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Note on Triple Aboodh Transform and Its Application

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Abstract— *In this paper, we introduce the definition of triple Aboodh transform, some properties for the transform are presented. Furthermore, several theorems dealing with the properties of the triple Aboodh transform are proved. In addition, we use this transform to solve partial differential equations with integer and non-integer orders.*

Keywords—triple Aboodh transform, double Aboodh transform, Caputo fractional derivative, Summed transform method.

1. INTRODUCTION

Many real world problems that arise in all the field of applied science are described by partial differential equation of integer and non-integer order. Many researchers have turned their attention to solve partial differential equation and to develop new methods for solving such equations, due to that many papers are published for developing methods for solving partial differential equation, integral equations, fractional differential equation and so on[1,2,3 ,5] "in press" [4]. One of the well-known methods for solving these equations is the integral transform methods, like Laplace transform method [6, 7], Summed transform method [8, 9], Natural transform [10], Ezaki transform method [11], and so on.

Khalid Aboodh[12] in 2013 introduced a new integral transform called Aboodh transform, which is derived from the Fourier integral and similar to Laplace transform, and applied it to solve ordinary differential equation , after that he introduced the double Aboodh transform and used it to solve Integral differential equation and partial differential equation[13]. Aboodh transform method proved very affection methods to solve partial differential equation, and fractional differential equation.

The objective of this article is to extend the Aboodh transform to the triple Aboodh transform, and discuss some theorems and properties about the triple Aboodh transform. To show the applicability and efficiency of this interesting transform we apply this transform to some test examples.

First, we recall the definition of first Aboodh and double Aboodh transforms given by Khalid Aboodh [12] the definition of first Aboodh transform is given by:

$$A\left[f(x)\right]=K(p)=\frac{1}{p}\int_0^\infty f(x)e^{-pt}dt,\ \text{x>0},$$

and the inverse Aboodh transform is defined by:

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})=\frac{1}{2\boldsymbol{\pi i}}\int_{\boldsymbol{\alpha}-\boldsymbol{i}\infty}^{\boldsymbol{\alpha}+\boldsymbol{i}\infty}\boldsymbol{p}\ \boldsymbol{e}^{\boldsymbol{px}}K(p)dp\ .$$

The double Aboodh transform [13] is defined by

$$A_xA_y\left[f(x,y),p,q\right]=K(p,q)=\frac{1}{pq}\int_0^\infty\int_0^\infty f(x,y)e^{-(px+qy)}dxdy,$$

where $f(x,y)$ is continuous function and $x,y>0$.

Moreover, the inverse of double Aboodh transform is given by:

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})=\frac{1}{2\boldsymbol{\pi i}}\int_{\boldsymbol{\alpha}-\boldsymbol{i}\infty}^{\boldsymbol{\alpha}+\boldsymbol{i}\infty}\boldsymbol{p}\ \boldsymbol{e}^{\boldsymbol{px}}\left[\frac{1}{2\boldsymbol{\pi i}}\int_{\boldsymbol{\beta}-\boldsymbol{i}\infty}^{\boldsymbol{\beta}+\boldsymbol{i}\infty}\boldsymbol{q}\ \boldsymbol{e}^{\boldsymbol{qy}}K(p,q)dq\right]dp$$

This article has been organized as follows: In Section 2 we introduce the definition of triple Aboodh transform, and we present the triple Aboodh transform of some partial derivative of function of three variables, in section 3 we present existence and uniqueness of triple Aboodh transform**.** In section 4, we state the convolution theorem of the triple Aboodh transform and its proof. Some theorems and properties of the triple Aboodh transform method are given in section 5. In section 3, we give an analysis of the proposed method. In section 6, we demonstrate the applicability of the triple Aboodh transform by presenting three examples. Finally, the conclusion follows in section 7.

2. Definition of the triple Aboodh Transform

In this section, we introduce the definition of triple Aboodh transform and triple Aboodh transform of partial and fractional derivatives which are used further in this paper, moreover we apply triple Aboodh transform for some basic functions.

