and the place in a structure that object might occupy is an office. According to Shapiro, the intuition that these are indeed two separate items boils down to a distinction in linguistic practice.

There are, he says, two different ways of talking about structures and their places, depending on where (or on what) your focus lies.

When your focus is on the exemplifications of a structure – that is, on the things that occupy its places – you speak as though there is a collection of things that can occupy or exemplify the places of that structure. This is Shapiro's 'places-are-offices' perspective. To take Shapiro's examples, we might focus on the objects – people in this case – that occupy the places of the structure of a baseball team. A comment that "the shortstop today was the second baseman yesterday" would be made from this perspective. Also from the 'places-are-offices' perspective, we could focus on the possible occupants of the structure of government, and note that "the current vice president is more intelligent than his predecessor". The idea that the Von-Neumann 2 has one more element than the Zermelo 2 is likewise a concept arising from the 'places-are-offices' perspective (all quotations from p. 82).

On the other hand, from the 'places-are-objects' perspective, statements are about structures as such, independent of any of their exemplifications (p. 83). To take Shapiro's examples again: "we say that the vice president is the president of the senate, [or] that the chess bishop moves on a diagonal" (p. 83).

In sum, the domain of discourse from the 'places-are-offices' perspective is the objects or things that occupy the places of a structure, whereas the domain of discourse from the 'places-are-objects' perspective is the places themselves, of a given structure.

For Shapiro, the slogan that mathematical objects are places in structures is to be interpreted from this latter perspective. So that when the structuralist says that sets are objects, he means that each place in the set theoretic structure is an object – at least grammatically and at most literally.

Arithmetic, then, is about the natural number structure, and its domain of discourse consists of the places in this structure, treated from the 'places-are-objects' perspective. The same goes for the other non-algebraic fields, such as real and complex analysis, Euclidean geometry, and perhaps set theory (p. 89).

The above distinction is important to Shapiro's account for a number of reasons, one of which is the articulation of his particular realism, particularly to his claim that structures are 'ante rem'. That is, Shapiro's realism takes the 'places-are-offices' perspective literally. For Shapiro, "bona fide singular terms, like 'Vice-President', 'short-stop', and '2', denote bona fide objects" (p. 83).

Likewise, for Shapiro:

> structures exist whether or not they are exemplified in a nonstructural realm. On this [interpretation] statements in the 'places-are-objects' mode are taken literally, at face value. In mathematics, anyway, the places of mathematical structures are as bona fide as any objects are (p. 89).

Another reason that this particular distinction is important to Shapiro's account is that he sees it as enabling his program both to commit to numbers as objects, and to resolve (or, more accurately, to resign) Frege's 'Caesar problem' via ontological, specifically interstructural relativity.

This second reason will be dealt with in some depth later, but is worth touching upon now. Recall that, in a nutshell, the Caesar problem asks how numbers can be objects (as Frege argued they were) when it seems that there is no way in principle of distinguishing between numbers and any other kind of object. How, for instance, can we tell that the object '2' is the same or different to the object 'Julius Caesar' (p. 78)?

Shapiro's solution uses the distinction between the 'places-are-offices' and the 'places-are-objects' perspectives by arguing that each perspective creates its own context for these kinds of identity questions, and it is only within the confines of one of these contexts at any one time that such questions can even make sense.

For instance, we may ask, from the 'places-are-objects' perspective, whether or not the object '2' is the same or different from the object '4'. And because this question is asked within the confines of only one of the two possible perspectives, we may expect to discover a definite answer. It is determinate, Shapiro argues, that '2' is identical to '2' and not to '4'. But he adds:

> it makes no sense to pursue the identity between a place in the natural number structure and some other object, expecting there to be a fact of the matter. Identity between natural numbers is determinate; identity between numbers and other sorts of objects is not, and neither is identity between numbers and the positions of other structures (p. 79).

According to Shapiro, these latter questions, rather than staying within the confines of only one or the other of the two perspectives, commit the error of taking both perspectives at once – that is, of trying to compare their different foci within one question.

Asking whether "the shortstop is Ozzie Smith" (p. 79) commits a similar mixing of perspectives, or crossing of contexts, and is therefore deemed by Shapiro to be a non-sensical question. He argues that in this case the person and the position share no criteria by which to establish or reject their identity. Of course, Ozzie Smith can take the position of shortstop, and so in this sense, 'be' that position, but according to Shapiro, this is not a question of identity. And it is in just this same way - that Ozzie Smith can be the shortstop, but can neither be nor not be identical to the shortstop - that Julius Caesar can be '2', but is neither identical nor non-identical to '2'. Shapiro's solution to the Caesar problem, then, is to ban the question it asks. Questions of identity like these, that mix perspectives, "do not have definitive answers, and they do not need them" (p. 80).

This solution though, in order to be consistent, has to apply to questions of identity between different structures as well. This is because the places of one structure can be the place holders of another, in the same way that Ozzie Smith can be the shortstop. I will be arguing (later) that it is this particular application of his general ban on mixing perspectives that threatens Shapiro's overall solution to the Caesar problem. To anticipate, I believe that there is something intuitively quite different between asking if Julius Caesar is identical to 2 and asking whether or not the 2 of the natural number structure is identical to the 2 of the real number structure. This latter sort of question is not so easily dismissed as non-sensical, or wholly indeterminate.

The important point for now though is that, from what we have looked at so far, it can be argued that the nature or degree of independence that Shapiro wants for structures is a significant one. For Shapiro, taking the 'places-are-offices' perspective literally means that structures exist before and apart from any non-structural exemplification in any realm whatsoever (p. 75). This could mean a couple of different things. It might mean that the existence of structures is an existence that is independent of the theories – mathematical or otherwise – that we have about them. It might also mean that their existence is independent of both the act (since they exist before exemplification) and objects (since they exist apart from any exemplification) of our comprehension or understanding.

The benefits of an ante rem structuralism, Shapiro believes, include "[the fact that this kind of structuralism is] the most perspicuous and least artificial ... It comes closest to capturing how mathematical theories are conceived" (p. 90). I agree with this claim only provided that by 'ante rem structuralism', Shapiro means a structuralism which includes both degrees of independence given above.

Even if this proviso is granted, though, and we conclude that the most perspicuous structuralism posits structures with this degree of independence, there still remains the question of how to accommodate this degree of independence in a structuralist account. This question will arise again later. First, I will further consider the evidence in support of the idea that Shapiro *does* mean to include this degree of independence for the structures in his account.

1.6. Ante Rem vs. In Re

When he considers the possible status of mathematical structures, Shapiro distinguishes his 'ante rem' stance from an 'in re' realism regarding structures. The former stance, he says, takes structures as "exist[ing] prior to and independent of any items that may instantiate them" (p. 84). The latter, by contrast, takes structures to be no more than a generalisation of those items. So, according to 'in re' realism, structures exist only as a result of their instances (specifically of the fact that those instances can be generalised).

Dummet's characterisation, and the fact that Shapiro reproduces it in order to help explain his 'ante rem' realism, is also useful here. Dummet's 'mystical structuralism' (congruent to Shapiro's 'ante rem' structuralism) distinguishes structures by what they are not; none of a structure's extrinsic instances or properties are what the structure itself is. Nor are they part of it. The structure is independent because it is not dependent on anything, not even on likely candidates such as sets or numbers.

By contrast, Dummet's 'hard headed' structuralism (congruent to Shapiro's 'in re' structuralism) identifies a structure as the common element of all its systems (p. 85).

So, the fact that Shapiro aligns his ante rem structuralism with Dummet's 'mystical structuralism' gives support to the idea that the independence he envisaged for those structures (at least at this stage) was a significant one. Indeed, Dummet's characterisation gives 'mystical/ante rem' structures complete independence from any of the ways we presumably come to know or 'see' them: i.e. via their extrinsic instances and properties. This suggests that Shapiro's structures *are* supposed to be independent of the acts and objects of our comprehension.

1.7. Not So Strong Independence / The Equivalence Solution

It could, however, be argued that the nature and degree of independence that Shapiro wants for his structures is not the sort of strong independence outlined above. Perhaps the independence he ascribes is not even supposed to be as strong as that which we usually ascribe to ordinary physical reality. This might be thought to be the case (or alternatively, it may be argued that Shapiro *can* no longer defend the strong ante rem stance he begins with) especially given chapter seven of Shapiro [1997]. In this chapter, Shapiro seeks to establish that an anti-realist who proposes to avoid a commitment to the existence of abstract objects such as sets or structures is ultimately no better off – particularly on the epistemic front - than a realist who embraces the notion that such objects actually exist.

In so far as it challenges the anti-realist notion that mathematics is about no more than linguistic or logical notions we are already familiar with, this line of thought is a powerful tool for the realist. But just so far as it challenges these anti-realist notions, to that same extent it also jeopardises the realist's own proposed notion that mathematics is about an actual rather than a possible reality, especially any kind of non-derivative, or strongly independent realm.

The argument that Shapiro puts forward begins by showing that the anti-realist programs he examines and the realist's commitment to real abstract objects are, in fact, equivalent. That is, he shows that "any insight that modalists [the particular group of anti-realists to whom Shapiro directs his argument] claim for their system can be immediately appropriated by realists and vice versa. Moreover, the epistemological problems get "translated" as well" (p. 219).[1]

Very basically rendered, the argument compares a possible mathematical realm (one built from a plural quantifier, a contractibility quantifier, or an operator for logical possibility) with a real mathematical realm (of structures, individual objects, or simply an independent reality). This comparison shows that each is as problematic as the other. Given this situation, Shapiro concludes that we might as well opt for the real realm. This is primarily because this option minimises the 'ideology' needed to explain the realm in question (given that a real realm takes mathematics at face value), is the 'most perspicuous and natural' interpretation and, not least, explains why mathematics is done as though it is about independently real objects.

I accept Shapiro's arguments here, and take it as true that modal anti-realism (where 'modal anti-realism' encompasses all the programs Shapiro reviews) is, in an important sense, equivalent to ante rem structuralism.

But Shapiro's argument shows more than this. After all, equivalence goes both ways. Unless something more than a reduced ideology sets it apart from its anti-realist counterparts, ante rem structuralism inherits all the problems of modal anti-realism, including, for example, the particular burden of explaining why mathematics can legitimately be taken at face value at all – how, for example, its objects are still as bona fide as any objects in the natural or physical world. And so, one of Shapiro's original desiderata is undermined. And these sorts of questions

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[1] For a full account of this argument see Shapiro [1997, 216–242].

mean that the realist camp now has to take on board precisely what Shapiro hoped to avoid – an increased ideology.

So the proposed "trade off ... between a vast ontology and an increased ideology" (p. 218) has more ramifications than are at first apparent. The trade-off does, as Shapiro claims, mean that any possible reality inherits all the problems of an actual reality, especially since a possible ontology is, in the end, no smaller or less problematic than its real counterpart.

But the trade-off also means that a real or actual ontology is, for all intents and purposes, equivalent to a possible ontology – unless something extra sets the former apart from the latter – and so perhaps not as real as, say, physical reality. Certainly, it is not easy to see how such an actual reality is in any way independent – either of the constructed or possible reality it is equivalent to or, by extension, of the quantifiers and concepts from which a possible ontology is derived. We are almost back to square one. The reality ascribed to ante rem structures now appears to be no more solid or objective than the reality ascribed to their modal counterparts. If this is indeed the case then the realist now shares the anti-realist's burdens and makes little or no gain over the anti-realist on such issues as justification, objectivity and how mathematical statements can convincingly – that is, at least as convincingly as statements about the physical world – be said to be true or false.

1.8. Independent Ante Rem Justification / Shapiro's Positive Approach to Realism

After outlining his ante rem stance, Shapiro proposes a possible anti-realist response asking how we are justified in our beliefs about mathematical objects. In response to this question Shapiro offers the theory that we can, through physical perception, "recognise", "apprehend" and "attain knowledge of" abstract patterns – hence also of abstract structures. Those structures not directly attainable through physical perception, he argues, "are in fact apprehended (but not perceived) by abstraction". Specifically, this theory says, "pattern recognition ... is a faculty ... leading to an apprehension of freestanding, ante rem structures" (all quotations p. 113).

As it stands, though, even another realist might feel that there is something missing from this theory. Imagine a basic physical realism; suppose that this realism is simply the belief that the physical realm is real. Such a realist might hold that ordinary physical objects have an existence independent of our own – i.e. they exist prior to and separate from us, and so are not dependent on us. In order to justify our beliefs about these sorts of objects, such a realist needs to show (at least) that what we 'perceive' as physical reality is in fact, or in some sense, the same reality that existed prior to, or independent of our perception of it. Note that in order to show this, the 'what-is-perceived' and 'what-existed-prior-to-that-perception' must (initially at least) be distinguishable from each other. If they are not distinguishable, then 'the tree I see' cannot, at some later stage, be *shown* to be identical to, or in any way the same as 'the tree that exists prior to and independently of my seeing it'. Yet this active, later identification is a necessary part of the justification Shapiro requires if his more perspicuous ante rem stance is to be maintained. Simply assuming that the 'what-is-perceived' and 'what-existed-prior-to-that-perception' are the same thing from the beginning, with no initial distinction between them, begs the question of how our beliefs about this previously existing independent reality are justified. And this reasoning can be applied to Shapiro's theory above, simply because nothing distinguishes his 'apprehended structure' from his 'independently existing structures'.

Shapiro does give this problem some attention, specifically by imagining and answering another possible objection to his theory – this time of pattern recognition:

> Somewhere along the line anti-realists might concede that pattern recognition and the other psycholinguistic mechanisms lead to a *belief* in (perhaps ante rem) structures, and they may concede that we have an ability to coherently discuss these structures. But anti-realists will maintain that these mechanisms do not yield *knowledge* unless the structures (or at least their places) exist. Have we established this last, ontological claim? Can this be done without begging the question?
>
> ... I present an account of the existence of structures according to which an ability to coherently discuss a structure is evidence that it exists ... This account is perspicuous and accounts for much of the 'data' – mathematical practice and common intuitions

about mathematical and ordinary objects. The argument for realism is an inference to the best explanation. The nature of structures guarantees that certain experiences count as evidence for their existence (p. 118).

Again, it is not only an anti-realist who could object here, but a realist as well. Specifically, the realist along with the anti-realist, could note that certain mechanisms do lead to things that look like the proposed ante rem structures, but nothing about the mechanisms themselves can establish that these things are indeed what they appear to be: independent, previously existing structures. The mechanisms lead perhaps to a belief in Shapiro's ante rem structures but not necessarily to those structures themselves. Further, evidence that an abstracted or psycholinguistically apprehended structure exists is not evidence that the same structure existed prior to, or independently of, the mechanism that led us to believe in it. This evidence only supports the existence of the *apprehended* structure, and without reason to believe that the structure is ante rem, evidence for the existence of what we see – even if this evidence is the ability to coherently discuss the object of our perception – is not evidence for its existence independent, or prior to, our perception of it. Again, these two objects – the object of our perception and the independent object existing prior to our perception – need to be distinguished before they can be shown to be the same, similar or strictly identical. You cannot prove that two things are in fact one thing simply by assuming that they are. So it still seems that Shapiro's structures have a degree of independence which is in fact less than his initial ante rem stance suggests.

Suppose, though, that we give up trying to maintain any initial separation between the two, and we do simply assume that the apprehended structure is one and the same as the independent structure. A question remains: in what *sense* is the yielded structure's existence something more than – above and beyond – the existence of the end result of the process of abstraction or psycholinguistic apprehension that yielded it? That is, does the idea of independent or prior existence attach naturally to the structures arrived at via abstraction? If so, how? What is it about the psycholinguistically yielded structures that makes it reasonable to ascribe independence or priority to their existence?

If this same question were applied to physical reality, one possible answer would be the apparent persistent existence of the objects when we close our eyes, walk away from them, etc. What sort of things can we say about abstract mathematical structures such that our experience of them gives evidence for their independence? There certainly are compelling features of mathematical reality we can appeal to that support the idea that it is independent and existed prior to our perception of it. The undeniable force of mathematical truth is one such feature. The point is that we can ask for this sort of evidence. And an appeal to our ability to discuss it coherently, by itself just doesn't seem to provide enough of the sort of evidence required for justification. And so the question remains.

All this is just to say that it is reasonable to ask what sort of existence the apprehended structures can naturally or reasonably be said to have – given only what we do know and experience of them. That is, we can ask: does independent or prior existence sit naturally with the structures arrived at via abstraction? Or is such an ascription of independence, in this case, somehow ad hoc?

1.9. The Natural/Least Artificial Attitude

How, then, do we accommodate strongly independent ante rem structures in a relatively natural – i.e. in a 'non-ad hoc' – way? One way of judging whether the inclusion of this sort of independence is ad hoc or not, is to assess how well the proposed independence sits with the account overall. This can be done, first of all, by assessing whether or not the account has 'room' (or enough room) for this sort of independence and, if it does, by assessing whether this room provides comfortable or forced accommodation.

But before thus assessing Shapiro's account, note that another occasion upon which a proposed independence might appear ad hoc is when its inclusion in an account is simply stipulated. I grant that a significant independence, such as Shapiro's ante rem structuralism involves, probably does have to be simply stipulated at some stage or other, if it is to be included in an account at all. Independence cannot, after all, be positively 'shown'. Generally, we establish that something is independent of something else by showing that it is not dependent on the something else, rather than by attempting to positively illustrate some inherent independent nature it might have. Independence is 'arrived at' (insofar as it is 'arrived at' at all) negatively, and so also by stipulation, rather than by proof. So the fact that something is simply stipulated in an account does not automatically render its inclusion ad hoc. I take it that we can suspect the idea of independence to be ad hoc when its setting (i.e. the rest of the account) renders its inclusion somehow unnatural or, at worst, if there is nothing in an account able to interpret or make some sense of such a stipulation.

Now recall that, even if we agree with Shapiro's claim that the ability to discuss a structure coherently is itself evidence that that structure exists, the issue of the nature of that structure's existence still remains untouched. At best, all we have is evidence that an abstracted or psycholinguistically apprehended structure exists (independent or otherwise). But it remains unclear how *this* evidence supports the *independent* existence of that same abstracted structure.

We can get some idea of how Shapiro regards this problem from his brief discussion of the ambiguity, inherent in his account, in the reference of numerals (and across mathematics in general). Shapiro points out that his structuralism entails that:

> '4', for example, denotes a place in the natural number structure, a finite cardinal structure, a finite ordinal structure, a place in the real number structure, a place in the integer structure, a place in the complex number structure and a place in the set theoretic hierarchy (p. 120).

Now recall that for Shapiro, structures exist before and apart from any of their exemplifications. But if we take this to mean that for Shapiro there is an independent structure that existed before our denoting term '4' did, then it seems that the above ambiguity ought to be able to be resolved. That is, we ought to be able to say whether or not the denoting term '4' picks out that particular pre-existing structure, or a place in that structure.

1.10. The Caesar Problem Revisited

This same problem can be approached from a number of different angles, and the Caesar problem is just one of these. In chapter four of [1997], Shapiro utilises

Robert Kraut's work, "Indiscernibility and Ontology", to provide an "epistemic route" to mathematical structures. Kraut considers an imaginary economist who speaks a version of impoverished English, a language that does not have the resources to distinguish between two people with the same income (p. 19). Where person P has the same income as person Q, anything that the speaker of this impoverished language can say about P applies equally to Q. That is, an interpreter of the economist's language could apply the Leibniz principle of the identity of indiscernibles that, for the economist, P=Q, since the two items cannot be distinguished. From the economist's point of view, they are treated as a single object. Shapiro gives the following scenario to illustrate:

> if the interpreter sees a certain woman [who earns $35,000 per annum], he might say (on behalf of the economist), "there is the $35,000." If a man with the same income walks by, the interpreter might remark, "There it is again." The economist herself might make the identification if she knows that the two are indiscernible and does not envision a framework for distinguishing them (p. 19).

Even at this stage, there is a natural ambiguity that threatens to undermine any potential realist stance regarding the objects proposed. So long as the economist's 'knowledge' that the two are indiscernible consists in her not being able to envision a framework in which they might be distinguished (in this case, the full background language, ordinary English), the object's existence could, in an important sense, be regarded as *given* by the language itself, or as prior to the existence of this particular language. This early ambiguity could be understood as highlighting the shortcomings of the Leibniz principle, specifically, the possibility that the principle by itself is not enough to establish true identity, i.e. objects that are able to be 'taken as' one rather than two, are not necessarily one rather than two in point of fact. An alternative understanding of the ambiguity might simply be that an application of the principle that results in identity relations between things on this basis is, in fact, a *mis*application.[2]

Shapiro makes two points, both of which can be used to support the idea that an object's existence is given by the (object) language itself. First, he notes that

[2] Suggestion by Dominic Hyde.

"nothing is lost by interpreting [the economist's] language as about income levels and not people [such that] ... A singular term, like "the Jones woman", denotes an income level" (p. 20). But if we were presented with this interpretation alone, we'd have a free-standing language, without the history from which it evolved, and in this case something more than history is lost. Specifically, the missing history takes with it an item of evidence that the object's (or at least *an* object's) existence predated the language being used to refer to those objects. Indeed Shapiro agrees that the only way we can understand examples like Kraut's economist is by possessing the 'background language' ourselves. This same point applies to the mathematician and to the 'number-person' that Shapiro models on Kraut's example. (This latter is someone who speaks an impoverished version of English in which equinumerous collections of objects are indiscernible.)

The second point Shapiro makes that appears to lend support to the idea that the object's existence arises from language, rather than the other way around, is the following: one way of understanding the thrust of the sub-languages examples, is through Frege's *Grundlagen* result (taken here from Coffa [1991]) that "a wide range of statements previously regarded as extralogical and involving an appeal to either empirical intuition (Mill) or to pure intuition (Kant) involved only reference to concepts" (p. 75). That is, 'number statements' make no sense so long as we try to understand them as being about objects, but they do make sense if we understand them as having a concept as their 'target'. Or, as Shapiro [1997] puts it:

> Suppose that our number-person looks at two decks of cards and we interpret him as saying "There is two". Then we assume that, at some level, the number-person knows that it is the decks and not the cards or the colours that are being counted. Nothing in the hunk of mass determines that it is 2, 104, or any other number for that matter. If the subject loses the use of sortals like "deck", he will not see the stuff as 2. To accomplish this feat, the subject must be aware of the decks and must distinguish the two decks from each other (pp. 20-21).

Frege's point (according to Coffa [1991]) is that since it is only when the concept is determined that the number attribute is fixed, regarding the concept itself as the topic of the number statement is natural. But Frege was sensitive to the possibility that this idea introduced an ambiguity between understanding and acquaintance.

He not only addressed this possibility; he went to great pains to avoid it by drawing a sharp distinction between sense and the real world, pointing out that the things we understand, be they concepts, universals, properties or essences, are something *altogether different* from the 'real world'. That is, "He denied that understanding is a glorified form of 'seeing' aimed at these entities...[His] point was that while understanding does involve "giving" the object in question, it need not necessarily be "given" in the mode of acquaintance" (p. 81).

Contra-Frege, and in defence of structuralism, Shapiro offers what he argues is a better way to understand the Kraut sub-language examples; namely, that what they show us is that income levels, numbers etc, are 'places in structures'. (Recall Shapiro's distinction, in linguistic practice, between an object, an office holder; and a place in a structure, an office.)

Although Shapiro acknowledges that the division between the places-are-offices perspective and the places-are-objects perspective is both relative and not sharp, its existence is nonetheless a crucial part of his structuralist interpretation of the sub-languages examples. This is because if the distinction is not drawn at some point, Shapiro's program becomes an eliminative program, whereby the places-are-objects perspective is reduced to the places-are-offices perspective – specifically by calling the former a generalisation over the latter. It is purposely to avoid this scenario that Shapiro gives a direct stipulation that at some point the places-are-objects perspective has to be taken literally. The problem then becomes whether, given the rest of his account, this perspective indeed can be taken literally and, if it can, whether the simple stipulation that it has to be taken literally and so therefore will be, can satisfy a realist for whom independence is a basic desiderata. That is, we now have some way of judging whether the independence of Shapiro's structures is somehow ad hoc. Recall that a realist for whom independence is a fundamental part of their philosophy may feel that simply *stipulating* a principle of independence does not do justice nor account for its central role in realism. Far from being fundamental, independence when simply stipulated can remain undefended, unaccounted for, and so relegated to a secondary, ad hoc position.

To further clarify these points, I propose the following as a working definition of the terms used above. A fundamental principle of independence is one which entails (or, perhaps more correctly, is burdened with) the idea that the existence of mathematical language itself depends on the previous existence of mathematical objects. That is, the list of things we can literally 'make up' or imagine, depends on the list of things already in existence. Further, if there were not an external reality we could talk about (including ourselves), there would be no talking. The fundamental principle, then, says that mathematical objects are not things we could have imagined. They are not part of the list of things we could have talked about were they not in existence. Hence, our ability to speak of them – mathematical language itself – depends on their existence. By contrast, a secondary or 'ad hoc' principle of independence is one which (at least at its extreme end) entails that mathematical language 'gives' the existence of mathematical objects – this is the idea that language and reality (minus mathematical objects or reality) is or was enough to give rise to mathematics itself, or the idea that the existence of mathematical objects depends on the previous or sometimes simultaneous existence of mathematical language. At least, this is the concept of an 'ad hoc' principle at its extreme end. At its less extreme end, an ad hoc principle is one which is ambiguous between the extreme ad hoc and fundamental definitions.

Suppose now that the independence sought is fundamental, and stipulated independence is consequently perceived as a problem. It is in the solution to this particular problem (i.e. the problem of stipulated independence) that a key difference between accounts such as Shapiro's, and accounts that run more along (what I have dubbed) Fregean lines, can be identified.

A Fregean-type answer argues not only that the places-are-objects perspective can indeed be taken literally, but also that the stipulation that it *ought* to be is neither simple nor ad hoc. This type of answer seeks to show that independence is fundamental in accordance with the definition above, and so can fulfil our proposed realist's desiderata. This type of answer represents (for our current purpose) a concerted effort to *retain* the realist idea that the truth or falsity of mathematical statements in some way depends on the presence or absence of the corresponding object in mathematical reality. In what follows I will argue that the cost of this effort – or the inescapable burden it entails – is the retention of 'the Caesar problem' (which is one reason for the dubbing of this sort of approach as 'Fregean').

A Shapiro–type answer also argues that the places-are-objects perspective can be taken literally, but this type of answer makes a concerted effort to avoid the Caesar problem. What I hope to show is that avoiding the problem entails the loss of the proposed realist's desiderata above – i.e. the notion of independence as fundamental.

1.11. The Fregean vs. the Shapiro approach

Frege's original Caesar problem came about via his belief that the 'real world' comes to us *already* divided up into separate elements: namely those elements that constitute a given claim's topic and determine its truth-value.

Coffa [1991] illustrates the Fregean approach with the following example. Whether the claim:

(A)The author of 'Waverley' is tall

is true or false depends entirely on whether a single object in the universe, namely Scott, has a single property, namely a certain height. The grammatical units:

"the author of 'Waverley'" and

"x is tall"

are associated with the truth-relevant parts of the world, i.e. with those elements of the world that are the only ones relevant to the determination of whether what (A) says is true. That is, the grammatical units of a claim are associated with the elements that constitute its topic and determine its truth-value.

The real world, then, includes all such elements: "all that we talk about when we don't talk about talking" (p. 80).

These elements, according to Frege, are themselves what is *meant* by a given claim. That is, they are a given claim's 'significance' or 'meaning' (p. 80).

So, grammatical units, effectively or ineffectively, correctly or incorrectly, sort the world into those elements that matter and those that don't as far as the truth of a given claim is concerned.

The following argument, also Coffa's, clarifies Frege's belief in an original, previously partitioned ontology (the real world), going on to show how this belief leads directly to the Caesar problem:

> All that is required for the purposes of communication or responsible discourse in general is that what we say be intelligible, and this has little to do with the possession of effective methods to identify either its referents or truth-values (p. 81).

But, as Coffa observes, sense in Frege's account is nonetheless intimately related to the truth-value-relevant elements in the world. This, Coffa notes, is strongly suggested by a seldom-noticed fact: Frege took it as self-evident that the grammatical analysis appropriate to the study of sense coincides with the grammatical analysis appropriate to the study of meaning. The understanding of the sense of a sentence is not, for Frege, a holistic phenomenon but comes about through the understanding of the sense of its parts.

> Notice that there is, in principle, no reason the grammatical units that provide the building blocks of propositional sense should be the same that provide the truth-relevant features of the world, *unless* sense (and therefore understanding) are essentially a matter of doing something with those worldly units (p. 81).

So, from the perspective of a Fregean approach, examples such as the Kraut sub-language procedure produce claims capable of being 'true' or 'false', *only* if the objects produced by such languages (indeed, by any singular term - which, for Frege, included numerals) are in some essential respect *not new* (this respect might be 'relevant to' or 'including something of' the old, or original, ontology).

For Frege, any language, including natural language, is capable of producing true claims only when it contains a device establishing a correlation, however ineffective, with objects of the original ontology – in this case, of the real world. That is, procedures such as Kraut's 'sub-language procedure', from the Fregean viewpoint would be capable of true claims only to the extent that they are capable of establishing a link between the original elements of the real world and the elements 'produced' by that language. Indeed, for Frege (according to Coffa), it is only in this way that language can be seen as a means by which we *understand* claims concerning the objects language produces.

So for Frege, whether or not our understanding is effective – that is, whether what we understand is capable of being true or false – depends upon the absence or presence of objects in the original ontology, the real world. Another way of putting this point is to say that our understanding depends on what is, and is not, the real world. I believe that this dependence of effective understanding on the existence of the relevant objects in the real world comes about directly as a result of Frege's (at least according to this interpretation) grasp and retention of the original realist intuitive notion of independence in a realist perspective. It is when the idea of a single fixed universe is retained that the problem of identifying all the items to be found in the original ontology presents itself. And so we arrive back at the Caesar problem, so named because Frege's account seems both to demand, and lack, a means by which to determine how and why each number is the same or different from any object whatsoever. Shapiro [1997] elaborates: Frege's account "has nothing to say about the truth-value of identity statements such as '(A) Julius Caesar = 2' or '2 = {{∅}}' etc." (p. 28).

Shapiro, along with commonsense, in fact agrees that we could not understand claims about objects which are themselves produced by a language without *an* original ontology. But, in contrast to the Fregean approach, he argues that there is not something that could be called *the* background ontology. For Shapiro, "the idea of a single, fixed universe, divided into objects *a priori*, is rejected" (p. 28).

Recall that Shapiro sets out to show that a "good philosophy of mathematics should show why questions as to the truth-value of statements like (A) need no answers, even if they are intelligible" (p. 13).

In order to establish this, Shapiro offers that in mathematics there are no objects *simpliciter* (p. 15). That is, there is no structure-independent answer to such questions. A mathematical object for Shapiro is a place in a fixed structure. So a number, for example, is a place in the natural number structure. According to

Shapiro, identity between objects within a given structure *is* determinate, but identity between numbers and other sorts of objects is not (p. 14).

Thus, according to Shapiro,

> ... one can ask about numerical relations between numbers, relations definable in the language of arithmetic, and we can expect definite answers to these questions. Thus, 1<4 and 1 evenly divides 4. These are questions internal to the natural number structure. But if one inquires whether 1 is an element of 4, there is no answer waiting to be discovered. It is similar to asking whether 2 is heavier than 4, or funnier (p. 14).

Thus, for Shapiro, only statements of the former kind – i.e. those *internal* to a single structure, have a determinate truth-value, so the truth-value of this (internal) sort of statement is *to be discovered*.

By contrast, Shapiro claims that the truth-value of statements like the latter is a matter of invention or stipulation. Hence there is no answer waiting to be *discovered* to the Caesar problem.

To delve further into the Kraut sub-language procedure, viewing the level-roles as played by groups of people, or by individual people, is taking the 'places are offices' perspective. But, "when we focus on the impoverished sub-languages and interpret them with the Leibniz principle, we take the places of the structure - income levels and numbers – as objects in their own right. This is the 'places are objects' perspective" (p. 21).

Now, the process by which we arrive at this 'impoverished sub-language' involves, first of all, taking a framework with objects already within the range of its variables – i.e. a framework that already has ontology. The next step is to

> focus on an equivalence relation over the ontology of the base language [which] ... divides its domain into mutually exclusive collections – called 'equivalence classes' ... The idea here is to see the equivalence classes as exemplifying a structure and to treat the places of this structure as objects (p. 21).

A sub-language is then formulated for which the equivalence is a congruence such that where two items are equivalent they are indiscernible. Shapiro suggests that:

> In such cases ... in the sub-language, the equivalence relation is the identity relation. The idea here is that the language and sub-language together characterise a structure, the structure exemplified by the equivalence classes and the relations between them formable in the sub-language. It is thus possible to invoke the places-are-objects orientation, in which the places in this structure are rightly taken to be its objects (p. 22).

The final step in Shapiro's argument is to introduce the idea that the framework of pure mathematics, along with that of his own 'structure theory', might allow sub-languages in which either isomorphism or structure equivalence is a congruence. In this way, structures themselves are to be seen as objects in their own right and, according perfectly with the structuralist's credo, mathematics is the science of structure.

From the perspective introduced above, the problem with this is that it can boil down to a claim that it follows from the coherence of a structure characterisation that there are things which satisfy it. Indeed, Shapiro believes that coherence just means satisfiability. And this means that coherence cannot be characterised in a non-circular way. In his defence, Shapiro argues that mathematics itself involves an inherent circularity born of the common mathematical practice of settling questions of coherence just by modelling one structure in another – for example, complex numbers are coherent just because they can be modelled in the real number structure. The final port of call, then, is set theory – in which all other mathematical structures can be modelled (p. 22).

