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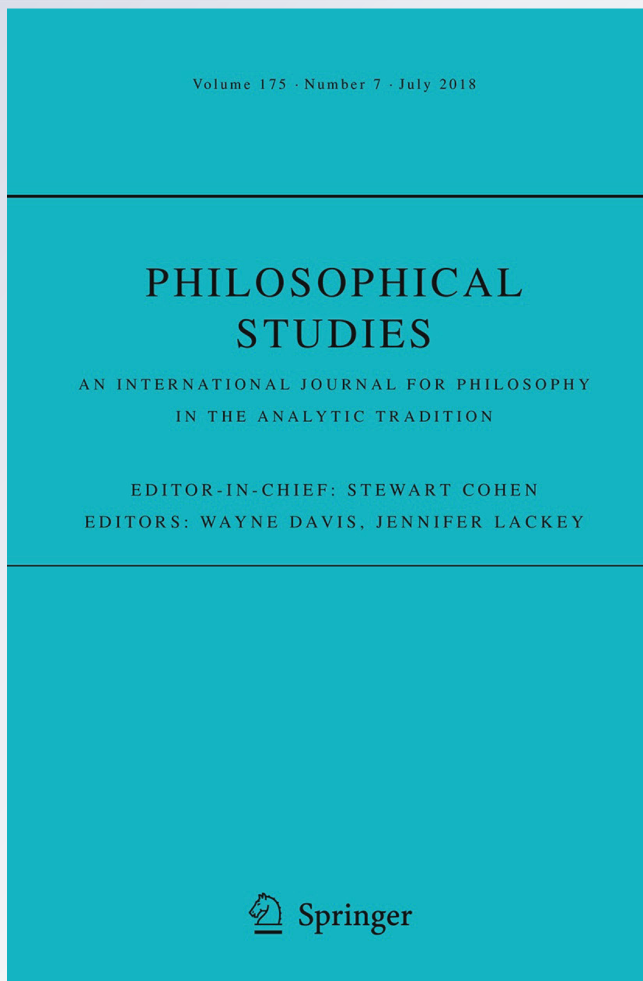
Nathan Salmón

Philosophical Studies
An International Journal for Philosophy
in the Analytic Tradition

ISSN 0031-8116
Volume 175
Number 7

Philos Stud (2018) 175:1631-1660
DOI 10.1007/s11098-017-0927-0



Springer

Errata for Nathan Salmón, "Julius Caesar and the Numbers"

Philosophical Studies, 175, 7 (July 2018), pp. 1631-1660, DOI 10.1007/s11098-017-0927-0

Page:	**Lines:**	**Current text:**	**Should instead be:**
1642	9-10 from bottom	should not have, and almost certainly did not, believe	should not have believed, and almost certainly did not believe,
1648	9 from bottom	render	renders
1649	4	unary concept	unary concept F
1652	24	definineda	definienda
1652	27	definiens	definientia
1653	1	lade	laid
1653	7-8	if there is exactly one F,	if there is not exactly one F,
1654	5	'№' should not entail that Caesar does not count as a number.	'№' *should not entail that Caesar does not count as a number*.
1655	13	fail	fails
1656	6	provide "criterion" for	provide a "criterion" for

Philos Stud (2018) 175:1631–1660
https://doi.org/10.1007/s11098-017-0927-0

Julius Caesar and the numbers

Nathan Salmón[1]

Published online: 6 June 2017

Abstract This article offers an interpretation of a controversial aspect of Frege's *The Foundations of Arithmetic*, the so-called Julius Caesar problem. Frege raises the Caesar problem against proposed purely logical definitions for '0', 'successor', and 'number', and also against a proposed definition for 'direction' as applied to lines in geometry. Dummett and other interpreters have seen in Frege's criticism a demanding requirement on such definitions, often put by saying that such definitions must provide a criterion of identity of a certain kind (for numbers or for linear directions). These interpretations are criticized and an alternative interpretation is defended. The Caesar problem is that the proposed definitions fail to well-define 'number' and 'direction'. That is, the proposed definitions, even when taken together with the extra-definitional facts (such as that Caesar is not a number and that England is not a direction), fail to fix unique semantic extensions for 'number' and 'direction', and thereby fail to fix unique truth-values for sentences like 'Caesar is a number' and 'England is a direction'. A minor modification of the criticized definitions well-defines '0', 'successor' and 'number', thereby avoiding the Caesar problem as Frege understands it, but without providing any criterion of number identity in the usual sense.

Keywords Abstraction principle · Julius Caesar problem · Frege · Hume's principle · Logicism

✉ Nathan Salmón
nsalmon@philosophy.ucsb.edu

[1] Department of Philosophy, University of California, Santa Barbara, CA 93106, USA

I

Gottlob Frege's *The Foundations of Arithmetic* (*Die Grundlagen der Arithmetik*, 1884—hereafter *FA*) is a rarity: On the one hand, it is unquestionably seriously flawed. On the other, it is unquestionably a work of unsurpassed philosophical genius and creativity. Its brilliance far outweighs its defects. Bertrand Russell, who was Frege's foremost booster and also his most formidable critic, said this of *FA*: "Although this book is quite short, not difficult, and of the very highest importance, it attracted almost no attention, and the definition of number which it contains remained practically unknown until it was rediscovered by the present author in 1901" (*Introduction to Mathematical Philosophy*, London: George Allen and Unwin, 1919, p. 11).[1]

Frege's objective in *FA* is to build a persuasive case for the analyticity of arithmetic. He does this in two stages. First, he provides purely logical definitions for basic arithmetic expressions, including the three primitives of what would come to be known as the (Dedekind-) Peano postulates: the numeral '0'; a general term for the natural numbers; and a functor for the successor function. This is followed by purely logical derivations of basic arithmetical truths—the Peano postulates themselves—from the definitions. *FA*§55 makes a preliminary stab at providing appropriate definitions. More exactly, §55 provides a system of three definitions, which ostensibly form a *recursive definition* (definition by mathematical induction[2]), simultaneously defining '0', '1', a successor-functor ('+1'), and a dyadic predicate (in English '__ has the number __') for the binary relation between a concept F and a cardinal number *n*, of there being exactly (no more and no fewer than) *n* F's.[3] In the next section Frege criticizes the proposed definitions, citing the notorious *Julius Caesar problem*.

In *FA*§§62–64 Frege proposes—tentatively and preliminarily—an alternate tack: to define a term for cardinal numbers by means of what in the literature is tendentiously called an 'abstraction principle'. An abstraction principle takes the particular form:

$$\forall x \forall y(\mathrm{F}x \; \& \; \mathrm{F}y \rightarrow [f(x) = f(y) \leftrightarrow x\phi_{eq}y]),$$

where ϕ_{eq} indicates an equivalence relation among F's. (An abstraction principle can also be of higher order.) Each of the following may be regarded as an abstraction principle:

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[1] The reader almost fails to notice that Russell said that he had independently discovered the central idea of a book that is of the highest importance. In point of fact, Russell had done exactly as he says. Frege and Russell were both creative philosophical geniuses of the very highest order.

[2] See Peter Aczel, "An Introduction to Inductive Definitions," in J. Barwise, ed., *Handbook of Mathematical Logic* (Amsterdam: North-Holland, 1977, 1983), chapter C.7, pp. 739–782.

[3] In Frege's terminology the word 'concept' does not mean *a concept* in the ordinary sense (a notion). By 'concept' (and by 'property') Frege means the characteristic function of a class. I use Alonzo Church's 'λ' throughout as a variable-binding functional abstraction operator, whereby the semantic extension of $\ulcorner\lambda\alpha_1\alpha_2\ldots\alpha_n[\phi]\urcorner$ is the *n*-ary function that assigns to $<x_1, x_2, \ldots, x_n>$, the semantic extension of ϕ under the assignment of $x_1, x_2, \ldots, x_n$ as values for the variables $\alpha_1, \alpha_2, \ldots, \alpha_n$, respectively.

Frege's locution '*dem Begriffe F kommt die Zahl n zu*'—very literally, *to the concept F comes the number n*—is typically translated as 'the number *n* belongs to the concept F'. The locution 'the number *n* belongs to F' would be less misleadingly formulated as '*n* is a number that belongs to F' or '*n* is a number that F has'. (*Cf. FA*§56.) Better yet is 'F has the number *n*' or 'F numerically has *n*'.

Lines have the same *direction(-al orientation)* iff they are parallel.
Triangles have the same *(triangular) shape* iff they are similar.
Expressions have the same *(semantic) meaning* iff they are synonymous.
Events have the same *time (of occurrence)* iff they are simultaneous.[4]
Creatures are of the same *(biological) species* iff they are conspecific.
Physical objects have the same *color* iff they are color-matching.
Concepts have the same *(metaphysical) extension* iff whatever falls under one also falls under the other.[5]

For some of these principles, it is not easy to imagine someone having the notion expressed on the right-hand side separately and independently of the notion expressed on the left, e.g., having the notion of simultaneity independently of the notion of the time of an event's occurrence. Provided one does have an antecedent and independent grasp of the notion expressed on the right-hand side (parallel lines, similar triangles, synonymous expressions, etc.), from this together with the relevant abstraction principle one can allegedly glean or "abstract" (extract, extrapolate, tease out) the notion expressed on the left. For example, provided one has an antecedent and independent grasp of the notion of parallel lines, using the relevant abstraction principle one can allegedly abstract the notion of directional orientation. In such cases, the abstraction principle supposedly provides a special kind of definition in which the relevant term (e.g., 'direction') is not defined directly and instead the semantic contents of whole sentences in which the term occurs is specified. Some writers call this a 'contextual definition'. That term, however, is at least highly suggestive of definitions of *incomplete symbols*, in Russell and Whitehead's sense, and hence at best misleading in discussing Frege's program. Presumably, in the general case one allegedly abstracts a complete notion of the functional attribute *f* as *that which, among* F*'s, all and only* ϕ_{eq}*-mates have in common*, e.g., that which, among lines, all and only parallel ones have in common. In *FA*§66 Frege points out that *definition by abstraction* (to use a term coined by Peano) is also subject to the Julius Caesar problem.

Most of the scholarly literature on the Julius Caesar problem focuses on the problem as it applies to definition by abstraction, despite the fact that the particular example of Caesar is cited only once and only in connection with a definition that proceeds by recursion (mathematical induction) rather than by abstraction. Many interpreters, notably Michael Dummett (see note 8), have misunderstood *FA*§§ 55–56. This has hampered attempts to interpret a good deal of the rest of the book. Here

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4 Following the special theory of relativity, the abstraction principle about time and simultaneity is tacitly relativized: Relative to any frame of reference, events occur at the same time iff they are simultaneous. If simultaneity is reference-frame-relative and the abstraction principle is true, an event's occurring at a particular time *t* is also relativized to a frame of reference.

5 See my *Reference and Essence* (Amherst, NY: Prometheus Books, 1981, 2005), at p. 46, on the distinction between *metaphysical extension* and *semantic extension*.

I propose an alternate interpretation of the Julius Caesar problem. The interpretation I propose runs against the grain of a good deal, if not quite all, of current commentary on the problem. The interpretation I favor is not the only coherent interpretation of the relevant passages. I do not urge the interpretation on the ground that it is more charitable than its rivals, although I do judge it to be so. I urge it, rather, on the ground that among the competing interpretations, it is the most plausible, and the one that has Frege focusing on the most glaring defect. I do not claim that my interpretation captures everything significant that Frege thought about the problem. I do believe that the interpretation I favor captures the core of Frege's thought, and makes coherent sense of Frege's writings as a whole. My principal reason is based not merely on the text of *FA* but also, indeed heavily, on general philosophical and logical considerations. What I take the Julius Caesar problem to be is in fact the most significant and most serious philosophico-logical defect with the definitions that Frege rejects. The principal consideration favoring the interpretation provided below is that it is what Frege *ought* to mean by his remarks. That Frege ought to mean something in particular by critical remarks is a compelling argument that he meant precisely that.