Definition 2.1 let f be a continuous function of three variables, then the triple Aboodh transform of $f\left(x,y,t\right)$ is defined by:

$$\boldsymbol{K}\left(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}\right)=\boldsymbol{A}_x\boldsymbol{A}_y\boldsymbol{A}_t\left(\boldsymbol{f}\left(\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}\right)\right)=\frac{1}{\boldsymbol{pqr}}\iint\int_0^\infty \boldsymbol{e}^{-(\boldsymbol{px}+\boldsymbol{qy}+\boldsymbol{st})}\boldsymbol{f}\left(\boldsymbol{x},\boldsymbol{y},t\right)\boldsymbol{dxdydt} \quad (1)$$

In addition, the inverse of triple Aboodh transform is given by:

$$\boldsymbol{f}\left(\boldsymbol{x},\boldsymbol{y},t\right)=A_x^{-1}A_y^{-1}A_t^{-1}=\frac{1}{2\boldsymbol{\pi i}}\int_{\alpha-i\infty}^{\alpha+i\infty}\boldsymbol{p}\ \boldsymbol{e}^{\boldsymbol{px}}\left[\frac{1}{2\boldsymbol{\pi i}}\int_{\beta-i\infty}^{\beta+i\infty}\boldsymbol{q}\ \boldsymbol{e}^{\boldsymbol{qy}}\left[\frac{1}{2\boldsymbol{\pi i}}\int_{\gamma-i\infty}^{\gamma+i\infty}\boldsymbol{r}\ \boldsymbol{e}^{\boldsymbol{rt}}K(p,q,r)dr\right]dq\right]dp \quad (2)$$

First of all, we find triple Aboodh transform for partial derivatives
The triple Aboodh transform of nth derivative of a function of three variables is given by:

$$A_xA_yA_t\left(\frac{\partial^n f(x,y,t)}{\partial x^n}\right)=p^nA_xA_yA_t(f(x,y,t))-\sum_{m=0}^{n-1}p^{n-m-2}A_yA_t\left(\frac{\partial^m f(0,y,t)}{\partial^m}\right),$$

$$A_xA_yA_t\left(\frac{\partial^n f(x,y,t)}{\partial y^n}\right)=q^nA_xA_yA_t(f(x,y,t))-\sum_{m=0}^{n-1}q^{n-m-2}A_xA_t\left(\frac{\partial^m f(x,0,t)}{\partial^m}\right),$$

$$A_xA_yA_t\left(\frac{\partial^n f(x,y,t)}{\partial t^n}\right)=r^nA_xA_yA_t(f(x,y,t))-\sum_{m=0}^{n-1}r^{n-m-2}A_xA_y\left(\frac{\partial^m f(x,y,0)}{\partial^m}\right).$$

- The triple Aboodh transform of mixed derivative of a function of three variables is given by:

$$A_xA_yA_t\left(\frac{\partial^3 f(x,y,z,t)}{\partial x\,\partial y\,\partial t}\right)=pqrK(p,q,r)-\frac{pq}{r}K(p,q,0)-\frac{pr}{q}K(p,0,r)$$
$$-\frac{qr}{p}K(0,q,r)+\frac{p}{qr}K(p,0,0)+\frac{q}{pr}K(0,q,0)+\frac{r}{pq}K(0,0,r)-\frac{1}{pqr}f(0,0,0)$$

$$A_xA_yA_t\left(\frac{\partial^3 f(x,y,t)}{\partial t\,\partial x^2}\right)=rp^2A_xA_yA_t\left(f(x,y,z,t)\right)-rA_yA_t\left(f(0,y,t)\right)$$
$$-\frac{r}{p}A_yA_t\left(\frac{\partial f(0,y,t)}{\partial x}\right)-\frac{p^2}{q}A_xA_y\left(f(x,y,0)\right)+\frac{1}{pr}A_y\left(\frac{\partial f(0,y,0)}{\partial x}\right)$$
$$+\frac{1}{r}A_2\left(f(0,y,0)\right)$$

