ARGUMENT
Vol. 4 (2/2014)
pp. 425–438

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Application of natural deduction in Renaissance geometry

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ABSTRACT

My goal here is to provide a detailed analysis of the methods of inference that are employed in *De prospectiva pingendi*. For this purpose, a method of natural deduction is proposed. The treatise by Piero della Francesca is a manifestation of a union between the fine arts and the mathematical sciences of arithmetic and geometry. He defines painting as a part of perspective and, speaking precisely, as a branch of geometry, which is why we find advanced geometrical exercises here.

KEYWORDS

philosophy of art; perspective in painting; geometry in art perception; reasoning and aesthetic perception

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 Published online: 25.06.2015

PROPOSITION 1.13

Piero della Francesca (1415–1492) refers directly or indirectly to Euclidean geometry. The proof of Proposition 1.13[1] refers to the similarity of the triangles. Piero does not mention a number of Euclidean proposition but he uses it in his proof. In Euclid's *Elements* (c. 300 BC), these issues are discussed in the Book VI, Proposition 4 to 8 (see Euclid). At this point it is worth recalling Proposition 1.12. It shows how to draw in perspective a surface of undefined shape, which is located in profile as a straight line. It means that a horizontal line *BC* can be foreshortened into a vertical line *EB*. The line *AD* represents a hypothetical observer and the point *A* is the position of the eye in relation to the line *EB*. The vertical line *BF* represents the picture plane (Fig. 1).

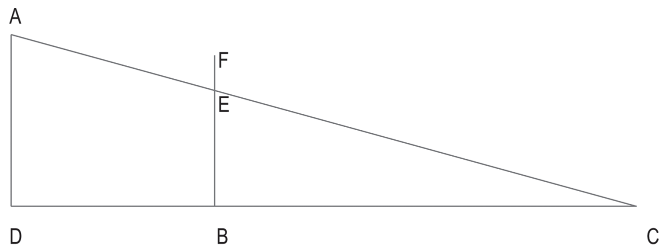


Fig. 1

According to Proposition 1.13 we add a square *BCGF* that represents the object to be drawn in reality in a horizontal plane (Field, 2005: app. 8). Then, we draw from the point *A* visual rays to the corners of the square (Fig. 2).

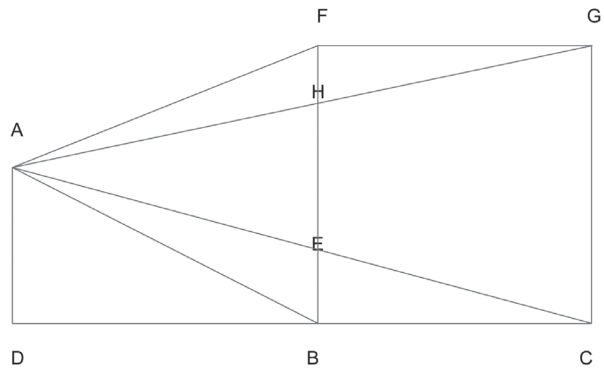


Fig. 2

[1] "1" refers to a number of the book *De prospectiva pingendi*, and the number "13" to the sequence of propositions.

Then we construct the parallel half-line from the point *A* to the line segment *BC* with no end point, and divide the line segment *BC* into two equal parts in *I*. From this point we construct the line perpendicular to the point *A′*, and then draw the line from the point *E* to the point *K*, again parallel to *BC* (Fig. 3).