As mentioned, *FA*§55 proposes a recursive definition for '0', a successor functor, and a predicate for the relation of a concept F having a particular cardinality n. I here use the neologism 'numerate' for the converse relation between n and F: n *numerates* F iff n is exactly how many F's there are. I shall use lower-case italic 's' as the successor-functor. The §55 definitional system may then be formulated as follows:

D0num $\quad 0 \textit{ numerates } \mathrm{F} =_{df} \forall a \sim \mathrm{F}a.$

D1num $\quad 1 \textit{ numerates } \mathrm{F} =_{df} \sim\forall a \sim \mathrm{F}a \ \&\ \forall a \forall b(\mathrm{F}a \ \&\ \mathrm{F}b \rightarrow a = b).$

Dsnum $\quad s(n) \textit{ numerates } \mathrm{F} =_{df} \exists a(\mathrm{F}a \ \&\ n \text{ numerates } \lambda b[\mathrm{F}b \ \&\ a \neq b]).$

The system *D0num–Dsnum* has four distinct definienda (defined terms): '0'; 'numerate'; '1'; and 's( )'. The following closure clause is an implicit supplement to *D0num–Dsnum*:

D$\sim$*num* $\quad n$ does not *numerate* F unless n *numerates* F according to *D0num–Dsnum*.[6]

Frege's terminology strongly suggests a second implicit supplementary definition:

D№ $\quad$ №$(n) =_{df} \exists \mathrm{G}(n \text{ numerates } \mathrm{G}),$

i.e., n is a *cardinal number* iff there are numerically exactly n G's for some concept G. The expanded system *D0num-D№* has a total of five definienda. Analogs of

[6] That is, where β is any singular term and Π is any monadic predicate, $\ulcorner\beta$ numerates $\Pi\urcorner$ is false unless one of the following is true: $\ulcorner[\beta = 0 \ \&\ \forall a \sim \Pi a]\urcorner$; $\ulcorner[\beta = 1 \ \&\ \sim\forall a \sim \Pi a \ \&\ \forall a \forall b(\Pi a \ \&\ \Pi b \rightarrow a = b)]\urcorner$; or $\ulcorner\exists m[\beta = s(m) \ \&\ \exists a(\Pi a \ \&\ m \text{ numerates } \lambda b[\Pi b \ \&\ a \neq b])]\urcorner$.

D1num can be given for ‘2 numerates F’, ‘3 numerates F’, etc., but *Dsnum* renders such definitions superfluous. Indeed, *D1num* is already superfluous since $1 = s(0)$.

Caveat emptor: The proposed definitional system is no ordinary recursive definition. As Frege notes in *FA*§56, *D0num–Dsnum* fall significantly short. The first objection he raises is that “the sense of the expression ‘the concept G has the number *n*’ is as unknown to us as is that of the expression ‘the concept F has the number $(n + 1)$’.” Here Frege points out that the definiens (the defining expression) of *Dsnum* includes an expression of the form ‘*n* numerates G’, which itself needs to be defined. Frege uses ‘G’ as a schematic letter for the position occupied by the particular concept-abstract ‘$\lambda b[Fb \;\&\; a \neq b]$’. His point is that *Dsnum* purports to use the locution ‘*n* numerates __’, with the variable ‘*n*’ ranging over cardinal numbers, to define ‘$s(n)$ numerates __’ *when both locutions stand equally in need of definition*. The §55 system of definitions suffers from a fatal flaw, that of vicious circularity. The definiendum ‘numerate’ (Frege’s phrase ‘__ is the number belonging to __’) occurs simultaneously among the definienda and also among the definientia, in such a way that the definitional system fails to assign it semantic content.[7] There would be no vicious circularity if *D0num–Dsnum* were put forward as a proper recursive definition of the single definiendum ‘numerate’—with ‘0’, ‘1’, and ‘*s*’ taken as among the antecedently understood definientia. But the definitional system is put forward as defining those same expressions as well as and in addition to the predicate ‘numerate’. The definitional system has too many definienda, not enough definientia, with one of the definienda appearing among the definientia. As Frege notes, vicious circularity is the inevitable and immediate result.

Frege notes furthermore in *FA*§56 that the definitional system does not enable the deduction that $\forall a \forall b(a$ numerates F & b numerates F $\rightarrow a = b)$. There is a closely related problem: Whereas it is deducible from these definitions that 1 numerates F $\leftrightarrow$ $s(0)$ numerates F, it is not deducible that $1 = s(0)$. Frege does not explicitly note this problem, though he was undoubtedly aware of it. By Frege’s lights, the proposed definitions fail to render the sentence ‘One is the successor of zero’ analytic. For Frege’s purposes this will not do.

Both of Frege’s criticisms are correct. Although he does not explicitly provide *D№*, Frege seems to have it very much in mind. (*Cf.* the official definition of ‘number’ in *FA*§72.) In §56 he also raises the Julius Caesar problem, which evidently presupposes *D№*. Frege points out that despite the system’s vicious circularity, *Dsnum* together with *D1num* provide semantic content (under the assignment of content to ‘F’) for the compound locutions ‘$s(1)$ numerates F’, ‘$s[s(1)]$ numerates F’, and so on. A proper recursive definition of ‘1’, ‘*s*’, and ‘numerate’ should do no less. Yet a significant problem remains. As Frege puts it,

> but we can—to give a crude example—never decide through our definitions whether a concept has the number Julius Caesar, whether this famous conqueror of Gaul is a number or not.

[7] I distinguish *vicious circularity*—an unacceptable defect—from the innocuous (and eliminable) sort of “circularity” that arises in a proper recursive definition, as with the use of ‘+’ among the definientia in the standard recursive definition of addition among natural numbers: $x + 0 =_{df} x$; $x + s(n) =_{df} s(x + n)$.

In his highly influential *Frege: Philosophy of Mathematics* (Harvard University Press, 1991), Dummett (at p. 101) dismisses this important passage as a "very bad" and "notoriously inept" rendering of Frege's first *FA*§56 objection to the discredited §55 definitions—which is, in fact, the vicious-circularity objection. Dummett goes even further, saying that "§56 may be stigmatized as the weakest in the whole of *Foundations*. The arguments lack all cogency: they more resemble sleight of hand. … Frege, impelled by his desire to establish that numbers are objects, seems to have been taken in by his own jargon" (pp. 105–106). This unduly harsh assessment is more indicative of a failure on Dummett's part than of any serious failure on Frege's. Frege's only mistake is in assuming that his reader will process his observation properly.[8] Intellectual history demands a more faithful and fair-minded rendering of Frege's text. What exactly does Frege mean when he observes that we cannot "decide through" *D0num–Dsnum* whether Caesar is a number? How is that a problem? Exactly what requirement does Frege mean to impose on his program with this Caesar objection to the §55 definitions?

A few sentences after presenting the Julius Caesar problem, Frege closes *FA*§56 by exposing a particular defect that might otherwise easily be overlooked: Contrary to their objective of defining '0' and '1', *D0num* and *D1num* actually provide content only for the compound locutions '0 numerates __' and '1 numerates __' (Frege's '0 is the number belonging to __' and '1 is the number belonging to __')—in effect, treating these phrases as if they were simple second-order predicates, i.e., as if they were single words like 'nothing' and 'exactlyonething'. Frege could have made an analogous criticism of *Dsnum*: Contrary to its objective of defining '*s*( )', it actually provides content only for the compound locutions '*s*(0) numerates __', '*s*(1) numerates __', '*s*[*s*(0)] numerates __', etc., treating these phrases as if they were infinitely many new single words ('*s*uccessorzero-thing', '*s*uccessorone-things', etc.). In general, the §55 definitions surreptitiously treat the occurrences of '0', '1',

[8] Dummett says, "No one reading the book for the first time can have seen this remark [the Julius Caesar passage in *FA*§56] as making much sense, let alone as relevant" (*op. cit.*, p. 101). On the contrary, since I first read the passage in 1971 as a student in Tyler Burge's undergraduate course on classical logicism, I have never found it puzzling and have given it the interpretation I provide below. I was surprised when I later learned that it is evidently not the standard interpretation.

Dummett (*op. cit.*) and numerous other interpreters miss a crucial feature of the *FA*§55 system of definitions, and the very feature that renders it subject to the Julius Caesar problem: It is intended to define simultaneously '0', '1', a successor-functor, and a separate dyadic predicate—'is a number belonging to' (see note 3)—for the binary relation of numerating between a number *n* and a concept F. Dummett incorrectly reconstructs the proposed definitions as defining numerically-definite quantifiers ('there are exactly *n* objects *x* such that …'), while omitting the last definiendum entirely (p. 100), thwarting Frege's intentions. Although Dummett endorses Frege's first objection, his reformulation omits the very definiendum that the original system defines circularly, rendering the objection unintelligible. Dummett identifies the first objection both with the Caesar problem and with the objection that the discredited definitional system treats the phrase 'has the number 0' as though it were a simple monadic predicate like 'is vacuous'. Neither of these difficulties applies to Dummett's reconstruction, which, unlike the §55 system, provides perfectly functional definitions for '0', '1' and a successor-functor, while making no claim of also defining the intended fourth definiendum. Dummett's misinterpretation leads him to misjudge both Frege's motive for discrediting the §55 definitions and the force of his objections (pp. 100–108). (Dummett, *ibid*., pp. 99–110, misreads *FA*§56 as raising the Caesar problem against Dummett's reconstruction, which is in fact not subject to the problem. He furthermore misinterprets §§56–61 as a botched attempt to discredit those same definitions).

'*s*', and 'numerate' as mere orthographic accidents—like the 'one' in 'telephone', the 'eight' in 'weight', the 'nine' in 'canine', the 'ten' in 'rotten'—while fostering the illusion of fulfilling the objective of defining those terms as separable, meaningful components. In fact, this very defect underlies the related flaw that Frege limns in his much discussed remark about Julius Caesar. (It is not the same flaw.) Some interpreters have taken it that the §55 definitions are put forward as merely providing semantic content for whole phrases of the appropriate forms ('0 numerates', '*s*(0) numerates', etc.), and are incapable of defining the term 'numerate' itself ('is the number belonging to'). In fact, however, the definitions *D0num*, *Dsnum*, and $D\sim num$ are perfectly serviceable as constituting a recursive definition for 'numerate' while avoiding the Julius Caesar problem, provided that '0' and '*s*' are antecedently and independently defined. (This would also involve replacing *FA*'s pivotal definition at §68 of the cardinality of a concept; see the appendix.) The Caesar problem arises from the fact that the §55 definitions are put forward as defining not only 'numerate' but also '0' and '*s*' simultaneously. The failure to recognize this has led to fanciful attributions with relatively scant textual evidence.

II

Some light is shed by Frege's reprising the Julius Caesar problem in the same work, and again in the first volume of his epic, flawed masterpiece, *Basic Laws of Arithmetic* (*Grundgesetze der Arithmetik*, 1893—hereafter *BLA*). Having discredited *D0num–Dsnum*, in *FA*§63 Frege considers adopting the Hume-Cantor abstraction principle (*HP*) as providing the needed definition of 'number' as a functor for the cardinality of a concept F.

HP: $\#(\mathrm{F}) = \#(\mathrm{G}) \leftrightarrow \mathrm{F} \approx \mathrm{G}$.

HP expresses that concepts F and G have the same cardinal number (i.e., F and G are equinumerous, there are exactly as many F's as there are G's) iff there is a *one-to-one correspondence* (a *bijection*) between them. In §64 Frege brilliantly illustrates the proposed strategy for defining '#' by means of a couple of geometric analogies: the directional orientation (*Richtung*) of a straight line and the shape of a triangle, both in Euclidean geometry. The analog of *HP* for directional orientation is the following:

DP: $\upharpoonleft\!\downharpoonright(a) = \upharpoonleft\!\downharpoonright(b) \leftrightarrow a \mathbin{/\!/} b$,

i.e., the directional orientation of (straight) line *a* and that of line *b* are the same iff *a* and *b* are either parallel or the same line. Interestingly, *DP* might be offered as providing a definition for the relational predicate '$/\!/$' or 'parallel'. For that matter, *HP* might likewise be offered as providing a *definition by concretion* for '$\approx$' or 'one-to-one correspondence'. (Compare our understanding of 'simultaneous' or 'synonymous'.) But for Frege's program this puts the cart before the horse. Instead '$\approx$' is defined independently, using only concepts of pure logic and, importantly, without invoking '#'. The intention, or at least the hope, is then to define '#' in terms of '$\approx$' using only concepts of pure logic (including that of one-to-one correspondence).