- The triple Aboodh transform of the partial fractional Caputo derivatives of a function of three variables is given by:

$$A_xA_yA_t\left(\frac{\partial^\alpha f(x,y,t)}{\partial x^\alpha}\right)=p^\alpha A_xA_yA_t(f(x,y,t))-\sum_{k=0}^{j-1}p^{\alpha-k-2}A_yA_t\left(\frac{\partial^k f(0,y,t)}{\partial x^k}\right),$$

$$A_xA_yA_t\left(\frac{\partial^\beta f(x,y,t)}{\partial y^\beta}\right)=q^\beta A_xA_yA_t(f(x,y,t))-\sum_{k=0}^{n-1}q^{\beta-k-2}A_xA_t\left(\frac{\partial^k f(x,0,t)}{\partial y^k}\right),$$

$$A_xA_yA_t\left(\frac{\partial^\gamma f(x,y,t)}{\partial t^\gamma}\right)=r^\gamma A_xA_yA_t(f(x,y,t))-\sum_{k=0}^{m-1}r^{\gamma-k-2}A_xA_y\left(\frac{\partial^k f(x,y,0)}{\partial t^k}\right).$$

Where the Caputo fractional derivative [14] of function $f(x,y,t)$ defined by:

- Aboodh transform of the Some Functions

a. If $f(x,y,t)=1$, for $x>0, y>0, t>0$, then $K(p,q,r)=\frac{1}{p^2q^2r^2}$.

b. If $f(x,y,t)=xyt$, then $K(p,q,r)=\frac{1}{p^3q^3r^3}$.

c. $A_xA_yA_t\left(e^{ax+by+ct}\right)=\frac{1}{p(p-a)q(q-b)r(r-c)}$.

d. $A_xA_yA_t\left(\sqrt{xyt}\right)=\frac{\pi\sqrt{\pi}}{8p^3q^3r^3}$.

e. $A_xA_yA_t\left(\cos(x+y+t)\right)=\frac{p+q+r-pqr}{\left(p+p^3\right)\left(q+q^3\right)\left(r+r^3\right)}$.

f. $\sin(x+y+t)=\frac{pq+pr++qr-1}{\left(p+p^3\right)\left(q+q^3\right)\left(r+r^3\right)}$

g. $A_xA_yA_t\left(\cos(ax)\cos(by)\cos(ct)\right)=\frac{1}{\left(p^2-a^2\right)\left(q^2-b^2\right)\left(r^2-c^2\right)}$.

h. $A_xA_yA_t\left(\sin(ax)\sin(by)\sin(ct)\right)=\frac{abc}{p\left(p^2-a^2\right)q\left(q^2-b^2\right)r\left(r^2-c^2\right)}$.

i. $A_xA_yA_t\left(x^my^nt^v\right)=\frac{\Gamma(m+1)\Gamma(n+1)\Gamma(v+1)}{p^{m+2}q^{n+2}r^{v+2}}$.

3. Existence and uniqueness of triple Aboodh transform

In this section we discuss the existence and uniqueness of the triple Aboodh transform, and we prove the uniqueness of the triple Aboodh transform.

Let $f(x,y,t)$ be a continuous function on the interval $[0,\infty)$, which is of exponential order, that is for some $a,b,c\in\Re$

$$\sup_{x,y,t>0}\frac{|f(x,y,t)|}{e^{(ax+by+ct)}}<\infty .$$

Under the above condition, the triple Aboodh transform exists for all $p>a, q>b, r>c$. In the next theorem, the uniqueness of the triple Aboodh transform is proven.

Theorem 3.1: let $h(x,y,t)$ and $l(x,y,t)$ be continuous functions defined for $x,y,t\geq 0$ and having the Aboodh transform $H(p,q,r)$ and $L(p,q,r)$ respectively. If $H(p,q,r)=L(p,q,r)$, then $h(x,y,t)=l(x,y,t)$.

Proof: If we assume α,β,γ to be sufficiently large, then since

$$f(x,y,t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}p\ e^{px}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}q\ e^{qy}\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}r\ e^{rt}K(p,q,r)dr\right]dq\right]dp$$

We deduce that

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$$h(x,y,t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} p\ e^{px}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} q\ e^{qy}\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} r\ e^{rt}H(p,q,r)dr\right]dq\right]dp$$

$$=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} p\ e^{px}\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} q\ e^{qy}\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} r\ e^{rt}L(p,q,r)dr\right]dq\right]$$

$$= l(x,y,t)$$

and the theorem is established.