According to Shapiro, set theory itself is satisfied because it is coherent, and is coherent because it is satisfied. But we can't have it both ways. If the former is the case, the (Fregean-type) realist notion of dependence of mathematical formalisms, proof, truth and coherence on mathematical reality is lost. Only if the latter holds is this important realist notion retained. And it is hard to see how Shapiro could argue that a structure is coherent because it is satisfied without a determinate background ontology of mathematical objects – or without the idea of a single, real mathematical universe that exists not because of, but before, our notion of mathematical coherence.

1.12. Structures, Realist and Otherwise

The idea that mathematical systems and concepts themselves are founded or depend on a pre-existing mathematical reality (and not the other way around) is one of the central realist notions given in the introduction as being worth trying to retain. But, for the realist seeking to retain those central notions, there are other features that a structure characterisation would need to posses before it could serve as any sort of foundation or 'final court of appeal' for mathematical theories and questions of mathematical existence.

Specifically, a foundation would need, for the realist, to be in some sense metaphysical, at least to the extent that existence questions are resolvable only by reference to a reality independent of the systems it grounds. That is, existence questions should, for the realist seeking to retain the central notions, be resolved by reference to a supposed, or stipulated (in a non-ad hoc fashion), abstract reality, as opposed to the coherent characterisation of a structure or indeed to any sort of a reality whose existence can likewise be attributed to ourselves, or to our ability to define or construct it. Indeed, without reference to a metaphysical independent reality, the line between construction and discovery – or in this case, between a coherent structure and a satisfied structure – is blurred.

The realist can grant that we can create coherent structures and say they are satisfied, or he can say that we can discover satisfied structures and show that they are coherent. But the term 'discover' and the idea that the structures are 'ante rem' both lose their force if, in the final analysis, each of these ideas is defined in terms of the other.

So, in order to retain the force of such terms and of the ideas they represent, the difference between the sort of foundation Shapiro envisages and what is perhaps a realist's ideal foundation for mathematics is worth exploring further.

This is where Maddy's Set Theoretic Naturalism provides a useful foil. Like Shapiro, Maddy suggests, in Maddy [1996], that set theory

> Perhaps most fundamentally ... provides a court of final appeal for questions of mathematical existence and proof; [that is] if you want to know if there is a

> mathematical object of a certain sort, you ask (ultimately) if there is a set theoretic surrogate of that sort: if you want to know if a given statement is provable or disprovable, you mean ultimately from the axioms of set theory (p. 26).

And, Maddy argues, that is *all* you mean, or all you need mean, for set theory to provide a secure foundation for mathematics or, more precisely, if set theory's claim to *be* a ground for mathematics is to be justified.

What you need not mean, in order for set theory to be such a foundation, is that set theory gives us the true nature of numbers, or that set theory is consistent or complete, or that there is a metaphysical justification for the foundational role it is supposed to play. For Maddy, set theory's mathematical role alone is enough.

2. Meaning and Justification

2.1. Introduction

Maddy's Set Theoretic Naturalism, as set out in Maddy [1998], attempts to provide a justification for the fundamental axioms of set theory and hence for the idea that set theory itself is a foundation or a ground for the rest of mathematics. The main point of Maddy's account is that the mathematical benefits of set theory achieve this justification on their own, making it unnecessary to look beyond mathematics itself in order to justify the axioms of set theory or to defend its foundational role.

In opposition to Maddy, one could argue that there *is* a need to look beyond the mathematical benefits of set theory in order to justify its axioms and its proposed role as the foundation for the rest of mathematics. Specifically, the opposing view I offer in the following is that there is a need to justify set theory philosophically or metaphysically justify set theory if it is to count as a foundation for mathematics – or at least there is for the mathematical realist. This chapter, then, primarily explores precisely what this opposing claim amounts to.

First, though, Maddy's claim itself needs examining. Among other things, Maddy [1998] argues that the primary job of set theoretic foundations is "to isolate the mathematically relevant features of a mathematical object and to find a set theoretic surrogate with those features" (p. 26).

This view, as Maddy points out, rejects the notion that there is any non-arbitrary or unique identification of numbers with sets, or for that matter, of integers, rationals, reals, functions, etc. with sets, such that one kind of set should be favoured over another as the true mathematical object or reality that numbers, integers, reals, etc. represent.

So, for instance, favouring von Neumann's identification of 2 with $\{\varnothing,\{\varnothing\}\}$ over Zermelo's identification of 2 with $\{\{\varnothing\}\}$, signifies "nothing deep enough to motivate a metaphysical argument" (p. 24) that von Neumann sets are the real sets or the ultimate reality underpinning mathematics. The same point, of course, applies to favouring Zermelo sets over von Neuman. Both claim to provide *a* set theoretic surrogate for mathematical objects, but neither need nor do claim to do more than that.

Equally, the notion that set theory's job is no more than to provide set theoretic surrogates for mathematical objects, involves the rejection of the idea that such surrogates eliminate or negate the original objects and so reduce the ontology of mathematics from numbers, reals, integers, etc. and sets, to sets alone. From her job description for set theory, Maddy points out, "no such strong ontological conclusion is [or need be] drawn" (p. 26).

With neither the 'unique identification', nor the 'reduced ontology' (both extra-mathematical) justifications claimed for set theory, and with no claim that set theory possesses any metaphysical characteristics the realist might desire or require of it in order to justify its foundational role, Maddy sets about arguing the case on set theory's mathematical merits alone.

These merits include the provision of a united arena of (set theoretic surrogate) mathematical objects in which their relationships and various interactions can be clearly laid out; the development of new set theoretic axioms as a consequence, and the provision of a stopping point for "doomed efforts at proof" (p. 28) until the status of these new axiom candidates is resolved.

In sum, for Maddy, set theoretic foundations:

> Play a strong unifying role: vague structures are made more precise, old theorems are given new proofs and unified with other theorems that previously seemed quite distinct, similar hypotheses are traced at the basis of disparate mathematical fields, existence questions are given explicit meaning, unprovable conjectures can be identified, new hypotheses can settle old open questions, and so on. That set theory plays this role is central to modern mathematics, that it is able to play this role is perhaps the most remarkable outcome of the search for foundations. No metaphysics, ontology, or epistemology is needed to sweeten this pot! (pp. 34-35)

2.2. Meaning and its Relationship to the Mechanical Systems of Mathematics, or Two Mathematical Meanings

There is, in recent literature, some question as to the relationship between what has been dubbed Gödel's philosophical, or extra-mathematical realism and his purely or practical mathematical concerns. Maddy's work addresses this particular issue quite frequently.[1] In order to discuss the view she presents more rigorously, I will for the moment assume that Gödel at least entertained the idea that purely mathematical issues in fact do share an important relationship with philosophical or 'extra-mathematical' issues. In this way, Gödel's thoughts – or, in this case, my own specific re-presentation of Gödel's thoughts – can provide a useful foil to Maddy's.

One of the things that I hope to highlight via this juxtapositioning, is the important effect on this issue and on wider issues of a specific ambiguity surrounding the separation between the formal or mechanical systems of mathematics and their meaning.

'Meaning' can be understood as referring to either to what we *take the facts to be* (the formalism[2]), that is, a certain common or individual understanding of what mathematical formalisms say; or *what the facts are* (independent reality), that is, the way the mathematical world really is, which (clearly enough) includes the realist notion that that reality exists quite apart from any collective or individual understanding of it.[3] The specific ambiguity that the juxtaposition between Maddy's and Gödel's work highlights, occurs when these two possible referents of the term 'meaning' are conflated. The fact that I do not understand such a conflation as an error, suggests that I take 'meaning' to be about more than language. In this way, the meaning of a term or sentence, and whether or not a term or sentence has meaning become ontological, as well as semantic issues.

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[1] Her [1998] and her [1996] both include discussions pertinent to this question.

[2] I use the term 'formalism' in much the same way as Frege uses the term 'sense'. That is, the formalism is the (correct) way of understanding a given mathematical symbol (or series of symbols, theorems, etc.). Importantly, "In grasping a sense [or formalism], one is not certainly assured of meaning anything" Frege [1984, 159]. There is more on this in the following section.

[3] This division is also discussed in my discussion of Folina's work in chapter 4. The two discussions taken together make the point that meaning cannot be thought of as something on the level of formalism, *unless* that level is identified with the level of independence. That is, Gödel's optimistic aligning of what the facts are and what we take them to be is one way of taking the problem of access as solved.

Gödel's writing conflates the two referents, and frequently.[4] I suppose (for the purposes of this chapter) that this conflation is due to Gödel's philosophical stance on the nature of mathematical truth. We could argue, for instance, that for Gödel, mathematical formalisms derive their meaning from the fact that they describe a pre-existing mathematical realm. The derived meaning and the pre-existing realm are at least intertwined; at most, indistinguishable. Indeed without this conflation, mathematics is, for Gödel, rendered meaningless, reduced to a 'mere game'.

In posing the problem of the justification of mathematical certainty in his [1995a], Gödel writes,

> [if] it is acknowledged that the truth of the axioms from which mathematics starts out cannot be justified or recognised in any way ... the drawing of consequences from them has meaning only in a hypothetical sense, whereby this drawing of consequences itself ... is construed as a mere game with symbols according to certain rules, likewise not [supported by] insight (p. 379, square brackets in text).

Further, the way in which Gödel poses the problem of justification above – specifically its ambiguous treatment of mathematical 'meaning' – might be understood as an expression of Gödel's own brand of mathematical realism, in particular of his faith in an independently existing 'mathematical realm'.[5] Another (for the sake of comparison) is his philosophical discussion of the Incompleteness Theorems, particularly in counting his results as evidence against the efficacy and expressive power of the formalisms rather than as evidence against the belief that what they express are the truths of an independently existing reality. But Maddy offers an alternate interpretation of Gödel's realism, and this alternative draws direct attention to the importance of the ambiguity between the two possible referents of the term 'meaning' in mathematics.

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[4] By this I mean that Gödel uses the term 'meaning' in both special and ordinary ways. Compare his special or particular use of 'meaning' in the quotation above (Gödel [1995a, 379]) with the following, more everyday use: " ... let A and B be two meaningful propositions ..." (Gödel [1995c, 196]).

[5] And, in fact, this is what I believe - that Gödel not only conflates the two possible referents of mathematical 'meaning' to this effect, but that this conflation itself is one of the primary forms of expression by which Gödel's brand of mathematical realism can and should be detected and understood.

2.3. Mathematical Meaning and Mathematical Reality

First note that by admitting the above ambiguity exists – i.e. by admitting that there is indeed a difference between what the facts are, and what we take them to be, we are admitting the presence of a three, rather than two-levelled separation, at least when the two referents of the term 'meaning' are *not* conflated. That is, we are acknowledging that there are three separate items in play: our individual understanding of the formalisms, the mathematical formalisms themselves (or, the correct way of understanding the mathematical symbols, sounds, theorems, etc.), and the independent reality that the formalisms (correctly or incorrectly) describe. After all, there could be no possibility of a conflation of the two referents of 'meaning' – in other words, there could be no way of expressing that 'the independent facts just are what we take them to be' – without there first being the possibility of two separate referents (the independent facts, and what we take them to be) and so also of this three-levelled separation. To clarify these three levels further, note that what I mean by 'our understanding of the formalisms' can be aligned with Frege's 'ideas' (Frege [1984]) – something wholly internal and subjective: "the same sense is not always connected, even in the same man, with the same idea. The idea is subjective: one man's idea is not that of another" (p. 160). Similarly, 'the formalisms themselves' can be aligned with Frege's 'sense' – something neither internal nor external, and so, according to Frege, both "actual" and "unactual" (p. 371): "a thought belongs neither to my inner world as an idea, nor yet to the external world, the world of things perceptible by the senses" (p. 369). Finally, my 'independent reality' can be aligned with Frege's 'referent' – which, for Frege, for a name at least, is an object. Frege also has truth-values as referents (the referents of sentences), but given his treatment of these as objects (pp. 163-164) it is, I believe, safe to attribute externality and independence to Frege's 'referent', and I align it with my 'independent reality' on that basis.

Gödel's conflation of the two possible referents of the term 'meaning' can be interpreted *either* as an expression of his philosophical realism *or* as the outcome of his desire to demonstrate the special nature of our *understanding* of mathematical truths – i.e. of what Shapiro (in the introduction to [1997]) calls 'optimism'. Which of these interpretations is favoured will depend on which of the two possible referents of meaning the interpreter takes Gödel as emphasising. Or, put another way, which interpretation is favoured will depend on the extent to which the interpreter agrees with my thesis that meaning is an ontological as well as a semantic issue. That is, it will depend on whether the interpreter sees Gödel's primary interest as located on the third level (of independent reality itself), or on the second (of what we understand of that reality, and so also our understanding itself). It is precisely this choice between interpretations and, more specifically again, the fact that the choice itself often remains unstated that creates the ambiguity presently under discussion.

Maddy [1998], makes a similar point:

> though one strain of Gödel's writings involves a truly philosophical Realism, another actually consists of a series of realistic-sounding methodological principles: allow infinitary methods in metamathematics, don't require existence proofs to provide constructions or definitions, allow impredicative definitions, regard axiomatic set-theories as extendable. All of these principles can be seen as arising out of practice and as guiding successful subsequent practice, in the manner of a true methodological maxim like mechanism (p. 36).

Thus Maddy sets out to explore a reading of Gödel's realism in which,

> the ultimate justification for these maxims is in the actual practice of mathematics rather than in philosophical realism. On this reading, Gödel's realism plays an inspirational role, like Einstein's Verificationism, while his various methodological realisms play the role of true methodological maxims, like Mechanism (p. 36).

Maddy's own interpretation, then, takes Gödel as emphasising an internal mathematical meaning related more to the nature of our understanding than to any metaphysical realm; over the sort of meaning associated with his philosophical realism – i.e. his belief in a pre-existing mathematical realm.

2.4. Maddy's Set Theoretic Naturalism

My own reading of Gödel's realism is in contrast to Maddy's, and is here identified by both Maddy and myself as 'philosophical realism'. A key principle guiding Maddy's Set Theoretic Naturalism is that of the primacy of mathematical practice over philosophy, or 'extra-mathematical' concerns. By contrast, a key principle underlying philosophical realism is that of the primacy of philosophy – in this case, a realist philosophy – over principles derived from mathematical practice, when it comes to the *justification* of mathematics itself. Note that I do not intend here to argue that philosophy is the court in which correct mathematical practice in general should be determined. Nor, though, do I wish to say that philosophy has nothing to say on such matters. Along with Shapiro [1997], I take it that "any principle used in mathematics is taken as correct by default but not incorrigibly. The correctness of the *bulk* of mathematics is [here] a well-entrenched, high-level theoretical principle" (p. 30).

Regarding the problem of justifying mathematics, Maddy's position, in [1996], excludes philosophy from any justification role whatsoever, in favour of mathematics itself:

> what I propose is a naturalistic picture: when pursuing methodological questions, forget the extra-mathematical philosophising and attend to internal mathematical considerations. This is not a philosophical position, on a par with realism and formalism; it is rather a meta-philosophical injunction about the proper relation between philosophy and mathematics, about the proper means by which a methodologist should approach the problem of justifying, organising or extending existing mathematical practice. The central tenet is that extra-mathematical philosophy plays an inspirational, not a justificational role (p. 502).

Indeed, Gödel is readily interpreted along these lines, such that the justification he seeks of the fundamental axioms of mathematics lies in the success of the set of axioms currently accepted. Undoubtedly, Gödel, as a mathematician, would have concerned himself with the sort of evidence for set-theoretic axiom candidates that Maddy is interested in. To illustrate the potency of this sort of evidence, and of the claim that Gödel would himself have appealed to that evidence, I include Maddy's list, given in [1988], of 'extrinsic' evidence (internal to mathematics) here:

1. Confirmation by instances (the implication of known lower-level results, as, for example, *reflection* implies weaker reflection principles known to be provable in ZFC.)

2. Prediction (the implication of previously unknown lower level results, as, for example, the Axiom of Measurable Cardinals implies the determinacy of Borel sets which is later proved from ZFC alone)
3. Providing new proofs of old theorems (as, for example, game-theoretic methods give new proofs of Solovay's older set-theoretic results)
4. Unifying new results with old, so that the old results become special cases of the new (as, for example, the proof of PWO(Π^1_1) becomes a special case of the periodicity theorem)
5. Extending patterns begun in weaker theorems (as, for example, the axiom of Measurable Cardinals allows Sovslins's theorem on the perfect subset property to be extended from Σ^1_1 to Σ^1_2)
6. Providing powerful new ways of solving old problems (as, for example, QPD settles questions left open by Luslin and Souslin)
7. Providing proofs of statements previously conjectured (as, for example, QPD implies there are no definable well-orderings of the reals)
8. Filling a gap in a previously conjectured "false, but natural" proof (as, for example, Det (Δ^1_2 filled the gap in Moschovakis's erroneous "sup" proof of PWO Π^1_3)
9. Explanatory power (as, for example, Silver's account of the indiscernibles in *L* provides an explanation of how and why V$\neq$ L)
10. Intertheoretic connections (as, for example, the connections between determinancy hypotheses and large cardinal assumptions count as evidence for each) (pp. 758-759).

Thus, in [1996], Maddy claims:

> if your interest is in set-theoretic methodology [as Maddy's is], and in particular, if you want to know how those methods can be justified and extended then you should attend to the details of practice. As a corollary, the demise of set-theoretic realism should not inspire a search for new extra-mathematical metaphysics (p. 503).

And this does seem a reasonable claim, given quotations from Gödel like the one she uses as further evidence here: "Such a belief [in the well-determined (mind independent) reality of sets] is by no means chimerical, since it is possible to point out ways in which the decision of a question, which is undecidable from the usual axioms, might nevertheless be obtained" (p. 497).

Further, Maddy's view offers a helpful way of understanding some of Gödel's seemingly paradoxical assertions regarding the status of the platonist position as justification for the axioms of set theory.[6] Not only this, but, by highlighting the ambiguity present in Gödel's writing, set-theoretic naturalism opens the door to responses to Gödel's problem of justification that are less traditionally or philosophically realist and more concerned with the internal workings of mathematics itself, and this is no bad thing.

The counter-point I want to make, though, is that there is (at least) some question, given that it can be argued that Gödel *does* conflate the two referents of the term 'meaning', as to which one of these referents Gödel was most interested in (or which, indeed, any philosophical realist might be). The issue of justification for Gödel can, in accordance with the ambiguity, be read as either (or both of) a concern to show that what we take the axioms to mean is in fact what the formalisms express, or that what the formalisms express are the truths of a mind-independent realm. I believe Gödel saw both of these as much the same question, given that for him, mathematics is true and independent. Which is why, for Gödel, showing the 'certainty of mathematical knowledge' most likely meant showing the *relevance* of the formalisms themselves by establishing the presence of a 'strong enough' link between formalised concepts and what they express. Which in turn explains his great interest in the idea of 'pre-theoretical' concepts, and the possibility of intuitive access to these as a process independent of the formalisms or 'mechanical systems' of mathematics themselves.[7]

[6] For example, "... our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent." Gödel [1995b, 50] versus his approval of Russell's Zoology analogy, namely that "[abstract studies, such as logic and mathematics, are] ... concerned with the real world just as truly as zoology, though with its more abstract and general features" Maddy [1997, 90 quoting Russell and Gödel].
[7] For an example, see Gödel [1995a, 380-383].

And so, in contrast with Maddy, my own interpretation of Gödel's realism entails the claim that any 'demise of set-theoretic realism', in fact *should* inspire a search for a new extra-mathematical metaphysics as an alternative justification or ground for mathematics itself; and that Gödel's work can be read as a search for just such a justification.

2.5. Maddy's Problem of Justification

Recall that Maddy's interpretation of Gödel's justification is along 'set theoretic naturalist' lines. What she hopes thereby to show is that it is somewhat misleading to characterise Gödel's realism *simply* as traditional platonism. She goes so far as to claim that any 'extra-mathematical' realism (the philosophical theorising about the nature of mathematical things) is perhaps better understood as largely irrelevant to Gödel's real concerns. As an alternative perspective on just how his realism *should* be understood, she offers, in [1996], the following naturalistic interpretation of Gödel's views:

> Insofar as extra-mathematical assumptions about the nature of mathematical things are relevant at all, they enter as secondary conclusions: such-and-such is a good mathematical method; therefore, realism is a better philosophical view than, say, constructivism (p. 497).

To defend this perspective, Maddy analyses samples of the reasoning process by which Gödel arrives at various conclusions which seem to her to show that his 'real concerns' are particular issues in the actual practice of mathematics, rather than the defence or even the expression of a philosophical realism. One such example is Gödel's argument that the Continuum Hypothesis is a meaningful question regardless of whether or not it turns out to be independent of ZFC. Maddy takes the following quotation as representative of his reasoning:

> ...the undecidability of Cantor's conjecture from the accepted axioms of set theory...would by no means solve the problem. For if the meanings of the primitive terms of set theory...are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. Hence its undecidability from which the axioms being assumed

today can only mean that these axioms do not contain a complete description of that reality (p. 496, quoting Gödel).

Maddy points out that our initial impression of what is going on here, namely an argument to the effect that there is a real world of sets in which CH (the Continuum Hypothesis) is true or false, therefore it remains a meaningful question, is too hasty. This is because, Maddy says, Gödel's next sentence (previously quoted) reads:

> Such a belief [in the well-determined reality of sets] is by no means chimeral, since it is possible to point out ways in which the decision of a question, which is decidable from the usual axioms, might nevertheless be obtained (p. 496, quoting Gödel).

On the strength of this sentence, Maddy claims that what is going on in fact is that,

> integral mathematical considerations are [being] used to defend the claim that CH is a legitimate question to which an answer might be found, and this conclusion, in turn, is [being] used to support the extra-mathematical belief in the objective reality of sets. In other words, the mathematics is supporting the philosophy, not the other way around (p. 497).

So the problem of justification for Maddy is a strictly mathematical one.

With this interpretation Maddy establishes what she believes are Gödel's real concerns. She then goes on to expand upon the central tenets of set theoretic naturalism and to fashion her own response to the problem of the justification of the fundamental axioms of mathematics.

2.6. The Metaphysical Problem of Justification

My interpretation, on the other hand, includes a principle of priority stating that of first importance to the realist (including Gödel) is the dependence of meaning upon the independent mathematical realm itself. This principle itself depends on the more general claim (contrary to Maddy's) that states that, for the realist, internal mathematical concerns are (at least at times) secondary to philosophical concerns.

The primary defence of my own interpretation is, like Maddy's, to be found in Gödel's work. In his [1995a], Gödel attempts to describe in terms of philosophical concepts the development of the foundational research in mathematics since around the turn of the 20th century, and to fit that description into a general schema of possible world views. The schema itself is developed by dividing possible world views up "according to the degree and manner of their affinity to or, respectively, turning away from metaphysics" (p. 375). But before looking at Gödel's discussion, more needs to be said about the term 'metaphysics'.

2.7. Metaphysics

When it comes to reading Gödel's later work, specifically his philosophy and his interest in Husserl, an interpretation can depend quite a bit on what the reader takes 'metaphysics' to mean: some questions have to be answered – e.g. to what extent is 'metaphysics' able to be aligned with 'philosophy', how or whether philosophy is different from metaphysics, and how or whether both are different from mathematics.

Maddy's Naturalism draws a clear line between mathematics and philosophy and her treatment of the term 'metaphysics' suggests that she takes metaphysics to be part of, if not identical to, philosophy.

I also interpret 'metaphysics' as part of, or the same as 'philosophy', specifically insofar as I agree that each term can be used in place of the other. Essential shared elements of 'metaphysics' and 'philosophy' are presented and enlarged upon below.

The primary difference between Maddy's and my own understanding of the terms involved is in our use of terms like 'mathematical realism' and 'justification'. Where Maddy draws a line between such terms and 'philosophy', I argue that so long as one's mathematics is underpinned by almost any sort of realism, no such line can be drawn.

2.8. The Definitional Framework

Derrida, in his [1962], points out that there are three levels of objectivity involved in any account of abstract or physical reality.

The first is the level of the "word's ideal objectivity". The word "set", for instance, is recognisable in several different languages, but is bound to those languages in which the word itself makes sense. The word's sense belongs on the second level, where 'sense' is the intended content or signification of the word. Sense is available to several languages also; for example, the sense of the word 'set' is available to all of those languages in which 'set' is able to be expressed – i.e. in which set is 'sayable'. The ideality signified at this level is free, Derrida says, "from all factual linguistic subjectivity".

Thirdly (and of first importance on my interpretation of Gödel's realism), there is the level of "absolute ideal objectivity", such as the independent idealities of Number, Set and the "free idealities" of geometry. The ideality in question here is that of 'the object itself'. On this level of objectivity, Derrida says, there is no adherence to any de facto language, only adherence to the possibility of language in general – or, as I will argue, to the possibility of meaning itself (all quotations above, pp. 12-13).

If we locate the recognisable, physical word or mathematical symbol on level one, and the independent mathematical realm (mathematical objects in themselves) on level three, and in between, on level two, we place something Gödel dubs 'the mathematical formalism' (thereby treating the formalism as a sort of meaning bearer); then we have a framework through which 'metaphysics' can be defined.

So, Derrida's three levels of objectivity not only fulfil the purpose for which they were intended (which was primarily to be a framework through which Husserl's work can be interpreted), but they can also be employed as a framework by which Gödel's writing can be interpreted.

The Husserlian equivalent to the Gödelian schema I am proposing has physical words, sounds and signs at level one, the object itself at level three and 'noemata' at level two. At this stage, 'noemata' can be loosely defined following Hintikka [1995], as "the sum total of what is thought or meant of an object in an act" (p. 88).

Added to their original purpose of the general elucidation of Husserl's work, Derrida's three levels have another purpose, when applied to Husserl. Witness Leavy:

> When Derrida elucidated these three [levels], it was specifically in order to show that when Husserl does not distinguish between the object itself and its noemata, this can *only* occur within the third region of ideal objectivity, the absolute free objectivity of language. (Derrida [1962, 14])

The three levels can be applied in exactly the same way to Gödel's work. Recall Gödel's conflation of 'what we take the facts to be' i.e. a common understanding of what mathematical formalisms say (or mean), with 'what the facts are', i.e. the way the independent mathematical realm really is, in and of itself. Derrida's and Leavy's point above applies equally here: this particular conflation can *only* occur within the third region of objectivity – the region of independent mathematical reality or of 'the thing in itself'.

The significance of this application can be seen when we note that it is precisely when the noemata 'reaches its object' (the intention is filled) that the Husserlian conflation occurs. The noemata 'reaches its object' when what is thought or meant of an object, and the object itself, are one and the same. The significance of this can be seen most clearly just by looking at what the absence of this conflation might mean to a realist. Recall, from the beginning of this chapter, Gödel's discussion of the *separation* of the formalism and the objective mathematical realm (Gödel, [1995a])– specifically in his posing of the problem of justification (see quotation, page 57 this text).

The reduction of mathematics to a 'game with symbols' results from mathematical formalisms being separated from their meaning (real as opposed to hypothetical). As such, Gödel's unjustified mathematics looks very much like the flip side of Husserl's 'filled intention'. Meaning which is not hypothetical: real, objective, or justified meaning presumably only occurs when the initial axioms are justifiably taken to be true.

So it seems that the justification or 'recognition' that Gödel seeks here can be interpreted as coinciding with the Husserlian 'filled intention'. The natural fit (one the flip side of the other) between the two scenarios backs this up, as does Gödel's well-documented interest in Husserl.[8] So also do the various attempts to resolve Gödel's problem through realist interpretations of Husserl's work, such as my own. Jakko Hintikka's [1995] is another such interpretation. In that piece, Hintikka puts the question "what is it that fills a noema?" (p. 81) and argues that the relationship between noemata and "the real world" leads, eventually, to a "phenomenological residuum" which is "an interface with reality" (p. 89) and "both a denizen of consciousness and a denizen of the real world" (p. 90). The resolution of the problem of justification I will propose echoes Hintikka's in this regard.

2.9. The Definition

A definition of metaphysics, or of a metaphysical enterprise, can now be attempted. Loosely, a given enterprise is metaphysical if, within that enterprise, a 'meaningful concept' or a 'meaningful formalism' is a concept or formalism conflatable or identifiable with what it is that is meant – i.e. the thing itself, or an independent realm or reality. According to this definition, then, metaphysics is an ontological exersize. The presumption of this definition is just that the metaphysician means to discuss what exists, and if what he discusses does not in fact (independently) exist, his words (and his concepts and formalisms) are meaningless.

In accordance with this definition, the ability of mathematical formalisms to be meaningful at all in fact depends on an initial identification between what we mean and what in fact is, or independently exists.

After, or dependent upon this initial identification, the formalisms can go on to re-present independent mathematical reality itself, and can do so meaningfully, without needing to re-establish the initial conflation; but simply by reference to it. The metaphysical realist thereby envisages an origin – a moment of 'pure presence' wherein the true, or original mathematical formalisms are fashioned.

Kevin Hart, in his [1989], clarifies the realist scenario with his exposition of Derrida's argument, namely: "that a discourse is metaphysical to the extent to which it claims that presence absolutely precedes representation" (p. 11).

[8] For an example, see Dagfinn Follisadal's introductory note to Gödel [1995a, 364-374].

My claim, then, is that Gödel's realism is, among other things, a metaphysical discourse – and thus the problem of justification Gödel poses is, regardless of what else it might be, importantly (and interestingly) *metaphysical*, as opposed, say, to methodological, mathematical or anything else.

Again, Hart clarifies this distinction:

> the distinction between the intelligible and sensible may not always be used to promote metaphysical positions, but it becomes metaphysical when intelligibility is taken to mean 'present to consciousness' ... if one uses 'signified' to mean a concept that is present to consciousness then one is complicit with metaphysics. *A discourse is metaphysical then, if the concept is fashioned as [therefore inextricably linked with] a moment of pure presence [here identified as an encounter with the real, independent mathematical realm] and the sign as representing the concept in its absence ...* within these terms, the sign *fails* to represent the concept purely and simply; it introduces the complicating elements of materiality and difference which serve to delay and defer the expected recovery of the concept. This, accordingly, is the basis of the case [by which metaphysics is recognised as such] for valuing the concept over the sign [or, on my interpretation, for giving priority to the dependence of meaning on the independent realm. That is on giving level three, on which meaning is located (since meaning is inextricable from that realm), first priority] (pp. 11-12, italics mine).

Derrida's deconstruction of the metaphysics of presence, aimed at all forms of metaphysical discourse, mathematical realism included, serves here to highlight and refine my realist interpretation of Gödel's metaphysics. Especially, Derrida offers a clear picture of the notion of the importance and of the nature of the order of dependence within this interpretation of Gödel's realism:

> Stated formally, Derrida's argument would run as follows. No context can circumscribe a sign's meaning; the sign's meaning will alter if repeated in a different context; but the sign is structurally open to repetition: therefore alterity is a structural feature of the sign. *The upshot of this argument is that in being subject to the second mode of representation, the sign must fail to perform its first and primary mode to the extent to which it does not signify the presence purely and simply* (p. 13, italics mine).

As Hart notes:

> but surely such words as 'primary' and 'fail' are beginning to lose their basis as the consequences of Derrida's argument above become evident. We have already admitted that by dint of its structure, the sign can always be repeated outside its original context, and that the sign's intelligible content is, therefore, always open to modification. However, if this possibility is *always* open, it is an essential possibility and therefore part of the sign's structure. We cannot [on Derrida's analysis] then speak of the sign's *failure* to repeat a presence, because the possibility of accidents enables the sign's intelligible content to emerge. *And if we are to talk of primacy [on Derrida's account], we are obliged to ascribe it to this condition of possibility rather than to presence of any kind* (p. 13, italics mine for final sentence only).

We now have a precise summary of the nature of metaphysics, namely that at which Derrida's critique above takes aim – i.e. that which ascribes priority to presence, and has the possibility of meaning dependent upon *this* priority. Now, when 'presence' is read as the 'independently existing, real, mathematical realm', and meaning is inextricably bound to this third level of ideal objectivity, and when 'mathematical formalism' is located on the second level, and physical word/mathematical symbol on the first (least important, or most dependent) level; then we have a summary of the interpretation of Gödel's realism offered here – this then, is a picture of philosophical mathematical realism.

2.10. Gödel's Philosophical Realism

The above explication of metaphysics will inform my reading of Gödel's views in what follows. In particular, this explication of metaphysics goes some way to establishing the validity of an interpretation arguing that Gödel held that philosophical [extra-mathematical] justification ought to be *prior* to extrinsic, mathematical justification of the sort outlined in Maddy's ten-point list, and prior also to internal mathematical justification in general, including practised or well established set-theoretic methodology.

The argument for the priority of philosophy over mathematics or methodology, at least when it comes to the question of justification, leads back naturally to Gödel's schema in [1995a] dividing possible world views up "according to the degree and the manner of their affinity to or, respectively, turning away from metaphysics". Gödel outlines his schema such that "skepticism, materialism and positivism stand on one side, spiritualism, idealism and theology on the other" (both quotations, p. 375).