Analogously, one defines ‘parallel lines’ independently—as lines that are co-planar and either non-intersecting or the same—with the hope of defining ‘directional orientation’ in terms of it.[9]

Barely sooner is the hope raised (*FA*§§ 62–65) than it is dashed (§§ 65–67). Some interpreters hold that Frege takes *HP* as providing a definition by abstraction for ‘#’. Others hold that he did not do this but should have, thereby avoiding reliance on naïve comprehension.[10] Frege would have emphatically disagreed with both claims, and for very good reason. Having sympathetically examined the prospect of defining ‘⥮’ by means of *DP*, and, by parity of form, of defining ‘#’ by means of *HP*, Frege concludes that “we thus can obtain no sharply demarcated concept of direction and for the same reasons none of number” (§68). His reasons for rejecting definition by abstraction include a reprise of the Caesar theme:

> we have in our definition the means to recognize this object [the direction of *a*] when it should occur in another guise as the direction of *b*. But this method is not sufficient for all cases. One cannot, e.g., decide whether England is the same as the direction of the Earth’s axis. Pardon this seemingly nonsensical example! Naturally, no one will confuse England with the direction of the Earth’s axis; but this is not to be credited to our explanation. (§66)

A great deal has been said about the Julius Caesar problem. Despite Frege’s clear remarks about two clear examples in *FA*, and his clear remarks in *BLA* (I§§10, 33; II§66), there is relatively little agreement concerning what the putative problem is, and exactly what Frege takes it to be.[11] Some writers take it that the so-called problem is really a plurality of problems misleadingly packaged as one. Perhaps most interpreters take their cue from *FA*§62, in which Frege rhetorically asks how numbers are to be “given to us” as self-subsistent objects if we have neither any experiential idea nor any Kantian intuition of them. There Frege says:

> If the sign *a* is to designate for us an object, we must have a criterion, which decides everywhere whether *b* is the same as *a*, even if it is not always in our

[9] It is unfortunate that Frege uses the word ‘equinumerous’ (‘*gleichzahlig*’ in Frege’s extension of German), rather than the phrase ‘corresponds 1–1’ (or even the symbol ‘≈’), as a shorthand term for the relation of concepts in one-to-one correspondence, while begging the reader to ignore the word’s etymology (*FA*§68). Etymologically, ‘equinumerous’ is in fact shorthand for the relation of being the same in number. In its literal sense, it is straightforwardly analytic that concepts are *equinumerous* iff they have the same number. Use of ‘equinumerous’ as a term for the relation of one-to-one correspondence is not justified until *HP* is proved from definitions and general logical laws (*FA*§73), and hence shown to be also analytic. Indeed, *HP* may be aptly paraphrased thus: *Concepts are equinumerous iff they are in 1–1 correspondence*. Analogously, *DP* may be paraphrased: *Straight lines are co-oriented iff they are parallel* (i.e., co-planar and either non-intersecting or the same).

[10] See for example Crispin Wright, *Frege’s Conception of Numbers as Objects* (Aberdeen University Press, 1983), at pp. 104–117, 140–142, 145–149; Ed Zalta, “Frege’s Theorem and Foundations for Arithmetic,” section 3.2 (“Contextual Definition of ‘The Number of Fs’: Hume’s Principle”), in the *Stanford Encyclopedia of Philosophy*; and William Demopoulos, *Logicism and its Philosophical Legacy* (Cambridge University Press, 2013), at p. 19.

[11] See for example “Dirk Greimann, “What is Frege’s Julius Caesar Problem?” *dialectica*, 57, 3 (September 2003), pp. 261–278; and Richard Heck, “The Julius Caesar Objection,” *loc. cit*. Issue no. 2 (June 2005) of *dialectica*, volume 59 is devoted entirely to the Julius Caesar problem.

> power to apply the criterion. … After we have thus acquired a means of apprehending a definite number and of recognizing it as the same, we can give it a number-word as its proper name.

These remarks, especially as they appear in the authoritative translations, are suggestive of a fairly demanding requirement Frege seems to impose on definitions of 'number' and its cognates. As present-day philosophers use the phrase, a *criterion of identity* for F's (e.g., for sets, or for events, artifacts, persons, etc.) is something along the lines of a binary relation (perhaps of qualitative-similarity, and preferably an equivalence relation) which "defines" the relation of *being the same F*, and/or to which the relation *same F* in some way "reduces," e.g., conceptually or metaphysically. Indeed, so interpreted Frege's remarks are suggestive of an unreasonable requirement. Many interpreters see in the Caesar problem a requirement that in order to use a term *a* to designate a number we should possess a criterion that decides for all number-designators *b* whether ⌜*b* is the same as *a*⌝ is true,[12] and an accompanying demand that we therefore define the phrase 'the number belonging to F' in such a way as to explain or reduce the sense of the equivalence locution 'The number belonging to F is the same as that belonging to G'.

In a couple of later passages of *Frege: Philosophy of Mathematics*, Dummett apparently changes his tune regarding the Julius Caesar problem. (See note 8.) Using single quotation marks in the manner of quasi-quotations marks, he writes:

> We can legitimately ask whether the object for which '*t*' stands is of a given kind, say an organism, because this is just to ask whether the sentence '*t* is an organism' is true; and it is to be answered by appeal solely to the principles that have been laid down for determining the truth-value of such a sentence. We can legitimately ask whether '*t*' has the same reference as some other term '*s*', because this is just to ask whether the sentence '*t* is the same as *s*' is true; and the question is to be answered in the same way as the previous one. …
>
> By the same token, however, we have not fixed the reference of the term until we have supplied the means of answering all those questions of this kind that *are* formulable within the language. This involves, in particular, that, to have fixed the reference of the term, we need to have laid down determinate conditions for the reference of the newly introduced term '*t*' to coincide with that of any other given term in the language: in other words, for the truth of any identity-statement formed by putting '*t*' on one side of the sign of identity and an arbitrary term of the language on the other. It may or may not be obvious that our intention, in introducing '*t*', included its not being taken to stand for the Moon; but, unless we have provided for the falsity (or, if we wish, for the truth) of the sentence '*t* is the Moon', we cannot claim to have fixed the reference of '*t*', since we have not stipulated whether or not it stands for the Moon.

[12] The original text does not employ quasi-quotation marks, a modern innovation invented by W. V. O. Quine, but their inclusion is indicative of Frege's intent.

> That is the fault that Frege finds with the purported contextual definition of the direction-operator. It affords us no means of determining the truth or falsity of a sentence like 'The direction of the Earth's axis is England'; and, in failing to do so, it fails to determine the references of the terms for direction. And that, by implication, is the fault he finds with the purported contextual definition of the cardinality operator; it affords us no means of determining the truth or falsity of a sentence like 'The number of planets is Julius Caesar', and thereby fails to determine the reference of numerical terms. … This means that we have not attained a unique specification of the reference of numerical terms formed with the cardinality operator: since we have failed to make any stipulation determining whether or not Julius Caesar is the number of planets, we have not said what the number of planets *is*, that is, what the term 'the number of planets' stands for. (pp. 156–157)
>
> … This is the Julius Caesar problem again. From the criterion of identity between numbers [*HP*], we cannot determine whether an object not given as a number, such as England or Julius Caesar, is a number at all, and if so, to what concept it belongs. … we cannot determine whether or not it [the number of planets] has a monarchy or was assassinated in the Capitol. (*ibid*., pp. 210–211)

Disagreements notwithstanding, there appears to be a consensus concerning one central point of exegesis. Perhaps most interpreters take it that Frege demanded of any proper, purely logical explanation of the notion of *belonging numerically to a concept* that it provide a criterion that decides (or alternatively, a means for determining) the truth-values of identity statements not only of the particular form ⌜The number of F's = the number of G's⌝ but also of the more general form ⌜The number of F's = q⌝ where q designates an object of any metaphysical category whatsoever—such as an historical figure or a monarchy.[13] Some interpreters see in the Caesar problem a demand that any explanation of number decide all statements that identify a particular number n, whether given by a description of the form ⌜the number of F's⌝ or by a canonical designator (such as a numeral), with any object x—whether x be that very number n, or a part of a sovereign state, or an equivalence class of concepts, or a famous conqueror of Gaul, or sealing wax.

A common view is that Frege identifies numbers with extensions of concepts precisely to meet the demand that his definitions provide the means for deciding an identity statement between a number-term and an object of any category. Frege supposedly believes it is suitably known that Caesar is not a concept-extension.[14] However, Frege explicitly says in *FA* that he "attaches no decisive importance" to employing concept-extensions (§107). In fact, in the note to §68 he says he believes

[13] *Cf. FA*§66. I believe this interpretation is due ultimately to Charles Parsons, "Frege's Theory of Number," in M. Black, ed., *Philosophy in America* (Ithaca, NY: Cornell University Press, 1965), pp. 180–203, reprinted in Parsons's *Mathematics in Philosophy: Selected Essays* (Cornell University Press, 1983), pp. 150–175, at sections IV and V.

[14] See for example Crispin Wright, *Frege's Conception of Numbers as Objects* (Aberdeen: Aberdeen University Press, 1983); and Joan Weiner, *Frege* (Oxford University Press, 1999), pp. 61, 82–83, 121–122.

the phrase 'the extension of' in his proposed identification of the number of F's with the extension of the concept $\lambda G[F \approx G]$ can be deleted. This would apparently result in his identifying the number with the relevant concept instead of its extension.[15] Although Frege clearly believes that numbers are objects, not functions, in *FA*, where the Julius Caesar objection is repeatedly pressed, he did not come to see concept-extensions as essential to his logicist program. As I shall argue below, Frege does not draw upon concept-extensions specifically to address the Caesar problem. I believe that problem is something else entirely. The Caesar problem is not that the definitions fail to decide the truth-values of every sentence of the form $\ulcorner\beta$ numerates F$\urcorner$. The Caesar problem is also not that the definitions fail to specify what kind or category of thing cardinal numbers are (concept-extensions rather than historical figures). The defect is much more serious than that, and the objection is much more decisive.

In *FA*§73 Frege justifies *HP* relying tacitly on naïve comprehension, in his infamous Basic Law V, which, as Russell pointed out, is inconsistent. A number of interpreters see the Caesar problem as a theoretical obstacle to Frege's substituting *HP* for Basic Law V, as some have recommended.[16] Richard Heck, for example, argues that the Caesar problem requires that the definition of 'number' also explain:

15 Recall his remark in "On Concept and Object" that the phrase 'the concept horse' designates not a concept but a corresponding object.

16 In addition to the works cited in notes 13 and 14 above see Richard Heck, "The Julius Caesar Objection," in R. Heck, *ed*., *Language, Thought, and Logic: Essays in Honor of Michael Dummett*, Oxford: Clarendon Press: 1997, pp. 273–308, at p. 276, as well as Heck's *Frege's Theorem* (Oxford University Press, 2011); Edward Zalta's entry on "Frege's Theorem and Foundations for Arithmetic," section 6.5, in *The Stanford Encyclopedia of Philosophy*; and the following works by George Boolos: "Saving Frege From Contradiction," *Proceedings of the Aristotelian Society*, 87 (1986/1987), pp. 137–151; "The Consistency of Frege's *Foundations of Arithmetic*," in J. J. Thomson, ed., *On Being and Saying: Essays for Richard Cartwright* (Cambridge, Mass.: The MIT Press, 1987), pp. 3–20; "The Standard of Equality of Numbers," in Boolos, ed., *Meaning and Method: Essays in Honor of Hilary Putnam* (Cambridge: Cambridge University Press, 1990), pp. 261–277; "Whence the Contradiction?" *Aristotelian Society Supplementary Volume*, 67 (1993), pp. 220–236; "Frege's Theorem and the Peano Postulates," *Bulletin of Symbolic Logic*, 1, 3 (1995), pp. 317–326; "Erratum: Frege's Theorem and the Peano Postulates," *Bulletin of Symbolic Logic*, 2, 1 (1996), p. 126; "Is Hume's Principle Analytic?" in R. Heck, ed., *Language, Thought, and Logic: Essays in Honor of Michael Dummett* (Oxford: Clarendon Press: 1997), pp. 245–262. For a detailed assessment of so-called neo-(Fregean) logicism see John Burgess, *Fixing Frege* (Princeton, NJ: Princeton University Press, 2005).