Theorem 3.2: If $K(p,q,r)=A_xA_yA_t\left[f(x,y,t)\right]$, Then

$$A_xA_yA_t\left[f(x-a,y-b,t-c)H(x-a,y-b,t-c)\right]=e^{-pa-qb-rc}K(p,q,r)$$

Where $H(x,y,t)$ is the Heaviside unit step function defined by :

$$H(x-a,y-b,t-c)=\begin{cases}1 & x>a,y>b,t>c\\ 0 & x<a,y<b,t<c\end{cases}$$

Proof: By defnintion we have

$$A_xA_yA_t\left[f(x-a,y-b,t-c)H(x-a,y-b,t-c)\right]=$$

$$\frac{1}{pqr}\iiint_0^\infty e^{-px-qy-rt}\left[f(x-a,y-b,t-c)H(x-a,y-b,t-c)\right]dxdydt$$

$$=\frac{1}{pqr}\int_a^\infty\int_b^\infty\int_c^\infty e^{-px-qy-rt}f(x-a,y-b,t-c)dxdydt$$

By letting $x-a=u_1$, $y-b=u_2$, $t-c=u_3$

$$A_xA_yA_t\left[f(x-a,y-b,t-c)H(x-a,y-b,t-c)\right]=\frac{1}{pqr}e^{-pa-qb-rc}\iiint_0^\infty e^{-pu_1-qu_2-ru_3}f(u_1,u_2,u_3,)du_1du_2du_3$$

$$=e^{-pa-qb-rc}K(p,q,r).$$

4. Convolution Theorem

In this section, we state and prove the convolution theorem of triple Aboodh transform.

Theorem 4.1: If at the point (p,q,r) the integral

$$G_1(p,q,r)=\frac{1}{pqr}\iiint_0^\infty e^{-(px+qy+rt)}g_1(x,y,t)dxdydt$$

is converge, and in addition if

$$G_2(p,q,r)=\frac{1}{pqr}\iiint_0^\infty e^{-(px+qy+rt)}g_2(x,y,t)dxdydt$$

is absolutely converge, then the following expression

$$G(p,q,r)=pqrG_1(p,q,r)G_2(p,q,r)$$

is the Aboodh transform of the function

$$g(x,y,t)=\int_0^t\int_0^y\int_0^x g_1(x-x_1,y-y_1,t-t_1)g_2(x_1,y_1,t_1)dx_1dy_1dt_1$$

and the integral

$$G(p,q,r)=\frac{1}{pqr}\iiint_0^\infty e^{-(px+qy+rt)}g(x,y,t)dxdydt$$

is converge at the point (p,q,r).

Proof:

$$G(p,q,r)=\frac{1}{pqr}\iiint_0^\infty e^{-(px+qy+rt)}g(x,y,t)dxdydt$$

$$\frac{1}{pqr}\iiint_0^\infty e^{-(px+qy+rt)}\left[\int_0^t\int_0^y\int_0^x g_1(x-x_1,y-y_1,t-t_1)g_2(x_1,y_1,t_1)dx_1dy_1dt_1\right]dxdydt$$

By using Heaviside unit step function

$$\iiint_0^\infty g_2(x_1,y_1,t_1)dx_1dy_1dt\left[\frac{1}{pqr}\iiint_0^\infty e^{-(px+qy+rt)}g_1(x-x_1,y-y_1,t-t_1)H(x-x_1,y-y_1,t-t_1)dxdydt\right]$$

$$=\iiint_0^\infty g_2(x_1,y_1,t_1)dx_1dy_1dt_1e^{-px_1-qy_1-rt_1}G_1(p,q,r)$$

$$=pqrG_1(p,q,r)G_2(p,q,r)$$

5. Some properties of triple Aboodh transform

In this section, we discuss and prove various properties of tripe Aboodh transform.