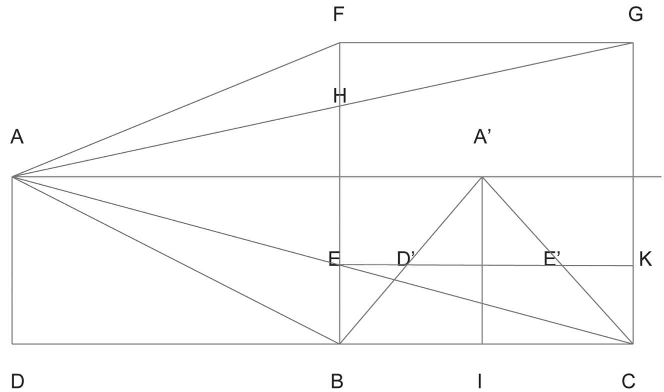


Fig. 3

Finally, we draw the line *A′B* and *A′C*. The goal of design is to show that from the point of view of *A* the perspective image of the side *BC* of the square *BCGF* is *EB*, of the side *FG* is *FH*, while the farthest side of the *CG* is *EH*. What may amaze is Piero's assertion that the line segment *D′E′* is also the perspective image of the side of *CG*, so *EH* = *D′E′* (1). Let us recall an approach proposed by Field, following today's convections of reasoning (Field, 2005:145). At the outset it should be noted, that the triangles *A′D′E′* and *A′BC* are similar, because the sides of *D′E′* and *BC* are parallel, and the triangles *CEE′* and *CAA′* are similar, because *EE′* and *AA′* are parallel[2]. On this basis, we obtain the equivalence:

$$\frac{A'C}{A'E'} = \frac{AC}{AE} \qquad (2).$$

The triangles *AEH* and *ACG* are similar, because the sides *HE* and *GC* are parallel:

$$\frac{AE}{AC} = \frac{EH}{CG} \qquad (3).$$

[2] Following Field's letters instead of He should be "*HE* is parallel to *GC*".

The right side of the equation (2) is the reverse of the left side of the equation (3). Both equations give so as a result of:

$$\frac{A'E'}{A'C} = \frac{EH}{CG} \qquad (4).$$

As already noted, triangles $A'D'E'$ and $A'BC$ are similar, so we have:

$$\frac{D'E'}{BC} = \frac{A'E'}{A'C} \qquad (5).$$

The left side of (4) is equal to the right side of (5), so we have:

$$\frac{D'E'}{BC} = \frac{EH}{CG} \qquad (6).$$

$BC = CG$ because $BCGF$ is a square, and therefore (6) is reduced to (1):

$EH=D'E'$, what was required to be proved.

As has been shown the proposition may be used in the interpretation of the painting of Piero della Francesca *The flagellation of Christ*. The final figure looks as follows:

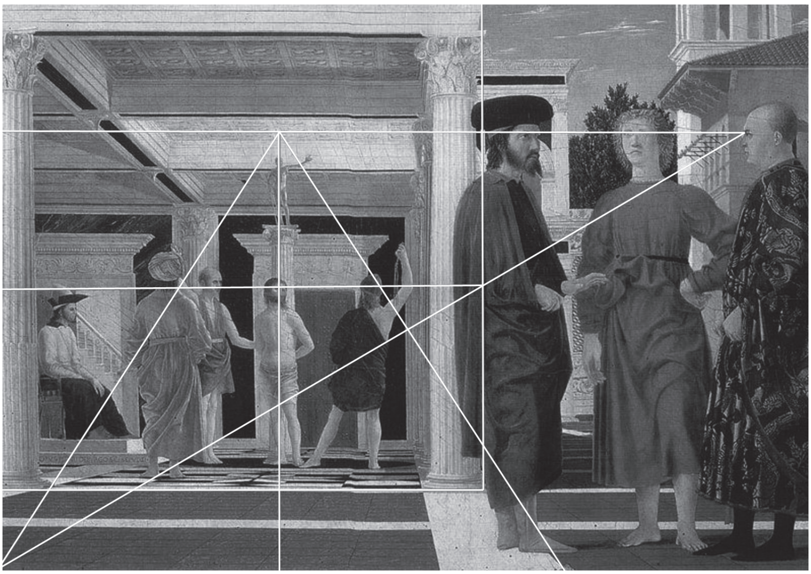


Fig. 4

However, what interests us is a logical system based on geometry. Such a system has been discussed in a separate text (Mirek, 2014). In a formal system *EF* one can use a natural deduction in the form proposed by Frederic. Fitch and Ludwik Borkowski, but for the first time was introduced, as the method of subordinate proofs, independently by Stanisław Jaśkowski and Gerhard Gentzen. The advantage of natural deduction is that it seems to present in a precise and visually readable way the Francesca's geometrical system. Let us start from the first part of proof:

P_1	$\Delta A'D'E' \sim \Delta A'BC$		
P_2		$\Delta CEE' \sim \Delta CAA'$	
3		$\Delta A'D'E' \sim \Delta A'BC$	$(R{:}P_1)$
4		$\Delta A'D'E' \sim \Delta A'BC \wedge \Delta CEE' \sim \Delta CAA'$	$(\wedge+{:}3,P_2)$
5		$\frac{A'C}{A'E'} = \frac{AC}{AE}$	(Fig. 3)
6	$\Delta A'D'E' \sim \Delta A'BC \wedge \Delta CEE' \sim \Delta CAA' \Rightarrow \frac{A'C}{A'E'} = \frac{AC}{AE}$		$(\Rightarrow+{:}P_2\text{-}5)$

Every proof within a natural deduction system begins with a hypothesis (premise), above marked as P_1 and P_2. The second premise is introduced within a subproof of the proof by means of repetition (reiteration). Generally, every subsequent step in the proof (subproof) is introduced by a hypothesis or it is a formula that is derived from previous steps using one of the rules of the system. Proofs and subproofs are marked out by vertical lines. Then we use the introduction rule for conjunction (∧+). For the rule implication an introduction (⇒+) is required a subproof from which we come back to the scope of the first vertical line. "Δ" means a triangle and "~" means a similarity. As a novelty is introduced the use of diagrams in a proof. This is consistent with the methods of inference that are employed in the *Elements* and particularly in *De prospectiva pingendi*. In the second example, we have one premise, and therefore we are within a subproof:

P_1		$\Delta AEH \sim \Delta ACG$	
2		$\frac{AE}{AC} = \frac{EH}{CG}$	(Fig. 3)
3	$\Delta AEH \sim \Delta ACG \Rightarrow \frac{AE}{AC} = \frac{EH}{CG}$		$(\Rightarrow+{:}P_1\text{-}2)$

In the third example, there are two premises, and in the fourth one:

P_1	$\frac{A'C}{A'E'} = \frac{AC}{AE}$	
P_2	$\frac{AE}{AC} = \frac{EH}{CG}$	
3	$\frac{A'C}{A'E'} = \frac{AC}{AE}$	$(R{:}P_1)$
4	$\frac{A'C}{A'E'} = \frac{AC}{AE} \wedge \frac{AE}{AC} = \frac{EH}{CG}$	$(\wedge+{:}3,P_2)$
5	$\frac{A'E'}{A'C} = \frac{EH}{CG}$	(Fig. 3)
6	$\frac{A'C}{A'E'} = \frac{AC}{AE} \wedge \frac{AE}{AC} = \frac{EH}{CG} \Rightarrow \frac{A'E'}{A'C} = \frac{EH}{CG}$	$(\Rightarrow+{:}P_2\text{-}5)$

P_1	$\Delta A'D'E' \sim \Delta A'BC$	
2	$\frac{D'E'}{BC} = \frac{A'E'}{A'C}$	(Fig. 3)
3	$\Delta A'D'E' \sim \Delta A'BC \Rightarrow \frac{D'E'}{BC} = \frac{A'E'}{A'C}$	$(\Rightarrow+{:}P_1\text{-}2)$

While the latter two are as follows:

P_1	$\frac{A'E'}{A'C} = \frac{EH}{CG}$	
P_2	$\frac{D'E'}{BC} = \frac{A'E'}{A'C}$	
3	$\frac{A'E'}{A'C} = \frac{EH}{CG}$	$(R{:}P_1)$
4	$\frac{A'E'}{A'C} = \frac{EH}{CG} \wedge \frac{D'E'}{BC} = \frac{A'E'}{A'C}$	$(\wedge+{:}3,P_2)$
5	$\frac{D'E'}{BC} = \frac{EH}{CG}$	(Fig. 3)
6	$\frac{A'E'}{A'C} = \frac{EH}{CG} \wedge \frac{D'E'}{BC} = \frac{A'E'}{A'C} \Rightarrow \frac{D'E'}{BC} = \frac{EH}{CG}$	$(\Rightarrow+{:}P_2\text{-}5)$

P_1	$\frac{D'E'}{BC} = \frac{EH}{CG}$	
P_2	$\quad BC = CG$	
3	$\quad \frac{D'E'}{BC} = \frac{EH}{CG}$	$(R{:}P_1)$
4	$\quad \frac{D'E'}{BC} = \frac{EH}{CG} \wedge BC = CG$	$(\wedge+{:}3,P_2)$
5	$\quad EH = D'E'$	(Fig. 3)
6	$\frac{D'E'}{BC} = \frac{EH}{CG} \wedge BC = CG \Rightarrow EH = D'E'$	$(\Rightarrow+{:}P_2\text{-}5)$

PROPOSITION 1.11

In turn, in the case of Proposition 1.11 we are referred to the problem that objects of the same size will appear in various proportions in the picture, depending on the distance of the eye. Once again let's use diagrams supplied by Field (2005: 101–103). Francesca proposes to draw four parallel lines, each 1 braccio long and they are one braccio apart. From the first line which is the picture line to the eye (A) is four bracci (Fig. 5).

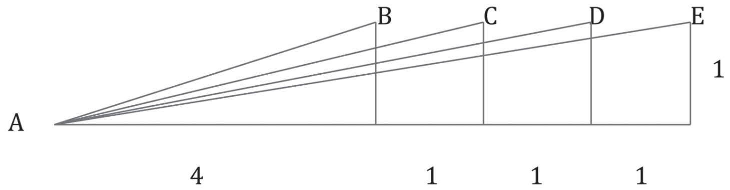


Fig. 5

According to Piero the proportion between the second one and the first is 5 to 4, the third and the second is 6 to 5, the fourth and the third is 7 to 6. These proportions are derived from pairs of similar triangles. As has been said, in *Elements* the issues can be found in Book VI, Proposition 4 to 8 (Euclid). In the case of the first and second line one can form the triangles by drawing an additional two lines that come to a point *A* (Fig. 6).

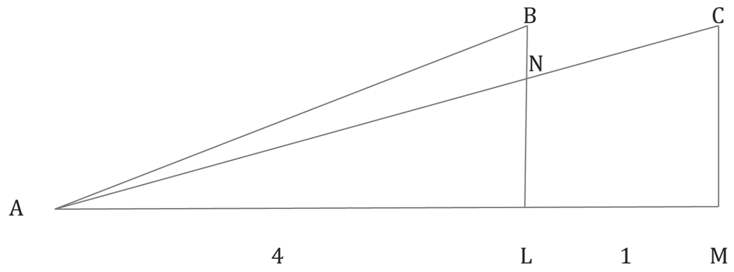


Fig. 6

The triangles *ACM* i *ANL* are similar and what we want is the ratio of *NL* to *BL*. Piero presents the proportion among those four lines is as of the four numbers, namely *105*, *84*, *70*, *60*. These ratios also depends on the distance of the first line to a point A. If we increase the distance to 6 bracci, the proportions will be like the four numbers *84*, *72*, *63*, *56*. While changing the height of the eye makes no difference to a ratio. In support of this thesis Field proposes the following figure:

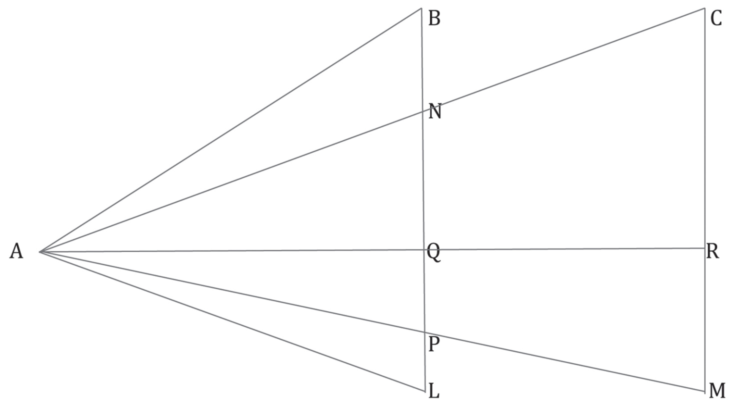


Fig. 7

It is worth noting that both the part of Figure 7 above and below the line segment *AQR* have the same form as Figure 5. Both triangles *ACM*, *ANP* and *ACR*, *ANQ* are similar. The aim is to find the ratio of *NP* to *BL*. Starting from triangles of *ACM* and *ANP*, we obtain the equivalence:

$$\frac{NP}{CM} = \frac{AN}{AC} \tag{7}$$

Likewise with triangles *ACR* i *ANQ* we have the equivalence:

$$\frac{AN}{AC} = \frac{AQ}{AR} \qquad (8).$$

By combining the left side of (7) with the right side of (8) and remembering that $CM = BL$, we have:

$$\frac{NP}{CM} = \frac{AQ}{AR} \qquad (9).$$

Again, let's use the method of natural deduction:

P_1		$\Delta\, ACM \sim \Delta\, ANP$	
2		$\frac{NP}{CM} = \frac{AN}{AC}$	(Fig. 7)
3	$\Delta\, ACM \sim \Delta\, ANP \Rightarrow \frac{NP}{CM} = \frac{AN}{AC}$		($\Rightarrow$+:P_1-2)

P_1		$\Delta\, ACR \sim \Delta\, ANQ$	
2		$\frac{AN}{AC} = \frac{AQ}{AR}$	(Fig. 7)
3	$\Delta\, ACR \sim \Delta\, ANQ \Rightarrow \frac{AN}{AC} = \frac{AQ}{AR}$		($\Rightarrow$+:P_1-2)

P_1	$\frac{NP}{CM} = \frac{AN}{AC}$		
P_2		$\frac{AN}{AC} = \frac{AQ}{AR}$	
3		$\frac{NP}{CM} = \frac{AN}{AC}$	(R:P_1)
4		$\frac{NP}{CM} = \frac{AN}{AC} \wedge \frac{AN}{AC} = \frac{AQ}{AR}$	($\wedge$+:3,P_2)
5		$\frac{NP}{CM} = \frac{AQ}{AR}$	(Fig. 7)
6	$\frac{NP}{CM} = \frac{AN}{AC} \wedge \frac{AN}{AC} = \frac{AQ}{AR} \Rightarrow \frac{NP}{CM} = \frac{AQ}{AR}$		($\Rightarrow$+:P_2-5)

As has been demonstrated, Proposition 1.11 can be used in the masterpieces painted by Piero, namely in his *The baptism of Christ* and *The resurrection*. In the former one can find the proportions between the trees, as well as between Christ and people on further plans (Fig. 8).