In the opening passage of "To Bury Caesar," in their *The Reason's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics* (Oxford University Press, 2001), chapter 14, Bob Hale and Crispin Wright argue on the basis of *FA*§62 that "the point of the Caesar objection" is that the §55 definitions fail to "establish a use for expressions of the form 'the number that belongs to the concept F' as Fregean *proper names* (singular terms)." I do not know what Hale and Wright mean by their locution 'to establish a use for __ as a singular term'. Perhaps it is simply to classify the designated expression as a singular term (rather than some other type of designator as, for example, a general term like 'horse') and to fix the semantic designation for the term. They say that to establish a use for the expression 'the concept F has the number a' the proposed definitions would need to "transform the hypothesis that the number a belongs to the concept F and the number b belongs to the concept F into the conjunction of identities: the number belonging to the concept F $= a$ and the number belonging to the concept F $= b$." In simpler terms, the definitions would need to transform the hypothesis that the number a numerates F into the hypothesis that a is the only number that does. But what exactly is it to "transform" one hypothesis into another one that is (at least ostensibly) stronger? And how could genuine definitions, which are given by mere analytic truths, do any such thing? (See note 18.).

(*i*) "our capacity to refer to numbers"; (*ii*) "how we apprehend numbers as objects"; and in particular, (*iii*) how we know that the number one is not Caesar. According to Heck, this requirement precluded Frege from adopting *HP* as an axiom.[17]

I shall make a couple of preliminary observations concerning the bearing of *FA*§62 on the Julius Caesar problem. First, even if Frege does impose a demanding requirement on definitions of the sort Dummett describes—a requirement that is not well motivated and that would be, in my judgment, quite unreasonable—he does so only some six sections *after* he initially raises the Caesar problem in §56. The Caesar problem is not the failure of the §55 definitions to meet a demanding, idiosyncratic requirement which Frege does not articulate until §62; it is something more basic, something that is, or ought to be, uncontroversial. Second, shortly after considering an alternative definition of 'number', one by means of *HP*, Frege replays the Caesar objection against the seemingly conforming definition. Again, the Caesar problem is something else, a failure of even the proposed "contextual" definition by abstraction.[18] I shall make a third observation, which is speculative, after I have laid out what I take the Caesar problem to be.

III

I am unpersuaded that Frege believed that a proper definition of arithmetical vocabulary must invoke identity criteria that "decide" cross-category formulations like that the number one is Julius Caesar, in the sense that the definitions, taken together with the premises that one and Caesar exist, either entail that the number one is Caesar or else entail that it is not. I am persuaded that even if Frege did believe this, the Julius Caesar problem, as Frege intends it, is not that the criticized definitions fail to meet such a condition. I maintain, contrary to the spirit of a number of existing interpretations, that Frege should not have, and almost certainly did not, believe that a proper, purely logical definition of 'number' needs to decide, in this sense, all statements that equate a number (given by a numeral, by its counterpart natural-language number-word, or as the number of such-and-such's) with some specified object. As Frege noted (*FA*§3, and *passim*), any sentence entailed by definitions that correctly specify semantic content is analytic, as 'All bachelors are unmarried' is entailed by the (let us assume) content-preserving definition 'x is a *bachelor* $=_{df}$ x is unmarried & x is a man'. Definitions do not decide (in this sense) synthetic statements of fact. It cannot be a requirement on an acceptable definition of '№', or of '1', that it decide every identity statement

[17] "The Julius Caesar Objection," *loc. cit.*; and "Julius Caesar and Basic Law V," *dialectica*, 59, 2 (June 2005), pp. 161–178.

[18] A definition by abstraction using *HP* apparently does exactly what Hale and Wright (note 16) seem to say Frege demands of a definition of 'number' in his Caesar objection. Yet Frege applies the objection against such a definition. For a trenchant critique of Hale and Wright, see Ignacio Angelelli, "On Neo-Fregeanism," *The Review of Modern Logic*, 3/4, 30 (December 2003–August 2004), pp. 87–97. (Angelelli's understanding of Frege on Julius Caesar is evidently similar to mine.) The appendix below uncovers that Frege need not have followed "the looking-around method" of seizing upon equivalence classes under 1–1 correspondence as cardinalities merely because they respect *HP*.

equating the number one with some specified object, or on a genuine definition of '$\upharpoonleft\!\downharpoonright$' that it decide every identity statement equating the directional orientation of the Earth's axis with some specified object. Some of Frege's remarks in *FA*§66, especially coupled with his remarks in §62, undoubtedly encouraged the widespread misinterpretation of Frege as imposing such a requirement. But those remarks do not support this interpretation. No better off is Dummett's claim that Frege demands of a definition of '#' that it "supply the means for determining the truth or falsity" of an arbitrary sentence of the form $\ulcorner\#(\mathrm{F}) = b\urcorner$. Given adequate linguistic resources, whatever decides every sentence of the form $\ulcorner\#(\mathrm{F}) = b\urcorner$ decides every expressible statement. It is a mathematical certainty that the sentence

$$\#(\lambda x[(x + 3 = 7)\ \&\ \phi]) = 1$$

is true (in standard notation) if and only if ϕ is. A means for deciding this equation is thus tantamount to a means for deciding ϕ, whatever sentence ϕ might be—whether it be exclusively about numbers, an undecided mathematical conjecture, not at all about numbers, or something else. It cannot be reasonably demanded of a definition of '#' that it provide the means for deciding the strengthened finite Ramsey theorem or Goldbach's conjecture.[19]

It should be presumed that Frege was well aware of all this. One might be tempted to suppose that whereas an acceptable purely logical definition of number must decide any identity statement that invokes a numeral or number-word on one side and a second proper name on the other (e.g., 'One is Julius Caesar'), but not that it must decide every identity statement invoking a numeral and a description like the one invoked here. But Frege did not distinguish between proper names and descriptions in a manner that could justify such a restriction. Quite the contrary, for theoretical reasons he assimilated proper names to definite descriptions. More to the point, though it is metaphysically necessary, and completely certain, that Caesar (the famous conqueror of Gaul) is not the number one in disguise (or vice versa), this is no truth of mathematics. No sentence expressing it is analytic. Exactly analogously, it is no truth of logic that Caesar is not an extension of a concept, and no sentence expressing that he is not is analytic. Insofar as they are well-formed, the sentences 'The number one conquered Gaul' and 'Caesar + 12 = 13' are mathematically consistent. Likewise, the sentence 'Caesar is the extension of a concept' is logically consistent. The exegetical hypothesis that Frege was unaware of these facts is seriously implausible on its face, and requires very substantial textual support to be justified. To comprehend Frege's intent behind the Caesar problem one needs instead to reconcile *FA*§§ 56 and 66 with a presumed unwillingness on Frege's part to declare 'Caesar is not a number', or 'England is not

[19] It should be noted that *HP* does not even decide all statements of the form $\ulcorner\#(\mathrm{F}) = \#(\mathrm{G})\urcorner$. For example, it fails to decide whether Frege and Einstein had the same number of children. Where ϕ is any sentence not decided by the Peano postulates (e.g., the strengthened finite Ramsey theorem or the Gödelian undecidable sentence), *HP* also does not decide $\ulcorner\#(\lambda x[(x + 3 = 7)\ \&\ \phi]) = \#(\lambda x[x + 3 = 7])\urcorner$. *Cf.* Donald Davidson, "The Individualtion of Events," in N. Rescher, *ed.*, *Essays in Honor of Carl G. Hempel* (Dordrecht: D. Rediel, 1969), pp. 216–234, at p. 226.

the directional orientation of the Earth's axis', or 'Caesar is not a concept-extension' analytic, i.e., corollaries of mere definitions.

Frege does not raise the particular example of Julius Caesar in connection with any abstraction principle. He cites the example of Julius Caesar only in connection with the *FA*§55 proposed system of definitions, which are not a definition by abstraction and are ostensibly instead a recursive definition. Frege then apparently finds the same general problem occurring again in the proposal to define a functor (e.g., '#' or '$\upharpoonleft\!\downharpoonleft$') by means of an abstraction principle. Frege understands the Caesar problem as a single problem specifically about improperly formed definitions—a problem that arises in both of these proposals but does not arise with *HP* or Basic Law V *per se*, insofar as they are not definitions.

What Frege intends by the Caesar problem is spelled out partially in *BLA*, I§33 and more fully in II§66, as specific strictures on proper definitions. I do not mean by this that one must read *BLA* in order to understand how Frege intends the Caesar problem in *FA*. *FA* is wholly self-contained on this score. In *BLA*, Frege goes the extra mile by articulating more fully the very point he already made in *FA*§56. Here is *BLA*-II§66 in its entirety:

> That the designation of an expression together with that of one of its parts does not always determine the designation of the remaining part is obvious. One must not, therefore, explain a sign or word by explaining an expression in which it occurs and whose remaining parts are known. For otherwise an investigation will first be required whether a solution to the unknown—to make use of a readily understood algebraic image—is possible, and whether the unknown is uniquely determined. Yet it is, as remarked above, infeasible to make the legitimacy of a definition depend on the results of such an investigation, one which, moreover, might not even be practicable. Rather, a definition must have the character of an equation, solved for the unknown, on whose other side nothing unknown occurs.
>
> Even less will it do to explain two things by means of a single definition; every definition must, on the contrary, contain a single sign whose designation it stipulates. After all, one cannot determine two unknowns by means of a single equation either.
>
> It sometimes occurs that a whole system of definitions is laid down, each containing several words that are to be explained, such that each of these words occurs in several of these definitions. This is comparable to a system of equations with several unknowns, where it is once again entirely left open to question whether there is a solution, and whether it is uniquely determined.
>
> To be sure, one may regard any sign, any word, as consisting of parts: however we deny it simplicity only if the designation of the whole would follow from the designations of the parts according to the general rules of grammar or of the notation, and if these parts also occur in other combinations and are treated as autonomous signs with their own designation. In this sense, therefore, we may say: the explained expression—the explained sign—must be simple. Otherwise, it might occur that the parts were also explained separately and that these explanations contradicted that of the whole.

> Names of functions cannot, of course, appear on their own on one side of a definitional equation, owing to their distinctive unsaturatedness; rather their argument-places have always somehow to be filled. This, as we have seen, is accomplished in *Begriffsschrift* by Roman letters, which must then also feature on the other side. In ordinary language, indeterminately indicating pronouns and particles ("something", "what", "it") fulfil this role. This constitutes no violation of our principle, since these letters, pronouns, particles refer to nothing, but merely indicate. (P. Ebert and M. Rossberg, trans., *Basic Laws of Arithmetic II*, Oxford University Press, 2013, pp. 79–80).[20]

HP does not provide a proper definition for the sentential matrix '#(F) = #(G)'. Given Frege's *BLA* strictures, nothing can. The sentential matrix is compound. It invokes both the identity predicate '=' and the cardinality functor '#' (twice) with their customary contents, in their default senses. (*Cf. FA*§§ 63–65.) The semantic contents of compound expressions are fixed not by definition, but by recursive semantic rules governing compositionality. Contentful simple (non-compound) expressions are either primitive (understood independently of verbal definition) or defined by means of an antecedently contentful expression, the definiens. The definiens gives the semantic content of the definiendum. Frege's views on proper definition undoubtedly underwent some refinement by the time he wrote *BLA*, but the central requirement that a definition must correctly capture semantic content was present already in *FA*.

At a minimum, the definiens together with the relevant extra-definitional facts—facts beyond what is given by the definitions, other than semantic facts concerning what the definiendum actually means, or the like—must determine a unique semantic extension (*Bedeutung*).[21] In short, the definition together with "the world" must fix, or pin down, the semantic extension. In contemporary mathematical terminology, a definition is required to *well-define* its definiendum. In a later piece Frege makes the point (nearly enough) explicit:

> I demand from a definition of a point that by means of it we be able to judge of any object whatever—e.g., my pocket watch—whether it is a point. … Mr. Korselt, however, misunderstands this to mean that I demand that the question be decidable from the definition alone, without the help of perceptions. … if it is merely on account of our incomplete knowledge of the object that we cannot answer the question, then the explanation is not to blame. If, however, the question must remain unanswered no matter how complete our knowledge, then the explanation is faulty. ("On the Foundations of Geometry: Second Series," 1906, in Frege's *Collected Papers on Mathematics, Logic, and*

[20] I have taken the liberty of replacing the translators' 'reference' with 'designation' and restoring Frege's term '*Begriffsschrift*' in place of its English rendering as 'concept-script'.