- The triple Aboodh transform is a linear operator, that is

$$A_xA_yA_t\left[(af+bg)(x,y,t)\right](p,q,r)=$$
$$aA_xA_yA_t\left[f(x,y,t)\right](p,q,r)+bA_xA_yA_t\left[g(x,y,t)\right](p,q,r)$$

Proof:

$$A_xA_yA_t\left[(af+bg)(x,y,t)\right](p,q,r)=\frac{1}{pqr}\iiint_0^\infty e^{-(px+qy+rt)}(af+bg)(x,y,t)dxdydt$$

$$=a.\frac{1}{pqr}\iiint_0^\infty e^{-(px+qy+rt)}f(x,y,t)dxdydt+b.\frac{1}{pqr}\iiint_0^\infty e^{-(px+qy+rt)}g(x,y,t)dxdydt$$

$$=a.A_xA_yA_t\left[f(x,y,t)\right](p,q,r)+b.A_xA_yA_t\left[g(x,y,t)\right](p,q,r)$$

- Changing of scale property:

If $A_xA_yA_t\left[f(x,y,t)\right]=K(p,q,r)$, then $A_xA_yA_t\left[f(ax,by,ct)\right]=\frac{1}{abc}K\left(\frac{p}{a},\frac{q}{b},\frac{r}{c}\right)$

Proof:

$$A_xA_yA_t\left[f(ax,by,ct)\right]=\frac{1}{pqr}\iiint_0^\infty e^{-(px+qy+rt)}f(ax,by,ct)dxdydt$$

$$=\frac{1}{p}\int_0^\infty e^{-(p+a)x}\left[\frac{1}{qr}\int_0^\infty\int_0^\infty e^{-qy-rt}f(ax,by,ct)dydt\right]dx$$

$$=\frac{1}{p}\int_0^\infty e^{-px}\frac{1}{bc}K\left(x,\frac{q}{b},\frac{r}{c}\right)dx$$

$$=\frac{1}{abc}K\left(\frac{p}{a},\frac{q}{b},\frac{r}{c}\right)$$

Note that the first and double Aboodh transforms satisfy the changing of scale property.

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- Shifting property:

If $A_x A_y A_t \left[f(x,y,t) \right] = K(p,q,r)$, then $A_x A_y A_t \left[e^{-ax-by-ct} f(x,y,t) \right] = K(p+a,q+b,r+c)$

Proof:

$$A_x A_y A_t \left[e^{-ax-by-ct} f(x,y,t) \right] = \frac{1}{pqr} \iiint_0^{\infty} \boldsymbol{e}^{-(\boldsymbol{px}+\boldsymbol{qy}+\boldsymbol{rt})} e^{-ax-by-ct} f(x,y,t) dxdydt$$

$$= \frac{1}{p} \int_0^{\infty} e^{-(p+a)x} \left[\frac{1}{qr} \int_0^{\infty} \int_0^{\infty} e^{-(q+b)y-(r+c)t} f(x,y,t) dydt \right] dx$$

$$= \frac{1}{p} \int_0^{\infty} e^{-(p+a)x} K(x,q+b,r+c) dx$$

$$= K(p+a,q+b,r+c)$$

Note that first and double Aboodh transforms satisfy the shifting property.

- Multiplying by $x^n y^m t^v$

$$A_x A_y A_t \left[x^n y^m t^v f(x,y,t) \right] = \frac{(-1)^{n+m+v}}{pqr} \frac{\partial^{n+m+v}}{\partial p^n q^m r^v} \left(pqrK(p,q,r) \right)$$

Proof:

$$A_x A_y A_t \left[x^n y^m t^v f(x,y,t) \right] = \frac{1}{pqr} \iiint_0^{\infty} \boldsymbol{e}^{-(\boldsymbol{px}+\boldsymbol{qy}+\boldsymbol{rt})} x^n y^m t^v f(x,y,t) dxdydt$$

$$= \frac{1}{p} \int_0^{\infty} x^n e^{-px} \left[\frac{1}{qr} \int_0^{\infty} \int_0^{\infty} e^{-qy-rt} y^m t^v f(x,y,t) dydt \right] dx$$

The expression in the bracket satisfies the property of the double Aboodh transform, that is

$$A_y A_t \left[y^m t^v f(x,y,t) \right] = \frac{(-1)^{m+v}}{qr} \frac{\partial^{m+v}}{q^m r^v} \left(qrK(x,q,r) \right),$$

thus,

$$\frac{1}{p} \int_0^{\infty} x^n e^{-px} \frac{(-1)^{m+v}}{qr} \frac{\partial^{m+v}}{q^m r^v} \left(qrK(x,q,r) \right) dx$$

$$= \frac{(-1)^{n+m+v}}{pqr} \frac{\partial^{n+m+v}}{\partial p^n q^m r^v} \left(pqrK(p,q,r) \right),$$

and this complete the proof.