In the following paragraph, Gödel briefly discusses 'mixed cases', noting that the analysis of 'mixed cases' consists in seeking out their materialistic and spiritualistic elements:

> thus one would, for example, say that a priorism belongs in principle on the right hand and empiricism on the left side...Furthermore one sees also that optimism belongs in principle toward the right and pessimism toward the left. For skepticism is certainly pessimism with regard to knowledge. [Moreover] materialism is inclined to regard the world as an unordered and therefore meaningless heaps of atoms [note that meaning is here , once again, associated, for Gödel, with the existence of an ordered, or structured realm] (p. 385).

Gödel then goes on to note what he sees as a general move from the right toward the left within philosophy since the Renaissance, adding:

> it would truly be a miracle if this (I would like to say rabid) development had not also begun to make itself felt in the conception of mathematics. Actually, mathematics, by its very nature as an a priori science, always has, in and of itself, an inclination toward the right, and for this reason, has long withstood the spirit of the time [Zeitgeist] that has ruled since the renaissance; i.e., the empiricists theory of mathematics, such as the one set forth by Mill, did not find much support. Indeed, mathematics has evolved into ever higher abstractions, away from matter and ever greater clarity in its foundations (e.g. by giving an exact foundation of the infinitesimal calculus [and] the complex numbers) thus, away from skeptisism (p. 385).

Note that Gödel does not regard the 'exact foundation' of the infinitesimal calculus and complex numbers referred to in the bracketed section – nor the general (increasing) clarity in mathematical foundations of which it is an instance – as a *justification* of those foundations.

In fact, Gödel plays down the significance of the 'antinomies of set theory' (contradictions included) on the one hand, and proclaims on the other that such antinomies "have been resolved in a manner that is completely satisfactory and, for everyone who understands the theory, nearly obvious" (p. 377).

It seems that Gödel does not appear to be overly concerned with whatever justifications for its foundations he sees as arising from mathematics itself. Gödel's concern – his posited *problem* of justification is (at least in the context of his schema and his perspective of the 'situation' mathematics is in) *extra-mathematical.*

His arguments are directed *against* the perspective (incorrectly held, he believed, by many of his fellow practising mathematicians) that

> denie[s] that mathematics, as it developed previously, represents a system of truth; [acknowledging, rather] this only for a part of mathematics (larger or smaller, according to [[their]] temperament) and retain[s] the rest in a hypothetical sense – namely [[one]] in which the theory properly asserts only that from certain assumptions (not themselves to be justified), we can justifiably draw certain conclusions (p. 377).

This opposes Maddy's Set Theoretic Naturalism. On Maddy's reading of Gödel, this process of conclusion drawing (a primary part of the 'if-thenism' that characterises mathematical – especially set-theoretical – practice) is precisely where the problem of justification *ought* to be located (and, presumably, solved). To Gödel, though, this sort of internal, mathematical justification is not only inadequate, but irrelevant to the problem of justification with which he is concerned:

> [Such mathematicians described above] thereby flattered themselves that everything essential has really been retained. [Since, after all, what interests the mathematician, in addition to drawing consequences from these assumptions, is what can be carried out] (p. 377, final sentence, in square brackets, Gödel's own).

It is worth continuing with Gödel's own words here:

> In truth, however, mathematics becomes in this way [outlined above] an empirical science. For if I somehow prove from the arbitrarily postulated axioms that every natural number is the sum of four squares, it does not at all follow with certainty that I will never find a counter example to this theorem, for my axioms could after all be inconsistent, and I can at most say that it follows with a certain high probability, because in spite of many deductions no contradiction has so far been discovered (p. 379).

2.11. The Problem of Arbitrariness

Gödel makes an implication here – that the fundamental axioms are rendered less 'arbitrary' when understood from a realist point of view, than when understood from a non-realist point of view.

His idea, I believe, is that the axioms are somehow more grounded, more 'foundational', if they embody facts about an independent reality, than if they are simply 'postulated' by mathematicians.

Indeed, this 'non-arbitrariness' or groundedness is a strength of the realist position – but it is not a guaranteed one. This is because the (realists') axioms are non-arbitrary only if the realists have got it right – that is, if the axioms really do embody the independent mathematical reality with which mathematicians are concerned. There is, of course, always the possibility that they do not.

The potential for such radical error seems built into realist accounts, which makes them vulnerable to a similar charge of arbitrariness, unless the realist can come up with a guiding principle, or set of principles, by which we can guarantee or at least increase our certainty that the axioms do indeed refer to or embody (or both) independent mathematical reality itself.

So the problem of justification, for the realist, now encompasses the problem of arbitrariness (at least insofar as the concept of arbitrariness is linked to that of radical error). For Gödel, the claim that the axioms are 'arbitrarily postulated' is linked with the undesirable result of mathematics consisting of 'certain conclusions ... not themselves to be justified'. That is, for Gödel, the idea of an 'arbitrary' axiom is if not identical to, then closely related to, the idea of an 'unjustified' axiom.

Recall again that Gödel levelled the charge of arbitrariness against accounts offered in 'the (leftward) spirit of the time' [Zeitgeist]. He went on to identify Hilbert's program as an attempt (perhaps the first) at a middle road account:

> and thus came into being that curiously hermaphrodite thing that Hilbert's formalism represents, which sought to do justice both to the spirit of the time and to the [rightward, optimistic] nature of mathematics (p. 379).

Note Gödel's exposition of Hilbert's attempt:

> It consists in the following: on the one hand, in conformity with the ideas prevailing in today's philosophy [the swing to the left identified above], it is acknowledged that the truth of the axioms from which mathematics starts out cannot be justified or recognised in any way, and therefore the drawing of the consequences from them *has meaning only* in a hypothetical sense, whereby this drawing of consequences itself (in order to satisfy even further the spirit of the time) is construed as a mere game with symbols according to certain rules, likewise not [[supported by]] insight.
>
> But, on the other hand, one [Hilbert] clung to the belief, corresponding to the earlier "rightward" philosophy of mathematics and to the mathematicians *instinct* [something different, presumably, from methodological dexterity, or internal, mathematical insight] that a proof for the correctness of such a proposition as the representibility of every number as a sum of four squares *must provide a secure* [extra-mathematical] grounding for that proposition (pp. 379-381).

What is important to note here is that one of the ways Maddy defends her theory is by the claim that it can be seen as responding to what were in fact Gödel's own concerns when he posed the problem of justification in the first place. Whether or not this is the case is not something she believes she needs to establish either way; as she points out, her theory stands separate from this particular defence, since it is nonetheless a response to *a* problem of justification, whether or not it is a response to *Gödel's* problem of justification. What is of interest here is the spirit of the defence itself, and the questions it raises. Can Maddy's or anyone's theory offer a response to Gödel's problem of justification? Exactly what are the issues it would have to address before it could? How does Gödel's problem of justification compare to other renderings of the problem of justification? And how crucial is Gödel's rendering of the problem to other ways in which it can be understood?

In response to such questions, I will argue that the Gödelian notion of justification underpins other renderings, just so long as those renderings can themselves be interpreted as philosophical. In other words, I will be arguing that a response to the problem of justification that does not address these of Gödel's concerns, cannot provide a viable (philosophical) solution.

In a sense, it is indeed Gödel's problem that Maddy seeks to address, just in so far as what she engages in is philosophy. And in that same sense, I will argue (in later chapters) that Maddy's response, along with other's, are viable responses to any problem of justification primarily to the extent that they do in fact address key elements of Gödel's concerns. To elaborate, I believe that a full interpretation of Gödel's problem of justification will establish that the key elements of Gödel's own understanding of the problem of justification he posed are the key elements of what Gödel dubs 'the rightward conception of mathematics', and that the concerns of this 'rightward conception of mathematics' cannot be left unaddressed if a theory is to properly count as a response to any (philosophical) problem of justification at all.

In the following, and in the remaining chapters, I take Maddy's response, try to strengthen it, and then argue that neither the original nor the strengthened versions can address Gödel's concerns, since neither can 'increase our certainty' that mathematics is – or is linked to – a reality wholly other than ourselves.

2.12. Justifying Mathematics – How Far Can we Go?

So, just what would make the axioms that Gödel refers to above less 'arbitrary', or, conversely, more justified? This question is addressed in the next chapter, but at this point I want just to outline a couple of responses that I think are worth considering. The first is that outlined by Maddy in [1988]. Besides her ten (ten!) point list of types of extrinsic evidence for set-theoretic axiom candidates, Maddy offers two further categories: intrinsic evidence and 'rules of thumb'.

Under the former, she places the perception of individual objects and sets: "[the belief], for example, that objects do not disappear when we are not looking at them, and [the belief that] the number of objects in a set does not change when we move the objects around" (p. 759). Thus, this category covers intuitive beliefs and pre-linguistic experience. The sort of evidence provided here is that described by Gödel as the axioms "[forcing] themselves upon us as being true" (p. 764).

Under the latter, Maddy places Comprehension principles – specifically full Comprehension, historically motivated by the appearance of the paradoxes of set theory:

> when uncritical, intuitive work with sets was interrupted by the appearance of the paradoxes, examination of previously unexamined practice revealed that full Comprehension was not in fact used. Rather, sets were thought of as being formed from objects already available. This led to the separation of sets from classes, and eventually to the development of the rule [of thumb] *iterative conception* (p. 759).

Other rules of thumb whose origins are "at least partly intuitive" (p. 759) are:

> Realism, maximise and richness [all of which are] closely tied to iterative conception. Finally, reflection is often claimed to be intuitive, perhaps with grounds in maximise as well. Inexhaustibility is just a special case of reflection, and resemblance is a consequence (p. 751).

Maddy's presentation of (the intuitive) rules of thumb points the way toward a further defence for set theoretic axioms, with which I believe her three categories can be augmented.

This further defence is best presented by Bernard Linsky and Edward N. Zalta [1995]. Linsky and Zalta offer their own comprehension principle for abstract individuals as the key constituent of what they dub 'principled platonism'. Principled platonism is opposed to 'piecemeal platonism'. The latter, Linsky and Zalta argue, describes traditional platonism which "typically assume[s] that their preferred abstract objects are "out there in a sparse way" waiting to be discovered and characterised by theories developed on a piecemeal basis" (pp. 532-533).

The next chapter will focus on the question of whether or not Linsky and Zalta have sufficiently addressed the problems of piecemeal platonism, specifically, whether or not their principled platonism, offered instead, adequately addresses the problem of arbitrariness for the realist.

The general comprehension principles Linsky and Zalta offer as the key constituent of a 'principled platonism' are those that "yield a *plenitude* of abstract objects". Such comprehension principles "assert that there are as many abstract objects of a certain sort as there could possibly be (without logical inconsistency); that is, ... these principles guarantee that the abstract objects in question constitute a plenum" (both quotations, p. 533).

The next chapter explores in depth the addition of such a comprehension principle to Maddy's three categories of evidence. Taken together, the two lines of attack form perhaps the most solid response to Gödel's problem of arbitrariness and so also to his problem of justification. It will turn out, though, that neither addresses the concerns Gödel raised, and that in fact they cannot. A philosophical justification demands more.

3. Platonism and Justification

3.1. Arbitrary vs. Piecemeal

This chapter takes as its starting point the supposition that one of the primary strengths associated with realist/platonist accounts of mathematical knowledge comes about by virtue of what these accounts are not: they are not 'arbitrary'.

Accordingly, the primary aim of this chapter is to understand precisely what this strength is, and to understand how to incorporate this strength into a modern account of mathematical knowledge that seeks to draw (as modern accounts tend to do) from both realist and constructivist strengths.

In order to achieve this, it is useful first to recall and establish some of the grounds upon which the charge of arbitrariness can be levelled against other (typically constructive) accounts, and thereby to negatively infer what precisely the mathematical realist/platonist might see in his own account as protecting it from the same charge.

Obviously, to call this strength 'non-arbitrariness' is not enough. In order for an account to preserve any supposed strength of realist accounts, that strength needs identification early in the piece, especially since once an account tries to preserve supposed strengths associated with either realism or constructivism, the *original* perceived strengths are too often, too easily lost. It seems to me that this occurs particularly where a modern solution is sought for traditional realist/platonist problems, such as the problem of how we come to know anything at all about the realist's independent, abstract realm.

Specifically, I hope to show that Edward N. Zalta and Bernard Linsky's [1995], in an effort to solve this age-old problem, along with other problems traditionally associated with platonism, ultimately sacrifices the particular realist strength under scrutiny here ('non-arbitrariness'), and in doing so loses something crucial, something its authors probably originally sought to retain.

For the purposes of this chapter, and in accordance with the previous chapter, arbitrariness will be defined from Gödel's [1995a]. Recall that in this work, Gödel identified the leftward 'spirit of the times' [Zeitgeist] with which he associates such positions as Hilbert's program, skepticism and empiricism. His interest, in that piece, is with the *true* 'inclination' of mathematics and with its 'nature', with which he associates such positions as a belief in a priori truth, realism and idealism (possibly referring here to social or theological idealism – as in the attainment of ideals – rather than to mathematical idealism, which is leftward). These typify the counter-Zeitgeist, called the 'rightward Zeitgeist'. Rather than try to re-define terms like 'constructivism', 'realism', 'platonism' and so on, I will take on the more general framework suggested by the terms here. Gödel's 'leftward' and 'rightward' spirits, or attitudes, will form (opposite ends of) the background scale into which the various positions discussed can fit, placed further leftward or rightward along the scale according to their various degrees of leftward and rightward 'spirit'.

From among the numerous possible interpretations of this general framework, I take the following to establish the scale against which the accounts I discuss will be set:

An account (of mathematical knowledge) is 'rightward' to the extent that it takes or stipulates the objects of mathematics to be independent from thinking beings, where 'full independence' (extreme right) takes the objects to be something(things) *other than* or different from that which thinking beings can, do or might comprehend.

An account is 'leftward' to the extent that it takes or stipulates the objects of mathematics to be dependent on thinking beings, where 'full dependence' (extreme left) takes these objects to *be,* or to be identical to, that which thinking beings can, do or might comprehend.

Where mathematical objects are taken or stipulated to *be* the objects of our comprehension (in accordance with the leftward attitude described above), one well-known result can be that there is no room left for a specific concept of their 'otherness': that which is in accordance with the rightward attitude described above. Another well-known problem associated with 'leftward' accounts is that of introducing the possibility of error. Where the objects of mathematics simply *are* what we can, do or might comprehend, a story needs to be told about how we can ever be in error – that is, how there can be any sort of difference between what we comprehend and what in fact is.

On the other hand, where these objects are taken or stipulated to be *other* than what we comprehend (a.k.a. 'rightward' accounts), a story needs to be told about how we can ever be correct – that is, how there can ever be any sort of identification between what we comprehend and what in fact is (if it is, indeed, something essentially other than what we comprehend). In this case, the question is: how can the difference (or a stipulated difference) between what we comprehend and what is, ever be overcome, and how can we tell when and if this difference is overcome?[1]

This, in fact, is just another rendering of the realists' problem of justification – the problem of how to show or ensure that mathematics is grounded by independent fact or, put another way, the problem of how to show or ensure that mathematical formalisms represent the (independent) truth. According to Gödel, this is the very 'nature' of mathematics itself - a rightward incline away from the leftward Zeitgeist, or from anything that "[denies] that mathematics...represents a system of truths" (p. 377).

Incorporating this idea with the framework suggested above gives an interpretation of the framework given, wherein it is 'the representation of a system of truths' itself that is the rightward 'nature' and 'inclination' of mathematics (away from which the leftward Zeitgeist inclines). And so, within the specific leftward/rightward interpretation offered here, the belief that in essence mathematics is 'a representation of a system of truths' is likewise to be regarded as essential to a rightward philosophy of mathematics.

To reiterate, an essential strength of realist, platonist, and accounts belonging to the 'rightward Zeitgeist' in general, according to this schema, is their association

[1] Note that at each extreme, there is a certain 'dependence' on thinking beings involved. It is the thinking beings, after all, who 'take' or stipulate to varying extents, the objects of mathematics to be other than or identical to the objects we comprehend. This kind of 'dependence' I take on board as a necessary evil. Where an object is taken to be other than what it is taken to be, there is at least a recognition that it may or may not *be* (able to be somehow identified with) what we take it to be. This is the most independence we can hope for, I fear, aside from remaining quiet on the issue altogether.

with the 'essential nature' of mathematics, which in turn is identified as the representation of a system of truths.

According to Gödel's schema, the rightward Zeitgeist is the context within which this essential nature of mathematics as the representation of a system of truths is to be understood. Gödel introduces the view that the nature of mathematics is *that which is* denied by the leftward Zeitgeist and embraced by the right. I take the same stance. The context I offer is specific, and is not here attributed to Gödel. But, following Gödel, the rightward Zeitgeist as I have interpreted it above is the context within which I mean the idea of 'mathematics as the representation of a system of truths' to be understood. That is, the extent to which an account takes the objects of mathematics to be other than what we comprehend is, according to my interpretation, the same as the extent to which an account takes mathematics itself as 'a representation of a system of truths', understood in this specific context.

On Gödel's schema, the leftward Zeitgeist acknowledges that mathematics represents a system of truths for a part of mathematics only – namely those assumptions which are "not themselves to be justified" (p. 377) – the fundamental axioms. According to Gödel, the focus of the leftward Zeitgeist is on the drawing of conclusions from the fundamental axioms rather than on the fundamental axioms themselves. The fundamental axioms are not, according to some of the programs belonging to the leftward Zeitgeist, (e.g. conventionalism) the sorts of things that need justification. They are not the sort of thing that requires it. Rather, they are more like a framework within which the main focus – the process of conclusion drawing – is understood.

Likewise, according to Gödel, in the context of the leftward Zeitgeist, the idea that mathematics (or *part* of mathematics) is 'a system of truths', demands or requires no justification. By contrast, in the context of the rightward Zeitgeist, the idea that mathematics is 'a system of truths' demands a very specific justification – precisely that which arises within the context or framework provided by the rightward Zeitgeist itself.

So, the meaning of Gödel's phrase, 'a system of truths', alters greatly depending on its context, so much so that the fundamental axioms when seen in the context of the leftward Zeitgeist, are called "arbitrarily postulated", but when seen in the context of the rightward Zeitgeist, these same axioms are called "a priori" (p. 377).

My claim here is that the specific strength of 'non-arbitrariness' associated with realism, is located, not only in the realist notion (inherent to the interpretation given above) that mathematics is ultimately justifiable, but also in its recognition of the problem in the first place: that is, in its recognition that there is a specific *problem* of justification inherent in the 'essential rightward nature' of mathematics itself. Justification is seen within the context of this interpretation as the provision of a ground outside of or other than the system of truths itself. Conversely, the 'system of truths' is rendered arbitrary within leftward conceptions of mathematics – typically in accounts without a *problem* of justification at all. That is, according to this particular aspect of the leftward Zeitgeist in general, no ground *need* be sought outside of the system of truths itself. The rightward Zeitgeist, on the other hand, according to the interpretation given above, is essentially associated with the need for, or at least the acknowledgment of, just this sort of outside ground. This means that the problem that this concept of an outside ground presents – i.e. the problem of seeking the ground itself – is, with all its seeming intractability and all its thorny consequences, nonetheless an identifiable strength of (the rightward Zeitgeist) realism itself (as I have construed it). Hopefully a (rightward) *solution* can be achieved at some stage, but at this stage it is enough to identify that, at the very least, the problem thus formulated is itself an essential part of the rightward position presented here.

Accordingly, for what follows, the trait 'arbitrariness' will be identified as a lack, either by (leftward) 'solution' or stipulation, of the specific rightward 'problem of justification' arising from the rightward Zeitgeist's understanding of mathematics as 'the representation of a system of truths.' 'Arbitrariness' is then, for the purposes of this paper, a rightward realist's 'groundlessness'. It is a trait identifiable - *from* a rightward context – as a (typically leftward) lack of concern for the problem of providing a ground – and this can include leftward solutions which seem to have misunderstood, or devalued the problem as it is in its rightward context. (Recall that Gödel offers the idea that "certain assumptions [need not] themselves [be] justified" (p. 371) as a case in point.)

Specifically, this is the problem of 'how we can be correct' that was mentioned earlier – the problem of how mathematical objects can somehow be identified with what we comprehend if they are taken as or stipulated to be *other* than what we comprehend. An inclusion of this problem *as* a problem can be taken as a fairly reliable indication of a belief in the 'otherness' (as defined earlier) of mathematical objects. Conversely, the lack of a belief in the 'otherness' of mathematical objects corresponds with a lack of the concept of the problem of justification *as* a problem (specifically understood in the same context) for that part of mathematics which 'represents a system of truths'. The contrary rightward trait – 'non-arbitrariness' – will be identified as both an awareness of the problem of justification (so specified) for all mathematics which itself 'represents a system of truths', and consequently the potential for the provision of the specific, rightward justification sought. (In fact I suspect that this particular problem must remain a problem if the 'otherness' of mathematical objects is to be retained in the specific sense given here, but this remains to be seen).

We now have a loose definition of 'arbitrariness' and 'non-arbitrariness'. This is enough with which to begin a comparison between arbitrariness and other perceived traits associated with accounts of mathematical knowledge in general.

Of specific interest here is the distinction between the concept of 'arbitrariness' as outlined above, and that of 'piecemealness' as outlined in Linsky and Zalta's [1995].

The first distinction to note is that, whilst arbitrariness is here associated with traditional constructivist, formalist and leftward accounts in general, Linsky and Zalta associate 'piecemealness' with traditional platonism – historically a rightward account of mathematical knowledge.

When they come to specify the sort of thing falling under this rubric, though, Zalta and Linsky offer a reading of 'traditional platonism' whose location in the general Zeitgeist schema is arguably further leftward than tradition would in fact place it.

3.2. Piecemeal Platonism

Piecemeal platonism, as presented by Linsky and Zalta, consists of three main principles that together constitute a model wherein the notion of an abstract object can be understood via an analogy to the notion of physical objects.

The objectivity and mind-independence of abstract objects is thus understood by analogy with the following three features of physical objects:

1. Physical objects are subject to an appearance/reality distinction. This distinction can be unpacked in two ways. (a) The properties physical objects have cannot be immediately inferred from the way they appear, nor can those properties be known in advance of empirical inquiry. Rather, they have to be discovered, and in the process of discovering we can be surprised by what features we find. The fact that you think of a physical object as having certain features is no guarantee that it does. (b) There is more to a physical object than that presented to us by its appearances, for example, we assume physical objects have "back-sides".
2. Physical objects are sparse. You can assert that they exist only after you discover them. This means they have to be discovered in a piecemeal fashion, and this is sometimes guided by direct observation, sometimes guided by theoretical need.
3. Physical objects are complete. We simply assume that physical objects have all sorts of properties we may not know about (indeed, more properties than we could ever know about), and that they are determinate down to the last physical detail. So, when we have a *bona fide* physical object x, then for every property F, either x has F or x has the negation of F (p. 530).

I believe that 1 to 3 above do form *a* picture of traditional platonism. I also believe, though, that there is another picture available – one which includes an overt expression of the rightward Zeitgeist presented above. This latter picture is an important one since it attempts to make explicit some of the original motives for adopting the sort of position that terms like 'traditional platonism' may have stood for in Gödel's schema, before the current debate and confusion over just what a traditional platonist looks like. So the latter picture will be put forward as a way of presenting a rightward position, as Gödel might have perceived it, and to do justice in a modern account to the possible real strengths associated with such traditional 'rightward' positions generally.

3.3. Piecemeal Platonism Revised

My revised piecemeal platonism, then, consists of the three principles above along with two more:

4. Statements about (real) physical objects are (relevantly) meaningful to the extent that they are about objects whose existence is independent of the statements themselves. For example, '(real) elephants are grey'; and (relevantly) meaningless to the extent that the statements themselves constitute/invent or imagine their own subject matter – for example, 'flying elephants are pink'.
5. The problem of how to justify principle 4 is a real problem. That is, there is or needs to be a justified connection between independent reality and the meaningful statements about that reality.

1 to 5, taken together, provide revised piecemeal platonism's physical model for abstract objects. To understand how 4 and 5 place revised piecemeal platonism further 'rightward' than its unrevised counterpart, recall the interpretation of Gödel's schema above. According to this interpretation, traditional platonism is rightward just in so far as it *essentially* involves the problem (and potential solution of the same) of justification, understood in the context of the rightward Zeitgeist. Indeed, according to my version of a traditional platonist (at least, perhaps, to Gödel) mathematical statements are meaning*less* without such a justification (see quotations on pages 57 and 74 of this text).

On the other hand, *with* such a (rightward) justification, "a proof *must* provide a secure grounding for a proposition and every precisely formulated yes-or-no question in mathematics *must* have a clear answer" (p.379).

In these passages, Gödel appears to be presenting the idea that mathematics as 'the representation of a system of truths' has an opposite counterpart: a (literally) meaningless game with symbols. This idea, coupled with the idea that a rightward philosophy of mathematics essentially involves the independence of mathematical reality – i.e. that mathematical objects are other than what we can, do or might comprehend – introduces an interesting problem.

If we grant both that mathematics has meaning only insofar as it is a representation of a system of truths (as opposed to being a system of truths in and of itself) and is without meaning insofar as it represents nothing outside itself (i.e. – seeks no 'ground' or represents no justifiable, recognisable truth), then where does mathematical meaning reside? Not, it appears, in mathematics itself. This is because the *representation* of a system of truths (mathematics itself) is excluded as a possible (permanent) residence of meaning, since, according to the rightward interpretation, this same representation can be with or without meaning, depending on the provision or lack of a 'ground'.

This means that the fundamental axioms, the process of drawing consequences, and mathematical symbols are all excluded as possible locations for mathematical meaning, since all are surely part of 'mathematics itself'. In fact, all of what we could imagine to be part of 'mathematics itself' ('the formalism'), is excluded as a possible source or residence of mathematical meaning (on this reading of Gödel).

Interpreting Gödel's use of 'mathematical formalism' in this way serves two purposes. It not only coheres with the interpretation I've given of the rightward Zeitgeist as essentially involving the independence of mathematical objects from what we can, do or might comprehend, it also leads naturally to a possible solution to the problem of justification.

It does this via the idea that meaning does not, according to this revised platonism, reside anywhere in mathematics itself, but in mathematical reality, which is independent of anything that we might want to call mathematics itself (or in anything we can, do or might mathematically comprehend).[2] In this way, we have introduced the concept of a fully comprehended, but possibly meaningless formalism, to which we will return in the following sections.

[2] I extend the use of the term 'the formalism' to cover all of mathematics itself on the basis that 'mathematical formalism' can and has been used (most notably by Gödel throughout [1995]) to refer to that which may or may not be mathematically meaningful ('the formalism' is also occasionally used interchangeably with 'the symbols', which are similarly rendered meaningless without a ground).

3.4. Piecemeal versus Arbitrary

For now, I argue that 1 to 3, without 4 and 5, form a platonism that is open to both the 'piecemeal' and 'arbitrary' charges, but that 'revised piecemeal platonism' forms a model for a mathematical realism (my own interpretation of a 'rightward platonism') that is open to only the former, and this turns out – once the latter is answered – not to be such a problem.[3]

To anticipate, I argue that what is missing from the Zalta/Linsky account is a specified dependence of the comprehended object on an entirely 'other' reality. Without this dependence, the difference between a meaningless formalism and a meaningful formalism is at best undefinable, and at worst, absent altogether.

The point is that there should, in a realist account, be an (at least) theoretical difference between an empty formalism (one unattached to, or not dependent in any way on mathematical reality) and its meaningful counterpart. Something should distinguish between the mathematicians having 'got it right', and, say, an entirely fictitious mathematical formalism.

If there is to be any hope of incorporating such an idea into a realist account, though, the relationship between what *we* do, can or might comprehend, and what is, cannot be a strict identity. The two cannot, at least not simply or only, be one and the same thing, even when we are correct. If they are, then there is no way, not even in theory, of distinguishing between meaningful and meaningless mathematics. Or, to put it in terms of the 'other' reality with which realism is concerned, then there is no way of distinguishing between mathematics that is reality-dependent, and mathematics that is radically divorced from reality.

We need an alternative to the simple (or isolated) identification of mathematics itself (literally, mathematics as we know it) – i.e. mathematical practice, or comprehension, or whatever – with mathematical reality. The alternative I suggest here is analogous to Husserl's filled and empty intentions: that is, that what we

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[3] Whether or not this reading constitutes 'traditional platonism' (historically speaking) is beside the point. Instead, I propose that if it does, then traditional platonism has been misunderstood. If it does not, then my reading is offered as a theory that seeks to preserve the original spirit of a platonism regarding mathematical objects.

comprehend is rendered an 'empty (or, using Gödel's term 'meaningless') formalism' *unless* the object 'intended' or comprehended is *present*. In this case, it seems that what holds between the comprehended object and the object itself is not an ordinary, or not only, an identity relation, but neither is it a relationship of representation.

Nonetheless, the closest analytic interpretation of this relation is in fact identity, just so long as we add some important provisos. The first proviso would have to be a specific notion stipulating the comprehended object's dependence (if it is to be anything more than an empty formalism) upon an entirely other reality. That is, when mathematicians are correct, then the comprehended object and the 'other' reality are identical in the sense provided by Husserl – there is not only the "full agreement of what is comprehended with what is present", but also the "'sameness' of the two being a theme of awareness in its own right" (Miller [1982, 36]).

This 'theme of awareness' is possible within the context provided by a 'rightward' traditional platonism, but it is not possible so long as Linsky and Zalta's comprehension principle, designed to supersede the traditional platonism they present, is left unamended.

Zalta and Linsky's [1995] comprehension principle[4] (principle (1) below) states that:

> an abstract object encodes exactly the properties used to specify it ... [and that] no matter what properties one brings to mind to conceive of a thing, there is something that encodes just the properties involved in that conception ... [which means that] there are as many abstract objects as there could possibly be (p. 536).

This is an existence claim "yielding" (p. 533) a plenitude of abstract objects.

Encoding is presented as a "mode of predication" (p. 536) and contrasted with exemplification. So an object can either encode or exemplify (or both) a property

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[4] I use the term 'comprehension' throughout this work, and the way in which I use it should not be confused with Zalta and Linsky's 'comprehension principle'. For me, comprehension, or comprehended objects refers to what we can grasp – literally, what we comprehend (or are capable of comprehending). While Zalta and Linsky's principle does have something to do with the way in which abstract objects are known or comprehended, it is not about comprehension as I use the term. It is about what exists and what does not (although, in the case of abstract objects – its point is that nothing does not exist).

'F'. If an object exemplifies 'F', we can say 'x is F'. With exemplification, we can be right about this, or wrong. That is, either 'x is F' or 'x is not F'. Encoding, on the other hand, transforms every possible property of x into a necessary property of x (see principle (2) below). The idea is that when we read ordinary mathematical statements like '2 is prime' as '2 encodes primeness', we are "provide[d] with a sense in which the ordinary sentence expresses a mathematical truth", and "given this reading, we can explain the *necessity* of ordinary mathematical statements by the fact that the encoding claims that provide the sense in which they are true are necessary" (p. 541). In their explanation of the notion of encoding, Zalta and Linsky highlight (as "most important" (p. 536)) the following three principles:

(1) For every condition on properties, there is an abstract individual that *encodes* exactly the properties satisfying the condition:

$\exists x(A!x \ \& \ \forall F(xF \equiv \varnothing))$, where x is not free in $\varnothing$

(2) If x possibly encodes a property F, it does so necessarily.

$\Diamond xF \rightarrow \Box xF$

(3) If x and y are abstract individuals, then they are identical if and only if they encode the same properties.

$A!x \ \& \ A!y \rightarrow (x = y \equiv \forall F(xF \equiv yF))$ (p. 536).

The problem with encoding is that it is hard to see how we could identify an abstract object that does not, or may not in fact exist. Indeed, according to this account even a round square exists (or "may" exist), as "the abstract object that encodes just being round and being square" (p. 537, footnote 52).

But I argue that it is just this possibility that needs to remain open if the 'sameness or otherness' of a fully comprehended abstract object with the independent object itself is to remain a 'theme of awareness' in its own right. Without this theme of awareness, an account is rendered arbitrary in precisely the way identified earlier – that is, it is cut off from an outside (an other) ground, upon the presence of which every comprehended mathematical object depends if it is to be something more than an empty (meaningless) formalism. A 'formalism' (comprehended object) without the presence of an 'other' reality is arbitrary in just this specific sense: it has no ground independent of that which we do, can or might comprehend.

Traditional platonism as construed by Zalta and Linsky – i.e. 1 to 3, lies further to the left than traditional platonism as construed by myself (and possibly Gödel) – i.e. 1 to 5. This being the case, it is arguable that 'piecemealness' is somewhat less of a problematic trait to the traditional platonist (as I have construed him) than Zalta and Linsky would have it, and it is the specific sort of arbitrariness given above which is, after all, the real problem.

A traditional platonist, as I have construed him, might reason thus: if Zalta and Linsky's traditional platonist is 'piecemeal', then so be it. After all, if it does not incorporate the primary strengths of the traditional platonist account (as I have construed it), then negative traits are to be expected. If it does incorporate these strengths, then perhaps 'piecemealness' is not such a negative trait after all. Especially if, as I argue here, arbitrariness is of greater concern.

I hope to show that Zalta and Linsky's own account, although it may not be 'piecemeal', does nonetheless suffer from the trait identified here as arbitrariness. Further, I hope to show that the inclusion of this trait in their account undermines their attempt to incorporate one of the essential strengths traditionally associated with platonism.