[21] See note 5. Extra-definitional facts exclude *inter alia* that the actual semantic extension is such-and-such.

There is an alternative kind of "definition" in which the semantic content is determined not by the definiens alone but by the definiens together with relevant extra-definitional facts. Saul Kripke calls this 'fixing the reference by a definite description'. Frege does not put forward definitions of this kind.

Philosophy, B. McGuinness, ed., E.–H. W. Kluge, tr., Oxford: Basil Blackwell, 1984, pp. 293–340, at pp. 303–304).

It is not required of a definition that it *identify* the extension, i.e. the definition need not specify *who* or *what* the extension is. It is enough for the definition to lay down conditions that, taken together with the extra-definitional facts, determine a unique extension. One could introduce a singular term, for example, by means of the following:

Victor2020 $=_{df}$ the person who will win the 2020 U.S. presidential election.

No particular person is identified by name as the designatum. Still less is it a consequence of the definition that Victor2020 is not Julius Caesar. But the definition together with future worldly facts determine a unique designatum—even though it is presently unknown who the designated person so determined is. An example of a definition that fails to meet the relevant requirement is the following:

Bill Clinton $=_{df}$ a particular former president of the United States.

This is not even well-formed. There are multiple former U.S. presidents, each a particular one. As a consequence, even when the attempted definition is taken together with the relevant extra-definitional facts of U.S. history, no semantic designatum for the definiendum is uniquely determined. The proposed explanation together with the world fail to select a unique semantic extension from among the candidates. The name is ill-defined.

Well-defined notation automatically satisfies a further desideratum on definitions, *the eliminability requirement*: the definiendum must be replaceable by the definiens in all extensional contexts, i.e., excluding quotation marks or the like. Indeed, in a proper content-preserving definition, the definiendum is replaceable by the definiens in all (singly) *ungerade* contexts as well as all extensional contexts. Whatever the truth-value is of 'Victor2020 is married to Bill Clinton', it is the same for 'The person who will win the 2020 U.S. presidential election is married to Bill Clinton'. As Frege says, "If on the left-hand side of a definitional equation we have a proper name that is correctly formed from our primitive names or defined names, then this will always have a designatum (*Bedeutung*) and we can put on the right a simple, hitherto unused sign which is now introduced by our definition as a co-designative proper name, so that this sign may in the future be replaced wherever it occurs by the name standing on the left." (*BLA*-I§33, stricture 4; throughout the present essay I have put the definiendum on the left-hand side, the definiens on the right.) Even before *BLA*, in fact before *FA*, Frege remarked in *Begriffschrifft* (1879) concerning a sample definitional sentence that the definiendum is a mere abbreviation (*Abkürzung*) of the definiens, hence eliminable (§24):

> We can do without the notation introduced by this sentence and hence without the sentence itself as its explanation; nothing follows from the sentence that

could not also be inferred without it. Such explanations have the sole purpose of bringing about an extrinsic simplification by fixing an abbreviation.[22]

Even a system of definitions that together simultaneously define a plurality of interconnected terms must fix unique semantic extensions for each of the definienda. The system must "triangulate," as it were, unique semantic extensions for each of the definienda in order that they be well-defined. The discredited definitional system *D0num–D ∼ num* does not have the proper form to define '0', '1', '*s*', and 'numerate' simultaneously. The *basic proper form* of a definition for '1' has the form '$1 =_{df} \ldots$' where the dots are replaced with a closed singular term for the number one. It would be natural, in fact, to replace *D1num* with

D1: $1 =_{df} s(0)$,

thereby recognizing the analyticity of 'One is the successor of zero'. This serves the purposes at hand if, but only if, the system *D0num–D ∼ num* determines unique extensions for '0', '*s*', and 'numerate' simultaneously. *D№* will then take care of '№'.

The basic proper form of a definition for '0' has the form '$0 =_{df} \ldots$' where the definiens (the right-hand side) is a closed term for the number zero. A basic proper form of a definition for '*s*' has the form '$s(n) =_{df} \ldots n \ldots$', where the definiens is an open singular term whose only free variable is '*n*' and whose designatum under the assignment of a value to '*n*' is precisely the successor of that assigned value. Alternatively (although it is tantamount to the same thing), '*s*' may be properly defined using Alonzo Church's important functional-abstraction operator, by means of a definition of the form '$s =_{df} \lambda n[\ldots n \ldots]$'.[23] A basic proper definition for the relational term 'numerate' is a definition of the form '*n numerates* F $=_{df} \ldots n \ldots$ F …' where the definiens is an open formula, with '*n*' and 'F' as its only free variables, satisfied by exactly those pairs of numbers and concepts such that exactly *that many* objects fall under *that concept*. Definitions of these basic proper forms satisfy the eliminability requirement. Proper definitions may deviate from the basic proper forms. As Frege notes, a plurality of terms may be properly defined simultaneously by a system of interconnected definitions. A definitional system should fix an interrelationship among the definienda in such a way that the definitions determine for each of the definienda a unique semantic content. At least the definitional system together with relevant extra-definitional facts must determine the semantic extensions. For example, the standard simultaneous recursive definition of 'formula' and 'singular-term' for a formal language in which compound formulae are built from singular terms and compound singular terms are built from formulae does just this. Insofar as the mission of *D0num–D ∼ num* is to define 'numerate', '0', '1', and '*s*' simultaneously, the system is a dismal failure. As Frege notes, *D0num* and *D1num* succeed only in defining simple expressions—in effect words, and highly misleading words at that—for the concepts of being a first-

[22] See also Frege's "Logic in Mathematics" (Spring 1914), in his *Posthumous Writings*, H. Hermes, F. Kambartel, and F. Kaulbach, eds (University of Chicago Press, 1979), pp. 203–250, at pp. 207–211; reprinted in M. Beaney, ed, *The Frege Reader* (Oxford: Blackwell, 1997), at pp. 313–318.

[23] Perhaps Frege sees functional abstraction as impossible. If so, he is mistaken.

level concept with nothing falling under it, and with exactly one thing falling under it, respectively. Frege's program needs something different: purely logical definitions for '0', '*s*', and '№'.

A definition for the functor '$\upharpoonleft\!\downharpoonright$' that satisfies the eliminability requirement may take the basic proper form '$\upharpoonleft\!\downharpoonright(l) =_{df} \ldots l \ldots$' where the definiens has the form of an open singular term whose only free variable is '*l*'. *DP* itself does not have this form, or any other form to provide a proper definition for '$\upharpoonleft\!\downharpoonright$'. If *DP* is offered as providing a definition in terms of '$/\!/$' or 'parallel' (which is taken as antecedently understood), at best it yields a definition for a syntactically simple (but deceptively orthographically complex) relational predicate that applies to pairs of lines that are co-oriented. It is all well and good to define 'co-oriented' independently of '$\upharpoonleft\!\downharpoonright$' as a synonym for '$/\!/$', but what we seek is a definition for '$\upharpoonleft\!\downharpoonright$' or, more precisely, for the functor 'the directional orientation of'. If the functor can be defined in terms of 'co-oriented', which is itself defined independently of '$\upharpoonleft\!\downharpoonright$', that would do splendidly. This *DP* fails to do. Taken as a definition of '$\upharpoonleft\!\downharpoonright$', *DP* does not meet the eliminability requirement. Analogously, if *HP* is offered as providing a definition, but not one for '$\approx$' (which is taken as antecedently understood), then at best it yields a definition for a syntactically simple (but orthographically complex) relational predicate that applies to concepts whose extensions have the same cardinality. It is all well and good to have a purely logical definition for 'equinumerous' independently of 'number', as *HP* provides. But what Frege seeks is a horse of a different color: purely logical definitions for '#' and ultimately for '№'. (This he provides prematurely but with a promissory note in *FA*§68, and ultimately in §72.)

IV

What has any of this to do with Julius Caesar?

As noted above, immediately before raising the Caesar problem Frege points out that the *FA*§55 system of definitions is viciously circular. As a direct consequence of the vicious circularity, the system does not simultaneously determine unique semantic extensions for all of '0', '1', '*s*', and 'numerate'(Frege's 'is the number belonging to'). The definitional system consisting of *D0num*, either *D1num* or *D1*, *Dsnum*, and $D \sim num$ render the definienda ill-defined.

We consider a gerrymandered relation (-in-extension) between objects *x* and unary concepts F. Let us say that *x Julius-numerates* F (alternatively, *x Julius-numerically belongs to* F) if either *x* is the number of F's or else there is exactly one F and *x* is Julius Caesar: $(\lambda x\mathrm{F})[(\text{№}(x)\ \&\ x$ numerates F$) \vee (\exists! y\mathrm{F}y\ \&\ x = \text{Caesar})]$.[24] ('$\exists! y\mathrm{F}y$' is used throughout as shorthand for '$\exists z\forall y(\mathrm{F}y \leftrightarrow z = y)$'.) Then each of the following is true:

Zero Julius-numerates $\lambda x[x$ is a moon of Venus$]$.
One Julius-numerates $\lambda x[x$ is a moon of the Earth$]$.

[24] The name 'Caesar' may be replaced by a non-identifying definite description provided it singles Caesar out (e.g., 'the famous conqueror of Gaul').

Two Julius-numerates $\lambda x[x$ is a moon of Mars].
Caesar Julius-numerates $\lambda x[x$ is a moon of the Earth].

The Julius-numerating relation is a slight expansion of the numerating relation. A natural number *n* Julius-numerates a unary concept iff there are exactly *n* F's. In fact, anything other than Caesar (including the number one) Julius-numerates a concept iff it numerates that concept. Caesar also Julius-numerates some concepts, a great many in fact. Specifically, Caesar Julius-numerates all, but only, those concepts under which exactly one thing falls. Let us say that something is a *Julian number* if there is a concept that it Julius-numerates. The Julian numbers are just Julius Caesar and the cardinal numbers. The definition of 'Julius-numerate' invokes a non-logical constant. The notion of a Julian number, then, is not one of pure logic. But it is a perfectly legitimate notion.

A flaw in the *FA*§55 definitions now emerges as a glaring defect: They fail to discriminate between the cardinal numbers and the Julian numbers. They do not preclude that 'numerate' is a predicate for Julius-numerating and '№' a general term applicable to Julius Caesar and the numbers. The definienda are thus ill-defined. The definitions constrain the semantic contents of '0', '1', '*s*', 'numerate', and '№', in that their semantic extensions are required to be interrelated in a particular way. But the semantic extensions themselves, as opposed to their interrelationships, are not uniquely determined. Even assuming that all the logical symbols are interpreted as having their usual meanings, the definitions do not determine unique semantic contents for the definienda, whose meanings are the "unknown quantities."

The inadequacy of the definitions can be demonstrated using model-theoretic machinery handed down to us by Alfred Tarski. Although Frege did not have model theory as we now know it, his point about Julius Caesar is very much a semantic point about definitions, one well explicated using model theory. A proper definition of a term must fix the semantic content of the term, and this requires that the term's semantic extension (*Bedeutung*) also be pinned down. This is no less true of recursive definitions. Consider the standard definition of the physical constant '*c*' occurring in the famous equation '$E = mc^2$':

Dc: $c =_{df}$ the velocity of light in a vacuum.

Being a definition, this should be regarded as placing a restriction on what is to count as an *admissible model*: The identity sentence incorporated by *Dc* is analytic, and hence is to be true in all admissible models; the definiendum '*c*' and the definiens 'the velocity of light in a vacuum' designate the same thing in every admissible model. The definition thus determines (fixes) the extension of the definiendum, in the sense that if in both of a pair of admissible models, $\mathfrak{m}$ and $\mathfrak{m}'$, the term 'velocity' has the same extension, and similarly for the terms 'light', and 'vacuum', respectively, then '*c*' likewise designates the same thing in $\mathfrak{m}$ as in $\mathfrak{m}'$. In particular, the designatum of '*c*' in any admissible model that correctly interprets 'the velocity of light in a vacuum' will *ipso facto* be the real velocity of light in a vacuum. Any admissible model that gives the definiens its intended interpretation *ipso facto* determines the correct semantic designatum of the definiendum. This is not to say that such models specify *what velocity* '*c*' designates (e.g., as approximately 3×10^8 meters/second). Models do not *specify* anything. They provide semantic extensions for non-logical vocabulary. (In specifying the model one may characterize it "by description" in Russell's sense.)