- If $f(x,y,t) = g(x)h(y)z(t)$, then $A_x A_y A_t \left[f(x,y,t) \right] = A_x \left[g(x) \right] A_y \left[h(y) \right] A_t \left[z(t) \right]$.

Proof:

$$A_x A_y A_t\left[f(x,y,t)\right]=\frac{1}{pqr}\iint\int_0^{\infty} e^{-(px+qy+rt)} f(x,y,t)dxdydt$$

$$=\frac{1}{pqr}\iint\int_0^{\infty} e^{-(px+qy+rt)} g(x)h(y)z(t)dxdydt$$

$$=\left[\frac{1}{p}\int_0^{\infty} e^{-px} g(x)dx\right]\left[\frac{1}{q}\int_0^{\infty} e^{-qy} h(y)dy\right]\left[\frac{1}{r}\int_0^{\infty} e^{-rt} z(t)dt\right]$$

$$=A_x\left[g(x)\right]A_y\left[h(y)\right]A_t\left[z(t)\right]$$

Theorem 5.1: An exponentially of order continuous function $f(x,y,t)$ on $\left[0,\infty\right)$ can be recovered from only $K(p,q,r)$ as:

$$f(x,y,t)=\lim_{n,m,v\to\infty}\frac{(-1)^{n+m+v}}{n!m!v!}\left(\frac{n}{x}\right)^{n+1}\left(\frac{m}{y}\right)^{m+1}\left(\frac{v}{t}\right)^{v+1}\frac{\partial^{n+m+v}}{\partial p^n\partial q^m\partial r^v}\left[pqrK(\frac{n}{x},\frac{m}{y},\frac{v}{t})\right].$$

Proof: The poof is similar to proof given by Abdon Atangana see [15].

To check the efficiency of the previous theorem, let us consider the following example:
Let $f(x,y,t)=e^{-ax-by-ct}$ for which the triple Aboodh transform can be found as:

$$K(p,q,r)=\frac{1}{p(p+a)q(q+b)r(r+1)},$$

by applying the high-order mixed derivative to the expression, we get the following:

$$\frac{\partial^{n+m+v}}{\partial p^n\partial q^m\partial r^v}\left[pqrK(p,q,r)\right]=\frac{n!m!v!(-1)^{n+m+v}}{(p+a)^{n+1}(q+b)^{m+1}(r+c)^{v+1}},$$

by using theorem(5.1), we get:

$$f(x,y,t)=\lim_{n,m,v\to\infty}\left(\frac{n}{x}\right)^{n+1}\left(\frac{m}{y}\right)^{m+1}\left(\frac{v}{t}\right)^{v+1}\left(a+\frac{n}{x}\right)^{-n-1}\left(b+\frac{m}{y}\right)^{-m-1}\left(c+\frac{v}{t}\right)^{-v-1}$$

$$=\lim_{n,m,v\to\infty}\left(1+\frac{ax}{n}\right)^{-n-1}\left(1+\frac{by}{m}\right)^{-m-1}\left(1+\frac{ct}{v}\right)^{-v-1},$$

by applying the logarithm and L'Hopital's rule we get:

$$\ln f(x,y,t)=-ax-by-ct,$$

$$f(x,y,t)=e^{-ax-by-ct}.$$

6. Application

In this section, we construct some different examples to illustrate the applicability and efficiency of the triple Aboodh transform.