3.5. Arbitrary Platonism

Note that Gödel's conception of arbitrariness is formed via the opposition of internal and external mathematical 'truth'. For instance, Gödel [1995a] writes:

> ...many or most mathematicians den[y] that mathematics, as it had developed previously [in accordance with the original rightward Zeitgeist] represents a system of truths; rather, they acknowledged this for only a part of mathematics (larger or smaller according to their temperament) and retained the rest at best in a hypothetical sense – namely one in which the theory properly asserts only that from certain assumptions (not themselves to be justified), we can justifiably draw certain conclusions. They thereby flatt[er] themselves that everything essential ha[s] really been retained. (Since, after all,

> what interests the mathematician, in addition to drawing these consequences from these assumptions, is what can be carried out) (p.377).

Gödel acknowledges here that the interest of most mathematicians is 'internal' – i.e. in the mathematical system itself, rather than in a system of truths which mathematics represents. This particular interest, though, is primarily practical, and accordingly its philosophical relevance is primarily to the mathematical system itself, rather than to anything outside it.

Gödel's idea here, I believe, is that the extent to which a mathematical system is arbitrary is closely related to the extent to which it refers only to itself. In accordance with this idea, the extent to which a system is relevant will be defined here as the extent to which it refers to something outside or strictly other than itself. The same idea is also applied here to accounts or philosophies of mathematics: the extent to which a given account is arbitrary is the extent to which it privileges what we comprehend over what is other than that which our comprehension grasps. The extent to which it is relevant is the extent to which it privileges what is external to our comprehension – the thing in itself, rather than the thing as we understand it.

Since, on Linsky and Zalta's account, one cannot specify a group of properties without thereby identifying an abstract object, the abstract object thereby identified cannot be mis-identified provided its properties are specified. That is, on Linsky and Zalta's account, specification 'gives' encoding, and encoding 'gives' what is comprehended or known, which is in itself the abstract object. The comprehended object simply is the real (abstract) object itself.

Zalta and Linsky do attempt to provide some room for error – specifically for the notion of error relative to a theory. Their case is that:

> [t]he mathematical objects of a theory encode the properties that genuinely follow from that theory. [And so] It is possible to make a mistake about the properties that a mathematical object encodes by making a mistake about what properties follow from the theory (p. 544).

Their conclusion follows:

[s]o we allow for error – a mistake about the objects of a theory is *not* a successful discovery of a truth about some different objects. Similarly, we allow for ignorance – mathematicians can form new judgments of the form 'In *T*, x is *F* without thereby thinking of objects of a different theory" (p. 544).

But the comprehension principle Zalta and Linsky offer and this conclusion are incompatible. It seems to me that they cannot be reconciled. The comprehension principle grants that for any group of properties there is an abstract object that encodes them. Restricting this plenum by adding that the group of properties must follow from a (presumably respectable mathematical) theory either changes the comprehension principle itself, or directly contradicts it.

It remains that, according to Linsky and Zalta's [1995] account, nothing is external to what we comprehend. Or, put another way, there is nothing beside or beyond what is comprehended. In this case, what we comprehend depends on nothing other than what we comprehend.

Zalta and Linsky's account does have practical relevance. In other words, it is relevant internally. But any account will automatically to some extent, perhaps even primarily, be relevant internally, given that (even where it is a strong realist account, attributing complete independence to the abstract realm with which it deals) it is still *our* account of *our* relationship with an independent reality. In this sense, even if an account has some sort of external relevance, such relevance is part of the account itself and so to that extent is internal also. My argument runs as follows: the best that any account can do, in terms of incorporating external relevance, is to *indicate,* or to leave room for, what is outside itself. And perhaps the only way of doing this is to include an incompleteness clause of some sort – stating that the account itself is not the whole story, or, going one better, is not the primary story.

In other words, an account can acknowledge its debts, which in this case can be done by indicating (as opposed to representing, or including) something beside itself upon which the account, and its degree of success in providing a correct description of the relationship between thinking beings and mathematical reality, depends.

Zalta and Linsky's conception of the existence of abstract objects depends only on itself rather than on something altogether other than what is given by that conception. And there is no room, in Zalta and Linsky's account, to indicate away from the account itself – to anything externally relevant to what is comprehended, since what is comprehended, on their account, is the whole story.

It is at precisely this point that the charge of arbitrariness can be levelled against their principled platonism.

Widening the scope of Gödel's accusation against the formalists whose fundamental assumptions or axioms are 'not themselves to be justified', I would include as another of its targets the assumption inherent in Zalta and Linsky's account that there is an abstract object for every group of properties, and *that* object (the comprehended object – complete or incomplete, with respect to exemplification and encoding) is identical to the abstract object in and of itself. As mentioned earlier, this means that the two objects, according to Zalta and Linsky's account, are in fact not two at all. There is only one object in play and that is the comprehended object; and this object is an object whose relevance to a reality external to any system or philosophy is 'not itself to be justified', and whose existence is grounded on principles internal to the account itself.

This assumption may perhaps be able to be read such that it does leave room for abstract objects existing independent of our comprehension itself – leaving room for a problem of justification whereby the ground that is sought is extra-mathematical and so is external to the formalism (thereby answering Gödel's original objection). But it does not leave room for abstract objects existing independently of thinking beings entirely. Nor does it leave room for the dependence of thinking beings, and the possibility of constructing any sort of account at all on an independent reality whose nature cannot be wholly captured or circumscribed by that account. That is, no room is left for a problem of justification whereby the ground that is sought is different from anything to do with us whatsoever: one in which the ground sought is strictly external – outside our comprehension.

A quick note: the foregoing is not meant to be an argument for a sort of representational realism. I believe a positive account can leave room for the literal identification (though not only identification) of what we comprehend with what independently is. An account can be given wherein the two are equivalent or equated when we are 'correct', but nonetheless remain separate – i.e. they remain *two*. (This matter is taken up in the final few chapters of this work). This sort of account, I believe, would preserve the realist intuition that mathematical reality and the nature of mathematical reality, are entirely independent of what we comprehend, even when what we comprehend *is* mathematical reality. To paraphrase Kevin Hart [1989] 'the two are the same, yet different'.

3.6. Principled Platonism

My argument here will be that there are two different kinds of principled platonism: one which seeks to overcome the problem of 'piecemealness' (of which Linsky and Zalta's is one), the other seeking to overcome the problem of arbitrariness (of which mine is one). The two aims are not necessarily incompatible, but nor are they identical. Addressing the one problem does not guarantee that the other is solved.

The first kind of principled platonism (from Linsky and Zalta's [1995]) has the assertion of the existence of a plenum of abstract objects in accordance with the comprehension principle "(1) For every condition on properties, there is an abstract individual that encodes exactly the properties satisfying the condition" (p. 537).

The second principled platonism has the assertion of the existence of abstract objects in accordance with themselves only, and this existence is something strictly outside of the existence asserted by any existence claim, or by any knowable or humanly accessible existence whatsoever.

The first entails that successful reference to abstract objects is determinate (i.e. what we refer to just is what exists[5]), the second has it that whatever it is we

[5] Specifically: "reference to abstract objects is ultimately based on descriptions alone ... [the comprehension principle and encoding] introduces abstract objects that may be incomplete with respect to the properties they encode, and this, together with the identity principle, ensures that incomplete descriptions will successfully refer" (Linsky and Zalta [1995, 546]).

ourselves access, it is not simply or not only the objects in themselves. The objects in themselves are something entirely other.

In fact, according to the second principled platonism, the formulation of the (or any kind of) comprehension principle itself depends on the independent reality of the abstract objects. That is, the fact that the abstract objects, in reality, are not what we comprehend *is* the fact that enables the formulation of any kind of access, or existence claims.

This does not mean that what is accessed by our comprehension cannot be identical with the objects (or reality) in themselves. Rather, it means that whatever identity we discover between the two must nonetheless also include or account for the *difference* between the two.

Note that these two principled platonisms both have the existence of abstract objects in some sense tied to our comprehension of them. The first sense (Linsky and Zalta's) is that what indeed whatsoever we comprehend is identical with what exists (abstractly). The second sense (mine) is that what we comprehend is strictly other than what exists (abstractly), although the two can be equated in a positive account, perhaps in accordance with a dependence principle which might read: the two can only be equated in accordance with a relation of dependence, such that whatever is comprehended depends on whatever exists independently.

In other words, I want to disagree with the claim that "an abstract object encodes exactly the properties used to specify it" (p. 537), which is a way of guaranteeing the existence of an abstract object with those specified properties. On the other hand, perhaps something like Linsky and Zalta's principle that "no matter what properties one brings to mind to conceive of a thing, there is something that encodes just the properties involved in that conception" (p. 536) can be upheld, provided the copula is rendered ambiguous (something like – the thing encoding the properties both is and is not what one brings to mind to conceive of that thing. But, again, there will be more on this later). I hope to argue my case by appealing to the notion of relevance. I argue that this notion and the idea of an independent 'other', are two important reasons why a Gödelian realism can accommodate the idea that one can specify a group of properties and still mis-identify an abstract object.

Put another way, specification does not 'give' encoding (it could, of course, if the concept of encoding was pulled back from a claim about the object itself, to a claim about what we grasp of that object. But, on Zalta and Linsky's account, what an object encodes is necessarily predicated of that object itself). Once again, this boils down to the importance of the order of dependence. Whereas Linsky and Zalta argue that:

> the comprehension principle [which I ultimately uphold – at least in a certain sense] and the logic in which it is framed are required for the proper analysis of natural language in general and mathematical language in particular" (p. 535).

I argue that the comprehension principle is required for the possibility of natural language itself, and that in particular the mathematical realm is required for the possibility of forming the comprehension principle itself.

3.7. Specification vs. Encoding

Recall that Linsky and Zalta begin their account of mathematical knowledge with the introduction of the primitive notion of encoding and that this contrasts with the "traditional exemplification mode of predication" (p. 536).

Just as there are (at least) two ways to understand 'traditional platonism', there are two ways of understanding the exemplification mode of predication.

The first is with an emphasis on perception. That is, exemplification may be understood as the logical distillation of the process of human cognition and perception, of the process by which we understand the world. Predicating a property of an object is indicative of the way we see that object, as well as of our own make-up. For example, from the perspective that, say, Poincaré's conventions (for an account of these, see Poincaré [1913a]) alone adequately describe the nature of thinking beings, the exemplification mode of predication could be understood as one logical distillation of the nature of thinking beings. From the perspective that Wittgenstein's rules alone adequately describes the nature of the language of thinking beings, the exemplification mode of predication could be understood as one logical distillation of the nature of the language of thinking beings.

That is, where the emphasis is placed upon perception in a given general background philosophical theory, there the exemplification mode of predication can (and probably will) be 'read' with the same emphasis. Forgetting for a moment the background philosophical theories likely to motivate different readings, and focusing on the particular understanding of exemplification thus motivated, notice that an understanding of exemplification can emphasize a thinking being's action upon the world, or the world's action upon thinking beings. That is, 'x exemplifies F (Fx)' can be read as 'there is a property F which x, in fact, has', or 'there is a property F with which one of the properties x in fact has, can be equated'.

The first reading has the existence of an x with the property F simply implied. The second has the property F's existence implied, but the existence of an x with that property as a further, separate problem. This is one way of formalising the position outlined earlier. That is, the notion that mathematical reality exists entirely separate from whatsoever we comprehend is specifically expressed in the second formulation of exemplification above.

The idea that the properties x has or does not have is an existence problem not immediately solved by the implied existence of an x exemplifying F, leads to the idea that encoding can be similarly understood. That is, it can be argued that exemplification itself entails a burden to show that the properties x has, independent of what we comprehend, are just those that we do comprehend, and so predicate of x. In much the same way, encoding can be understood as giving abstract objects themselves just in so far as the identification of the 'given' abstract object and the independent abstract object remains a problem. For instance, the encoded object is 'the same' as the object itself only if the object itself is in fact present. Where the presence or absence of the object itself *as something other than the object encoded or specified* is in this way a theme of awareness in itself, *there* is the possibility of a non-arbitrary account of mathematical knowledge.

3.8. Some Thoughts on the Equivalence Relationship

Formalising the concept 'other than, yet identifiable with' within a philosophy of mathematics is difficult. Nonetheless, I don't believe it is impossible. Below is just a sketch of some of what my own attempt will entail.

Taking Husserl's lead, I begin with the concepts of filled and empty intentions. One way to understand these concepts is via a formal equivalence relationship like that offered by Shapiro [1997] with his "sub-language procedure" (pp. 120-126):

> Imagine a mathematician who decides to speak an impoverished language that cannot distinguish two integers if their difference is divisible by 7 (or, equivalently, if the numbers produce the same remainder when divided by 7). On her behalf, we make the indicated identifications: 2 is identified with 9, 16, -5, and so forth, whereas 3 is identified with 10. We interpret our mathematician as saying that 5+4=2. Of course, 5+4=9 as well, because in her system, 2 and 9 are indiscernible and thus identical (pp. 121-122).

As Shapiro goes on to note, this sort of equivalence relationship can be understood with Putnam's "conceptual relativity" in mind. It is worth quoting a large portion of this section:

> [this sort of equivalence relationship can be understood such that] the device of reinterpretation ... recognises that one person's existence claim might be another person's something else ... that in going from framework to framework, we are reinterpreting the *logical* terminology, the identity sign and the existence quantifier in particular ... the notions of object and existence are not treated as sacrosanct, as having just one possible use ... the existence quantifier can be used in different ways – ways consonant with the rules of formal logic ... the point is that when we interpret one discourse in another framework, the 'translation' of the logical terminology is not homophonic. *One person's identity is another's equivalence. The equivalence relationship has the logic of identity in the sub-language [only] because the equivalence is a congruence there* (p. 128, italics mine).

For Putnam, this means that singling out one particular use of the existence quantifier "as the only metaphysically serious one" (p. 128), is a mistake. Whereas I take the same thing to mean that an account of mathematical knowledge can retain the concept of an original ontology and recognise this identity, (rendered here as something like a 'layered', or aware identity) as the relationship between its objects and the objects comprehended. And I argue that such an account is to be preferred over the same thing without this 'theme of awareness'. Without this theme of awareness, Shapiro's mathematician, for example, 'knows' that 2 is 9, without any awareness that there is (or even might be) more to the story. The person able to make the identifications in the first place, the one in possession of the (knowledge of) the full background ontology, is in a preferable position.

The possibility or impossibility of attaining this preferable position is, of course, another important issue. In the case of mathematics as we know it, perhaps the best we can do is remain aware that the object within the scope of our comprehension is not the end of the story, even when our identification of any given mathematical object is in fact correct in the full realist sense outlined above – that is, even when the comprehended object is in fact equated with an external reality.

Against Putnam, Linsky and Zalta, I propose that where the presence or absence of (an) original ontology is not a theme of awareness, the comprehended mathematical object is at worst an empty formalism. At best, an account of mathematical knowledge minus this theme of awareness is vulnerable to the charge that it is arbitrary. Certainly, the term 'arbitrary' is here given a very specific meaning. But I don't think it is such a stretch to take this specific sense as consonant with Gödel's complaint against accounts without such an awareness. Recall Gödel's [1995a]:

> [against the result that] mathematicians [are denying that] mathematics represents a system of truths [aside from] only a part of mathematics (larger or smaller according to temperament) and retain[ing] the rest at best in a hypothetical sense – namely one in which the theory properly asserts only that from certain assumptions (not themselves to be justified), we can justifiably draw certain conclusions. [It is this that renders the axioms] arbitrarily postulated. ... If one wishes to justify [mathematical assertions] with any certainty, a certain part of mathematics must be acknowledged as true *in the sense of the old rightward philosophy* (p. 377, italics mine).

The inclusion of an awareness of the need (or even of the possible need) for at least a certain part of mathematics to be 'justified' in a specific way, is what, according to Gödel, saves an account from being rendered 'arbitrary'. I argue that the inclusion of a specific awareness at least begins to save an account in the way Gödel probably had in mind when he wrote the above, namely that the inclusion of the awareness that the objects within the scope of our comprehension are not themselves the end of the story.

4. On the Attainability of Objectivity

4.1. Introduction

This chapter discusses Janet Folina's Poincaréan philosophy of mathematics. Although this is the last of the programs other than my own touched on in this work, the set of programs addressed is not in any way supposed to round out an exhaustive list of programs relevant to my account. There are a number of clearly relevant yet overlooked programs (mentioned in the introduction). Nonetheless, since each of the programs discussed thus far has been chosen in order to highlight various subtle manifestations, appropriations and interpretations of the realist desiderata under scrutiny, Folina's account suggests itself as an appropriate final foil. This is because her account – or, more correctly, the basis or framework upon which her account is built - is (transparently) representative of a widely employed general approach to the host of realist and anti-realist desiderata. And this general approach is, I believe, itself a specific interpretation of these desiderata, and as such deserves further analysis.

In her [1993-4] paper, Folina offers a solution to a problem widely understood to be, if not the greatest problem realism (and, on its flip side, constructivism) faces, then one of the big ones. The problem itself is engendered by the opposition between a widely accepted, yet very specific conception of the 'greatest strength' and 'greatest weakness' of realism, and of the 'greatest strength' and 'greatest weakness' of constructivism.

The solution she proposes to the problem thereby engendered is, I believe, one of the best available – but in what follows, I argue that no solution to this particular problem can gain an advance over the traditional realist response upon which such solutions are designed to improve. The traditional response can be summed up thus: 'we don't know'. This chapter shows that modern attempts to improve on this must ultimately fail.

4.2. Folina's Poincaréan account

Folina's account addresses what she sees as a "fundamental dilemma" (p. 202), for philosophers of mathematics hoping to choose between realism and anti-realism.

Of course, her particular perception of this dilemma informs her perceptions of just what might constitute its solution. As such, attention will be given, in what follows, not only to the 'fundamental dilemma' itself, but to Folina's particular setting up of that dilemma.

Folina begins by opposing platonism: "a theory about the literal nature of mathematical statements and the reality of mathematical objects" (p. 202), to that group of constructive theories, "according to which mathematical objects are constructions in the minds of mathematicians" (p. 202). She then argues that platonism's main strength is in its account of mathematical truth – namely a correspondence theory enabling

> a picture of mathematical truth as completely objective ... because the mathematical facts (which true mathematical propositions reflect) are (mind-) independent; and the facts are independent because mathematical objects (which constitute the facts) are also (mind-) independent (p. 203).

For Folina, the most crucial aspect of this picture of mathematical truth is the "difference, or gap" (p. 203) between the independent facts of mathematics and our understanding of those facts. It is this gap, according to Folina, that enables platonists to claim that theirs is an account of *objective* mathematical truth and knowledge. Folina also identifies a set of primary problems for the platonist. These are the same, primarily epistemological concerns identified earlier (chapter 1).

In turn, the primary strengths of constructivist accounts, according to Folina, is their "appealing epistemology" (p. 202) and relative lack of metaphysical mystery. Constructivist accounts typically claim to incorporate a tenable explanation for how we know mathematics. One of the most common forms that this explanation takes is: "we know mathematics just by knowing how to construct mathematical objects and proofs" (p. 203). The idea that we construct or create mathematical objects, though, also gives rise to constructivism's main problem; i.e. the apparent lack of a gap between the facts of mathematics and our understanding of those facts.

Folina summarises the situation as follows:

> In general, lack of an epistemological gap is both what is attractive and what is problematic about constructivism; and the existence of an epistemological gap is both what is attractive and what is problematic about platonism (p. 204).

Folina's own account, which she hopes will be a middle road between realist and anti-realist philosophies of mathematics, is primarily informed by this initial conception of the nature of the divide she hopes her account will bridge between these opposing views.

It is her initial conception of the problems and strengths of the opposing views that will come under scrutiny here. This is not only because it forms the basis upon which the rest of her account is built, but also because any outline of the divide between realism and anti-realism is open to be misconstrued and potential misunderstandings, especially given that the titles involved – realism, platonism, constructivism, anti-realism, idealism, etc. – are now, in effect, up for grabs thanks to years of overuse.

Any judgment concerning whether or not an account has managed to run a successful middle road between realist and anti-realist philosophies of mathematics will depend on what is first specifically identified as realism and anti-realism and, specifically, as their associated strengths and weaknesses.

One of the things I hope to show in this chapter is that some of the disagreements over the definition of terms like 'realism', 'anti-realism', etc. are themselves based, at least in part, on still more fundamental confusions whose reach extends across philosophical discourse in general. In particular, at the fundamental level the philosophy of mathematics utilises a host of different terms to discuss its subject – nearly all of which can be (and have been) confused with one another. Yet what these different terms are taken to stand for determines, by and large, our working definitions of realism, anti-realism, etc. For example, just what we understand by 'symbol', 'meaning', 'formal', 'intuitive', 'the sign', 'the signified', 'representation' and 'presentation'; will directly inform just what we understand by 'realism' and 'anti-realism'. And yet the precise definition of such terms is, by and large, left unaddressed within the philosophy of mathematics.

Putting it another way, note that in almost any articulation of a mathematical realism, mathematical anti-realism, etc. terms such as 'formalism', 'meaning', 'sense' and 'real world' are used. And the way in which they are used cannot help but determine our perception or interpretation of the account in which they are used. In what follows, I will present a model of interpretation for some such terms, and oppose it to the model Folina offers. I hope thereby to show how the initial interpretation of such fundamental terms is crucial to the establishment of any account claiming to run a middle road between realism and anti-realism.

In giving two models of interpretation, I hope also to show that my own model of interpretation more thoroughly and accurately captures the strengths of realism (outlined in the introduction) than Folina's model. Of course, whether or not the reader agrees that it does will depend on their own interpretation of the strengths and weaknesses of realism and anti-realism (and so also of their main problems). My model is offered as just one possibility, and should be understood in that spirit. In accordance with the method established in previous chapters, the model itself will come about via one possible reading of Gödel's philosophical remarks.[1]

4.3. Reading Gödel

Recall that when discussing the problem of giving a justification for the axioms and rules of inference of mathematics, Gödel [1995b] says:

> And as to this question it must be said that the situation is extremely unsatisfactory. Our formalism works perfectly well and is perfectly unobjectionable as long as we consider it as a mere game with symbols, but as soon as we come to attach a meaning to our symbols, serious difficulties arise (p. 49).

The particular reading of Gödel's philosophical remarks from which my model of interpretation comes about, takes it that for Gödel the real or objective meaning of the fundamental axioms of mathematics is something that existed prior to their conception or construction. Gödel's problem of justification then, on this reading, was a 'pre-axiomatic' search for that which justifies the link between the formalised

[1] The aspects focused on here will be different to the aspects focused on in earlier chapters insofar as they address different opposing accounts.

concepts (the axioms themselves) and what we take them to mean, or what we understand them to mean. Ideally this search would find that what we understand the axioms, or indeed any part of mathematics, to mean (and what we understand mathematics to mean is what I call 'the formalism', and Frege calls the 'sense') is inextricably linked to their 'real' or objective meaning. That is, the meaning *is* the independent object, rather than something to which the independent object needs to be linked. So, the link required is not that between the axioms and the objects that exist independently of the axioms. Rather, it is that between what we think the axioms mean, and their objective meaning. Again recall my previously stated thesis that meaning is both an ontological and semantic notion for the realist presented here.

I believe that Gödel would not have foreseen any serious difficulty with the process of 'attaching meaning' to our symbols if he did not take the notion of a pre-existing objective meaning seriously. Yet, he did foresee such difficulties, and he pinpointed their origins. He effectively named the realist notions of non-constructive existence and impredicative definition as two such origins, noting that the axioms allow us to talk about mathematical objects whose existence or non-existence is impossible to determine and that a property may be defined as follows: "an integer x shall possess the property p if for all properties (including p itself) some statement about x is true" (p. 50).

Both of these 'serious difficulties' arise as a result of the presupposition of an objective realm "existing somehow independently of our knowledge and our definitions." Indeed, for Gödel, "our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of platonism" (both quotations p. 50). Recall, also, that Gödel [1995a] believed

> [if] the truth of the axioms from which mathematics starts out cannot be justified or recognised in any way ... the drawing of consequences from them has meaning only in a hypothetical sense (p. 379).

This 'recognition' of the truth of at least some axioms is in accordance with Gödel's realist 'spirit'. This spirit incorporates both a faith in mathematics being true, and in the realist idea that being true essentially involves our understanding of mathematics accessing a mind-independent realm. That is, Gödel's realist 'spirit' has it that at least for some section of mathematics, what we take mathematics to mean is in fact what it objectively (i.e. in reality) means.[2]

These brief considerations alone suggest that the possibility of error in (this interpretation of) Gödel's realism has more to do with the 'link' between the realm *wherein* meaning resides, and the axioms as formalised concepts, than with an affirmation of Folina's platonist principle that what we take the facts to be is strictly different to what the facts objectively are. Indeed, affirmation of this latter principle can be seen as contrary to Gödel's realist 'spirit' of mathematics.

To see this we need to ask what Gödel might have meant when he referred to the problem of "justifying the truth of the axioms from which mathematics starts out" (p. 379). For Gödel, it was true that mathematics represents a system of truths and that 'the truth of this system of truths' can in fact be justified. In accordance with this idea, I interpret the problem of justification that Gödel puts forward thus: that for Gödel the fundamental axioms *are* in fact justified and the specific 'problem' is just how to demonstrate that fact. That is, he does not ask the question 'is mathematics justified?', but presumes that it is, then seeks to justify this presumption.

I believe that it is in accordance with this strategy that much of Gödel's work focuses on the process by which we can be said to *recognise* the truth of the axioms, as opposed to a process by which we could *establish* the truth of the axioms. (Maddy [1998, 89-92] presents a good case for this claim).

I take it that it was his search for such a process that led Gödel to conclude that there is no real possibility of a meaningful mathematical statement or formalism whose meaning discords with the way the mathematical realm objectively is. That is, for Gödel, a formalism that is not justified is in fact meaning*less* (recall Gödel's

[2] The denial that mathematics represents a system of truths, runs contrary to the realist spirit. Recall: "[anti-realist or sceptical] mathematicians denied that mathematics, as it had developed previously, represents a system of truths; rather, they acknowledged this only for a part of mathematics ... and retained the rest at best in a hypothetical sense – namely [one] in which the theory properly asserts only that from certain assumptions (not themselves to be justified), we can justifiably draw certain conclusions. They thereby flattered themselves that everything essential had really been retained ... in truth, however, mathematics becomes in this way an empirical science" Gödel [1995a, 377].

quotation on page 74 of this text). And so it seems appropriate to suppose that for Gödel, where there is no link between our understanding of a given formalism and the independent mathematical realm, there is no real meaning (only, at best, hypothetical meaning). On this interpretation then, meaning is closely associated with, if not the same as, objective truth in the independent mathematical realm.

Compare this interpretation with Folina's. Folina has it that the possibility of error depends, for the platonist, on 'what we take the facts to be', a.k.a. 'the meaning of mathematical formalisms', being essentially different from 'what the facts are', a.k.a. 'what is true in the independent realm'. Her interpretation, in other words, strictly separates semantic truth (meaning) from objective truth (i.e., in Fregean terms, Frege strictly separates sense from reference). This sort of interpretation in general, places meaning firmly in humanity's court – at least insofar as 'what we (humanity) take the facts to be' is equated with the meaning of those facts. That is, it seems that for Folina the meaning we attach to mathematical statements and mathematical formalisms can be incorrect just because we can create a meaningful, yet factually incorrect, mathematical statement or formalism.

My Gödel-based interpretation, on the other hand, has it that this latter scenario is impossible. That is, I interpret Gödel as holding that if an independent objective realm is not accessed, then mathematics is not incorrect yet meaningful. Rather it is, in a literal sense, meaning*less*. This is because, for Gödel, the possibility of error depends (as, indeed, it does for Folina) on the possibility of mathematical statements being incorrect. But for Gödel, where a mathematical formalism is incorrect, it is also *really* (or in actual fact) meaningless. That is, my interpretation of Gödel deliberately and significantly places *more* emphasis on the independent mathematical realm itself than on what we take this realm to be. Indeed, under my interpretation the latter literally depends on the former (see chapter seven).

Recall that for Folina [1993-4], mathematical realism enables a picture of mathematical truth as completely objective, just because true mathematical propositions reflect mind-independent mathematical facts, and the facts are independent just because mathematical objects (which constitute the facts) are also mind-independent:

> Since the mathematical objects exist independently of the knowers of mathematics, there is a clear difference between the facts and what we take them to be. Arguably, some such gap is essential for a domain of objective knowledge (p. 203).

To recap: for Folina, the difference between *being* right and merely *seeming* right depends, for the realist, upon the correspondence or discordance respectively of the meaning we give a mathematical statement or formalism, and the way the mathematical realm really is. For Gödel, on the other hand, *we* cannot give a mathematical statement (real) meaning; only independent reality and the nature of the mathematical realm can do that. Accordingly, we can *understand* a mathematical formalism incorrectly, but this incorrect understanding would constitute a failure to recognise the (true, i.e. objective) *meaning* of that formalism (along, of course, with any independent fact with which it is associated).

4.4. Possible Confusions and Conflations

Note that the model of interpretation presented here (Folina's) – wherein 'meaning' is strictly separate from our understanding (or from 'what we take the facts to be') – might easily be conflated with a (perhaps less contentious) model wherein a formal mathematical expression is separate from its relevant pre-theoretical notion. This division has meaning disassociated from formalism, and associated with pre-theory or with the informal ideas a formal system expresses, rather than disassociated from what we understand (or from the formalism) and associated with the independent realm. On the face of it, the former division may seem less contentious than the latter. And certainly, Gödel's realism can be interpreted such that the possibility of error depends on this former division rather than on my separation of meaning and what we understand.

For example, one of the key differences between Hilbert and Gödel (Gödel [1995a]) lies in the means they believed would 'secure' the certainty of the belief that "for clear questions posed by reason, reason can also find clear answers" (p. 381). For Hilbert the means was "proving certain properties of a projection onto material systems" (p. 383). For Gödel, the means consisted in admitting that some concepts cannot be formalised, and "cultivating (deepening) knowledge of the abstract concepts themselves, which lead to the setting up of [those systems that express pre-theoretical concepts, which can be formalised]" (p. 383). In this quotation, Gödel is quite clearly talking about the conceptual separation of pre-theoretical concepts from their theoretical formalisation. Preferring, or even conflating this particular division over the division between our understanding of a formalism and its meaning, is in itself instructive.

I believe that in fact each division should properly be seen as contentious as the other. This will be argued for in due course. Perhaps the reason the former is more generally accepted is that it is confused with yet another division: that between physical symbols (e.g. markings on a page) and the (usually formal) concepts they refer to. This further possible conflation will be elaborated upon later also. In the meantime it will be sufficient to note that Gödel's idea of "attaching meaning" (p. 372) to formalisms remains interesting and fruitful, whether it is interpreted as something as (supposedly) uncontentious as establishing a connection between theory and pre-theory, or in accordance with the interpretation offered here.

Once again, Leavy's preface to Derrida's [1962] provides a framework to help demonstrate this claim. Recall that in the preface, Leavy notes that Derrida distinguishes three levels of 'ideal objectivity' in Husserl's analysis (p. 13). The first is that of the word itself, which is bound to the languages in which it 'makes sense'. This level can be thought of as analogous to the level of the physical mathematical symbol, which is bound to the language in which it 'makes sense' - namely the language of mathematics. The second is the level of the word's (or, analogously, of the symbol's) sense itself, 'theory' and 'formalism' belong on this level, wherein "the ideality signified (the intended content or signification of the word or symbol) is free 'from all factual linguistic subjectivity'" (p. 13). The third level is the only level upon which a conflation (or, better to describe Gödel's case, a deliberate alignment) between what the facts are and 'what we take them to be' is possible. That is, it is only if it is placed on this third level that meaning can be understood as objective. As Derrida puts it, this is the level of "absolute ideal objectivity, such as the free idealities of geometry. The ideality in question here is that of 'the object itself.' On this level of objectivity, there is no adherence to any de facto language, only adherence to the possibility of language in general" (p. 14).

To further elaborate this point, recall Leavy's remark, that "Derrida has elucidated these three degrees in order to show that when Husserl ... does not distinguish between the object itself and its sense, this can only occur within the third region of ideal objectivity" (p. 14).

My model takes it that when Gödel does not distinguish between the independent mathematical realm and (real) mathematical meaning, then he expresses his faith that the two coincide, and this coincidence itself can only occur within the third region of objectivity. My interpretation has it that Gödel is (correctly) most interested in the nature of the link between the third and the second regions, most specifically when it comes to the problem of justification referred to earlier. Once *that* link is established and shown to be 'strong enough', a further question may remain about a Folina-style coincidence of our understanding of the facts and the facts themselves, but this latter coincidence would have to depend on the quality or success of the fulfilment of the initial task. Whatever other ambiguities exist in Gödel's platonism, the centrality of the idea of a division or difference between the formal system and its meaning remains. And so, the possibility that Gödel can be interpreted according to the model I offer here also remains. Specifically, it remains feasible to interpret the difference between a formal system and its meaning as an essential part of any possible solution that seeks to address this particular interpretation of Gödel's realism. It remains to explore that division and determine its fruitfulness – this will occur in later chapters.