The situation is more delicate regarding contextual definitions. Russell brilliantly made use of contextual definition as part of his overarching view that the definiendum itself—most famously, a definite description—is an incomplete symbol, having no semantic content (no "meaning in isolation"). This cannot be part of Frege's understanding of the proposal to define '#' by means of *HP*. Frege is emphatic that cardinality terms like '#(horse)' are contentful. Insofar as Frege sees the proposal as providing a definition for cardinality terms, he would insist that the definition, if proper, likewise pins down the term's semantic extension. Consider the following illustration: If one holds (against Russell) that a set-theoretic description $\ulcorner\{\alpha \mid \phi\}\urcorner$ is a genuine singular term, one might provide what might be called a 'contextual definition' as follows:

$$D\{\}: \quad \psi(\{\alpha \mid \phi\}) =_{df} \psi(\ \imath\beta[\beta \text{ is a set } \& \ \forall\alpha(\alpha \in \beta \leftrightarrow \phi)]\ ),$$

where the left-hand side is an arbitrary sentence in which a set-theoretic description $\ulcorner\{\alpha \mid \phi\}\urcorner$ occurs (not within quotation marks or the like), and the right-hand side is the result of uniformly substituting the definite description $\ulcorner \imath\beta[\beta$ is a set & $\forall\alpha(\alpha \in \beta \leftrightarrow \phi)]\urcorner$. Let α be '*x*' and ϕ be the open formula '*x* has read *Waverley*'. Then every biconditional of the form $\ulcorner\psi(\{x \mid x$ has read *Waverley*$\}) \leftrightarrow \psi(\imath y[y$ is a set & $\forall x(x \in y \leftrightarrow x$ has read *Waverley*$)])\urcorner$ is true in every admissible model. Now let $\mathfrak{M}$ be an admissible model that assigns to '*Waverley*' and to 'has read' their proper English extensions. If '$\{x \mid x$ has read *Waverley*$\}$' designates anything in $\mathfrak{M}$, it designates the set of all those who have read *Waverley*, or at least something that is indiscernible from it as regards the properties that are expressible in the object language. In particular, '$\{x \mid x$ has read *Waverley*$\}$' cannot designate Julius Caesar in such a model. The crucial point is this: Any admissible model (i.e., any model of *D{}*) that gives the definientia—here ψ, ϕ, 'set', and the logical symbols—their intended interpretation determines the correct interpretation of the definiendum, or at least constrains the interpretation by excluding incorrect semantic extensions for the description. The definition's purpose is precisely to achieve this very result.

Within a model-theoretic framework, Frege's objection to the *FA*§55 and §§ 62–64 proposals is that there are models of the definitions (i.e., admissible models) which correctly interpret the definientia but which nevertheless misinterpret some of the definienda. Indeed, the definitions *D0num-D№*, *even on their intended interpretation*, do not logically impose any relevant constraint on the semantic extension of '№'. The definientia in this case are purely logical ('$\forall$', '$\sim$', etc.), so that any model automatically gives all the definientia their intended interpretations —other than 'numerate', which is a definiendum. One such unintended admissible model takes the variables '*a*' and '*b*' to range over any definite set of individuals and the variable 'n' to range over $\mathbb{N}_0$ (the natural numbers including zero), and assigns semantic extensions as follows[25]:

$$\begin{array}{rl} \text{`0'}: & \text{one} \\ \text{`1'}: & \text{two} \\ \text{`}s\text{'}: & \lambda n[n+1] \\ \text{`numerate'}: & (\lambda n\mathrm{F})[\text{there are exactly}(\mathrm{n}-1)\ \mathrm{F's}] \end{array}$$

This model assigns something other than the intended semantic extension to three of the definienda—'0', '1', and 'numerate'. Only the functor '*s*' (besides the definientia) is assigned its intended extension. Yet each of the §55 definitions and *D1* is true on this model. The definitions fail to decide between the intended model and this unintended model in their attempt to assign the intended semantic extensions to the definienda. That attempt thereby fails.

That the definitional system *D0num–D$\sim$num* does not simultaneously well-define '0', '1', '*s*', and 'numerate' is dramatically illustrated by means of a much cruder misinterpretation that the definitions permit by not precluding it. Consider the model that assigns '0' zero and '1' one, but assigns 'numerate' the Julius-numerating relation, '№' the Julian numbers, and '*s*' the partial *Julius-successor function*: $\lambda x[\ \imath y(\ [x = 0\ \&\ y = \text{Caesar}] \vee [x = \text{Caesar}\ \&\ y = 2] \vee [x \in \mathbb{N}_1\ \&\ x > 1\ \&\ y = x + 1])]$.[26] This unintended model satisfies the §55 definitions and automatically correctly interprets their definientia, which are entirely logical. More to the point, the model simultaneously satisfies *D0num–Dsnum* together with the implicit definition *D*№ and even *D$\sim$num*. (See note 6.) It is therefore admissible. The definitional system does not decide between the intended model and this unintended alternative, or among infinitely many similar admissible models. Frege's objection is not the nowadays familiar point that there are non-standard models of the Peano postulates. The point, rather, is that even with all the definientia receiving their intended interpretation, and even taken together with all of the extra-definitional facts, the definitions fail to determine a unique semantic extension for 'numerate' and '№'. For all that the definitions accomplish—even when properly interpreted and even when taken together with the worldly facts—it remains undetermined whether '№' applies to Caesar, whether Caesar counts as a "№."

Looked at another way, *FA*§55 contemplates expanding a language *L* that lacks '0', '1', '*s*', 'numerate', and '№' into a language L^+ that is suitable for Peano arithmetic, by adding those expressions and by stipulating semantic postulates governing their use. The §55 "definitions" may be reformulated as the following semantic postulates, which supplement the standard clauses of a Tarskian definition of 'true-in-*L*' (e.g., ''Caesar' designates-in-*L* Julius Caesar'):

[25] A definition $\ulcorner \chi =_{df} \zeta \urcorner$ is true in (satisfied by) a model iff the definiendum χ and the defining expression ζ are universally co-extensional in the model. Models for the *FA*§55 definitions are unusual in that the definienda are intended to be purely logical expressions (as the definientia are in fact). The mere fact that there are models that simultaneously satisfy the definitions while assigning the definienda different, non-logical semantic extensions shows that the definitions do not well-define the definienda. Consider by contrast the following properly purely logical definition: $(\phi \mathbin{\bar{\vee}} \psi) =_{df} [(\phi \vee \psi)\ \&\ \sim(\phi\ \&\ \psi)]$.

[26] This is the function s' defined as follows: (*i*) $s'(0) = \text{Caesar}$; (*ii*) $s'(\text{Caesar}) = 2$; and (*iii*) if $n \in \mathbb{N}_1$ and $n > 1$, then $s'(n) = n + 1$. This is easily expanded into a fully defined function.

P: Every formula that is true-in-L is also true-in-L^+.

Where Π is a monadic-predicate and α is a singular term:

P0num	$\ulcorner$0 numerates $\Pi\urcorner$ is true-in-L^+ iff $\ulcorner\forall a \sim \Pi a\urcorner$ is true-in-L.
P1num	$\ulcorner$1 numerates $\Pi\urcorner$ is true-in-L^+ iff $\ulcorner \sim\forall a\sim\Pi a$ & $\forall a\forall b(\Pi a$ & $\Pi b \rightarrow a = b)\urcorner$ is true-in-L.
Psnum	$\ulcorner s(\alpha)$ numerates $\Pi\urcorner$ is true-in-L^+ iff $\ulcorner\ \exists a(\Pi a$ & α numerates $\lambda b[\Pi b$ & $a \neq b])\urcorner$ is true-in-L^+.
$P\sim$*num*	$\ulcorner\alpha$ numerates $\Pi\urcorner$ is not true-in-L^+ unless it is true-in-L^+ according to *P-Psnum*.
P№	$\ulcorner$№(α)$\urcorner$ is true-in-L^+ iff $\ulcorner\exists$G(α numerates G)$\urcorner$ is true-in-L^+.

The problem now is that these postulates do not rule out that 'numerate' is a term for the Julius-numerating relation in L^+ and '№' a term for the Julian numbers. In particular, the postulates fail to deliver the following *T*–sentence:

'№(Caesar)' is true-in-L^+ iff Julius Caesar is a cardinal number.

Indeed, the postulates are perfectly compatible with:

'№(Caesar)' is true-in-L^+ iff Julius Caesar is a Julian number.

As far as *P*–*P*№ go, even given the fact that Caesar is not a number (and a Tarskian definition of 'true-in-L'), it remains undecided whether '№(Caesar)' is true or false in L^+.

With the Julius-numerating relation in hand, the content of the Julius Caesar problem clearly emerges: The *FA*§55 definitions fail to well-define 'numerate'. Even when the definientia are correctly interpreted, and even when taken together with the relevant extra-definitional facts, the definitions do not determine a unique semantic extension for each of the definineda. This defect is forcefully, albeit crassly, illustrated by the fact that the general term '№' can be misinterpreted as applying to the Julian numbers so that '№(Caesar)' is true while still satisfying *D0num*-*D*№, including $D\sim$*num*, correctly interpreting the definiens, correctly interpreting 'Caesar', and respecting the non-logical fact that Caesar is no number.

The alternative proposal to define '#' by means of *HP*, combined with the *FA*§72 definition of 'number' as the cardinality of a concept, has the same fatal defect, although for a different reason. Although the proposed definition of '#' by means of *HP* is not viciously circular, it together with the extra-definitional facts fails to determine a unique semantic extension for '#(F)' under the assignment of a particular concept to 'F'. The cardinality functor is not well-defined by *HP*. To illustrate, Frege first observes that *DP* fails to define a functor for directional orientation. Consider the gerrymandered function that assigns to any infinitely extended line a its directional orientation if a is not parallel to the Earth's axis, but assigns England to any line parallel to the Earth's axis: $\lambda a[\ \iota x(\ [a$ is not parallel to the Earth's axis & $x = \upharpoonleft\downharpoonright(a)] \vee [a$ is parallel to the Earth's axis & $x =$ England])]. Let us call this feature of a line its *English direction*. The English direction of any line not parallel to the Earth's axis is simply the line's direction, whereas the English direction of the Earth's axis, as well as of any line parallel to it, is England. The failure of *DP* to provide a definition for ‘⥮’ is now lade bare: *DP* does not discriminate between the direction of a line and the English direction. Even given the extra-definitional facts, including the fact that England is not a direction, for all that *DP* imposes on ‘⥮’, that symbol could be a term for the English direction function instead of the real direction function. Taking it as a term for English direction fully accords with *DP*. By analogy, consider the gerrymandered function on unary concepts that assigns to any unary concept F its cardinality if there is exactly one F, but assigns England to any concept F such that there is exactly one F: $\lambda \mathrm{F}[\ \iota x(\ [\sim\exists! y\mathrm{F}y\ \&\ x$ numerates F$] \vee [\exists! y\mathrm{F}y\ \&\ x =$ England$])]$. Let us call this feature of a unary concept its *English cardinality*. Note that whereas the cardinality of the concept $\lambda x[x$ is a planet of the Solar System$]$ and the English cardinality of the same concept are the same (currently believed to be eight), the cardinality of $\lambda x[x$ is a moon of the Earth$]$ is one while the English cardinality is England. Even taken together with the extra-definitional facts, including the fact that England is not a number, *HP* does not decide between the cardinality and English cardinality functions. *HP* and the worldly facts thereby fail to determine whether ‘$\exists \mathrm{F}$ [England $= \#(\mathrm{F})$]’ is true, whether England counts in the object-language as a “number.”