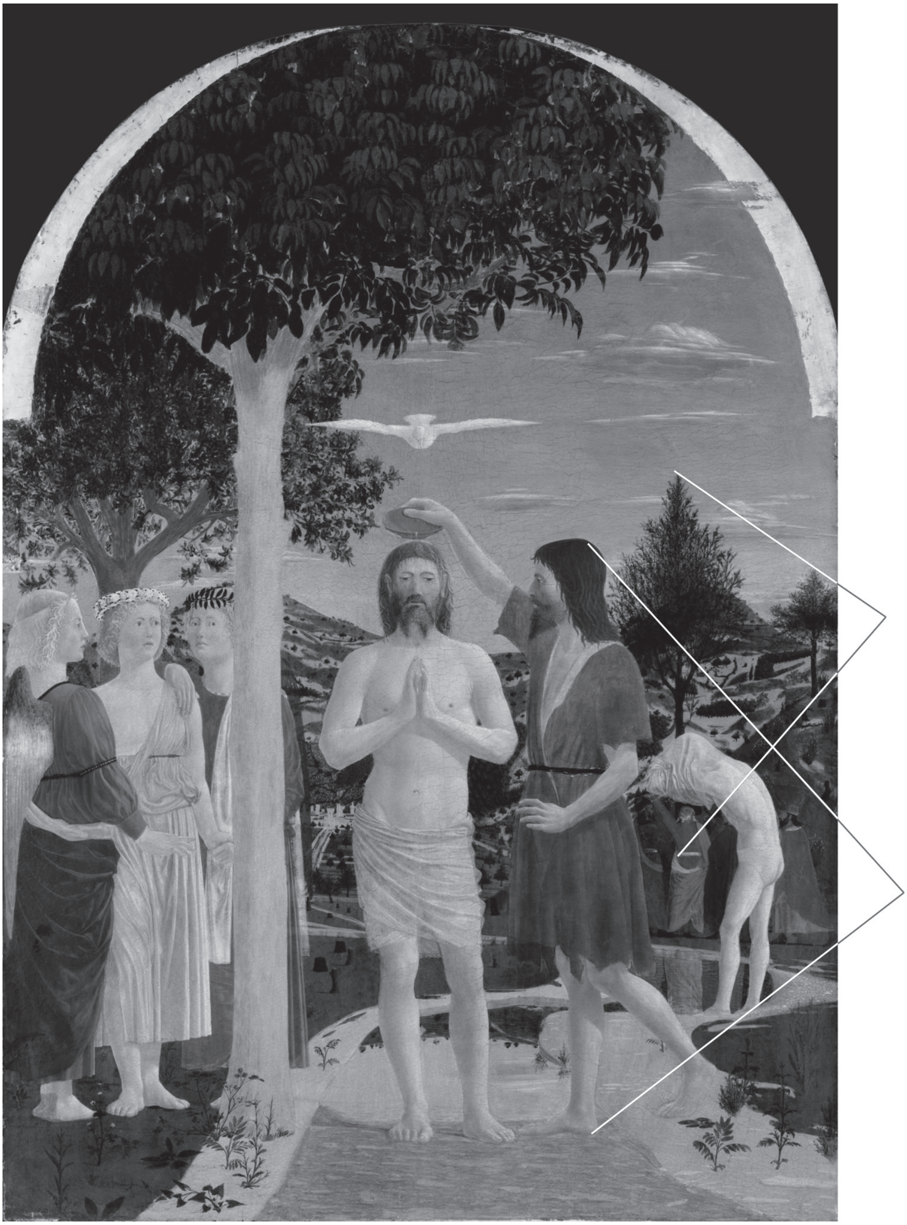
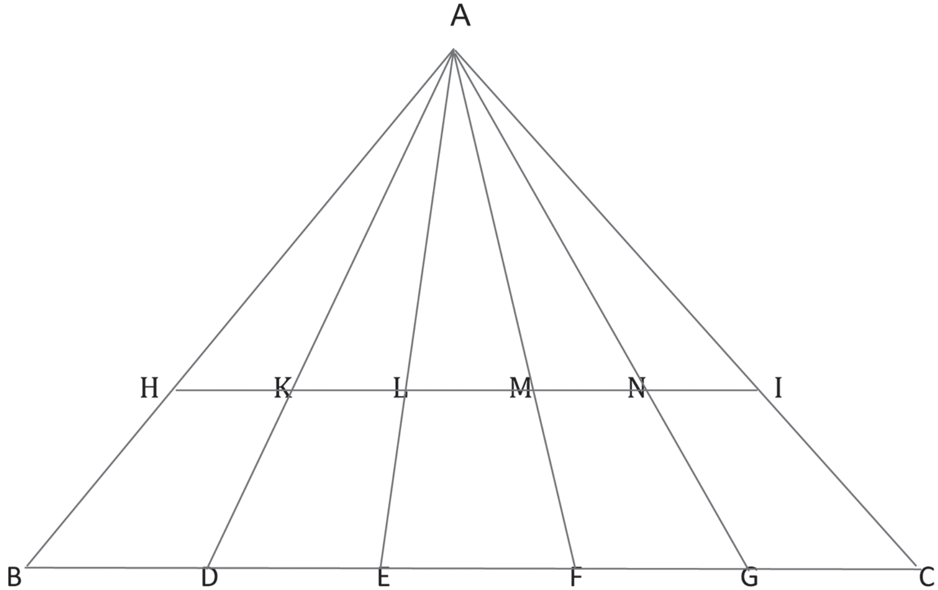