Note that my reading of Gödel need not entail that, for Gödel, incorrect mathematical statements are incomprehensible. Insofar as comprehensibility entails some sort of subjective meaning, incorrect mathematical formalisms may be seen as having just this sort of 'meaning' (analogous to Frege's 'idea'), but this is not the sort of meaning that has the possibility of being true or false within mathematics. (The apparent circularity of this claim can be overcome via the establishment of an order of dependence, set out in the next chapter). So, my claim is not that subjective 'meaning' or 'comprehensibility' is non-existent when there is no independent fact of the matter, rather that it is largely irrelevant, or not genuine. That is, if there is no link between the second and third levels of objectivity above, then there is no possibility of genuine meaning, as opposed to simple comprehensibility. If, on the other hand, there *is* a link between the second and third levels, then there is the possibility of genuine meaning and hence also the possibility of genuine error (the nature of that error is here seen as something akin to poor reception on our behalf, something coming between, or corrupting the essential link upon which the possibility of meaning depends). In other words, on this interpretation, the notion of primary importance to the realist and thus to Gödel is that meaning is inextricably bound to the existence of the independent mathematical realm. This theory is further fleshed out in the next few chapters.

Note also that my interpretation does not deny a link between our subjective or individual understanding and the formalism, or *what* we understand (levels one and two above); rather, it stipulates that such a link, if it is to be relevant, depends upon the first: that is, the principle stipulating the inextricability of meaning and the real, independent mathematical realm has priority. Literally, this principle must be in place *prior* to any other possible links between levels.

By contrast, Folina offers a middle road account whose formulation relies on an interpretation of realism in which, although meaning is inextricably linked to the formalism, it is *this* link that has priority over the link between meaning and the real realm. That is, on Folina's interpretation, we understand what we do by virtue of our make-up – or the way we are – as opposed to understanding what we do by virtue of the way mathematics is. I claim that both of these positions are true, but that the second has to be, at least for the realist I am interested in, a more important part of their overall philosophy. For example, it should be treated (at least) as a more effective indicator of the nature of independent mathematical reality than the first.

4.5. A Poincaréan Middle Road

Having argued that Folina's initial interpretation forms the basis of her following strategies, we need now to look at the middle road account she offers on that basis.

The primary goal for Folina in the construction of her account, is to retain the attraction of the constructivist position – namely, a good explanation of epistemological access, and to retain this attraction whilst overcoming the drawbacks of the same – namely, the (general) inability of such accounts to provide satisfactory accounts of mathematical truth – i.e. to provide an account of mathematical truth *as* objective.

To recap, recall that the conception of the divide between realism and anti-realism upon which Folina's middle road account, in her [1993-4], depends is summarised as follows:

> in general, lack of an epistemological gap is both what is attractive and what is problematic about constructions; and the existence of an epistemological gap is both what is attractive and what is problematic about platonism (p. 204).

It is this conception of the divide between realism and anti-realism that I take here to be representative of a widely accepted and commonly employed general approach to the problems associated with each.

Recall that this is opposed to my interpretation wherein such a gap is acknowledged as important to realism's strength, but only with the proviso that that strength in fact derives from the priority that realism (at least, my Gödelian interpretation of realism) gives to the independent mathematical realm itself. That is, under my interpretation, even the possibility that the gap Folina discusses exists in actual fact – i.e. in any sense relevant to mathematical truth and mathematical fact – itself depends upon the existence of the real mathematical realm.

To put the same point another way, on my reading, the meaning of the formalism (at level two) is only relevant – or genuine – if the mathematical realm does in fact exist, and exist independently. Therefore, the independent mathematical realm itself has ultimate priority; the dependence of the formalism's meaning upon this realm is secondary, and the dependence of the formalism's meaning upon ourselves (or on our understanding) is least important. Only once this order of dependence (and priority) is established can this last dependence-set be said to be at all relevant to the independent mathematical realm, or to the nature of that mathematical realm. And even then, the fact that the last dependence-set is relevant to the independent mathematical realm cannot determine or affect the nature of that realm. That is, the relevance is not a determining relevance. It is, rather, a relevance of association, such that without this order of priority, the last dependence-set has no genuine relationship at all with any of the prior dependence-sets.

So asking, as Folina does, whether or not our comprehension of a mathematical formalism or any kind of intersubjective meaning (a.k.a. 'what we take the facts to be') and the objective facts *correspond* is, according to my interpretation of realism, rather beside the point. If, instead, the point is that there is an order of dependence of the former on the latter and without this order of dependence, such realist stalwarts (or bugbears, depending on your point of view) as correspondence theories, or indeed any theory built in response to Folina's realist's 'gap' identified by Folina as the source of strength in realist's accounts, are rendered quite literally irrelevant on my interpretation (there will be more on why this is so in following chapters).

Nonetheless, Folina builds a strong philosophy of mathematics around the gap with which she begins, arguing that Poincaré's philosophy of mathematics successfully runs a middle line through the divide she has identified between realism and anti-realism. Particularly, she argues that Poincaré's philosophy provides the constructivists with what was previously identified as a realist strength – namely, a framework capable of distinguishing between the facts themselves and what we take them to be.

One question here, then, is whether or not Folina's account is successful in reaping the benefits of the primary strengths of realism: and, more specifically, whether or not her account incorporates *both* the strong objectivity of realism and the 'appealing epistemology' of constructive accounts. I argue that in the end it does not, because it cannot, due to Folina's initial conception of the realist's problematic divide as that between the facts and our understanding of them. On my interpretation, the divide (if it can be called a divide) needing to be bridged in order to incorporate the strong objectivity of realism, is that between the formalisms and independent meaning.

4.6. Poincaré

Poincaré's ideas though, and Folina's interpretation of them, take us further toward 'the other' than the other accounts discussed so far. This is because, although transcendentalism is primarily about the way we are, it leaves room for the notion that the way we are is a result of the way (transcendent) reality is. The difference between a transcendental and a phenomenological account of our relationship to an 'other' could fill another thesis. My own stance, though, is that a phenomenological account is more capable of an overt and optimistic 'space' for the other as a knowable reality, whereas transcendental accounts retain their mystery at the expense of the possibility of acquaintance. Nonetheless, given that they engender a solution to the problem of access which does not seek to incorporate, reach or demystify the 'other', it is worth looking at those of Poincaré's ideas that are most central to Folina's program.

Poincaré offered a Kantian philosophy of mathematics, specifically one which took Kant's original ideas and sought to strengthen them against the logicist program. A fundamental motivation underlying both Poincaré's and Kant's philosophies – and one which, it could be argued, underlies transcendental arguments in general – is the belief that certain aspects of our human experience are to be explained by broadening our idea of what it is to be human. The thought is that a description of our five senses alone (and, indeed, of whatever faculties it is generally accepted that humans posses) cannot account for what we know. In fact, Kant argued that this sort of approach cannot even account for our most basic experience of the world. So rather than attempt to explain the apparently unarguable fact that we posses something we want to call 'mathematical knowledge' with the resources afforded by widely accepted accounts of knowledge acquisition (based mainly on sense perception and causal theories), the Kantian picture proposes an explanation based on a more detailed and broader examination of what being human necessarily involves. It is this sort of consideration that drives both Poincaré and Kant to posit a faculty of mathematical intuition to explain mathematical knowledge acquisition. The existence of this mathematical intuition is then defended via more thorough accounts of what it (necessarily) is to be human, or to have the 'form of experience' that all humanity shares.

Poincaré's account of what it is to be human is broader than Kant's and therefore, Folina argues, offers a stronger objectivity (strong enough to compare favourably with that of a more traditional realism) for mathematical truth. This is because Poincaré describes a set of necessary preconditions for what it is to be a finite thinking being, and for finite thinking beings collectively, whereas Kant focuses on humankind in particular. For Kant, experience has an a priori form, and that form is spatio-temporality, whereas for Poincaré, intuitions are the a priori form of experience.

To see this, note that it was against the Leibnitzian theory that space is a mere system of relations and therefore a mere concept (like, for example, the relation 'larger than') that Kant set his theory that space is an (a priori) intuition. The concept of space, for Kant, is an instantiated one; space is an 'actual individual', a thing in itself, rather than a concept relating things. Kant reasoned thus: geometry is a science, which determines the properties of space synthetically, and yet its truths seem a priori. What then must our representation of space be in order that such knowledge of it is possible? He concluded that it must in its origin be intuition since, according to Kant, no propositions can be derived from a mere concept which go beyond the concept itself, as happens in geometry, unless that concept is an instantiated one.

Similarly, the intuition Poincaré posits is an 'intuition-of' something true, rather than an 'intuition-that' something is true. Note too that Poincaré offers arguments similar to Kant's from the existence of science to the existence of something else (arguments taking as given a certain 'way of being' which scientific thought embodies or is based on). An example is Poincaré's argument for the existence of a coherent hierarchy of facts: "scientists believe that there is a hierarchy of facts, and that a judicious selection can be made. They are right, for otherwise there would be no science, and science does exist." (Poincaré [1982, 363]). And again, Poincaré's argument for the 'intuition of indefinite iteration' follows Kant's transcendentalism: "[the mind is] capable of conceiving the indefinite repetition of the same act when once this act is possible. The mind has a direct intuition of this power, and experience only can give occasion for using it and thereby becoming conscious of it" (Folina [1996, 421], quoting Poincaré). Indeed, for Poincaré the intuitions necessary for "coherent, conceptualisable experience" (Folina [1996, 418]) are the intuition of indefinite iteration and the intuition of continuity. The latter, for Poincaré, is "the awareness that we posses [the] faculty" (Folina [1996, 421], quoting Poincaré) of constructing a physical and mathematical continuum.

These two intuitions in particular, Poincaré argues, are essential for both logic and mathematics. Note that Poincaré's point is that mathematical knowledge is inexplicable without the positing of intuitions. The logicist's claim that a knowledge of logic, even if it is coupled with an account of the role of sense-perception in the acquisition of mathematical knowledge cannot, according to Poincaré, account for mathematical knowledge.

For example, Poincaré argues that the principle of induction cannot be explained by reduction to logic nor by our experience of the world. It is from this that he arrives at the conclusion that induction is synthetic a priori.

Folina clarifies Poincaré's position as follows: "for Poincaré, induction is a truth about a priori intuition. The judgment that induction is true is synthetic since it requires synthesis via the intuition of indefinite iteration" (p. 423). This kind of argument – for the indispensability of the intuitions for mathematical/logical thought and, in the final analysis, for any kind of thought at all – is typical of Poincaré's general strategy.

4.7. Folina on Poincaré

To examine Folina's claim that Poincaré's philosophy of mathematics steers a safe course between the problems she identifies for both mathematical realism and anti-realism, we need now to highlight the main attraction of Poincaré's arguments. To do this we initially need to revisit Folina's summary of these problems as, respectively, the problem of an epistemological 'gap' between what we 'take to be true' and what in fact is true, and the lack of any such gap. To elaborate further on the motivation of her account, we need to explore the merits of the perspective from which Folina builds her case.

Note again that the realist's gap, according to Folina, is introduced by the positing of mind-independent objects, or an abstract mathematical realm, and enables a picture of mathematical truth which, in some important sense, satisfactorily expresses the standard of objectivity we commonly associate with mathematical statements. This is widely accepted to be the main attraction of realist philosophies: their ability to thus capture the instinctual or natural thought that mathematical statements are true or false, despite or 'outside of', what we believe about them, and that there are, therefore, as yet undiscovered mathematical truths, even undiscovered (even undiscoverable) axiomatic truths. So let's now suppose that it is indeed this gap that presents the largest obstacle for realist programs, regarding the question of how we come to know about the abstract realm. The problem now before the platonists is how to give an account of knowledge-acquisition that similarly satisfies our instincts – how can we picture ourselves learning about a realm so mysterious and seemingly inaccessible?

It is not enough for platonism to say that we 'just do' have access to this realm, or that we 'just do' intuit the objects within it. What is required is an account of *how* we, as finite thinking beings, interact at all with a realm that we cannot see, touch, taste, hear or smell. A satisfactory account of (mathematical) knowledge acquisition would include viable descriptions of the working of our minds and an explanation of the link between these workings and the facts we came to know. But there is, to date, no clear or obvious candidate account able to provide this. The objectivity of platonism, it seems, is gained only at the cost of opening an epistemic mystery, one whose various resolutions do not seem to fit naturally with any intuitive or pre-theoretical beliefs we have regarding knowledge acquisition.

So, one of the issues here is whether Poincaré's (or anyone's) philosophy offers a better account of knowledge acquisition (by the standards set forth here) than platonism has traditionally done (i.e. the 'we just do' response, and variations of it), since each program can be seen as offering an account that appeals primarily to the way we 'just are'. A possible response to this is to note that the transcendental philosophy put forward by Kant and Poincaré maintains a strong tie to empirical knowledge, and to experience. Derk Pereboom [1990] addresses precisely this point. Pereboom argues that according to Kant, transcendental philosophy is justified a priori in the sense that the only empirical information needed for its justification can be derived from any possible human experience (p. 25). According to Pereboom, Kant's restrictions on what kind of knowledge the transcendental point of view can provide ensure that the kind of objectivity transcendental philosophy provides is neither beyond human capacity nor illusory. That is, Kant's restriction on the transcendental point of view involves the notion of 'stepping back'. And, in one sense, to take this point of view is indeed to take a step back from the empirical. This step involves

> not only remov[ing] oneself from unreflective involvement with one's ordinary experiences and proceed[ing] to see them as elements in the empirical world, but also com[ing] to see the whole empirical world, including one's ordinary experiences, as produced by the self's organisation of sensation by the categories and the forms of intuition (p. 29).

The kind of objectivity Kant's philosophy affords, then, is more inclusive of, and further removed from experience than that of the empirical standpoint, but significantly *less* removed from experience than that of platonism. But the transcendental point of view is "not a step back with respect to the criteria for the justification of knowledge" (p. 29). Rather, Kant applies the same limiting condition to the knowledge and cognition available from the transcendental point of view as to that available from the empirical standpoint; namely that no knowledge is possible without intuition – direct awareness or cognition of objects. So in this respect the transcendental point of view is, after all, *not* a step back from the empirical.

Poincaré's objectivity is relative only to the form of experience of any finite thinking being. Thus his defence of Kant moves a further step back from ordinary human experience and a further step closer to the strong objectivity of platonism.

Nonetheless, much the same point as is made above about Kant's philosophy can be made about Poincaré's. Poincaré's transcendental knowledge is a priori because it is independent of particular experience or observations for its justification, but it is still dependent on information derivable from any possible human experience, insofar as our concept of any possible finite thinking being must be derived from our own instantiation of that concept.

4.8. Identity Realised

In fact, it is just this problem that constructivism tries to get around by describing our ability to construct mathematical objects and proofs and linking this with the mathematical 'realm', simply by identifying the two: i.e. according to constructivist accounts, our mathematical constructions themselves constitute the mathematical realm. Chihara [1990], for example, proposes replacing the existential theorems of traditional mathematics with constructability theorems such as "where, in traditional mathematics it is asserted that such and such exists, in this system it will be asserted that such and such can be constructed" (p. 25). He then goes on to give an account of mathematical truth by means of the constructability quantifier, 'C'. The question of how we come to know mathematical truths is then readily answered by the intuitively accessible picture provided by an account of how we are able to construct mathematical objects and proofs via their connection to 'real world' construction.

This, though, not only closes the gap between 'the facts' and what we 'take to be true' but it does so on level 1, which is at some remove from reality itself; indeed this constructive sort of gap closing ignores independent reality altogether, since here what we take to be true is given by what we could construct, and what we could construct precisely is the mathematical realm. The strong tie to the empirical is clearly evident here. But using the constructive abilities of finite thinking beings to close the epistemological gap can be regarded as a tactic similar to Kant's proposal regarding 'intuition', and hence similar to Poincaré's response to the problem of knowledge acquisition. Predictably, the objectivity provided by the more blatant constructive approaches is weaker than both Poincaré's and Kant's. In fact, it is better described as an 'intersubjectivity', dependent upon "actual communal consensus in drawing the bounds of what is objectively true" (Folina [1993-4, 224]). There is, however, another general approach available to the constructivist. This involves idealising the notion of construction so that what we can construct is equated, or nearly so, with what is objectively true. But idealising the constructor in this way introduces problems normally associated with platonism, namely by raising the question: how can an abstract 'object' or concept explain anything? Note, though, that the same question can be asked of Poincarés idealisation across all possible finite thinking beings. Where an abstract concept is used to describe our ability to have knowledge of abstract concepts, the natural problem that arises is how we come to know anything of the former or initial abstract concept.

Another way of characterising this impasse is as follows: closing the epistemological gap means that non-revisionist anti-realist programs, including Chihara's constructivism, struggle to provide a satisfactory account of the objectivity of mathematical truth. There are two reasons for this. The first is that it is at best hard to see how such programs can lay claim to the mathematical and logical notions they wish to preserve, without expressing them in terms of a background ontology such as that of the set theoretic hierarchy. There is a dilemma confronting these types of programs that can be expressed as follows.[3] Either the programs succeed in laying claim to the language used or they do not. If success *is* achieved, the quantifiers will have to be explicated such that the level of rigour they can accommodate matches that of mathematics itself. In this case, though, the concepts – either of constructability or of possibility – will have had to have come loose from their epistemologically graspable moorings. From being everyday concepts whose workings we can understand, they will have become technical notions whose relationship to the real world and hence to easily understandable or everyday processes of knowledge acquisition is at best strained and, more probably, lost altogether. But perhaps the more serious worry here is that this kind of success can only ever be an illusion, achieved by sleight of hand. Shapiro's [1997] argument against constructivism is easily adapted to modalism and to mathematical anti-realism in general:

> we only understand how the constructibility locution works in Chihara's application to mathematics because we have a well-developed theory of logical possibility, satisfiability etc. and, again, this well-developed explication is not primitive or pre-theoretic. The articulated understanding is rooted in set theory, via model theory. Set theory is the source of the precision we bring to the modal locutions (p. 25).

As we have seen, Shapiro goes on to offer another angle on the dilemma, via the principle of inter-translatability. He shows that each of the anti-realist programs he

[3] The dilemma is a version of Bob Hale's two-horned dilemma confronting Blackburn's Quasi-realism. To read about this, see Blackburn [1993].

addresses is intertranslatable with classical logic, or traditional mathematics complete with its realist commitments. He then argues that the problems associated with each program are intertranslatable also. So constructivism, for instance, ends up saddled with the same problems regarding epistemological access as traditional realism. This is all due to the fact that each non-revisionist anti-realist program has to posit abstracts or use the notion of 'possibility' in a semi-concrete sense. The point here is that Poincaré's philosophy, although perhaps not intertranslatable in the same way, is vulnerable to the same, or at least to a similar point. The fundamental problems associated with positing the abstract objects of traditional realism are surely present when all possible finite thinking beings are under consideration. Interestingly, this concern itself can be addressed transcendentally, by supposing that the ability to run through possible worlds is another part of the preconditions for a finite thinking being – along with the Poincaréan preconditions – or is explicable by reference to these. But at this point, we reach another impasse, insofar as platonism could adopt a similar approach to justify the necessity of realistic thinking in general.

The second reason anti-realist programs have trouble giving a satisfactory picture of mathematical objectivity is that in order to close the epistemological gap, these accounts seek to show that mathematics depends somehow upon us and our creative abilities (Folina [1993-4, 223]). But if this is the case, it is hard to see how mathematical truths could be necessary, rather than conventional or linguistic. With talk of intersubjectivity or "shared standards of admissibility and error" (p. 223) the gap between what the facts are and what we take them to be is forced open a little, but not enough to guarantee the sort of objectivity we would like. The necessity of mathematical truth is an intrinsic part of our understanding of its objectivity, and so accounts of mathematical truth upon which we can still conceive of finite thinking beings, other human beings, or other 'possible communities' for whom these primitives are not foundational to, or even intrinsic to, mathematical thought, falls short of the necessity requirement.

Recall the question at hand: given Folina's initial characterisation of the respective strengths and weaknesses of realism and constructivism, can there be an account of mathematics which preserves the gap between the facts and what we take them to be without failing the necessity requirement (inherent in our conception of objectivity) and without being impaled on either horn of Hale's dilemma, or something like it?

Poincaré's philosophy does seem promising. It seeks to explicate the concept of mathematical intuition in such a way as to answer epistemological concerns, and to demonstrate how we need not look to any abstract realm beyond ourselves, or at least beyond our understanding of 'finite thinking being' in order to accommodate the sort of objectivity we require of mathematical truth. It also seems (given the brief considerations two paragraphs above) that Poincaré's philosophy could have a transcendental solution – albeit a self-referential one – to Hale's dilemma. Further, the dilemma is activated by those anti-realist programs which seek to show some underlying 'deep structure' behind mathematical language by providing, to take the current examples, a constructive or modal 'backdrop' and arguing that this is prior to the 'surface' semantics, or to the form of mathematical language itself. Although Poincaré's philosophy does run up against a hybrid of the dilemma, it nonetheless incorporates no such program.

The Kantian method of obtaining the sort of objectivity we require of mathematics is to argue that mathematical truth is universally (or potentially universally) intersubjective. Poincaré expands Kant's universe and so strengthens the transcendental claim to objectivity, but at what cost? Is Poincaré's advance over constructivism any less expensive in the long run than the platonist's? Specifically, does Poincaré's account have any great advantage over the proposed realist desiderata that real meaning is located on the objective level? More generally, how do the philosophical systems in which intuition and transcendental arguments play a key role compare with philosophical systems in which an abstract mathematical realm provides the main foundation? It seems that any good account of how we come to know mathematics – including Poincaré's intuition and strengthened version of objectivity – must, in the end, be at least as mysterious as the platonist's independent realm.

In fact, I think that Poincaré's positing of intuition does provide a good account of knowledge acquisition, and manages, too, to incorporate a strong degree of objectivity. But the issues raised as a result still need a great deal of attention before Poincaré's philosophy can be said to have made significant gains over either constructivism or platonism. In the end, Poincaré's intuitions are no less mysterious than the idea of a fully independent mathematical reality, and do less justificatory or grounding work. The idea of an independent ground is (for a good many realists at least) a powerful, perhaps inescapable intuition in itself. Attempts to circumvent it seem only to replace mystery with mystery. Doing away with the mystery of an 'other', though, does away with the most potent expression of necessary mathematical truth and objectivity that we can hope to imagine – namely, the idea of a relationship between us and an 'other', of a ground, and of the notion that we 'discover' rather than create mathematics.

5. Dependence

5.1. Abstract Objects/Ordinary Objects

The previous chapters have gone over various different perceptions of, ways to model, and perceived associated problems with, the strengths and weaknesses of realism and anti-realism in general. I have sought to show that each of the formulations discussed falls short in some way of attaining either the realist desiderata outlined in the introduction, or the authors' own realist ideals, or both. In these final few chapters, I attempt to capture and fully express the ideals of a full-blown realism, incorporating, undistilled, the set of desiderata with which we began. How the resulting position compares to those discussed so far remains to be seen. The principal task at hand is not to compete with the accounts presented but to articulate a realism which does not undermine or sacrifice any of the ideals outlined, and so to discover precisely what is retained, what is lost and exactly which problems remain for a realist account with this particular agenda. Hopefully, it will become apparent that the realism I arrive at is no more problematic than those discussed so far, and perhaps it has an advantage over them simply by being to a certain extent, an ideal version of them – at least insofar as it retains some of the more crucial components of the stated principles and/or beliefs of realism in general.

5.2. By Contrast

The realist accounts touched on so far have all attempted, in one way or another, to overcome a set of identified problems for realism. And this attempt has been, for each, a principal task. It seems to me that the full expression of the realist principles or ideals underlying this attempt is thus relegated to a secondary task, or is balanced against the principal task so that each is forced to accommodate the other. My method takes its inspiration from this observation, and reverses the order of importance of the tasks at hand. That is, my initial and primary aim is to come up with a realism that retains, as completely as possible, the set of desiderata outlined, and to see what problems remain. I then turn, as a secondary consideration, to the question of whether those problems can be addressed or solved and, finally, to whether they undermine my account any more than the problems left outstanding in the other accounts discussed. In the final chapter, I make use of some ideas from phenomenology to argue that my mathematical realism is no more problematic than an analogous physical realism, which in turn is no more problematic than our relationship to the physical world itself, even accounting for the best of our scientific theories and the vast reach of our experience.

I begin, then, by retreading ground already laid down in the discussions of some of the accounts represented thus far. Recall some of the problems motivating realist accounts in general: what are abstract objects? Are they real, and in what sense? If they exist at all, then how? And, given that we cannot see, taste, touch, smell or hear them, how do we come to know anything at all about them?

These questions, and more like them, are classically the bane of realist philosophy across the board. It seems that just as soon as one adopts the belief – and wishes to articulate this belief sensibly – that the abstract is as real as the ordinary (or at least, as real as the physical is commonly presumed to be), one immediately takes on the burden of providing answers to questions such as these.

But, rather than form my own expression of this realist belief (applying in particular to the abstract objects of mathematics) around an attempt to address such questions, what I am proposing is, essentially, a new set of questions with which to begin – one which shifts the emphasis away from ourselves and on, instead, to the abstract itself, at least in much the same way as an emphasis is often already placed upon the objects themselves when we come to inquire after ordinary objects.

Rather than ask regarding a thing that we cannot receive via our five senses how then we come to know anything about it, or have anything at all to do with it I propose that we look at the sort of underlying assumptions that drive us to ask these sorts of questions *before* any others. The classical problems (e.g. those given above as the problems motivating realist accounts, and also in Benaceraff's [1965]) need not, I think, come *first* in our investigations. I think that our inclination to give priority to such questions is initially more attractive than prioritising other, perhaps equally important questions, only because of an initial priority given to an immediate disanalogy between abstract and ordinary objects: namely that ordinary objects are accessible via our five senses and abstract ones are not.

Instead of beginning with this *dis*analogy (and others), and with the classical questions it inspires, my expression of the belief in the reality of abstract objects begins with a scrutiny of the nature of, and our relationship to, ordinary objects. In this way I hope to have possible analogous features of the two sorts of things brought to our attention *before* their admittedly compelling disanalogous features. I will take my lead from the questions that arise as a result of this strategy, and accordingly form my own expression of the belief in the reality of abstract objects around these, rather than the classical questions.

This approach is, prima facie, a kind of naturalism, in so far as it appears to use only our more established (scientific, psychological and experience-based) theories about the world and from there to answer questions about the mathematical realm using these tools of inquiry. But what I am proposing is in fact akin to naturalistic approaches only insofar as it adopts a kind of blanket attitude to physical and abstract objects alike. Contra naturalism, though, this is less because I believe abstract objects are like physical objects, and more the other way around. Physical objects, in their fundamentally (metaphysically) inscrutable nature, their unknowableness and their mysteriousness, share a list of important features with abstract objects. To ask of abstract objects, but not of physical objects (whose nature is presumed to be eternal, unchanging, acausal etc.) just what they are, how we come to know anything at all about them, etc., is to approach the general problem of knowledge acquisition with a bias already firmly in place – one that assumes that such questions have already been resolved, or are at least less pertinent for physical objects. I hope to show (via a phenomenological approach to physical objects) that such questions have not been resolved for physical objects – so to demand they be resolved for abstract objects is inappropriate, and is equally as futile (or fruitful, according to your particular degree of optimism) an exercise as trying to prove that anything whatsoever exists besides, or external to ourselves.

My project, then, reasons that the main problem for a realism regarding physical objects is not the problem of access. (We presume that problem is solved with the descriptions of how our five senses access reality.) The main problem for a physical realism is, rather, the problem of how the accessed reality is (or is the same as a reality which is) independent of that access. Put another way, the problem is one of access on another level: how does the accessed object relate to (or access) independent reality? So, even if the story we told of just how we access mathematical objects was as satisfactory as that which we tell of how we access physical objects (i.e. even if mathematical intuition, or our mathematical comprehension was as clear and as acceptable a method of knowledge acquisition as the five senses), even then, a fundamental realist problem remains. And this problem – that presented by the gap between what we access and what independently is – is no less a problem for physical reality than it is for mathematical reality. Our senses 'give' something we call physical reality. Our mathematical comprehension 'gives' something we call mathematical reality. No matter how effective or satisfactory a story we can tell about just how these realities are given, this further, seemingly intractable, problem still remains, at least for the realist. Put another way, the realist still has to ask just how the something that is 'given' is independent of ourselves and our methods of access. This amounts to asking how what the senses or our intuition gives is also other than what our senses or intuition gives. This is the problem that unites physical and abstract entities, and so it is the problem that occupies this author. That there remains the classic problem of access is granted. But since the solution to this problem gets us only as far as it gets the physical realist, which is no further than the problem of independence, the former and not the latter problem is the primary focus of this work.

It is probably already clear that of the features of ordinary objects upon which I hope to focus, I want primarily to focus on our attitudes toward them. In particular I want to focus on the way we tend initially to emphasise the objects themselves, rather than any set of problems surrounding them. That is, we tend, in our inquiries regarding ordinary objects, to direct our attention toward the objects themselves as the source, or at least a primary source from which a solution ought to be informed. Science, after all, often seeks to eliminate the observer and aims to direct attention to the observable objects themselves. What if our initial approach toward abstract objects was no different? Answering this question means beginning my own expression of realism regarding (mathematical) abstract objects with an emphasis similar to that with which our investigations of the ordinary begins. And this means, initially, focusing on the objects themselves rather than on the observer – ourselves.

With the objects themselves as my starting and focal point, I will, I hope, come up with some interesting new solutions to the questions with which this chapter began: i.e. those classically provoked by the articulation of realist philosophy in general. I will also show that one of the results of this approach is that many such problems can be, or ought to be, avoided by means of highlighting their inappropriateness and, finally, even their inapplicability to the realist position I present.

5.3. A New Approach

Grant, then, that a primary motivation for the set of problems that *classically* arise for the realist regarding the abstract is the initial idea that there is a stark disanalogy (beginning with that already touched upon – the accessibility of the ordinary versus the inaccessibility of the abstract via our five senses) between the (proposed) real abstract things and real ordinary or material things. So compelling is this immediate disanalogy that the term 'real', uniting the two kinds of things just as much as it does, can seem entirely misappropriated when applied to abstract objects. Abstract objects can seem to be not 'real' at all, not at least in any way similar to the way in which ordinary objects are.

One response that a realist regarding the abstract can make to this train of thought is to try to make the abstract seem more ordinary. Maddy's Set Theoretic Naturalism is a case in point, as is Shapiro's Structuralism. Another possible response for such a realist is to grant that the abstract and the ordinary are indeed two different sorts of things, but then to argue that this does not mean that abstract things are any less deserving of being called 'real' than ordinary things are. Zalta and Linsky's 'principled platonism' makes this response.

My own approach, broadly stated, is the reverse of the first. I do not attempt to overcome the immediate dissimilarities between the two sorts of things, or attempt to make the abstract seem more ordinary. That is, I do not begin with the abstract realm and search there for ways in which that sort of thing, or our relationship to it, is, after all, similar to the ordinary. Instead, I begin with the ordinary and try to form judgments as to the possible analogous and disanalogous aspects of the abstract and the ordinary, via a close/phenomenological analysis of *ordinary* things and our relationship with them. In particular, I propose that we re-examine our relationship to ordinary things and the nature of the existence of ordinary things themselves, and search there for features from which analogies can be drawn with our relationship to, and the nature of, the existence of abstract things.

I begin, then, with an immediate consideration of the ordinary, and move to a consideration of the possible analogous aspects of the ordinary and the abstract. After all, the question 'in which ways, if any, are the abstract and the ordinary analogous?' offers as interesting a place to start as its perhaps more obvious, inverted twin. This move is not new. It is, in fact, this sort of move that can get a realist about the abstract into trouble in the first place, principally by (once again) inviting the contrary consideration. What is new is the call back to the prior consideration of the analogous aspects of the ordinary and the abstract, based on an investigation of the ordinary and our relationship to it, rather than on an initial attempt to overcome the obvious problematic features of the abstract. Also new is the particular analogy between the ordinary and the abstract that I propose.

5.4. How Are Ordinary Objects Analogous to Abstract Objects?

One of the ways in which the abstract and the ordinary are analogous is that they are both able to be spoken of. The most obvious instance of this is just that they *can* be compared. This is an assumption of both the considerations tabled above. That is, consideration of the possible disanalogous aspects of the ordinary and the abstract, and consideration of the possible analogous aspects of the two, both admit this much. There would be literally nothing to speak of if this were not the case.

This immediate similarity between the two suggests a line of investigation along which many more such (immediately linguistic) similarities appear. Of both abstract and ordinary objects we are able to speak. About both we are able sensibly to ask similar sorts of questions. Crucially, of our relationship to both we can ask how much of what we understand (the things themselves to be, and of the things themselves) is somehow received from the things themselves, and how much we ourselves have somehow created, or constituted. This question is applicable both to the abstract and to the ordinary, and ought, therefore, to be among the first focused upon, in accordance with the strategy outlined above.

Certainly, when asked of our relationship to the ordinary, this is not a vacuous question. Consider for a moment the legitimacy or otherwise of our assumptions regarding the ordinary. Is there reason to suppose that ordinary objects exist independently of us – or our creations, definitions, constructions etc. – so that our understanding of these objects is somehow more received than created? And, if there is reason, how much?

Gödel suggests that it is precisely here, at the applicability of this question to both, that the ordinary and the abstract are in fact the same, or at least analogous: "It seems to me [Gödel] that the assumption of such [abstract] objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence", (Maddy [1998, 92], quoting Gödel) and "the objects and theorems of mathematics are as objective and independent of our free choice and our creative acts as is the physical world" (Maddy [1998, 89], quoting Gödel).

I think that Gödel's point could be put this way: in as much as we have reason to believe in the existence of the ordinary, we have reason to believe in the existence of the abstract. And this point can be argued by trying to show something stronger: that we have as much reason to believe in the *independent* existence of abstract objects as we do to believe in the *independent* existence of ordinary objects.