The principal problem with the prospect of defining a term for a functional attribute f (e.g., directional orientation or cardinality) by means of an abstraction principle can be thrown into even sharper relief. The proposed definition of f by abstraction is most naturally converted into an explicit definition as follows:

Dabs $\quad f =_{df} (\iota f')[\ \forall x \forall y(\mathrm{F}x\ \&\ \mathrm{F}y \rightarrow [f'(x) = f'(y) \leftrightarrow x\ \phi_{eq}\ y])\ ].$

This proposed definition captures the notion of *that which, among* F*’s, all and only* ϕ_{eq}*-mates have in common*. The problem now is that the definite description on the right-hand side is improper. (This problem remains even if the ‘f'’ is taken as ranging over functions-in-extension rather than functions-in-intension.) For example, synonyms have in common not only their meaning but also their *synonymy class*, i.e., their shared set of synonyms. All and only home-mates have several distinct functional attributes in common: their residence, their household, their address, the roof over their heads, etc. In general, there are infinitely many functions f'—and two is too many—that, among F’s, yield the same value for all and only ϕ_{eq}-mates. One-to-one corresponding concepts share the same cardinality, but they also share the same English cardinality (and much else besides).[27]

[27] *Cf.* Gary Kemp, “Julius Caesar from Frege’s Perspective,” *dialectica*, 59, 2 (June 2005), pp. 179–199. Kemp proffers an interpretation of the Caesar problem close to the one offered here. (See also Angelelli’s paper cited in note 18 above. My own interpretation was arrived at independently; see note 8 above.) Although Kemp’s emphasis is different from my own (for example, he relegates to a parenthetical remark what I take to be Frege’s central criticism), I am largely sympathetic to his observations. It is important to recognize that *D*№, which is an intended, tacit supplement to *D0num–D* ~ *num*, is meant to provide the very predicate that Kemp suggests would rule out that Caesar is a number. But, as Kemp seems to say, *D* № fails to rule this out.

Definition by abstraction might be repaired as follows: $f =_{df}$ (the *most salient* f')$[\forall x \forall y(\mathrm{F}x\ \&\ \mathrm{F}y \rightarrow [f'(x) = f'(y) \leftrightarrow x\ \phi_{eq}\ y])]$, e.g., *direction* $=_{df}$ the most salient feature that, among lines, all and only parallel ones have in common. (*Cf.* Rudolf Carnap, *Meaning and Necessity*, University of Chiacago Press, 1947, One exegetical point warrants special emphasis. As I interpret Frege, the Julius Caesar problem is not that the definitions do not entail that the number one is not Caesar. The problem is not even that the definitions do not entail that Caesar is not a number at all. *Indeed, a correct system of definitions for '0', '1', 's', 'numerate', and '№'* should not entail that Caesar does not count as a number. A proposed definition of 'number' that does so oversteps its bounds, assigning more semantic content to the word than it has. There is no actual inconsistency in the surreal notion that Caesar is a number. Notoriously, Frege believes that the English sentence 'Julius Caesar, if he existed, conquered Gaul' is analytic in some idiolects. This is a significant philosophical error on his part. But it presents no compelling reason to suppose he also mistakenly believes that there is an actual contradiction in the bizarre idea that a number conquered Gaul. The definitions need not, and indeed should not, decide the actual truth-value of '№(Caesar)'. It would be enough if the definitions determined its necessary-and-sufficient semantic *truth-conditions*, i.e., that '№(Caesar)' is true-in-the-object-language iff Caesar is a number—allowing for the logical possibility (albeit metaphysical impossibility) that Caesar is in fact a number. But the definitions fail to do even that much. The Julius Caesar problem is that even when the definientia are correctly interpreted, *and even taken together with the relevant extra-definitional facts, including that Caesar is no number*, both the *FA*§55 definitions and *HP*, construed as a definition of '#' and supplemented by the appropriate analog of *D*№, fail to determine a unique truth-value for '№(Caesar)'. In short, both of the proposed definitional systems render '№' ill-defined.[28]

The failure of *D0num-D*№ to well-define 'numerate' and '№' has nothing to do with the need for an identity criterion for numbers that is applicable to all cases. Nor is the problem that the proposed definitional systems fail to specify or constrain what kind or category of thing a number is. Rather, the Julius Caesar problem is that the particular definitional systems fail to provide non-circular, necessary and sufficient conditions for the semantic application of 'numerate' or '#', and so do not

Footnote 27 continued
1956: "If two designators are equivalent [co-extensional], we say also that they have the same *extension*. If they are, moreover, L-equivalent, we say that they have also the same *intension*. Then we look around for entities which might be taken as extensions or as intensions for the various kinds of designators," at p. 1.) This proposed fix is not suitable for Frege's purpose. Even if the cardinality of a concept is more salient than its English cardinality, that it is so is no truth of logic.

[28] *Cf.* C. Parsons, "Frege's Theory of Number" (*loc. cit.* in note 13 above), §V. Dirk Greimann sees the Caesar problem as partly a failure by definitions to fix truth-conditions for certain sentences. (See note 11 above.) However, he also sees the problem as a failure to fix truth-values, or more generally, *Bedeutungen* (*op. cit.*, pp. 269, 271, 272, 274). (Greimann does not sharply distinguish between fixing truth-conditions and fixing truth-values.) He furthermore sees the problem as a failure to satisfy more controversial requirements. He concludes, contrary to my interpretation, that

> the ultimate root of the Caesar Problem in *Grundgesetze* is that the objective content of the notions of number and value-course does not determine a unique sort of object. In order to overcome this kind of referential indeterminacy, we must fix the transsortal identity-conditions of numbers and value-courses, i.e., we must determine whether the numbers are value-courses, whether some value-courses are truth-values, and so on. This is precisely what Frege actually does in order to overcome the Caesar problem. (p. 276).

pin down or even constrain a unique semantic extension for '№', not even a semantic extension that is beyond our knowledge. In particular, even taken in conjunction with the whole of pure mathematics, *D0num-D№* fail to decide between the numerating and the Julius-numerating relations as the semantic extension of the predicate 'numerate'. The supplementary definition *D№* thereby fails to assign to the term '№' a unique extension which *as a matter of fact* excludes Caesar. It is in this sense that "we can never decide through our definitions whether the famous conqueror of Gaul is a number or not." Even taken together with the extra-definitional facts, the *FA*§55 definitions do not "decide" [*entscheiden*]—they leave it open—whether Caesar counts as a thing that "numerates."

To repeat, the Caesar problem is a failure by particular flawed definitions to well-define a term or terms. It is a problem with definitions that are defective in a particular way: the definiens itself, and even taken together with the extra-definitional facts, fail to determine a unique semantic extension for the definiendum. The Caesar problem is in no way a problem for the official definitions that Frege provides in *FA* and *BLA*. Neither is it a problem with axioms that are not put forward as definitions. In particular, it is in no way a problem with either *DP*, *HP*, or the inconsistent Basic Law V, as long as these principles are not taken as defining '⥮', '#', and 'extension', respectively. Nor is it especially a problem about the metaphysical category of the natural numbers or about deciding cross-categorical identity statements. Some interpreters have been derailed by the fact that Caesar is of a completely different metaphysical category from numbers. Frege's choice of an historical figure as an exemplar non-number is, at least in part, for dramatic effect. The Julius Caesar problem is equally illustrated through consideration of the alternative relation $(\lambda x\mathrm{F})[(x = \varnothing \,\&\, \exists! y\mathrm{F}y) \vee (\sim \exists! y\mathrm{F}y \,\&\, \text{№}(x) \,\&$ x numerates F)]. The fatal flaw is exactly the same as before, only with Caesar's role now played by a mathematical object.

A well-known case of a term that is ill-defined arises in connection with the imaginary unit i of complex-number theory. The familiar explanation is that i is a non-real number whose square is –1: $i^2 =_{df} -1$. This definition does not satisfy Frege's strictures; it does not have the proper form to well-define 'i'. Solving for the unknown yields:

$$i = \pm\sqrt{-1}.$$

This is not a single equation, but a disjunction of equations. The familiar definition renders 'i' ill-defined. It is tantamount to a system of definitions intended to fix the designations of both 'i' and 'j' simultaneously:

Di²j² $\quad i^2 = j^2 = -1$
Dij $\quad i + j = 0$

According to complex-number theory, there is a pair of imaginary numbers that satisfy these definitions. But no way is given of singling one out except by reference to the other (e.g., as the product of the other with –1). The definitional system is entirely symmetric with respect to 'i' and 'j'. Just as *D0num–D*$\sim$*num* fail to decide between the numerating and Julius-numerating relations as the semantic extension of 'numerate', *Di²j²-Dij* fail to decide between the two candidate designata for 'i'.

The definitions thereby fail to determine a unique designation for '*i*'. Even when taken in conjunction with the whole of mathematics, the question of which of the distinct numbers whose square is –1 '*i*' designates has been given no definitive answer.[29]

Frege's *FA*§62 remarks quoted above concerning the need for a definition for numbers to provide "criterion" for the identity of particular numbers need not be seen as imposing any unreasonable requirement on an appropriate definition of 'number', or even any particularly demanding requirement. It should be noted that Frege does not use the word '*Kriterium*', the German word closest to the English 'criterion'. Instead he uses '*Kennzeichen*', German for a *characteristic feature* or *distinguishing mark*. This suggests not so much a reductionist notion of an identity criterion as it does Frege's later notion of *sense* (*Sinn*). Both notions are theory-laden, to be sure, but they are laden with different theories.[30] Instead of imposing a controversial requirement, Frege's remarks may be read as demanding merely that a proper definition of a numeral or a number-word lay down conditions that are sufficient, when taken together with the extra-definitional facts, to zero in on (if the reader will pardon the expression) a unique number as the semantic designatum for the defined number-term. It is not permissible, for example, to define '3' by writing: $3 =_{df}$ a prime number less than 4. Such a "definition" is a disgrace. Even when taken together with all of the facts of mathematics, this fails to decide between two and three as the designatum of '3', and thus renders '3' ill-defined. (Notice that what is at issue here is not the identity of three with an object, such as three itself, but the designation of the numeral '3'.) This observation is significant, but it is also philosophically modest. It carries with it no substantive theses about reductionist "identity criteria" or the like.

Frege's *FA*§62 remarks might be fairly translated thus:

> If the sign *a* is to designate for us an object, we must have a distinguishing feature, which determines everywhere whether *b* is the same as *a*, even if it is not always in our power to use the distinguishing feature. … After we have thus acquired a means of getting at a definite number and of recognizing it as the same, we can give it a number-word as its proper name.

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[29] Under the given system of definitions nothing decides which of the pair of non-real numbers whose square is –1 is designated by '*i*' and which by '–*i*'. Instead, the depiction of one as positive and the other as negative is merely an artifact of the notation. It cannot be said, for example, that one of *i* and –*i* is greater, and the other less, than zero. In the greater-than relation among real numbers, each of *i* and –*i* is incommensurate with zero. Instead, given those definitions we operate under the pretense that '*i*' designates one of the two candidates and '–*i*' the other. The symbol '*i*' would thus be not a genuine constant, but a free variable obtained through existential instantiation. The pretense that the instantial term '*i*' is a constant is not problematic, however, since it makes no algebraic difference which of the two candidates is the value of '*i*' and which is the value of '–*i*'.

On the other hand, such unresolved semantic issues are to be avoided in the theoretical structure of a demonstrative science and ought not to occur in a perfect language. Evidently Frege had intended to extend his logicist program to complex numbers but ultimately gave up on the project, at least as it was conceived in *FA* and *BLA*.

[30] Ironically, the philosophical thesis that definitions (or theories, etc.) must provide identity criteria might have originated in a misinterpretation of *FA*§62, or a mistranslation (or both).

These remarks might be seen as saying something specifically about definition, something that should be uncontroversial—something significant to be sure, but nothing more than *BLA*-II§66 says, and nothing issuing a demand for identity criteria in the contemporary sense.

Acknowledgements I am indebted to Harry Deutsch, Steven Humphrey, Gary Kemp, Oliver Marshall, Michael Rescorla, Teresa Robertson, and Clark Sexton for discussion and comments, to Jutta Schamp for assistance in translation, and to Yoko Graham Ishii (age 6) for correcting a transcription error.

Appendix: Quantifier numbers versus *FA*-numbers

In *FA*§56 Frege immediately scraps the §55 definitions, at least in part because of the Julius Caesar problem, i.e., because of their failure to well-define the definienda. In §66 he dismisses for the same reason the proposal to define by abstraction a functor for the function that assigns to any concept its numerater. Ultimately he chooses instead to define the cardinality of a concept as the equivalence class of concepts in 1–1 correspondence with the concept. Whereas the Caesar problem is indeed fatal to definition by abstraction, Frege is not forced to abandon the §55 definitions. The discredited definitions are easily retrofitted to be made Caesar-proof.