Fig. 8

3. PROPOSITION 1.8

In turn, Proposition 1.8 is known as "Piero's theorem". Piero shows that the perspective images of orthogonals converge to a centric point (Fig. 9).



What we want to prove is that if *BH*, *DK*, *EL*, *FM*, *GN*, *CI* can all be extended to meet at *A*, then the pattern of ratios defined by the points *HKLMNI* on the second transversal, *HI*, is the same as that defined by *BDEFGC* on the first one, *BC*. According to Francesca the triangles *ABD* i *AHK* are similar, the triangles *ADE* i *AKL* are similar and *AEF* i *ALM* are similar etc. What is more, the angles of the triangle *ABD* are similar to the angles of the triangle *AHK*, what follows from Euclid's theorem from the Book VI, Proposition 21. Therefore, *BD* to *DE* is in the same ratio as *HK* to *KL*, *EF* to *FG* is the same ratio as *LM* to *MN*, and *FG* to *GC*, as *MN* to *NI*. Translation into the language of natural deduction takes the form:

P_1	$\Delta ABD \sim \Delta AHK$	
P_2	$\sphericalangle ABD \sim \sphericalangle AHK$	
P_3	$\Delta AGC \sim \Delta ANI$	
P_4	$\sphericalangle AGC \sim \sphericalangle ANI$	
5	$\Delta ABD \sim \Delta AHK$	$(R{:}P_1)$
6	$\sphericalangle ABD \sim \sphericalangle AHK$	$(R{:}P_2)$
7	$\Delta ABD \sim \Delta AHK \wedge \Delta AGC \sim \Delta ANI$	$(\wedge+{:}5,P_3)$
8	$\sphericalangle ABD \sim \sphericalangle AHK \wedge \sphericalangle AGC \sim \sphericalangle ANI$	$(\wedge+{:}6,P_4)$
9	$(\Delta ABD \sim \Delta AHK \wedge \Delta AGC \sim \Delta ANI) \wedge (\sphericalangle ABD \sim \sphericalangle AHK \wedge \sphericalangle AGC \sim \sphericalangle ANI)$	$(\wedge+{:}7,8)$
10	$\frac{FG}{GC} = \frac{HK}{KL}$	(Fig. 9)
11	$(\Delta ADE \sim \Delta AKL \wedge \Delta ABD \sim \Delta AHK) \wedge (\sphericalangle ADE \sim \sphericalangle AKL \wedge \sphericalangle ABD \sim \sphericalangle AHK) \Rightarrow \frac{BD}{DE} = \frac{HK}{KL}$	$(\Rightarrow+{:}P_3\text{-}10)$

“$\sphericalangle$” means that all angles of both triangles are similar. In the rest part the proof proceeds by analogy.