This approach works *from* our beliefs regarding the ordinary, and attempts to show that our opinions regarding the reasonableness or otherwise of these beliefs can quite legitimately be mapped without too significant a change onto our opinions regarding the reasonableness or otherwise of our beliefs regarding the abstract. Again, rather than making the abstract seem more ordinary, such an approach can make the ordinary seem more abstract. But that is no bad thing, or at least it ought not be assumed to be a bad thing at the outset without some further argument.

5.5. Understanding Ordinary (and Abstract) Objects

That it is not a bad thing is, in fact, the approach I wish to adopt, specifically in response to the question touched on earlier, regarding the independence or otherwise of abstract objects. This sort of inquiry into the independence or otherwise of abstract objects can itself be refined or represented to reveal further analogous features of the ordinary and the abstract. To be more specific, we can ask, 'upon what does our understanding of such things (both abstract and ordinary) depend?' – or, (put another way) 'what are the preconditions of our understanding?' Note that this question can be put in terms of our apprehension, our perception, and even our consciousness of, abstract and ordinary things alike (for now I use the term 'understanding' to cover all of the former concepts, and do so to encapsulate a broader scope of our approach to, and acquaintance with, objects in general).

When applied to the ordinary, this is a compelling, problematic and complex question. Again, in accordance with the strategy outlined, I will try to show what happens when we suppose that this question is at least as compelling, problematic and complex when applied to the abstract, and that it is so in just the same (or isomorphic) ways. To anticipate (and reiterate), what happens is that we arrive at an expression of realism about abstract objects that is able to cast a new light on the 'classic' sorts of problems raised earlier.

Consider the host of possible factors upon which our understanding of the ordinary, material world might depend. The presence or existence of the ordinary itself is a possible factor, or at least our reception of something that looks, feels, tastes, sounds or smells like the ordinary. This is the factor I will concentrate on in what follows. I hope to show that the way in which our understanding of ordinary things depends upon the existence of those things themselves, is very much like the way in which our understanding of mathematical things depends upon mathematical things themselves. The realism thereby constructed will, hopefully, fulfil the desiderata for realism outlined earlier. Other possible factors might be our upbringing, or the social constructs that seem to inform our perception. The list goes on.

In order to more fully articulate the first factor (that of the presence or absence of the objects themselves) and in order to further explore the questions posed, I focus on one particular response to such questions – the phenomenological response. That is, I plan to use phenomenology as a vehicle for my own ideas. I do not plan to give an exposition of the phenomenological approach, per se. Rather, I offer an outline of my own interpretation of those sections of phenomenological theory that seem to aid expression of my own theory. Phenomenology, then, is largely treated as a framework upon which to hang my account. The same approach is taken toward ideas dubbed 'Fregean'. Again, these ideas are not supposed to be an exposition of Frege's philosophy. In sum, I use the terms 'phenomenology', or 'Husserlian', or 'Fregean' simply to acknowledge the inspiration for the ideas presented, and the authors whose original ideas I am reinterpreting, and nothing more.

Returning now to the question 'upon what does our understanding of ordinary objects depend?', phenomenology's short answer is 'consciousness'. Everything, according to Husserl, depends on our individual consciousness, or on our own undeniable ego. Before examining phenomenology's longer answer, note again that I will not be following the theory of phenomenology as a whole or per se. For a start, phenomenology treats abstract and ordinary objects differently.[1] By contrast, I want to map the phenomenological approach regarding ordinary objects onto a realist approach to abstract objects. Note that my argument will not seek initially (or perhaps even primarily) to show that the phenomenological approach to ordinary objects *is* in fact directly applicable to abstract objects (and, in accordance with this, that that approach is an analogous 'feature' of the ordinary with the abstract), but just that it is at least as reasonable as it is not to start with an initial supposition that it is, and *if* we suppose it is, then we arrive at an expression of a realist belief (or at least a belief that abstract objects are *as* real as ordinary ones) that avoids some of the problems outlined earlier, classically associated with realist positions in general.

[1] For a good explication and defence of this claim, see Tieszen's "Mathematics", in Smith [1995, 438-463], specifically pp. 447-449.

5.6. A Phenomenological/Fregean Approach

Now for phenomenology's longer answer. Ordinary material things (physical reality, all of 'nature', etc) are, according to this theory, 'given to' consciousness, and so, in a certain sense depend on consciousness for their 'being'. There is a distinction to be drawn here between our consciousness that is part of nature (or tied to ordinary material things such as our brain and body) and the consciousness which phenomenology studies. The latter is independent of nature. It is, according to Lévinas, [1998], "philosophy's starting point", and it starts by "exclud[ing] all propositions which have nature for their object" (p. 20), or which have anything other than this pure, transcendental consciousness as their object.

The place to begin then, according to Husserl, is with consciousness. In particular, with its aspect of being "the relationship to an object", the aspect called "intentionality" (p. 20). And yet, considering intentionality itself as the object of study brings into play almost every other factor upon which we might suppose our understanding of ordinary things to depend:

> Intention, the relation of consciousness to the object, is not an empty look upon the object whose sole function is to supply that object ... in order to study how the object is given to consciousness it does not suffice to study this 'empty look' (p. 20).

The factor now drawn into play that I'm particularly interested in is that of the presence of the ordinary thing itself. If 'relation to an object' is the primitive phenomenon that phenomenology studies, then so is the object of this relation. This is made clearer when we consider the 'empty look': that aspect of the relation consisting of the various attitudes we, the ego, might have toward the object. If this aspect were all there was to the relation, then joy, desire, judgment, perception, representation and the like would be the only possible factors phenomenology introduces as candidates upon which our comprehension of the object might depend. But these attitudes, according to Husserl, all have "the function of giving the object" and "it is precisely for this reason that the problem of knowledge - the study of the relationship to the object – opens up an infinite field of research" (all quotations p. 21).

Husserl goes on to say that the 'giving of the object' in a complex of intentions gives rise to such 'constitutional problems' as "how these intentions are bound together when the object constituted by them is given as *existing*, as known with *reason*, and what the acts are that give it as pure appearance" (p. 21). Here Husserl makes a point similar to one Frege (from Coffa [1991]) made earlier: "that while understanding does involve "giving" the object in question, it need not necessarily be "given" in the mode of acquaintance" (p. 81). Husserl's similar point is that there is a distinction between objects given *as existing,* and objects given as 'pure appearance'. This distinction invites a problem: which are the acts that transcend themselves and reach their object? The problem Frege faces, on the other hand (and put in Fregean terms) is: "under what circumstances does understanding take place (or how do we distinguish apparent propositional understanding from the *real thing*)?" (p. 81).

Frege, in his solution to the latter problem, assumes the independent existence of the 'real world' of ordinary material objects, and then asks how this existing world is related to what we understand: that is, he offers an exploration that emphasises the importance of the (assumed) *existing* object to a 'real' understanding of it.

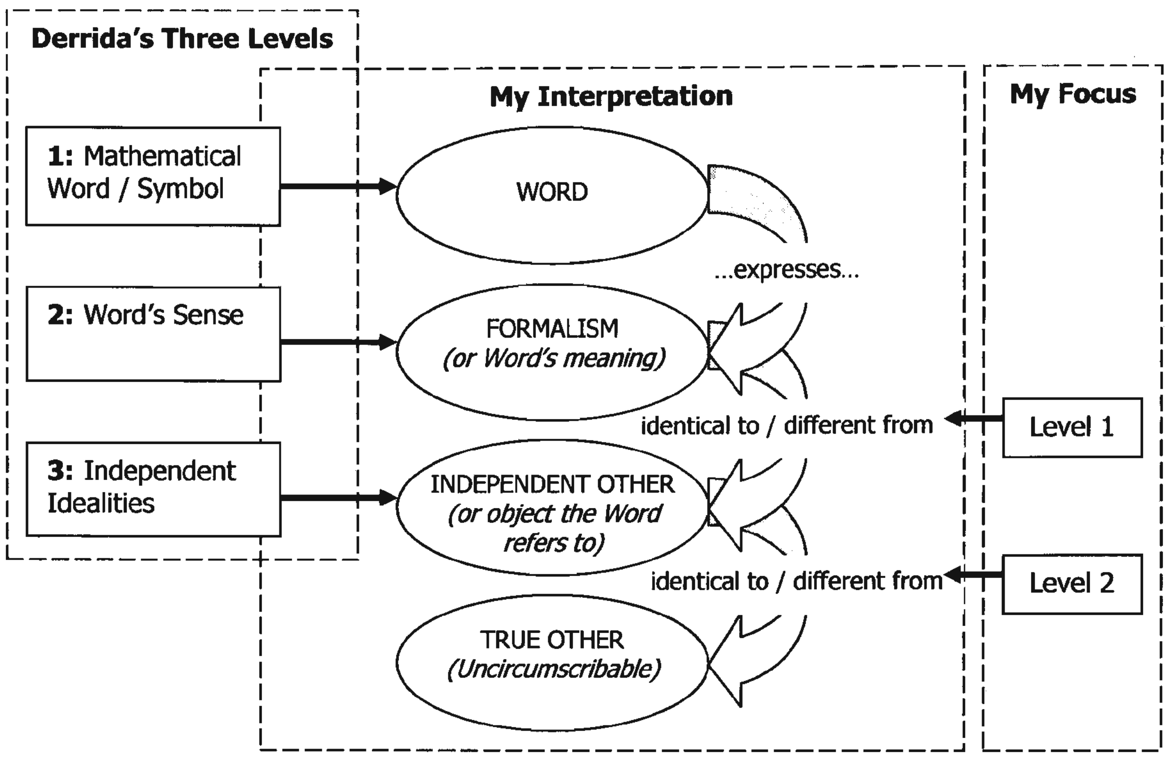
Husserl, in his solution to the former problem, examines the ways in which the 'real world' of ordinary material objects is given to consciousness *as existing*. That is, he offers an analysis of the nature of the existence assumption itself. So, Husserl's theory seeks to establish just when the Fregean assumption is justified.

I take Husserl's solution to the first problem as a foundation for Frege's solution to the second. It is possible thereby to construct the theory that real (or successful) understanding of an ordinary object depends on (at least) two things: (1) the assumption of an object whose existence is independent of our understanding, and (2) the accuracy of that assumption, i.e. upon the object *justifiably* being given to consciousness *as* existing independent of our understanding. According to my theory, then, understanding depends on there being a link of some sort between our understanding of something and that particular thing itself. If this link is to be a link between two different things (ourselves and reality) rather than a link between ourselves and the constructions of our own minds – a kind of metamorphosis of ourselves *into* another reality – then the link itself has to depend on the thing actually being there, actually existing independently; even of its role in the forgoing relationship as 'the thing itself'.

There are two relationships present in the foregoing proposition. To see this, recall that I have previously talked of 'levels', outlining (Derrida's) three levels: (1) 'the word's ideal objectivity' (the recognisable physical word or mathematical symbol); (2) 'the word's sense' (the meaning, or what I call the mathematical formalism) and (3) 'independent idealities' (what I have called independent reality). The two relationships to which I now (and from now on) refer are those between levels (2) and (3), and between (3) and 'the other'. In what follows, the 'first level', or 'level 1' is the place wherein the first of these relationships occurs, and the 'second level' or 'level 2' is the place for the second.

That (3) and 'the other' are two separate things can be grasped, at least intuitively, by noting that what we call or perceive as independent reality is not necessarily 'what is out there' at all. We have to assume at some point (if we're realist), that there is independent reality, and that we can name, perceive or relate to it – but I argue that we should not suppose that 'independent reality' simply is reality itself, as it exists apart from us. What is truly 'other' than us cannot be contained in our own assumptions. And yet, of course, those assumptions are vital to any realist philosophy, and indeed to our ability to exist in a world containing 'others' at all; and so I affirm both that the 'other' cannot be contained in our assumptions nor grasped by our comprehension, and that it must be.

There is, on the first level, a relationship between what is accessed by our understanding and the thing itself. But there is a second level, occupied by the relationship between the thing itself *of* the relationship on the first level and the thing itself, *simpliciter* – i.e. between the *assumed* or circumscribed existing object and the independent, actual existing object. I will argue that successful or real understanding and, by extension, understanding in general, depends on the possibility of the relationships on both levels being identity relationships (with some important provisos). That is, I will argue that successful understanding takes place when our understanding of a thing is identical to the thing itself, and the thing itself is identical to the assumed 'thing itself' of the previous identity.



(Recall that 'formalism' can be seen as Frege's 'thought', and 'the independent other' can be seen as Frege's 'referent' – provided, in the latter case, that one reads Frege as ascribing independence to the referent.)

5.7. Identity

Phenomenology goes some way toward clarifying these prospective identities. Although the phenomenological reduction (of which more will be said in later chapters) initially excludes ordinary things as the object of study and focuses instead on consciousness, the things themselves are "recovered" in the phenomenological "attitude" *as* whatever they were for consciousness. Their specific "way of being an object" (all quotations Lévinas [1998, 22]) now properly becomes the focus of study. The object as perceived, judged, desired etc. is called the *noema*. The *noema* is both different from, as well as inseparable from, the act of consciousness (called the *noesis)* in which it is given. This idea, of the co-existence of difference and identity at this level – the first level – is important to my theory. In particular, I apply the idea on the second level as well, to clarify the phrase 'the independent existence of the object'. That is, I take the phrase 'the independent existence of the object', when applied to abstract objects, to mean much the same thing as 'the object pole of intention' when (phenomenologically) applied to ordinary objects. The 'object pole of intention' is different from, and yet in an important sense, identical to the *noema.* According to phenomenology, the way ordinary objects are *given as* existing is both the same as and different from the existing object itself. This is at the level of the object. But in order to understand the co-existence of identity and difference, or the idea 'same yet different' at the level of the object, we need to briefly look again at the same idea as it appears at the first level – the level of consciousness, or of the object as perceived.

Lévinas [1998] puts it this way:

> The world, the objects of the natural sciences and ontologies, excluded by "reduction", reappear – "in quotes" as Husserl puts it – within the immanent sphere of consciousness, where they are studied as *noemata* ... But this means that the study of consciousness permits us to grasp the way of being of each category of objects in consciousness, and thus to study the meaning of the existence of things....*Everywhere the noesis is given with its intentional correlate, the noema: the noesis of desire with the noema "the desired as desired", the noesis of willing with the noema "the willed as willed" etc. A study of the nature of consciousness shows a rigorous parallelism between the noemata and the noesis in all the domains of consciousness* (pp. 22-23, italics mine, apart from those underlined).

Lévinas' ideas offer a position from which to further clarify the term 'understanding' and to separate it from the elements upon which it might depend. I take 'understanding' to be the 'act of consciousness' itself – the *noesis* of the quotation above. But that is not the end of the story. Recall that the phenomenological theory is here being taken only as a foundation. The framework takes its lead from Frege, and as such it already assumes what phenomenology seeks to establish – the existence of an independent reality. [2]

That is, phenomenological theory has established that there is 'a rigorous parallelism' between our 'understanding' of ordinary objects (noesis) and the way in which these objects are given to consciousness. I add a Fregean framework to this result by supposing that there are ordinary objects already existing. I also suppose that the existence of these ordinary objects is what *produces* this 'rigorous parallelism' and that this occurs by dint of the fact that their existence is independent of the existence that is given to consciousness (that which I call the mathematical formalism – and Husserl calls 'noemata').

5.8. The Bare Bones

According to my account, then, the 'rigorous parallelism' between noemata and noesis occurs when there is a *prior* identity relation between the noemata and the independently existing object – i.e. when the noemata *is* the independent object itself. In other words, my approach takes the focus off what is given to consciousness (the 'constituted' object) and shifts it, instead, onto the object itself. The object is taken as existing independently of the 'constituted object' given to consciousness, but (when successful understanding takes place) being identical with it. Although this is a move that phenomenology in some sense disallows, it is phenomenology that provides the means by which such a move can be best described and justified. And it is upon the assumption that this move is justified that my theory can be built.

Returning, then, to the (now foundational) level of the object simpliciter. Note that the predicates that form the noemata's 'nucleus' (that which is common to different acts of consciousness directed to the same object) are necessarily predicates *of* something. This something phenomenology introduces as "the object simply" – that which "cannot be omitted by a description that would be exact" (p. 25). It is

[2] Specifically, my theory is only Fregean insofar as it is a theory of how *real* understanding depends on the object's *real* existence itself. This is not to say that, for Frege, sense depends on reference, but that his sense fixes a referent – that there is, for Frege 'a thing meant': "The regular connection between a sign, its sense, and what it means is of such a kind that to the sign there corresponds a definite sense and to that in turn a definite thing meant" (Frege [1984, 159]).

specifically this - the object simply – that is called the "object pole of intention" (p. 26).

I take phenomenology's 'object pole of intention' to be the independent object itself. So the independent other can be thought of as the object of the predicates forming the noemata's (or the formalism's) nucleus. And it is this 'independent object' that can be identical to that which is given to consciousness – the formalism – (although this identity has provisos, and the identification itself needs to be justified – more of this later). Put another way, the independent object can be identified with – i.e. literally identical to – the noemata (more specifically, the noemata's nucleus). And it is when this identity can *justifiably* be drawn that further identities can also be drawn, using this identity as their base. The resulting account then has the capacity to connect the 'real world' with our understanding of it.

So, the pieces are now in place to develop a theory of how we understand when we (successfully or 'really') understand an independently existing object. Particularly, we can develop a theory of how this understanding depends both upon the *reception* of the independent object itself, and our *perception* of it *as* an independently existing object. Real understanding, according to Husserl, "can only consist in the claim of not only positing [the] object as existing, but of having a right to do so" (p. 26). This is about *reception* of the independent object itself. This question asks when the identity relationship holds at the second level – between the object simpliciter (or 'the other') and the object within the grasp of our understanding (or the 'independent other', which is able to be identified with 'the formalism').

As Lévinas puts it, "The question confronting us now is: How is the act characterised in which this claim is justified? In what does this right of consciousness to posit its object as existing consist?" (p. 26). This, in fact, is exactly how I have interpreted Gödel's problem of justification. This interpretation directly determines just what counts and does not count as a solution to the problem, just because the problem itself can only be expressed within a framework that differentiates our own understanding, the thing understood (or collective understanding – a.k.a. the formalism), the independent object, and the other – (that which is uncircumscribable but also identifiable as the independent other).

Ultimately, the problem does need to ask the question, 'does this independent object exist?' But I believe that the answer to such a question is something we can only guess at, so, in the end, we can not know or prove that a mathematical claim is justified. What we can reasonably present, though, is what we and the world, and to a certain extent, mathematics, would look like if (most) mathematical claims were just that – justified. What I am interested in is the role of the assumption that the other exists. Phenomenology leaves a space where 'the other' might be; it shows us what we look like in relation to an other and of what, precisely, that relationship consists. It includes the other as a denizen of (our) real world, as well as excluding it from our world. It allows both that we 'access' and do not 'access' the other. My more Fregean sentiment simply assumes that the space is filled. And my more Fregean point is that if the latter sentiment is correct, then phenomenology is complete. From being a picture of what could be, it becomes a picture of what is.

Frege (Coffa [1991]) introduced the realm of sense as a response to the two major problems of semantics: "what is it that we understand, and under what circumstances does understanding take place?" (p. 81). Sense, for Frege, was what we understand. So, in accordance with the above, sense is 'the noemata'. In opposition to the phenomenological model though, sense was also for Frege a way (though not necessarily an effective way) of reaching the referent, or the 'object itself'. Sense, in Fregean terms, should ideally select the truth-relevant elements of 'the real world' that correspond to the grammatical units of a claim. On this sort of schema, when the sense of a claim does in fact correspond to its referent, then *real* understanding takes place. And this is opposed to the *apparent* understanding that takes place when the proposition's sense does not correspond to its referent. So, *real* understanding, Frege-wise, has more to do with the noemata than the noesis of phenomenology – specifically with whether the noemata is filled or empty (or, in my terms, with whether the independent object exists, and so can 'fill' or be identified with the formalism). And so recruiting Fregean notions enables the retention of an emphasis on the independent object itself (with which we began), and in this way the importance of that independent object is reasserted.

6. Meaning

6.1. Three - Tiered Identity

The object itself is also of primary importance for the theory of meaning in my account. I have already gone some way toward establishing the core of the theory of meaning I put forward in this chapter: namely that the presence of the independent object is a prerequisite for the presence of real meaning. In what follows, this idea is elucidated and enlarged upon.

The previous chapter put in place the relationship between the object of our understanding/the formalism and 'the thing itself', or 'the thing simpliciter', which, when it exists and is present, is also the thing that exists independently even of its role in the foregoing relationship *as* 'the thing itself'.

In particular, the previous chapter put forward the theory that this relationship itself – and indeed the possibility of such a relationship – depends on the *existence* of an independent other – a thing wholly outside its role (or name) in the relationship. Beside 'the thing itself', names or descriptions of this role tabled thus far have included phrases such as 'the referent', 'the independently existing object' and 'the object pole of intention'. Further phrases might be added to the list: 'the unnameable', 'the self-sufficient', 'that which is outside of language', and so on.

My theory has proceeded on the supposition that despite its independence of all we might say or even think about it, the existence of such a thing is nonetheless a necessary precondition for the meaningfulness of all we might say or think about it – in particular, to the relationship between it and any role it might play in our language and our thought.

Two levels of operation have been identified and, at each level, the possibility of a relationship between two separate things was established. Put briefly, there is a relationship at the first level between what we understand – the formalism (the object of our comprehension, or what we comprehend as mathematics), and the 'thing itself'. The 'thing itself' here is not necessarily the (fully) independent other per se, but it is something like (to borrow a phrase from Shapiro) the 'office' of the independent other. It is the place where the thing itself is, only when the thing itself in fact does occupy the office we have supposed for it, i.e. when there is, and we do indeed access, an independent, wholly other reality.

And so, the second relationship (i.e. the relationship on the second level), is a relationship between the office of 'the thing itself' and its potential occupant, the thing itself simpliciter. This can be thought of as the relationship between the assumed existing object and the actual existing object – the object whose existence is strictly independent, particularly of the office it can occupy.

That is:

- At the first level, there is a relationship between understood, or comprehended mathematics – the formalism - and the 'independent object'.
- At the second level, there is a relationship between the 'independent object' and the (uncircumscribable) independent object.

The formalism is the office of the independent object, and the independent object is the office of 'the other'.

That is:

[[[ ]]] = | The formalism contains the independent other contains/(is the space for) the uncontainable. |
|---|

On my model, the suppositions that mathematics is correct and that mathematical reality is independent, taken together amount to the resolution of these two relationships into identities.

The relationships are identity relationships.

That is:

[ ] = | 1(the formalism)*1(the independent object)*1(the other) |
|---|

That is, on this interpretation, when a mathematician has 'got it right', the office 'the thing itself', is occupied, and so just *is* the thing itself simpliciter. That is, the office just is the independent mathematical reality meant or indicated by the office. Further, the same mathematician, if his realism is in accordance with the realism put forward here, would also, in supposing he 'got it right', hold that his understanding of 'the thing itself' is the same as 'the thing itself': the independent object fills the formalism, and the uncircumscribable 'other' fills the independent object. In this case, all three may be identified as 'the same', and the two pertinent relationships are identities.

Regarding this relationship, the previous chapter put forward the theory that the object we understand/perceive/intend, etc. might be identical to the 'independently existing object', but with the important proviso that even the possibility of this identity depends on the independent object's self-sufficiency. But, the *possibility* of 'an identity relationship' holding between what we understand and what is independent of that understanding, can only arise if the identity of that relationship is *not* identity or, at least, not only identity. That is, something is needed to differentiate the identity proposed here from identity per se. This needs to be done, not by changing the ordinary notion of identity into something more palatable, but by saying something more.[1] Generally, an identity holding between two things is not in fact a relationship between two *separate* or different things – rather, the situation is one of two separate names, or ways of approaching, one and the same thing. The mathematical realist's situation outlined above is, by contrast, that of three *different* things that can be (and, when mathematics is correct, in fact *are*) identical. Note, though, that for the realist, identity holding between them does not mean the three are in fact one and the same thing. So, for my mathematical realist,

[1] I am using the ordinary notion of identity here, with all its standard properties – e.g. if K =G, then everything that holds true for K holds true for G. Watering down or tweaking identity invites representational and correspondence realisms into the picture, along with all the troubles therein – but, by the same token, if we stick only to identity as we know it, something is missed. One of the things that gets missed is the difference between a trivial and an interesting identification. Saying 'I am me' is not as interesting as saying 'I am matter' – but in both cases, ordinary identity applies. I don't think that the information carried by the latter can be expressed simply as the provision of a new name for an old object. (Pinker [1997] gives a thought experiment that lends some intuitive credence to the idea that ordinary identity misses something: "suppose an infinite white plane contains nothing but two identical circles. One of them slides over and superimposes itself on the second for a few moments, then proceeds on its way. I don't think anyone has trouble conceiving of the circles as distinct entities even in the moments in which they are in the same place at the same time" (p. 116).)

the mathematical formalism itself (or the office of the mathematical object itself), the independent mathematical reality accessed by that formalism, and that reality as it exists in and of itself, beyond the reach of any theory or name, are always three *different* things, *even* when they are identical. This is not as extraordinary or as radical an idea as it might at first seem, particularly for the philosopher of mathematics. The '2' of the natural numbers is not simply and only identical to the '2' of the reals. The claim that they are identical (at least for all mathematical intents and purposes) is indeed true, but so is the claim that they are different (even if only because they inhabit different structures).[2] There is also, of course, an entire literature covering just these sorts of problems – or 'inconsistent identity', under the general heading 'paraconsistency' (see, for example, Priest [2002, 287-393]).

To elaborate further: the thing that prevents this identity from being an identity only, is the same thing that gives rise to the possibility of 'an identity relationship' between the three levels in the first place – namely, that the independent thing is something different from anything we can hope to contain in a name or a referring term. It is independent in the sense that it in itself does not exist contained within the confines of our understanding – even our understanding *of it.*

This might seem to amount to the theory that the formalism, and the independent object can not, after all, really be identical to the mathematical reality that exists independent of both these things. I believe that it is indeed possible to build a theory according to which we can maintain that an identity does in fact hold. But, due to such considerations as those above, I ultimately arrive at the conclusion that this identity must paradoxically be accompanied by its opposite – i.e. a statement that the three are different – if it is to reflect the mathematical realist's belief. This will be discussed further in the next chapter.

[2] Although there is no determinate answer to the question of whether the '2' of the natural numbers is identical to the '2' of the reals, mathematicians do not habitually refer to the two '2's as different things. The potential difference between the '2's, and their potential sameness are both (mathematically) viable ideas. This is a direct and practical use of an identity paradox.

6.2. Two-Levelled Dependence

The idea so far is that the possibility of *any* sort of identity at all between the office of the thing itself and what exists independent of that office, itself depends on there being an independent object in the first place. That is, without the existence of independent place holders for this office, no identity can be drawn between the office and the existing 'other' (this is at level 2). In this section, I will explore the relationship between our understanding of x and the office of x (the level 1 identity). I will then explore the idea that this (level 1) relationship is in fact an effective model for the (level 2) relationship between the office and the independently existing object.

To anticipate, my model has is that on the second level, the office of the independent object depends for its existence on the existence of the independent object. And, on the first level, the existence of what we access when we refer (or intend to refer) to the thing itself – i.e. the formalism – depends on the existence of the *office* of the independent object. The formalism is empty or full (i.e. it has meaning or it does not) depending on whether or not it successfully refers to (or is identifiable with/filled by) the object itself.[3] The object itself is empty or full (exists or does not) depending on whether or not it is identified with/filled by the uncircumscribable other.

These ideas have an important corollary, one which sets them apart from similar previously tabled theories of their kind in the literature. It is that ultimately, our referential activity or statements depend for their success on the existence of something entirely independent even from 'the referent', 'the thing itself', or 'the independent other' (i.e. independent of the thing being referred to). My proposition is that this corollary, as problematic as it is, need not be discarded as too intractable or too difficult, at least not before weighing the consequences of its absence against the difficulties of its presence.

[3] Note that this is where my 'meaning' differs from Frege's 'sense'. For Frege, although the sense fixes the reference, it is not determined by it. That is, the sense can give the reference, but the reference cannot give the sense. So, for Frege, whether or not a sentence has sense does not depend on whether or not it has a referent. (For more, see Frege [1970].) My account on the other hand, says that whether or not a formalism has meaning does indeed depend on whether or not it has a referent (or, in my parlance – an independent object) since meaning 'resides' on the level of the independent object, and not on the level of the formalism.

We have already touched on one of the consequences of the absence of this corollary, namely that mathematical realism – particularly in a modern account – tends, under scrutiny, at best to be able to be aligned with, and at worst, to be indistinguishable from what are meant to be totally opposite philosophies. A mathematical realist account (without this sort of corollary) that sets out to defend or to articulate the belief that mathematical objects exist independently of our ability to comprehend them, can wind up virtually indistinguishable from accounts defending the belief that mathematical objects are dependent on our ability to comprehend them. The accounts previously discussed have run this risk to varying degrees (in fact the risk ultimately attaches to any account attempting to articulate the idea of an independent reality. But perhaps this is just another way of saying that the idea of independence is notoriously difficult to articulate).

Tieszen's 'Intuitionism' and Wright's 'Minimalism' and 'Neo-Fregean platonism'[4], are also examples of realist-sounding language which can be appropriated (to varying degrees) for non-realist, and even anti-realist ends. I have previously noted that the appropriation of mathematical realism in particular can itself be accomplished in much the same way as Blackburn's 'Quasi-Realism' appropriates realism in general.[5] In fact, all realist accounts without an explicit principle highlighting the importance of an independent reality are vulnerable to a hybrid of 'Hale's dilemma' whereby the absence of such a principle itself begs a question, or the account is vulnerable to charges normally only effective against constructive accounts.[6] Put another way, the hybrid dilemma runs as follows: if an account is realist, then there is a problem caused by the positing of an independent 'other'. If there is not a problem caused by the independent other within an account, how can that account be realist?

So far, I have used the phrase 'depends on' or the notion of dependence to express the corollary under discussion (that our referential activity depends on something outside the thing itself). The previous chapter goes some way toward clarifying this notion and this chapter will attempt to clarify it further. That is, a specific aim of this chapter is to set up a framework (regarding level 2) whereby the existence of

[4] Tieszen [1989], Wright [1987], Wright [1992].
[5] Blackburn's arguments for Quasi-Realism appropriation are given in his [1993].
[6] Again, for an account of 'Hale's dilemma', see Blackburn [1993].

the thing itself (i.e. the *office* of the thing itself) depends on the independent existence of something other than 'the thing itself', just because the two (the office and the independent reality) are inextricable. Likewise (regarding level 1) I aim to show that whether a referential statement or activity has meaning depends on the existence of the office of the thing itself, just because 'meaning' and this office are inextricable. The idea, then, is to elucidate the notion of dependence via an elucidation of this notion of 'inextricability', particularly, in this section, as it pertains to meaning and referent (or the independent other), or in other words, to meaning and to what is meant.

6.3. Inextricability

When two things are inextricable from one another, the two may be taken together – indeed, in some sense, the two have to be taken together – to form a third thing made up of the combination of the initial two. When we speak of one part of this combination, the other is in some sense, included, since one of its properties is that of inextricability from the other. Speaking of, identifying, referring to, indicating and so on, one part of the combination successfully, involves speaking of, identifying etc. a thing of this nature – a thing with the property of inextricability from the other.

This does not mean that anything which may be said about one part of the combination may be said about the other. It means only what has been said, that each has as a fundamental part of its nature, inextricability from the other. So, speaking of, indicating, etc. something which (say, it turns out on further inquiry) does not have the property of inextricability from the other, is in fact speaking of, indicating etc. something else altogether. The property of inextricability with another thing is one of the defining factors we may use to identify just what it is that is being spoken of.

6.4. A Theory of Meaning

For an example of inextricability, imagine hearing someone's voice, but not seeing the person calling out. If Joe stands outside my room and calls out, then I hear Joe's voice. I can properly say that I hear Joe just so long as he himself is physically present producing his voice. In this case, hearing Joe's voice is inextricable from Joe himself.

Notice, though, that Joe can extract himself from the picture. He could have set up a tape recording of his voice and placed that outside the room, removing himself to another area altogether – say, an area too far away for me to possibly hear him (or, having recorded his voice, he may have died). In this case, I could not say I was hearing Joe, at least not without further explanation. That is, the event 'hearing Joe' depends on Joe being present outside my room – but Joe's presence or absence does not depend on the event 'hearing Joe'. The dependence only goes one way.

To extend the example a little, note that we can only explain Joe's absence by some kind of trickery on his part, or mistake on my part. All things being equal, when someone calls me from outside my room, I will quite naturally and properly presume that that person is indeed present, producing his voice at that moment.

We might go so far as to say that I cannot fully understand Joe's absence without this initial natural presumption of his presence. If I fail to make the presumption in the first place, I am missing the simplest explanation of my experience of the event 'hearing Joe'. On the other hand, if this natural presumption proves wrong, the event 'hearing Joe' appears to lose some of its inherent meaning. The only meaning left in the event is now a sort of pedantic, technical meaning, likely to frustrate further communication.

Regarding mathematical objects, I suppose that the relationship between meaning and 'referent' is similar. The mathematical formalism, on this model, is akin to the event 'hearing Joe' and as such is inextricable from the office of the independent object, or 'the referent', or the thing that we presume is producing the event. That is, my presumption that Joe is present outside my room is equated here with my presumption that there is an existing referent for a given mathematical formalism.