Frege's analogies of the notion of the cardinality of a concept to that of the directional orientation of a line and the shape of a triangle, although brilliant, also do a disservice to his program. The *FA*§68 identification of a line's directional orientation, and of a triangle's shape, with an equivalence class of things having that direction or shape in common is conceptually dissonant. Directional orientations and triangular shapes are *properties*, *qualities*, or *features* of things, not sets of the things with those properties. Analogously, the meaning of an expression is not its synonymy class. The situation is very different with regard to the natural numbers. Words like 'three' and 'thirty-seven' are unlike terms for directional orientations (e g., 'north–south') or triangular shapes ('equiangular'). Number-words are *determiners*. As such, they are naturally regarded as quantifiers. A quantifier is essentially a quantitative second-order predicate (typically monadic) combined with concept-abstraction (in Frege's idiosyncratic sense of 'concept'). The classical logicist program basically casts numerals as numerically-definite second-order predicates ('nothing', 'exactly one thing', 'exactly two things', etc.). The identification of natural numbers with equinumerous equivalence classes then simply follows.[31]

The following modification of the *FA*§55 definitions is decidedly Fregean in spirit. First a variable-binding term for a zero-concept is defined thus:

$D\nexists \quad \nexists a \mathrm{F}a =_{df} \forall a \sim \mathrm{F}a.$

The operator '$\nexists$' is, in effect, the negated-existential quantifier—in English, the 'nothing' in 'Nothing is without mass', the 'none' in 'None deserves the fair'. Its semantic extension is the function that assigns truth to the empty concept (i.e., to the constant function to falsity) and falsity to everything else.

[31] Compare my "Numbers *versus* Nominalists," *Analysis*, 68, 3 (July 2008), pp. 177–182.

A special functor 'I+' can be defined for the basic arithmetical function that assigns to any quantificational concept Φ the quantificational concept that "immediately succeeds" it:

$$D\mathrm{I}+ \quad [\mathrm{I} + \Phi]a\mathrm{F}a =_{df} \exists a(\mathrm{F}a \mathbin{\&} \Phi b[\mathrm{F}b \mathbin{\&} a \neq b]).$$

These definitions are purely logical. They determine unique semantic extensions for '$\nexists$' and 'I+' respectively. Each of '$\nexists$', '[I + $\nexists$]', '[I + [I + $\nexists$]]', etc. is a numerically-definite quantifier: *nothing*; *exactly one more thing than nothing*; *exactly one more thing than exactly one more thing than nothing*; etc. Let us abbreviate '[I + $\nexists$]' as 'I', '[I + [I + $\nexists$]]' as 'II', '[I + [I + [I + $\nexists$]]]' as 'III', etc. Thus: $\vdash \mathrm{I}x\mathrm{F}x \leftrightarrow \exists!x\mathrm{F}x$; $\vdash \mathrm{II}x\mathrm{F}x \leftrightarrow \exists y\exists z(y \neq z \mathbin{\&} \forall x[\mathrm{F}x \leftrightarrow x = y \vee x = z])$, etc. Each of these numerically-definite quantifiers has its semantic extension the characteristic function of an equivalence class of equinumerous concepts. Those characteristic functions constitute *the quantifier-numbers*: $\nexists$, I, II, III, IV, … .

$$DQ\text{№} \quad Q\text{№}(\Phi) =_{df} \forall\Im[\Im(\nexists) \mathbin{\&} \forall\Psi(\Im(\Psi) \rightarrow \Im[\mathrm{I} + \Psi]) \rightarrow \Im(\Phi)].$$

That is, a second-level concept Φ is a *quantifier-number* iff Φ falls under every (including the most restrictive) third-level concept $\Im$ under which $\nexists$ falls and which is closed under I+.

The notation introduced by *D*$\nexists$-*DQ№* is well-defined. The definitions do not provide identity criteria for numbers, in the sense that previous interpreters think Frege requires, but the Caesar problem, as Frege means it, does not arise.[32] That Caesar is not among the quantifier-numbers is no truth of logic—it is not an analytic truth—but *D*$\nexists$ and *D*I+ together with the facts of logic and the facts about Caesar determine that 'Q№(Caesar)' is false.

For idiosyncratic reasons Frege evidently rejects as impossible any function like I+ whose values are themselves functions. (See note 23 above.) Frege's scruple against functions to functions poses an inconvenience, but not an insurmountable obstacle. There is a straightforward workaround. In lieu of the offending function I+, we define instead the corresponding immediately-preceding relation between quantifier-numbers:

$$D\wp \quad \wp(\Phi, \Psi) =_{df} \forall\mathrm{F}[\Psi x\mathrm{F}x \leftrightarrow \exists a(\mathrm{F}a \mathbin{\&} \Phi b[\mathrm{F}b \mathbin{\&} a \neq b])].$$

The following relationships obtain: $\wp(\nexists, \mathrm{I})$; $\wp(\mathrm{I}, \mathrm{II})$;$\wp(\mathrm{II}, \mathrm{III})$, etc. It follows immediately from $D\wp$ that $\wp(\Phi, \Psi) \mathbin{\&} \wp(\Phi, \mathrm{X}) \rightarrow \forall\mathrm{F}(\Psi x\mathrm{F}x \leftrightarrow \mathrm{X}y\mathrm{F}y)$.[33]

As noted above, on the one hand Frege appears to be content in *FA* to identify numbers with the quantifier-numbers. On the other hand, he sees concepts as functions and therefore not objects, and he is very clear that numbers are objects and therefore not functions. I do not share Frege's view on this issue—the quantifier-numbers are numbers enough for the purposes of classical logicism—but to a

[32] See again note 8 above. Dummett (*op. cit.*, pp. 99–110) misreads *FA*§55 as proffering something like *D*$\nexists$-*D*I+ instead of *D0num–Dsnum*, and misreads *FA*§56 as raising the Caesar problem against *D*$\nexists$-*D*I+. He furthermore misinterprets §§56–61 as a botched attempt to discredit *D*$\nexists$-*D*I+.

[33] The construction offered here does not obviate the need, recognized by Whitehead and Russell, for an axiom of infinity. The Caesar problem is separate from the need for an axiom of infinity.

significant extent his preference can be accommodated.[34] Insofar as Frege insists that numbers be objects, the numeral '0' and the functor '*s*' are easily introduced as the extension-oriented analogs of $\nexists$ and I+. Terms for these analogs are defined using Frege's variable-binding extension-abstraction operator '$\grave{\varepsilon}$', which he regards as a device of pure logic. For good measure, the familiar dyadic predicate '$\in$' of set theory is first introduced as a purely logical term for the binary relation between an object *x* and the extension *y* of a concept G under which *x* falls.

D$\in$ $x \in y =_{df} \exists \mathrm{G}[\mathrm{G}x \mathbin{\&} y = \grave{\varepsilon}\mathrm{G}\varepsilon]$.
D0 $0 =_{df} \grave{\varepsilon}\nexists a\varepsilon(a)$.
Ds $s(n) =_{df} \grave{\varepsilon}\exists a(\varepsilon(a) \mathbin{\&} \lambda b[\varepsilon(b) \mathbin{\&} a \neq b] \in n)$.

The FA-numbers, to be distinguished from the quantifier-numbers, are exactly the elements of the sequence: 0, $s(0)$, $s[s(0)]$, $s(s[s(0)])$, … . These are the numeraters of *FA*, the objects that numerically "belong" to concepts.

With *D0* and *Ds* in play, Frege's *D0num*, *Dsnum*, and *D*$\sim$*num* may be pressed into service to provide a legitimate recursive definition for the relational predicate 'numerate' as a term for a particular third-level relation (and without pretending also to define thereby '0' and a successor-functor—see note 8). The recursive definition is easily converted into a fourth-order "explicit" definition:

Dnum n *numerates* $\mathrm{F} =_{df} \forall\phi[\forall \mathrm{G}[\phi(0, \mathrm{G}) \leftrightarrow \forall x \sim \mathrm{G}x] \mathbin{\&} \forall m \forall \mathrm{H}(\phi[s(m), \mathrm{H}] \leftrightarrow \exists a[\mathrm{H}a \mathbin{\&} \phi(m, \lambda b[\mathrm{H}b \mathbin{\&} a \neq b])]) \rightarrow \phi(n, \mathrm{F})]$.

That is, a class *n* of first-level concepts is said to *numerate* (or to *be a number that belongs to*) a first-level concept F iff *n* stands to F in every (including the most restrictive) third-level binary relation ϕ such that (*i*) 0 stands in ϕ to just the vacuous concepts; and (*ii*) the successor $s(m)$ of a class *m* of concepts stands in ϕ to a concept H iff there is an object *a* that falls under H and such that *m* stands in ϕ to the concept $\lambda b[\mathrm{H}b \mathbin{\&} a \neq b]$. One third-level binary relation that satisfies (*i*) and (*ii*) is the Julius-numerating relation. But *D*$\sim$*num* quickly dispels any fear on this score. The most restrictive relation satisfying (*i*) and (*ii*) is that of an *FA*-number *n* to a concept F of there being exactly *n* F's. Caesar Julius-numerates but he does not numerate.

The implicit definition *D*№ is retained intact. (Alternatively, *DQ*№ may be imitated for the *FA*-numbers.) The cardinality functor '#' is defined in terms of 'numerate':

[34] The logicist view of natural numbers as equivalence classes of equinumerous sets (or analogs thereof) is incorrect. *Cf.* Robert Hambourger, "A Difficulty with the Frege-Russell Definition of Number," *The Journal of Philosophy*, 74, 7 (July 1977), pp. 409–414. A natural number is more accurately a property of a plurality of things—of a multiplicity, of a "many" as opposed to a set—and then only *under a property* (or equivalently, a binary relation between a plurality of things and a property of those things). A week is a whole week; yet the whole weeks in June 2014 are four in total, whereas the weeks in June 2014 are slightly more numerous (approximately 4.2857). *Cf.* my "Wholes, Parts, and Numbers" in J. E. Tomberlin, *ed.*, *Philosophical Perspectives 11: Mind, Causation, and World* (Atascadero, Ca.: Ridgeview, 1997), pp. 1–15.

D# $\#(\mathrm{F}) =_{df} \mathfrak{n}n(n \text{ numerates F})$.

This completes the system of purely logical definitions that replaces the discredited *FA*§55 definitional system as well as *FA*'s key definition (§68) of *the number of* F's as the class of concepts in one-to-one correspondence with F. That Caesar is not among the *FA*-numbers is no truth of arithmetic, but the replacement definitions together with the relevant facts determine that '№(Caesar)' is false. The notation is well-defined. The Caesar problem does not arise.[35]

None of the definitions directly employs the notion of one-to-one correspondence. For Frege's purposes, *HP* must be derivable from them as an analytic truth. Moreover, the use of '#' is justified if it can be proved, as a matter of pure logic, that $\forall n \forall m(n \text{ numerates F} \;\&\; m \text{ numerates F} \rightarrow n = m)$. Here Basic Law V rears its ugly head. Contrary to a widely held view, the logical inconsistency of Frege's program is not due to his identification of the natural numbers with the *FA*-numbers (equivalence classes of equinumerous concepts). An exactly analogous inconsistency awaits the philosophically more natural identification of the natural numbers with the quantifier-numbers. Inconsistency results if the concept-calculus is untyped and the classical λ-conversion rule of λ-expansion is applied across the board directly to unreduced paradoxical constructions like '$\lambda\mathrm{F}[\sim \mathrm{F}(\mathrm{F})]$', which purports to designate the Russellian putative concept, *concept that does not fall under itself.* (*Cf.* the derivation of *HP* in *FA*§73.) The disease is borne not by the identifications but by naïve comprehension, whether right-to-left Basic Law V or unrestricted λ-expansion in an untyped *Begriffsschrift*.[36]

[35] An anonymous referee objected that the Caesar problem likely arises in connection with the corrected definitions proposed here since they employ Frege's extension-abstraction operator '$\grave{\varepsilon}$', which he defines by means of an abstraction principle. Quite the contrary, Frege does not so define extension abstraction, precisely because the Caesar problem would otherwise arise. *Cf. BLA*-I§10.

[36] I explore a variety of unrestricted comprehension principles in "Russell's Law: An Ode to a Logical Theorem," forthcoming in a festschrift edited by Joseph Almog and Jessica Alden Pepp in honor of David Kaplan's 80th birthday.