P_1	$\Delta AEF \sim \Delta ALM$	
P_2	$\sphericalangle AEF \sim \sphericalangle ALM$	
P_3	$\quad \Delta AFG \sim \Delta AMN$	
P_4	$\quad \sphericalangle AFG \sim \sphericalangle AMN$	
5	$\quad \Delta AEF \sim \Delta ALM$	$(R{:}P_1)$
6	$\quad \sphericalangle AEF \sim \sphericalangle ALM$	$(R{:}P_2)$
7	$\quad \Delta AEF \sim \Delta ALM \wedge \Delta AFG \sim \Delta AMN$	$(\wedge+{:}5,P_3)$
8	$\quad \sphericalangle AEF \sim \sphericalangle ALM \wedge \sphericalangle AFG \sim \sphericalangle AMN$	$(\wedge+{:}6,P_4)$
9	$\quad (\Delta AEF \sim \Delta ALM \wedge \Delta AFG \sim \Delta AMN) \wedge (\sphericalangle AEF \sim \sphericalangle ALM \wedge \sphericalangle AFG \sim \sphericalangle AMN)$	$(\wedge+{:}8,7)$
10	$\quad \frac{FG}{GC} = \frac{LM}{MN}$	(Fig. 9)
11	$(\Delta AEF \sim \Delta ALM \wedge \Delta AFG \sim \Delta AMN) \wedge (\sphericalangle AEF \sim \sphericalangle ALM \wedge \sphericalangle AFG \sim \sphericalangle AMN) \Rightarrow \frac{FG}{GC} = \frac{LM}{MN}$	$(\Rightarrow+{:}P_3\text{-}10)$

P_1	$\Delta AFG \sim \Delta AMN$	
P_2	$\sphericalangle AFG \sim \sphericalangle AMN$	
P_3	$\quad \Delta AFG \sim \Delta AMN$	
P_4	$\quad \sphericalangle AFG \sim \sphericalangle AMN$	
5	$\quad \Delta AEF \sim \Delta ALM$	$(R{:}P_1)$
6	$\quad \sphericalangle AEF \sim \sphericalangle ALM$	$(R{:}P_2)$
7	$\quad \Delta AEF \sim \Delta ALM \wedge \Delta AFG \sim \Delta AMN$	$(\wedge+{:}5,P_3)$
8	$\quad \sphericalangle AEF \sim \sphericalangle ALM \wedge \sphericalangle AFG \sim \sphericalangle AMN$	$(\wedge+{:}6,P_4)$
9	$\quad (\Delta AEF \sim \Delta ALM \wedge \Delta AFG \sim \Delta AMN) \wedge (\sphericalangle AEF \sim \sphericalangle ALM \wedge \sphericalangle AFG \sim \sphericalangle AMN)$	$(\wedge+{:}8,7)$
10	$\quad \frac{EF}{FG} = \frac{MN}{NI}$	(Fig. 9)
11	$(\Delta AEF \sim \Delta ALM \wedge \Delta AFG \sim \Delta AMN) \wedge (\sphericalangle AEF \sim \sphericalangle ALM \wedge \sphericalangle AFG \sim \sphericalangle AMN) \Rightarrow \frac{EF}{FG} = \frac{MN}{NI}$	$(\Rightarrow+{:}P_3\text{-}10)$

4. CONCLUSION

As has been noted, the advantage of natural deduction is that it seems to present in a precise and visually readable way a geometrical system. In Francesca's *De prospectiva pingendi* one can find 48 propositions along with the diagrams. The use of diagrams in a Piero's geometry as in a Euclidean one is governed by a discernible logic. An attempt to include it in a logical system with the use of diagrams it seems so obvious and indicated.

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