Even in the case where Joe is not present, my experience of the event 'hearing Joe's voice' can properly be said to have everyday or natural meaning only when Joe is presumed present. In the same way, a given mathematical formalism can properly be said to have meaning only when a referent is supposed. If, upon further investigation, it is discovered that no referent exists, or is not present, i.e. if there is in fact no independent existing mathematical reality producing the formalism, then the meaning of the mathematical formalism is changed. The formalism may still be said to have *some* meaning, but it is no longer the natural meaning with which we were working, all things being equal, to begin with. In the case of hearing Joe's voice, the natural meaning is replaced by a more shallow or pedantic meaning – one needing special clarification before the event can be communicated effectively: "I heard Joe outside my room" no longer suffices. An effective communication of the event must add "but in fact it was not Joe himself that I was hearing". Without the addendum, the natural presumption of Joe's presence properly prevails.

So while a given mathematical formalism may still be said to possess *some* meaning when there is in fact no independent object, or none present, to which it refers, that meaning is something different – and (so far as simple or natural communication is concerned) less desirable – than the meaning arising from the object's existence.

To further clarify this distinction between natural and pedantic or cumbersome meaning, we need to look at the relationship between 'the independent object' (office) and the other itself (the independently existing object). Recall that this relationship is to be modelled after the relationship we've just set up between (any kind of) meaning and 'the independent object'. On this model, the independent object (office) is inextricable from the other, but the other is extricable from the independent object.

Whether or not there is objective or real meaning depends on the independent object, but whether or not there is an independent object depends on the other. Part of the nature of natural or real meaning is inextricability from the independent object, whose nature in turn includes its inextricability from the other. And so, part of the nature of natural meaning itself is inextricability from the other.

A formalism, on this account, has natural meaning (henceforth simply called 'meaning') if and only if it has meaning so defined; if the 'other' exists, and so our supposition that it exists is legitimate. Again, note that this makes the presence or absence of meaning objective. This is so because the presence or absence of meaning now depends (to this extent) on something entirely apart from what we understand. That is, something completely outside the bounds of our comprehension is now a contributing factor determining the presence or absence of meaning. The extent of its objectivity is precisely the extent to which the nature of meaning includes its inextricability from the independent object, which includes its inextricability from the other.

This theory commits its subscriber to a robust theory of meaning without eliminating the possibility of a mathematical formalism possessing something *like* natural meaning, but still turning out, or discovered to be 'meaningless' (say, upon further investigation). And more importantly, a formalism may be 'rendered meaningless' by the absence of the supposition of an (entirely) independently existing mathematical reality upon which the formalism depends, and is assumed to depend.

6.5. Frege

This theory may seem to give meaning a role similar to that of 'sense' (regarding proper names) in Frege's [1970]. For Frege, sense is that which lies between the idea and the reference (which I have called the independently existing object). To explain the phrase 'lies between' in this context, Frege gives the analogy of somebody observing the moon through a telescope. The moon itself is the reference, the 'real image' projected by the object glass in the interior of the telescope is the sense, and the retinal image of the observer is the wholly subjective idea.

If indeed we take sense to be analogous to the image on the object glass in this example, then we take its nature to be (at least to some extent) determined by the reference itself (the moon itself). But, following this analogy also gives sense to be, to some extent, determined by us – perhaps by our conventions. That is, the nature of sense may be dependent on the conventions or activities of society, just as the image of the moon is dependent on the apparatus which 'gives' that image.

So, for Frege, when it comes to ideas, the constructed or conventional sense lies, as in the example, between a reference and our idea. For myself on the other hand, reference – the independent object – depends for its nature on an entirely 'other' reality. According to my theory, sense or meaning cannot be extricated from reference without thereby altering the nature of that 'sense' altogether: the image of the moon on the telescope lens is in fact *not* the image of the moon on the telescope lens if no moon is present, and the moon itself is in fact not only or simply the moon itself; it is also a mystery. To reiterate the paradox, it is also what it is not. In the case where the moon is not present, the image is not meaning, it is something of a different nature altogether – an illusion perhaps, or a construction alone. So sense or meaning, when it is taken apart from its reference – or, when it is considered extricable from its reference, is no longer natural. Just as (in the event of the absence of the moon) the image of the moon on the telescope becomes something else, so too, taken apart from the reference, natural sense or meaning becomes something else.

This is not to say that sense cannot be *considered* separately from reference. It is to say that sense cannot be considered *as* separate from reference. Sense, provided inextricability from reference is taken as part of its nature, can be considered or dealt with in isolation from the reference, just as the image of the moon can be considered in isolation from the moon itself. But if sense is considered *as* extricable from the reference – by its nature – then what is under consideration is no longer, properly, sense at all (according to my theory). That is, my theory interprets Frege's example as follows: if the image of the moon on the glass is considered as something which by its nature is extricable from the moon itself, then we are either mistaken about the thing under consideration, or considering something which we take to be (the same as) the thing under consideration, but which in actual fact, isn't. To reiterate, considering the image in isolation from the moon is one thing, but taking the image *as* isolated from the moon is quite another. Just as 'hearing Joe' is one thing and taking that event as isolated from Joe is another.

All this means that we can *render* something meaningless, which in fact has meaning. For instance, you could render the image of the moon meaningless, even in the case where the independent moon is in fact present by, say, taking the image of the moon on the lens to be all there is, perhaps by supposing that no independent moon was present. But taking the image of the moon this way does not entail that the image has thereby in actuality become all there is to the moon, or able to be extricated meaningfully from the moon. Rather, the situation could be accurately described in one of (at least) two ways.

First, it may be that the thing under consideration is indeed meaning (the moon is indeed present, producing its image), but that meaning is in actual fact something different from what we believe it to be (recall that the belief in this case is that the moon is not present). That is, we may have it right, but not completely right. We may, as it were, have meaning in our sights, but not appreciate or recognise it entirely for what it is. Thus, in the case of a mathematical formalism, we would understand the meaning of the formalism, but not the nature of that meaning.

For example, according to one position, the Axiom of Choice means that a certain class exists, even though it cannot be characterised. For another, it means mathematical language may be identified by the inclusion (or exclusion) of the axiom. For yet another, it means that a certain class does not exist because it cannot be characterised. But for all three positions, the Axiom of Choice also means the following: that for an infinite family of non-empty classes, another class can be made up of exactly one element from each of the classes in the family. This latter meaning (which is in fact the formalism) potentially has 'in its sight' all of the three positions above. It is, I think, safe to say that this is an important but relatively shallow meaning of the Axiom of Choice. That is to say that the Axiom of Choice may be viewed, manipulated, studied and so on with only this latter meaning in hand. Again, with only this meaning in hand, the Axiom of Choice may be employed in the practice of mathematics itself – in the embarking upon of mathematics, in the furthering of mathematics, in the judging of it, the acquiring of its principles, and so on. Just as in the case of the image of the moon, that image may be discussed, manipulated and enlarged upon, all without necessarily appreciating its inextricability and dependence on the independent moon itself. So far as this point goes, I agree with the formalists that whatever real meaning mathematics may have, it can (or at least some part of it can) at least survive without. But, if indeed there is an independent mathematical reality, then mathematics done without an appreciation of the inextricability and dependence of mathematical formalisms *on* this independent reality is drastically impoverished. Indeed it is impoverished in much the same way as an observation of the image in a telescope *as* the moon, is an impoverished observation *of* the moon.

To further the Axiom of Choice example, we have called the mathematical definition of the Axiom of Choice 'the Axiom of Choice formalism'. This formalism has meaning (according to my account) if its objects exist independent of our consideration of them *as* its objects. So, the first interpretation of the axiom (that a certain class exists, even though it cannot be characterised) has, as one of its existing objects, an uncharacterisable class. If, outside (independent) of its role as the referent of that claim such an object does not, in actual fact exist, then that claim is, as an objective matter of fact, meaningless. The third interpretation above, though, *renders* the Axiom of Choice formalism meaningless.

This brings us to the other possible explanation of the above scenario (wherein the image of the moon is taken to be all there is): it could be that the thing under consideration has no (relevant) meaning at all. This would be where, in actual fact, the image under consideration is something different from the moon's image. In this case, we are not 'seeing' the moon in any sense whatsoever. The thing 'in our sight' has blocked the possibility of meaning (perhaps, to extend the analogy even further, like a smear on the lens of the telescope being taken for 'the image of the moon'. In this case, meaning is objectively present, but not noticed or accessed by us. That is, the moon does in fact exist and can be viewed through a telescope). I think that in the case of mathematics, and in accordance with the example given above, the former possibility occurs in the study of, practice of, and the philosophy of, mathematics far more frequently than the latter.

Indeed, that it is possible to identify a formalism's meaning, but not thereby to perceive or to comprehend the nature of that meaning, is an important feature of the theory put forward here. This is because I want my theory to accommodate the scenario that the widely accepted meaning (or meanings) of, say, '2' is enough to identify (or to 'have within our sight') its real meaning. This is just another version of the presumption that mathematics is correct most of the time. More specifically, mathematical formalisms are meaningful, or at least they are most of the time. That is, in accordance with the view that most of mathematics is correct most of the time, I hold that, most of the time, the formalism's objects are independently there. Therefore, most of the time, the formalisms *are* (objectively) meaningful.

6.6. Identity Revisited

Recall that just as I wish to say that the object we call or assume to be the independently existing object, can, at times, be the same as the genuinely independently existing object, so I wish to say that the formalism can, at times, be the same as the assumed independent object. When these identities hold, then the formalism has (objective) meaning. I have already noted that this entails that meaning is dependent on a genuinely independently existing object. And I have added that the widely accepted meaning of a given mathematical formalism is (more often than not) enough to identify its real or objective meaning. That is, even where the meaning of a given formalism is perceived or understood in a way entirely contrary to the theory I present here (say, for instance, a given formalism and its meaning is understood to stand alone, not dependent on any 'other' reality at all – such as my 'independently existing object', or 'the object pole of intention' – or even perhaps on the referent) what is thereby accessed – the formalism – may, in fact, have the same meaning as I have in mind – i.e. it may nonetheless still possess what I have called the real or natural meaning which, according to my theory, *does* depend on the independently existing object. All this is obvious enough, especially if you believe that the independent objects actually do exist.

This is just to say that, according to my theory, there is a fact of the matter as to whether or not meaning is present, and that this fact is itself independent of the belief, or 'idea' of the perceiver – and, indeed, of the entire body of perceivers. Because an independent object's presence or absence is an objective matter, and because meaning depends on the presence or absence of the independent object, the meaning of a given formalism is present or absent objectively.

There is, on this theory of meaning, still an element that is subjective. That is, there is an element of meaning that does depend, at least to some degree, on the social apparatus of perception and understanding (analogous to the telescope in Frege's moon example). But describing this element does not amount to 'giving' the independent object, any more than describing a telescope 'gives' us the moon. Rather, it amounts to a description of the way the object can be received, when the above identities in fact hold. The object itself is something other than its name or role in our theory, which again is something other than the comprehended object. The element that depends on a social apparatus – on us – is just our understanding of the formalism, and, to some extent, the formalism itself: i.e. the formalism, just like the image of the moon, is perceived as well as received. The meaning of a mathematical formalism is determined objectively (i.e. depending on the absence or presence of the independent object) but our perspective, or our own unique understanding of that mathematical formalism is determined by the way we are.

Our perspective is always just that – ours. And so it is always different from something which is not ours – the independent object. Different, and yet when we are 'correct', the same. But this is, in itself, a whole separate topic, to be explored next. The next chapter explores how the formalism can be the independent other, as well as that which is outside the independent other. That is, it explores the 'level 1' relationship outlined in the diagram on page 136 of this text. It presumes, for the most part, that the 'independent other' is in fact filled by the other. (That is it takes the 'level 2' relationship as identity. It assumes that the uncircumscribable other exists, and is identical to – or 'fills' – the independent other.) With this assumption in place, it asks how what we comprehend (the formalism) can be identified with what we assume to be an independent other. So, the term 'independent other', in what follows, includes 'other', or is identified with what it also is not.

7. Justification (or Mathematical Realism: The Theological Model)

7.1. Introduction

A primary aim of this chapter is, finally, to articulate and defend a particular version of mathematical realism. But rather than do this by further defining terms or elaborating on the various versions of realism already offered, I will characterise the realism I am interested in defending by defining and hopefully also beginning to solve one of the central problems such a realism encounters. This problem, already encountered in various places throughout this work, is (loosely) called 'The Problem of Justification'.

It seems to me that this particular problem itself almost entirely constitutes the realism I am interested in. This, I believe, is partly because so long as someone holds this realism, we can be quite certain that they will encounter this particular problem.

The problem itself can be outlined as follows: first, by stating three suppositions that generate the problem. These (in barest outline) are:

1. That there is an abstract mathematical realm that is wholly independent of us – i.e. independent of our beliefs, sensory perceptions, even of our comprehension or our understanding of this realm.
2. That this realm's reality is at least as robust as the physical world's.
3. That we can access this realm (despite 1) and that most of, or correct mathematics (assuming that most of mathematics is correct) is the result of this access.

It is upon the assumption of these three suppositions that the problem of justification arises. Simply put, the problem is the question "what then, (given 1-3), is the precise nature of our relationship with this mathematical realm?"

The problem is a problem of *justification* because a good answer will (ideally) show that it is this relationship that justifies our mathematical knowledge (given 1-3) – i.e. a good answer will show that or how our mathematical knowledge is correct and true. Note that I do not claim that a good answer will decide *whether* or not our mathematical knowledge is justified, but that it will show that it is. Recall that, along with Shapiro [1997], I take it "as 'data' that most contemporary mathematics is correct" (p. 4).

If we cast the same problem in terms of physical reality, an ideal solution (and one that is analogous to the ideal solution for abstract mathematical reality) would demonstrate that there is a relationship between us and physical reality such that our perceptions (by and large, most of the time) are of what is really, independently there (which itself means that what we perceive is also received from something other than us). That is, an ideal solution to the problem of justification for physical reality would show that our physical perceptions are correct and true.

So, the problem I am interested in involves this particular analogy between mathematical and physical reality, the most important feature of which is the idea that mathematical and physical objects are both types of objects to which the problem of justification can apply. One reason for the universal applicability of this problem is the universal applicability of the notion that there are things, not just things in an abstract realm, but in the physical realm as well, that are independent of our experience of them. This is the notion that there is a dichotomy between the objects within the scope of our experience and the grasp of our comprehension and the objects as they exist in themselves – namely, whatever it was that existed before our consciousness came into being, and that will exist after our consciousness has gone.

This dichotomy can be applied to both physical and abstract reality and the fact that it can, unites these realities. This common ground is an important part of the solution I propose in what follows and, as such, it deserves a closer look now.

7.2. Physical and Abstract Independence

So long as one does acknowledge something of the former preamble, and that the mathematical realism which includes the particular problem of justification outlined is an important, or indeed even just an interesting problem, one implicitly acknowledges a number of further points. Or, more pertinently, I believe that these further points are implicitly entailed in such an acknowledgment and that they ought to be made explicit. This 'ought' is, in fact, in aid of the original stated aim of this chapter – to ensure the articulation and defence of the particular version of realism I am interested in.

The first of these further points is simply an elaboration of supposition 1 – it states that the reality posited, or indicated, is indeed independent. Recall that the attribute of independence supposed by this particular version of mathematical realism, according to supposition 1, is such that the independent reality is indeed not circumscribable by any theory. Such a degree of independence means that the reality cannot simply be 'the object of our comprehension', nor simply 'the referent', nor even simply 'the other'.

It cannot be any such thing, just because the independence supposed is complete. In terms of the dichotomy drawn above, the point here is that one of the important generators of the problem of justification (supposition 1) disallows a certain move, at least at the outset, or as an early unsupported supposition. This move is the simple identification as precisely one and the same thing, of what is accessible (by our consciousness, our understanding or whatever) with what exists before and after that access. The independent mathematical reality supposed here must, if indeed it must be anything at all, be a mystery.

To reiterate the point, supposition 1 states that just what the reality that mathematics describes *is,* is literally beyond our comprehension, and so is, in this sense, a mystery.

This point, of course, raises another: what of supposition 3? That is, we need to address just what this extreme degree of independence means regarding mathematics as we *do* know it – i.e. the mathematical reality which *is* the object of our comprehension. For a start, supposition 1 means we can never suppose that what we comprehend or see or know of mathematics (supposition 3) itself simply *is* the independent mathematical realm, end of story.

Nonetheless, given supposition 3 – and the presumption that most of mathematics is correct – I suppose that direct comprehension of that which we have just stated is beyond our comprehension, must in fact be possible. Direct comprehension is the most obvious (perhaps the only) scenario able to fulfil supposition 3, given supposition 1 – since neither can be sacrificed in favour of the other (because both are generators of the problem posed). That is, supposition 3 suggests that there is no 'problem of access' (because access is assumed), so a correspondence-type theory, or indeed any theory seeking to *show* access, is not an option here. Supposition 1, on the other hand, precludes any kind of constructive account, wherein the problem of access is solved, but only by undermining this first supposition. Direct comprehension of an independent reality (which, at this stage we need only think of as some kind of 'seeing', analogous to physical perception, but which, later, I will argue amounts to a phenomenological reception of reality) enables that reality to remain independent and also to be, in some sense at least, 'the object of our comprehension'. But for the reasons outlined above, the sense in which this would be true must always preclude the comprehended, seen, perceived or intuited reality itself *being (or being strictly identical to)* the *independent* reality that happens, in this case, to be comprehended, seen, perceived or intuited.

And so it appears that, in order to maintain both 1 and 3, even direct access must always be conditional. In fact the situation amounts to this: the precluding of a strict identity between what is perceived/comprehended/etc. and what exists independently, is a condition that needs to be met before the independent reality supposed or indicated can, *in any sense,* be claimed to be directly comprehended or accessed. Or, put another way, we can say that what we comprehend just is what is independently there, only if we also maintain that it is not.

Perhaps independent reality can be comprehended reality for the particular realist who holds 1-3, just in case, and only after, the problem of justification is entirely solved – i.e. after 1 and 3 are reconciled. But exactly what sort of a scenario would constitute a solution to the problem generated by 1-3? Perhaps a scenario whereby direct access of independent reality was proved. But, even then, such a scenario could constitute a solution to the problem *only* provided more was said. That is, the problem in its original form – most notably supposition 1 – cannot be sustained against an isolated statement or solution that what we see or comprehend mathematical reality to be *is* itself the supposed independent mathematical realm, *even* if this statement or solution was proved true. This is because by itself or with no more to the story, such a solution automatically undermines the very suppositions that generate the problem it is meant to solve – which ultimately makes the problem solved different from the problem posed, which is no help at all.

A brief exploration of this conundrum irresistibly yields the following possibility for the mathematical realist holding 1 - 3: that the mathematical reality that we *do* see/comprehend/perceive/intuit can itself be said to be identical with or the same as independent mathematical reality just in case the independent mathematical reality is *not* the thing that we see/comprehend/perceive/intuit – i.e., just in case mathematical reality is still always irreducibly *other than the* object(s) of our comprehension/perception, etc.

This is a paradox.[1] But it is a paradox which serves to highlight the point from which it leads – namely that a realist for whom mathematical reality is wholly independent of us and entirely as real as we are, just *is* committed – at least to some extent – to maintaining the mysteriousness of mathematical reality.

Rather than 'mysterious', though, which can imply necessary inaccessibility, or even irrelevancy, I adopt the term 'theological' for the sort of reality with which, it so far appears, suppositions 1 – 3 demand that we deal. So, for this chapter, a 'theological' reality or object is one capable of being both within and without our comprehension *at the same time*. A theological reality or object is whatever it is that the above paradox indicates. It is something we *can* see/perceive/comprehend, etc. just in case it is also *other* than what we see/perceive/comprehend, etc.

Theology, after all, typically deals with just these kinds of objects and this kind of reality all the time. God/theological reality is an object *outside* our comprehension as the beginning and the end, "the first other", "the other par excellence", "the absolutely other" and "the limit". But at the same time, God/theological reality is *within* our comprehension as "the core of the world's reality", the word, life itself and as the creator. (All quotations from Hart [2002, 165-166].)

[1] The paradox can be seen as a 'reductio ad absurdum' of the realist beliefs put forward.

In fact, theology often grounds its accounts on just this kind of reality – just as mathematical realism is wont to ground its accounts on an independent, abstract mathematical realm. And, as the foregoing chapters have sought to show, as a fundamental foundation, mathematical reality has no distinct advantage over that proposed by faith. The two 'realities' are (at least in view of the particular angle I am taking here) equally paradoxical. To understand more fully just why this is so, it is helpful to look again at Derrida's work (as will be done presently). Derrida comprehensively argues that the fundamental foundations of *any* philosophical account are also equally paradoxical – just so long as the account posits an independent 'other' at all.

An immediate response to all this might be to suggest that the paradox is perhaps not as alarming as it at first appears, that perhaps we can soften it by adding terms like 'aspect' and 'only' – e.g. 'what we comprehend is an aspect of mathematical reality just in case mathematical reality is not only what we comprehend.' Or alternatively, 'there is more to mathematical reality than meets the eye'. But such softened versions do not, it seems to me, adequately convey the other as *really* other – i.e. the other as strictly separable from whatever it is that we do comprehend.

It seems that what is needed to solve the problem of justification is a framework which can encompass both the mysteriousness of the other, as well as its potential 'knowability' (its potential to *be* that which is the object of our comprehension). What is needed, in other words, is a framework to accommodate the paradox as it stands in its most severe form – a framework able to include an entirely 'other' (therefore fundamentally mysterious) reality that is also accessible.

An ideal solution then to the problem as posed is one which explains our relationship to mathematical objects as *theological* objects, and one (if the kind of realism described here is to be defended) which is as good as the best explanations of that relationship currently available.

It is in this spirit that the following solution is offered. It is given as a solution in which no more questions are left unanswered than in the best philosophical accounts of our relationship to any objects or reality whatsoever (in particular, to physical objects).

So, how does theology relate to the philosophy of mathematics? Not too surprisingly, it does so via strong realist claims like the suppositions above. It has long been noted in the literature that realist/platonist accounts of mathematical objects render those objects mysterious or supernatural. Historically, realists have tried to defend their accounts against such charges. I propose instead to accept the charges, and show why they are not necessarily a liability.

I begin by alternately offering a quotation from Derrida and my own interpretation of that quotation. What is not direct interpretation in what follows is an attempt to adapt Derrida's comments, broadly to the philosophy of mathematics in general, and specifically to the problem of justification I have defined above.

7.3. A Conversation/Conversion

Recall the first supposition above:

1. That there is an abstract realm that is wholly independent of us, our beliefs, our sensory perceptions, even of our comprehension of it.

This, on the whole, is a negative proposition. It supposes that there is a mathematical reality (a positive proposition), but adds that it is strictly beyond our comprehension (a negative proposition).

Now Derrida [1962] notes (and I agree), that "From the moment a proposition takes a negative form, the negativity that manifests itself need only be pushed to the limit, and it at least resembles an apophathic theology" (p. 76).

I take this to mean something like 'a negative statement can also positively outline what it denies.' So, the statement that mathematical reality is beyond our comprehension is also the statement that there *is* a mathematical reality, and at least this (albeit negative) thing can be said about it. Our comprehension forms a border outside of which mathematical reality itself must be.

I am, in following this line of thought, opening up the possibility of asking positive questions of such a mathematical reality, such as what this reality is, rather than focusing simply on what it is not. Positively considering a reality that is stipulated to be outside the grasp of our comprehension amounts, I believe, to engaging in a theological-like activity.

Derrida says:

> Everytime I say: X is neither this nor that, neither contrary of this nor of that, neither the simple neutralisation of this nor that with which it has nothing in common, being absolutely heterogeneous to or incommensurable with them, I would start to speak of God, under this name or another (p. 76).

My interpretation of this is as follows: I can say of mathematical reality that it is not the object of our comprehension, but I can also say that mathematical reality is not *not that* which *is* the object of our comprehension (the mathematical formalism). Put another way, the formalism, or that which we comprehend as 'mathematics itself', *can* be equated with mathematical reality. For instance, the two can coincide and in fact be identified with one another. This means there is at least the possibility of *a sense* in which I can say that the formalism *is* independent mathematical reality. Such, as I believe, is indeed the case. But this sense is precisely the sense in which the two are the same, yet different.

According to my account, mathematical reality is certainly not the 'neutralisation' of the formalism, although it is absolutely heterogeneous and can be (but is not necessarily) incommensurable with the formalism.

I have begun then, according to Derrida, to speak of God.

Actually, I believe that to say I am 'speaking of God' is a little too strong in this case. Nonetheless, I do still contend that the way 'mathematical reality' functions in a realist's account (that includes 1-3) is analogous to the way 'God' functions in some theological accounts of divine reality. So while I am talking of mathematics, not God, my account being realist and stipulating supposition 1, can nonetheless be seen as a type of theology.

This is simply because my account introduces as an article for consideration something strictly outside consideration – something always different from what our comprehension can grasp. For the sake of comparison, note that accounts that include incomprehensibility or mysteriousness as a feature of mathematical reality, can go in a few directions, different to that which I'm taking here. They can, for instance, be driven back a step toward what *is* within the grasp of our comprehension, say by adopting a version of direct realism (e.g. stating that what we comprehend simply *is* the independent mathematical reality and that that is the end of the story). On the other hand, such an acknowledgment might prompt resignation – an idea, for example, that what is 'outside' comprehension is also beyond the scope of any philosophy, rendered irrelevant.

I hope in my response to avoid both of the above positions, and to do so while retaining the ideas with which this thesis began – specifically supposition 1: the notion of an entirely independent reality – and so to explore the relationship between mathematical reality and our comprehension of it when 'mathematical reality' and 'the object of our comprehension' are defined by 1-3, and left unamended. The theological model offers what I believe is the most illuminating account of this relationship, under just these circumstances.

The theological problem of our relationship to God, and the mathematical problem of our relationship to mathematical reality do share a number of important features. They both *can* ascribe independence to the reality they consider. That reality is, in both cases, abstract. Both can include some version of supposition 3 – i.e. the assumption that there is a relationship between our comprehension of the reality, and the reality itself. And, in fact, both often do begin thus, and cite the prima facie value or truth of their formal systems of belief as a reason to do so.

Derrida again:

> God's name would then be the hyperbolic effect of that negativity or all negativity that is consistent in its discourse. God's name would suit everything that may not be broached, approached, or designated, except in an indirect and negative manner. Every negative sentence would then be haunted by God or by the name of God, the distinction between God and God's name opening up the very space of this enigma (p. 76).

What I am claiming, then, is that the 'very space of this enigma' for theology is analogous to the space that the problem of justification opens up for mathematical realism. Equally, though, the distinction between mathematical reality and our comprehension of it (i.e. the formalism) 'opens the space' of the problem of justification as it has been defined here. Simply put – there is a gap between God and our comprehension of God which theology seeks to overcome, understand, or perhaps simply to describe. This gap *resembles* the gap between mathematical reality and our comprehension of it which the philosophy of mathematics (particularly realist philosophies of mathematics) seeks to overcome. I do not expect here to overcome the gap – especially given the nature of suppositions 1-3 – but I hope to understand it better, and perhaps to describe it. I hope also that this will clarify a particular realist position within the philosophy of mathematics, and that such a clarification will itself present an argument for the importance of the position thereby described, at least insofar as it will show that the position is too interesting to do away with, or to ignore.

Back to Derrida: "If there is a work of negativity in discourse and predication, it will produce divinity" (p. 76).

This point can be read as a summary of the ideas presented so far. It states that if you wish to retain supposition 1 – i.e. if your account maintains that mathematical reality is something different from what we comprehend – then your account has in some sense 'produced' a divinity. You are not thereby committed to equating mathematical reality with divine reality, but the way 'mathematical reality' functions in your account is analogous to the way 'God' functions in theological accounts.

A 'produced divinity', though, is not what most mathematical realists have in mind when they posit mathematical reality. Divinity aside, the mathematical realist's reality is not produced at all. It is independent. This independence includes independence from anything we ourselves are able to produce, including the idea or name of an independent reality and, more obviously, the formalism (our own understanding of mathematics, given that it is ours and so to some extent 'produced by us', or at least produced as a result of us) – assuming supposition 1 is just one way of stipulating precisely that degree of independence. But before relegating the theological model to the list of accounts fated to an incorporation or sublimation of the other, there is another move to make within this model, which may save it as an expression of this particular realism.

Derrida: "It would then suffice to change a sign (or rather to show, something easy and classical enough, that this inversion has always already taken place, that it is the essential movement of thought) in order to say that divinity is not produced but productive" (p. 76).

This is the move I make in the preceding two chapters, and in what follows. According to my account, then, independent or mathematical reality itself is different from comprehendible mathematical reality, and part of the difference between the two lies in the fact that the former is, to some extent, responsible for the production of the latter. The relationship between the two, then, begins with production. The first produces the second. How? Primarily, I would hazard, by being the way it is. That is, mathematical reality produces mathematical formalisms in some way analogous to the way that physical reality produces our comprehension of physical reality. This is not to deny that we also produce our comprehension, and that our comprehension of a thing might in fact be far removed from the way that thing is in and of itself. It is just to say that there is production both ways. And the primary producer, so to speak, is the independent reality, not us.

Back to Derrida:

> infinitely productive, Hegel would say, for example. God would be not merely the end, but the origin of this work of the negative. Not only would atheism not be the truth of negative theology; rather, God would be the truth of all negativity. One would thus arrive at a kind of proof of God – not a proof of the existence of God, but a proof of God by His effects, or more precisely a proof of what one calls God... (p. 76).

Mathematical reality itself, according to this account, has an effect on us. Mathematics is the result of this effect. But of course, this idea – that mathematical reality is productive and effective, rather than a constructed or a passive recipient of our own interpretations – is notoriously difficult to present in a clear and comprehensive manner, let alone to argue for.

My account uses deconstruction and theology primarily in order to address precisely this difficulty. Deconstruction, in making metaphysics its foil, tends to describe, painstakingly, as part of its project, just what is involved in taking a metaphysical stance. Derrida's work does this. And it is this painstaking description I want to use - albeit for an entirely different purpose than the one Derrida probably had in mind.

Indeed, any attempt, including Derrida's, to capture the essence of metaphysics can be taken as an argument to the effect that metaphysics and theology share many features. These features can be stated very generally, so that each in fact includes a broad spectrum of beliefs, ranging from anti-realism and constructivism through to strong realism – in other words, ranging over most, if not all, of philosophy. After all, for Derrida 'philosophy' and 'metaphysics' are practically interchangeable terms.

The features themselves include:

1. The idea that there is something external to our own understanding of a given subject (most often, this is the subject itself, or a 'presence' of some kind – not necessarily a fully independent reality) upon which our understanding is based. This idea can take the form of a division between sense and referent, word and object, form and content or formalism and reality. Also within the scope of this feature is the strong realist belief I'm putting forward: that there is, somewhere, somehow, an external reality, unconstructed and ultimately unaffected by our own ideas – a reality (or a 'presence') that is what it is quite apart from, or despite, what we perceive it to be.
2. The idea is that our 'sign system' – e.g. words, symbols, mathematical formalisms etc. – is a kind of 'passage' between us and the 'reality' or 'presence' upon which it is based. The phrase 'sign system' can cover a great deal here. It includes written and spoken words, concepts, ideas and thoughts. It also includes 'text' – a term covering not only writing, but consciousness, an epoch or a society. In fact, 'text' covers knowledge in general. Essentially, this feature of metaphysics is the admission of fallibility: the idea that what we know, see or comprehend may fail to be what really is the case, really is present or really exists. But it is also, importantly, the idea that we can 'get it right'. The idea

that, given the 'right sign system', we can see, know, etc. what really exists and what really is the case.

3. The idea that a sign's meaning can be given only by referring to something external to the sign, or sign system.
4. The idea that the 'something external'/ reality/ presence/ concept/ thought/ consciousness, etc. forms the ground upon which signs and sign systems are based.
5. The prioritising of the 'something external' over the sign or sign system as a result of the preceding points.

These features do not form an exhaustive account of the nature of 'metaphysics' and 'theology', nor does a given philosophy have to include all, or even most of them to come within the scope of either term. Instead, taken together, they serve to illustrate the common tendency of metaphysics and theology to privilege, seek, or to secure a ground for the systems of knowledge they consider. The function of such a ground can vary – e.g. at one end of the spectrum it might have a justificatory function, and at the other its function might be simply explanatory. Regardless, the idea of a ground is the main similarity between metaphysics and theology.

Certainly, my account includes the idea of a ground. But Derrida's point is that most of philosophy to date does as well. Indeed, realist philosophies of mathematics in general are probably some of the most obvious or notorious examples of philosophies that include the idea of a ground. But, it is when it comes to the explication of this idea that mathematical realism, also notoriously, often flounders.

In theology, though, we have a host of possible explications of just what it is to have this idea. What is a ground? Taking our cue from theology, we could answer with what was outlined earlier: a ground is a first principle, a totalising force, the beginning and the end, etc. Moreover, recall that we could also speak of this other's role in life or ordinary reality with such ideas as: the ground is the immanent transcendent, the "very core of the world's reality" (Hart [2002, 166]), the word and the creator.

"For a limit is both within and without an event, and its uncertain status requires us to think 'experience of God' as a flaring of what is outside human understanding as well as within it" (Hart [2002, 166]).

The 'ground' in my account is the mathematical realm itself. The notion (now revealed as 'theological') that this ground is both outside and inside human understanding, knowledge, perception, etc. is a crucial component of my proposed (theological) solution to the problem of justification. Another crucial notion for my solution that can be clearly categorised as theological is that of *responding* to the mathematical realm (as opposed to constituting or constructing it) or *receiving* the mathematical realm (as opposed to perceiving it only). The notion of reception thus coalesces with the idea that the independent realm is a productive one.

In theology, such notions legitimately serve as (full or part) explanations or descriptions of our relationship to theological phenomena. But a simple substitution from talk of theological objects to talk of mathematical objects will not, I think, suffice. In order to justify my proposed employment of theological notions to describe or explain our relationship with mathematical objects, I need to show a number of things.

First, I need to show that the theological notions I wish to employ to explain our relationship to mathematical objects are at least as good as the best explanations of that relationship currently available. Second, I need to show that theological notions 'work' as a solution to the problem of justification – i.e. as a tenable, literally 'workable' explanation of the relationship between us and mathematical objects. Third, and finally, I need to show that the employment of theological notions genuinely enhances or improves upon our current understanding of that relationship.

7.4. 'At Least as Good as'

To address each in turn; to show the first point, I appeal to the foregoing conversation/conversion, which, I believe, establishes that:

a) philosophy itself is theological and consequently philosophical accounts of our relationship to anything whatsoever already entail most if not all of the mysteries and intractable open questions inherent to theology. This particularly, or more obviously, applies to philosophical realism and so my task here is made easier just because I am focusing on this part of philosophy.

There is a related argument, (b), highlighted earlier, some of which can be seen as a consequence of (a). (b), then, is the argument that:

b) no more questions are left unanswered in my account than in the best philosophical account of our relationship to physical objects. I take it as given that the best explanation of our relationship to abstract/mathematical objects is one which explains this relationship as well as the best explanation of our relationship to physical objects explains that relationship.

I have used Derrida's work in order to establish (a) in the preceding comments, and I will use phenomenology[2], specifically Husserl's 'epistemological reduction', to establish (b) (or the bits of (b) that don't follow straightforwardly from (a)) in what follows.

Note that I have employed Derrida's work only insofar as it helps to establish the fundamental similarities between theology and philosophy, but not much further. For instance Derrida argues, and I agree, that acknowledging that certain components of my solution are theological amounts only (or 'ultimately', depending on which way you view it) to acknowledging that my solution is properly a part of philosophy. Derrida also argues, but here I do not agree, that this means my account can never address the problems it seeks to address. That is, Derrida believes that the tasks philosophy sets for itself – the problem of justification included – are unachievable. I do not – at least no more than are any of the tasks we undertake as humans trying to understand what is not us (although I do believe that the unachievable is written in to realist accounts. It is when the idea of unachievability is thought to imply futility, or a complete lack of progress, that I disagree.) This brings us back to (b), and to phenomenology.

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[2] What I hope to show, though, is that it does not undermine those beliefs. I use phenomenology as a framework only, upon which to hang my own ideas. In other words, Husserl's ideas enable me to speak my own – the latter are in no way intended to be attributed to Husserl himself.

7.5. 'These Notions Work'

Recall that the specific aspect of phenomenology I wish to employ in order to establish (b) is what Husserl calls 'the epistemological reduction'. Note also that while (b) addresses the first point above, it also addresses the second. Indeed, my method will be to focus, initially, on the second problem listed above. That is, in arguing (b) I will be giving my proposed workable solution. I will then defend that solution by revisiting the first and, finally, the third point.

To establish the second point, then – (namely) that theological notions 'work' as a solution to the problem of justification – I look again at what is involved in any account of our relationship to any thing, in having that thing as an 'other', i.e., in differentiating that thing from ourselves, or from anything that can be reduced to ourselves.

We have seen that including an 'other' in such an account can be achieved in a number of ways. For example, the 'other' can simply be named or stipulated 'other'. Call this approach 'Fregean'. Frege gives the problem of access short shrift and has the 'other' (the referent) as a theoretically definable, knowable, as well as real and independent object – simply (or primarily) by stipulation. Coffa [1991] notes that for Frege the situation is obvious: "Objective representations can be divided into concepts and objects" (p. 67). Objects, although they are 'other' than ourselves, are, as Coffa puts it, "no problem for Frege – they are the tables and chairs of everyday experience, the numbers and classes of mathematical knowledge, the truth-values of his logic, and so on" (p. 67).

At the opposite end of the scale, the 'other' can be interpreted as essentially inaccessible, even impossible. In this case, the other can only be brought into an account of our relationship to a thing as a boundary or a limit of that relationship – what lies beyond. Call this approach 'Wittgensteinian' or 'Kantian'. Wittgenstein, for example, (according to Coffa [1991]) "banishes to the ghetto of suspect notions" anything, including the notion of an 'object', "intended to apply to things out there in the world". All reference to the 'real world' is "studiously avoided" (both quotations p. 242). This treatment of 'other' amounts to ruling it out altogether. The idea here is that we can never transcend experience. So, if there is an 'other' separate to our experience of it, it is impossible to 'reach'. Talk about such a thing is in fact talk about our experience, and so not about an 'other' at all. Essentially, this position holds that "the world we live in is the world of sense-data; but the world we talk about is the world of physical objects". That is, "mind and matter is a division *in* experience" (both quotations p. 245).

There are well known problems with both these positions, but of specific interest to me are those problems that a realist might identify for each position. Bear in mind that the 'realist' here is one whose position incorporates the three suppositions given at the beginning of this chapter (similar arguments could be given regarding a realist holding something akin or derivative of the same three suppositions), and for whom the problem of justification arising as a result of these suppositions is a real, if not essential, question.

Such a realist might have some problems with the positions outlined above. One important problem they might have with the former position – i.e. with taking the 'other' as a Fregean-type referent – is that in this case not enough 'space' is left in the account for a genuinely mysterious, foreign, or theological object. A literally commonsense approach is taken to tell the whole story. Conversely, the problem with a Wittgenstienien/Kantian exclusion of the 'other' is that this position attributes too *much* mysteriousness to the other – so much so that the 'other' remains entirely unknowable, completely outside our grasp and so impossible to 'relate to' at all. This renders all of theology/metaphysics/philosophy as positive endeavours to describe our relationship with the other, impossible.

I take Husserl's phenomenology to be an account of our relationship to pure phenomena which both includes a space (or, putting it another way, leaves room for) the mysteriousness of the 'other', and tells a story of just how our comprehension or perception can accommodate this mystery.

What I have been calling 'other', Husserl [1983] calls "transcendent", "objective reality", "the objectivity of the object", "spatial actuality" and "real reality" (pp. 365-367). The realm that this variety of names refers to encompasses anything in objective reality (including ordinary physical objects), anything in an ego (including 'psychological objects' such as thinking, feeling, willing, etc.) and anything in the temporal world (including physical and psychological objects). Anything, that is, whose objective or independent existence we can doubt (via his 'epistemological reduction').

The way in which Husserl makes room in his account for this 'other' is, first of all, by ruling it out. That is, Husserl [1970] begins by aiming his account not at any kind of actual phenomena (where actual phenomena includes ordinary physical things and 'external' reality in general) but at "the descriptive character of the phenomena, as experienced by us". The 'pure phenomena' Husserl targets, though, are not the objects of individual experience, nor of any sort of experience belonging to an 'ego' or a self. They are not the objects of perception, or cognition or any psychological fact "in objective time, belonging to the experiencing ego, the ego that is in the world, and endures for a time" (both quotations p. 856).

Rather, the pure phenomena Husserl [1964] studies are arrived at by

> plac[ing] the ego and the world and the experience of the ego as such in question, then [by] reflection upon what is given in the apperception of the relevant experience, upon my ego – a reflection that simply 'sees' – [this] yields the *phenomenon* of this apperception: the phenomenon, roughly, of "perception apprehended as my perception (p. 14).

Because this 'apperception' still involves the ego (i.e. it is perception apprehended as *my* perception), there is one further step to take before we arrive at just what it is that phenomenology studies. According to Husserl [1964], the objects of phenomenology – 'pure phenomena' – are arrived at by "ignoring its [the apperception's] relation to the ego, or abstracting from that relation".

The point of this entire procedure (called 'the phenomenological reduction') is to somehow get at objects that are 'absolutely given' as opposed to transcendent or psychological: "Thus to every psychological experience [including the experience of transcendent reality] there corresponds, by way of the phenomenological reduction, a pure phenomenon that exhibits its essence ... as an absolute giveness".

It is only at this stage that the 'other' as described above, is 'ruled out': "All positing of a "non-immanent reality", a reality not contained in the phenomenon and therefore not given ... is shut off" (all quotations above p. 34)

But it is precisely this 'shutting off' or ruling out of the 'other' that also rules it in. Here's how. According to Husserl [1983], the phenomenological reduction as described above, i.e. "the description on essential lines of the nature of consciousness ... leads us back to the corresponding description of the object consciously known" (p. 359). Acknowledging this point leads in turn to the following question (and related questions): "what [does] the "claim" of consciousness to be really "related to" an objective, to have an objective "reference", properly come to?" (p. 360).

There are a number of different characterisations of Husserl's answer to this question, in his own writings as well as in the work of his readers. I will give my own interpretation taken from two such characterisations, the answer as given by the later Husserl, and the answer as given by the earlier Husserl, both of which, I believe, express the same idea.

7.6. Noetic and Noematic

The main difference between the two characterisations is simply that the later Husserl employs special terms such as 'noesis', 'noemata' and 'the object pole of intention', while the earlier Husserl does not.

Using the later terminology, the answer Husserl [1983] gives is built around a "cleavage between two radically opposed and yet essentially interrelated regions of Being ... [i.e. between] real and intentional, [or] noetic and noematic analysis" (p. 359).

His beginning point is just this. When we come to exploring 'intentionally', i.e. the nature of consciousness, specifically the relation of consciousness to an object, we find that the 'pure phenomena' of intentionality itself has two sides. There is a persistent 'parallelism' between the act of consciousness, and the object of consciousness. The former, called 'noesis' and the latter, called 'noema', correspond. That is, (Husserl says) "the essences, noema and noesis, are mutually inseparable: every lowest difference on the noematic side points ... back to lowest differences of the noetic" (p. 359). Or (Husserl [1999]) "corresponding in every case to the multiplicity of Data pertaining to the really inherent noetic content, there is a multiplicity of Data, demonstrable in actual pure intuition, in a correlative "*noetic content*" or, in short, in the "*noema*"" (p. 88).

The 'parallelism' can also be expressed as 'coincidence' (Husserl [1970]):

> In the ideal limiting case of adequate perception, this self – presenting sensed content coincides with the perceived object. ... [that is] If we may conceive of a percept put into a relation of fulfilment to the adequate percept that would offer us the object itself, in the ideally strict and most authentic sense, then we may say that a percept so intends its object that this ideal synthesis would have the character of a partial coincidence of the purely perceptual contents of intending and fulfilling acts, and also the character of complete coincidence of both complete perceptual intentions (p. 713).

The difference between perceptual *contents* and perceptual *intentions* here can, I believe, be stated in the following way. First note that 'adequate perception' is the case in which it is correct to describe a given perception of an object to be a perception of that object as it really is, in and of itself. In other words, adequate perception is where what we see *is* the object that exists independent of our perceiving it.

In the case of adequate perception, the perceptual content of the intending act and the perceptual content of the fulfilling act *partially* coincide. Husserl's 'partial coincidence' and parallelism describes the same feature of adequate perception that I have been describing with the idea that the two contents, in this case, are 'the same but different'. Husserl is highlighting just this – that the two contents are the same, indeed identical, but are also, and equally importantly, not the same.

One of the partially coinciding contents is the noema – what my consciousness brings to the object or what I perceive – the other is what the object brings to my consciousness or what I *receive*. According to Husserl, what I receive can only ever partially coincide with what I perceive because (i.e. the two contents are not the same inasmuch as) a given perception cannot completely encompass every aspect of the object itself that there is to receive. Nonetheless the coincidence is still a true coincidence because (i.e. the two contents are the same inasmuch as) an adequate perception of an object necessarily entails that object itself. If this were not so, we could never identify anything we perceive, we could only perceive disparate sensory events belonging to no unified self-identical thing.

In short, Husserl seems to be saying that when I perceive an object I am only, properly speaking, perceiving an aspect of the object. Nonetheless, I can still properly claim that I am perceiving *that* object, by which I mean the object in its entirety. Therefore in adequate perception, the content of what I perceive and the content of what I receive are essentially the same but also essentially different.

This discussion is touching a deeper distinction – namely that between the content of what I receive from the object and the independent object itself. So far in this chapter I have been discussing the 'level 1' relationship between the formalism and the independent object (see diagram on page 136 of this text) and speaking of the independent object as though it were both the assumed existing object and the actual existing object (i.e. I have been assuming that the office 'independent object' is filled). I now consider the 'level 2' relationship between the 'independent object' nameable or circumscribable in a theory, and the other itself – the mystery. Husserl's distinction above introduces yet another nucleus within the layers of consciousness already delineated, namely the thing Husserl [1983] calls the "content of the object-nucleus of the noema" or the "predicate calculus" (p. 365 and p. 366 respectively).

Now, even before exploring this last nucleus further, we have a picture of just how 'the other' *can* be something like the theological objects examined and duly admitted earlier – that is, the other can be at once inside and outside our comprehension. As such, phenomenology is one *philosophical* system in which such objects can be properly accommodated – where a philosophical system is a system wherein 'the other' can be separated from 'the object of consciousness', and a system that involves the positing of 'a ground'. Objects, both physical and mathematical, are admitted to be theological just as, according to Derrida and deconstruction, the objects of any philosophical account must be. Nonetheless, phenomenology provides a lucid and straightforward description of how we come to grasp the objects in question. It offers an epistemological solution to the problem of access for any theological/philosophical objects whatsoever. In fact the phenomenological solution covers all objects or reality not entirely given in or given by our own access.

More specific to the realism I am exploring, there is nothing in phenomenology that excludes the reading of 'the other' as something quite beyond the processes of our consciousness – but at the same time, phenomenology does allow the 'other', in some crucial sense, to *be* the object of our comprehension. To elaborate this point, more needs to be said about the 'content of the object-nucleus of the noema' itself.

This last nucleus is the group of properties properly 'belonging' to the object itself. I take it that this last 'object itself' is comparable to the uncircumscribable 'other'. Husserl [1970] writes: "[the object itself, or this last nucleus is] the synthesis of identical thinghood [wherein] the thing establishes itself through its very self, in so far as it shows itself from varying sides while remaining one and the same" (p. 712).

Elsewhere (Husserl [1983)] he says:

> A fully dependable object is marked off in every noema. Every consciousness has its "that" ... and means its objective: it is evident in the case of every such consciousness we must be able as a matter of principle to carry out a noematic description of this same objective "exactly as it is meant", through development and conceptual apprehension of our data we acquire a definite system of predicates either form or material, determined in the positive form or left 'indeterminate' – and these predicates in their modified conceptual sense determine the "content" of a core identity (p. 364).

And yet, this nucleus itself is still not 'the other', or the 'theological object' I am proposing as 'the other' in a mathematical realist's account. It cannot be. Rather, Husserl adds, this nucleus has "set alongside it" or "not separable from it" or "belonging to it" or "disconnected from it" or "detached but not separable" that which *can* be thought of as 'the other' in a realist's account – namely "that which the predicates are inconceivable without and yet distinguishable from". Husserl also calls this – "The object" or "the objective unity" or "the self-same" or "the determinable subject of its possible predicates" or "the pure X in abstraction from all predicates" or "the determinable which lies concealed in every nucleus and is consciously grasped as self-identical" or "the object pole of intention" (all quotations, pp. 365-367).

7.7. 'Getting it Right' (or, Justification – the Theological Solution)

According to Lévinas [1998], a solution to the problem of justification would, in Husserlian terms, be a claim of mathematical knowledge to truth – i.e. a claim that mathematical knowledge is true, which consists of (or even, in virtue of) "positing the object pole of intention *as* existing and having a right to do so" (p. 26).

A solution, then, could read as follows: the relationship that holds between us and mathematical reality when our mathematical knowledge is true or correct (i.e. when we are 'seeing correctly' or 'getting it right') consists both in (our) positing the object pole of intention *as* existing, and in our having a right to do so.

The former can be characterised as our role in the relationship, and the latter as 'the reality's' role in the relationship. It can be characterised this way because we "have a right" to posit the object pole of intention as existing, just when it in fact exists. In accordance with the mysteriousness of the reality concerned, the relationship between it and us goes both ways: mathematical knowledge is justified when both the constitutive and the receptive components of that relationship are fulfilled. More simply put, mathematical knowledge is justified when what we see is what is there. A realist (along with most if not all practising mathematicians) sees mathematical reality as a set, a number, a structure etc. *and* as an independently existing reality. When what is seen as a set is in fact a set, and does in fact exist independently (especially of this fact), then mathematical knowledge of this particular set is justified. And so on. Just as with physical objects. Physical or perceptual knowledge of a tree is justified just when what we see as a tree is in fact an independently existing tree and not an illusion.

Now, if we simply assume that the 'group of predicates'/'pure x'/'the uncircumscribable other' is in fact filling the office 'the independent other' (given the framework we have in place identifying this independent other with the formalism) then we have an account of exactly what 'having a right to do so' involves. It involves the existence of this particular other, upon which the 'independent other' (of the foregoing theory) depends, upon which, in turn, the formalism depends. Mathematics is justified when these three items are identical – only provided they are also different. We know now that what we can know is also what we cannot.

8. Conclusion

My account of mathematical realism includes the idea that mathematical reality can be both independent and knowable only if it can be both within and outside of our consciousness at the same time.

I have argued that certain fundamental realist premises (1 to 3 of the preceding chapter) entail this paradox and that if the paradox is resolved, the premises themselves collapse. In other words, the paradox must be retained in order to retain the premises.

I have presented an account of how we come to know mathematical reality that leaves room for the unknowableness of that same reality. I have done this using a phenomenology-inspired description of what is involved in our seeing that reality as it is – given both that mysteriousness is part of its nature, and that its separateness from ourselves needs to remain a 'theme of awareness' even when that separateness appears to be overcome (in other words, even when we can confidently say that what we understand is identical to what independently *is*).

8.1. Recapping Chapters 1 – 4

The idea of independence is built into my account, and is a fundamental part of it, rather than an added or stipulated ('ad hoc') part. In fact independence is included foundationally – it is a ground and, as such, everything depends on it, even including the account itself.

This notion is built in without its perpetual bugbear – the burden to provide a link between the independent reality and mathematics as we know and do it. This is because I have argued that the independent entity or reality is in fact identical with what we do know – via the phenomenological/Fregean theories I've appropriated for the purpose. Arguing that what is independent is the same as what we know is not new. The theories I've covered have all made a move of a similar sort (with varying degrees of identity, and of the burden to show a link). My own theory, though, is notably different, not only because the identity is recognised and made overt, but also because of its recognition of the paradox that must result from this identification. The paradox is inherent – where a realist identifies what we know with independent reality, he must also (at least in order to retain the fundamental realist notions mentioned above) maintain their separateness. In particular, the expression of a fully independent reality is compromised unless this paradox is accommodated.

I have also argued that mathematics is metaphysical if it is realist and that therefore metaphysics can provide its ground. Or, put another way, the very notion of a ground is metaphysical, so arguing that an external or 'other' reality grounds an account moves the account no further afield than positing the notion of a ground in the first place. The ground I provide addresses both the problem of arbitrariness, and the problem of justification by showing how the fundamental axioms do indeed embody independent reality itself.

Finally, I have attempted to redraw the division between 'what we take the facts to be' and 'what the facts are', associating meaning with the latter rather than the former. I have argued that, as a result, establishing a correspondence of objective fact with intersubjective meaning is not the principal problem for the realist. Instead, I argue that what we know or comprehend of mathematics is meaningful only if it is identifiable with what independently exists. Given this, 'what we take the facts to be' (the formalism) is itself meaningless without this identification. So the primary problem for the realist is that posed by the notion of independence – specifically, in this case, that posed by an independent reality upon which meaning depends.

8.2. A Final Word

In his [2000], Shapiro writes,

> The job of a philosopher is to give an account of mathematics and its place in our intellectual lives. What is the subject-matter of mathematics (ontology)? What is the relationship between the subject-matter of mathematics and the subject-matter of science which allows such extensive application and cross-fertilisation? How do we manage to do and know mathematics (epistemology)? How can mathematics be taught? How is mathematical language to be understood (semantics)? (pp. 15-16).

As a brief final defence of my view I will take each of these questions in turn, and look at how they are answered in the account I have offered.

According to my account, the subject matter of mathematics is and ought to be a mystery, as well as being exactly what we think it is. I have contended that the most important feature of mathematical reality is its independence, because so long as it is independent, the problem of determining (everything else about) its nature is a problem only as much, and no more so, as the problem of determining the 'true' nature of physical reality (supposing physical reality is also independent). And this problem, as Husserl shows, is as impossible and as simple as seeing what is really there.[1]

There is then on my account a relationship between the subject matter of mathematical reality and that of physical reality. The relationship is defined first of all by the similarities between the two, particularly the analogous ways in which we receive them as other than us.

I have argued that a description of how we come to know anything, including physical reality, is not complete and acceptable nor provably correct just because a causal story can be told about how we came by that description, even if that story involves all of the five senses. Just what the five senses access, how they do it, and whether the accessed reality is reality as it is, in and of itself (independent of the existence of those five senses), is as complex a problem as that of defining mathematical intuition. Accordingly, constructing an account of how we interact with an 'other' is just as simple or complex a feat when it is about physical reality as it is when it is about abstract reality. In either case, we receive something – stimuli, pattern, structure, data, etc. and (in acknowledgment of the notion of reception) we grant that it is other than us (or we do if we are realist). And so long as there is in actual fact an independent reality producing what we receive, all of

[1] Pinker [1997] provides a vivid illustration of the problem as it looks when applied to physical objects: "of course, the world *does* have surfaces and chairs and rabbits and minds. They are knots and patterns and vortices of matter and energy that obey their own laws and ripple through the sector of space-time in which we spend our days. They are not social constructions, nor the bits of undigested beef that Scrooge blamed for his vision of Marley's ghost. But to a mind unequipped to find them, they might as well not exist at all. As the psychologist George Miller has put it, "The crowning intellectual accomplishment of the brain is the real world ... [A]ll [the] fundamental aspects of the real world of our experience are the adaptive interpretations of the really real world of physics" (p. 333, brackets Pinker's).

this – including the ability to make such an acknowledgment of an 'other' – is done simply *by* receiving it. Upon this reception, we believe that what we see either is in fact independent reality as it really is (but it still is not) or is not *really* independent reality as it is in and of itself (but it still may well be, or indeed is in the case of genuine reception).

I have proposed that our confidence that there is a relationship between 'other' and us is justified if there is a two-way relationship between that other and us, and if we are justified in being confident about anything at all.

Phenomenology, just by being a description of a two-way (rather than one-way) relationship between an other and ourselves, provides a good framework for the positing of such proposals. A phenomenological relationship can be described as a two-way relationship or, more exactly, as a correlation (co-relation) for a number a reasons. The first of these is that it describes experience. And here I echo Hart [2002] "what is experience if not a correlation?" (p. 165). But phenomenology can be described as a two-way relationship also insofar as it allows room for an 'other' which is both knowable and beyond knowing.

Given that we cannot possibly say, on a given reality's behalf, just what it looks like or how it behaves when we are correct about it, we cannot say just what mathematics is or does when we see mathematics as it really is. All this is for the simple reason that we are not mathematics. It is not us. Which is another reason why the paradox outlined above is inherent in mathematical realism. It is inherent because we cannot turn ourselves inside out. We cannot be not us. And yet we can reasonably assume that we see or access what is not us, and that mathematics is the result of this access. And we can describe what we look like when we can (justifiably) say that we are seeing mathematics as it really is. If this picture is a detailed and rigorous one, it will include a space for the 'other'. If we then assume that this space is filled, we encounter the problem of whether or not this assumption is justified. Justification in this case involves both our own reasons to assume its existence, and the actual existence of the other itself. A phenomenological account does give us good reason to suppose the existence of the other. The most obvious example of this is the fact that it opens a space for the other and is a convincing account of intentionality (or the nature of our own consciousness), or intentional correlation – both as a perception and as a reception of reality.

In short, I have argued that mathematical and physical reality are not all that different and that the assumption that we access physical reality via our five senses is analogous to the assumption that we access mathematical reality via intuition (or whatever) – unprovable, but reasonable. What we understand when we 'look' at mathematics is the formalism. The formalism is full if there is an independent reality it embodies (or is identical to), empty if not. Much as what we see when we look at physical reality is full (or identical to) what exists independently so long as what exists independently is in fact present.

The teaching of mathematics is not a concern for this thesis – but in accordance with the account I have offered, I would suggest that mathematics ought to be taught as though it were not us, with discovery in mind, as opposed to construction.

My theory of meaning takes its lead from the above considerations, and is built around the notion that whether or not a given formalism has meaning is an objective question. My semantic model can account for Gödel's idea of a 'meaningless' yet understandable formalism, and captures the realist's intuitive notion that a fully constructed mathematics has no real meaning.

All this suggests that mathematics done without an awareness of the other *as* other is drastically impoverished – similar to taking what we physically see to be all there is. Exploration is stunted, responsiveness is dimmed, and the search for an origin is stopped.

Finally, in the interest of indicating beyond this work and toward future possible applications of its proposals, I include another quotation from the field into which I have delved for its phenomenological/theological aspects. It is followed by my own interpretation/appropriation.

> To resolve [the] tension in the word "experience" Lévinas distinguishes "phenomena" from "enigma". For a phenomenologist, experience turns on the transcendental

reduction; transcendence is bracketed, and what remains is a phenomenon for me ... Unable to be brought into an intentional correlation with me, the other ... cannot be an object of my experience. Instead I am faced with an enigma that disturbs the settled order of my life. It is not a permanent confrontation; for ... the other ... irrupts in my world as sheer transcendence while retreating into the assurance of a said. ... [there is then] the trace of the infinite that approaches, never arrives, but summons me, in an unthematic manner, to responsibility. The trace signifies "beyond being," not in the Platonic sense, which calls for a negative theology, but beyond the metaphysics of knowledge implied by the transcendental reduction. "Beyond," here, indicates an immemorial past: the other is always and already within me, as a trace, imposing a meaning – responsibility for the other – that cannot be reduced to either knowing or being. (Hart [2002, 165])

This, as I understand it, vindicates my appropriation of phenomenological theory as a framework able to address and respond to the other, while at the same time ruling it out.

The notion of responding to the other, and indeed that response itself as an article of consideration, has been studied for years, and has attached to it a language far richer than I have used in this work. Concepts like 'trace', 'summons' and 'responsibility' all promise a discourse capable of a variety of expressions of our relationship with the other. And this itself indicates that perhaps the paradox need not remain. Perhaps whatever it is that is 'beyond the metaphysics of knowledge' will produce more subtle and detailed descriptions of experience and these will capture the same aspect of experience so far (at least in this work) captured only via the paradox.

The idea of the (mathematical) other as something 'already and always within' is one such description, as is the idea that the (mathematical) other is irreducible to either being (itself) or knowing (ourselves). In a sense, both are refinements of what has been claimed in this work, and so both suggest possible refinements or reductions of the paradox.

Note though that 'irreducible' also suggests 'unavoidable': the other as irreducibly other is unavoidable, at least to the realist seeking to retain (literal) responsibility for (and to) what is not us.

9. Bibliography

Benacerraf, P. [1965], "What numbers could not be", *Philosophical Review*, vol. 74, pp.47-73.

Blackburn, S. [1993], *Essays in Quasi-Realism*, Oxford University Press, New York.

Caputo, J. D. (ed.) [2002], *The Religious*, Blackwell Readings in Continental Philosophy, Blackwell, Malden.

Chihara, C. [1990], *Constructibility and Mathematical Existence*, Oxford University Press, Oxford.

Coffa, A. [1991], *The Semantic Tradition from Kant to Carnap*, Cambridge University Press, Cambridge.

Derrida, J. [1962], *Edmund Husserl's 'Origin of Geometry' An Introduction*, University of Nebraska Press, Lincoln.

Field, H. [2001], *Truth and the Absence of Fact*, Clarendon Press, Oxford.

Folina, J. [1993–4], "Poincaré on Objectivity", *Philosophica Mathematica*, ser. 3, vol. 1-2, 1993-4.

Folina, J. [1996], "Logic and Intuition in Poincaré's Philosophy of Mathematics", in *Selected Papers from the International Congress on Henri Poincaré*, Akademic Verlag and Albert Blanchard.

Frege, G. [1970], *On Sense and Reference*, in Black, M. (trans.), *Translations from the Philosophical Writings of Gottlob Frege*, Blackwell Oxford.

Frege, G. [1984], *Collected Papers on Mathematics, Logic, and Philosophy*, McGuinness, B. (ed.), Black, M. et al. (trans.), Basil Blackwell, Oxford.

Gödel, K. [1990], *Collected Works 2*, Feferman, S. et al. (eds.), Oxford University Press, Oxford.

Gödel, K. [1990a], "Russell's Mathematical Logic", in Gödel, K, *Collected Works 2*, Oxford University Press, Oxford.

Gödel, K. [1990b], "What is Cantors' Continuum Problem?", in Gödel, K, *Collected Works 2*, Oxford University Press, Oxford.

Gödel, K. [1995], *Collected Works 3*, Feferman, S. et al (eds.), Oxford University Press, Oxford.

Gödel, K. [1995a], "The Modern Development of the Foundations of Mathematics", in Gödel, K, *Collected Works 3*, Oxford University Press, Oxford.

Gödel, K. [1995b], "The Present Situation in the Foundations of Mathematics", in Gödel, K, *Collected Works 3*, Oxford University Press, Oxford.

Gödel, K. [1995c], "In What Sense is Intuitionistic Logic Constructive?", in Gödel, K, *Collected Works 3,* Oxford University Press, Oxford.

Hart, K. [1989], *The Trespass of the Sign,* Cambridge University Press, Cambridge.

Hart, K. [2002],"Experience of God", in Caputo, J. D, *The Religious,* Blackwell Readings in Continental Philosophy, Villanova University, Pennsylvania.

Hintikka, J. [1995], "The Phenomenological Dimension", in Smith, B. and Woodruff Smith, D (eds.), *The Cambridge Companion to Husserl,* Cambridge University Press, Cambridge.

Husserl, E. [1964], *The Idea of Phenomenology,* William P, Alston and Nakhnikian, G (trans.), Nijhoff, The Hague.

Husserl, E. [1970], *Logical Investigations VII,* Findlay J. N (trans.), Humanities Press, New York.

Husserl, E. [1983], *Edmund Husserl: Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy (first book),* Kersten, F. (trans.), Nijhoff, Boston.

Husserl, E. [1999], *The Essential Husserl, Basic Writings in Transcendental Phenomenology,* Welton, D. (ed.), Indiana University Press, Bloomington.

Lévinas, E. [1998], *Discovering Existence with Husserl,* Cohen, R. and Smith, M. B. (trans.), Northwestern University Press, Evanston.

Maddy, P. [1988], "Believing the Axioms, II," *Journal of Symbolic Logic,* vol. 53, no. 3, pp. 736–764.

Maddy, P. [1996], "Set Theoretic Naturalism", *Journal of Symbolic Logic,* vol. 61, pp. 490–514.

Maddy, P. [1998], *Naturalism in Mathematics,* Oxford University Press, Oxford.

Miller, P. J. [1982], *Numbers in Presence and Absence: a Study of Husserl's Philosophy of Mathematics,* M. Nijhoff Publishers, The Hague.

Pereboom, D. [1990], "Kant on Justification" *Sythese,* vol. 85. no. 1, pp. 25-54.

Pinker, S. [1997], *How the Mind Works,* Penguin, London.

Poincaré, H. [1982], *The Foundations of Science: Science and Hypothesis, The Value of Science, Science and Method,* Halsted, G. B (trans.), University Press of America, Lanham.

Priest, G. [2002], "Paraconsistent Logic", in Gabbay, D. and Guenthnen, F. (eds.), *Handbook of Philosophical Logic,* vol. 6, 2nd edn, Kluwer Academic Publishers, Dordredst.

Putnam, H. [1987], *The Many Faces of Realism,* Open Court, La Salle, Illinois.

Russell, B. [1920], *Introduction to Mathematical Philosophy*, Simon and Schuster, New York.

Shapiro, S. [1997], *Philosophy of Mathematics, Structure and Ontology,* Oxford University Press, New York.

Shapiro, S. [2000], *Thinking about Mathematics, The Philosophy of Mathematics,* Oxford University Press, New York.

Smith, B. and Woodruff Smith, D. (eds.) [1995] *The Cambridge Companion to Husserl,* Cambridge University Press, Cambridge.

Tieszen, R. [1989], *Mathematical Intuition: Phenomenology and Mathematical Knowledge,* Kluwer Academic Publishers, Dordrecht, Boston.

Wright, C. [1983], *Frege's Conception of Numbers as Objects,* Aberdeen University Press, Aberdeen.

Wright, C. [1987], *Realism, Meaning and Truth,* B. Blackwell, Oxford.

Wright, C. [1992], *Truth and Objectivity,* Harvard University Press, Cambridge, MA.

Zalta, E. and Linsky, B. [1995], "Naturalized Platonism vs. Platonized Naturalism", *The Journal of Philosophy,* vol. 92 no. 10, (Oct., 1995) pp. 525-555.

Zalta, E. [2000], "Neo-Logicism? An Ontological Reduction of Mathematics to Metaphysics", *Erkenntnis* vol. 53, no. 1 –2. pp. 219-265.