Example 1: Consider the following fractional partial differential equation:

$$D_t^{\alpha} f(x,y,t)=\frac{\partial^2 f(x,y,t)}{\partial x^2},\quad 0<\alpha\le 1, \qquad (3) \quad ,$$

with the following initial and boundary values:

$$f(0,y,t)=0,\ f_x(0,y,t)=\sin(y)E_{\alpha}(-t^{\alpha})$$

$$f(x,y,0)=\sin(x)\sin(y)$$

Solution: Applying the triple Aboodh transform to equation (3) and for the initial and boundary conditions we get:

$$r^{\alpha}A_xA_yA_t\left(f\,(x,y,t)\right)-r^{\alpha-2}A_xA_y\left(f\,(x,y,0)\right)=p^2A_xA_yA_t\left(f\,(x,y,t)\right)$$
$$-A_yA_t\left(f\,(0,y,t)\right)-\frac{1}{p}A_yA_t\left(\frac{\partial f\,(0,y,t)}{\partial x}\right),$$

Where,

$$A_xA_y\left(f\,(x,y,0)\right)=A_xA_y\left(\sin x\,\sin y\right)=\frac{1}{p(p+1)}\frac{1}{q(q+1)}$$
$$A_yA_t\left(f\,(0,y,t)\right)=A_yA_t\,(0)=0$$
$$A_yA_t\left(\frac{\partial f\,(0,y,t)}{\partial x}\right)=\frac{r^{\alpha-2}}{1+r^{\alpha}}\frac{1}{q(q+1)},$$

thus,

$$\left(r^{\alpha}-p^2\right)A_xA_yA_t\left(f\,(x,y,t)\right)=\frac{r^{\alpha-2}}{pq(q^2+1)}\left[\frac{1}{p^2+1}-\frac{1}{r^{\alpha}+1}\right]$$
$$A_xA_yA_t\left(f\,(x,y,t)\right)=\frac{r^{\alpha-2}}{r^{\alpha}+1}\frac{1}{p\left(p^2+1\right)}\frac{1}{q(q^2+1)},$$

by applying the inverse of triple Aboodh transform, we get:

$$f\,(x,y,t)=A_x^{-1}A_y^{-1}A_t^{-1}\left(\frac{r^{\alpha-2}}{r^{\alpha}+1}\frac{1}{p\left(p^2+1\right)}\frac{1}{q(q^2+1)}\right)=\sin(x)\sin(y)E_{\alpha}(-t^{\alpha})\,.$$

Example 2: Consider the following nonhomogeneous third-order Mboctara partial differential equation:

$$\frac{\partial^3 f\,(x,y,t)}{\partial x\,\partial y\,\partial t}+f\,(x,y,t)=3e^{-x-2y+t}, \qquad (4)$$

subject to the following initial and boundary conditions:

$$f\,(0,y,t)=e^{-2y+t},\ f\,(x,0,t)=e^{-x+t},\ f\,(x,y,0)=e^{-x-2y}\,.$$

Solution: Applying the triple Aboodh transform to equation (4), we get:

$$(pqr+1)K\,(p,q,r)=U\,(p,q,r)+\frac{1}{p(p+1)q(q+2)r(r-1)}, \qquad (5)$$

Where,

$$U\,(p,q,r)=\frac{pq}{r}K\,(p,q,0)+\frac{pr}{q}K\,(p,0,r)+\frac{qr}{p}K\,(0,q,r)$$
$$-\frac{p}{qr}K\,(p,0,0)-\frac{q}{pr}K\,(0,q,0)-\frac{r}{pq}K\,(0,0,r)+\frac{1}{pqr}f\,(0,0,0)$$
$$=\frac{pqr-2}{p(p+1)q(q+2)r(r-1)},$$

By substituting the value of $U\,(p,q,r)$ in equation (5), we get:

$$K\,(p,q,r)=\frac{1}{p(p+1)q(q+2)r(r-1)},$$

by applying the triple inverse Aboodh transform we get:

$$f\,(x,y,t)=e^{-x-2y+t}.$$

Example 3: Consider the following diffusion equations:

$$\frac{\partial^2 f\,(x,y,t)}{\partial x^2}+\frac{\partial^2 f\,(x,y,t)}{\partial y^2}-5\frac{\partial f\,(x,y,t)}{\partial t}=0, \qquad (6)$$

Subject to the following initial and boundary conditions:

$$f(0,y,t)=e^{2y+t}, f(1,y,t)=e^{1+2y+t}$$
$$f(x,0,t)=e^{x+2y}, f(x,0.5,t)=e^{x+1}$$
$$f(x,y,0)=e^{x+2y},$$

Solution: By applying the triple Aboodh transform to equation(6), we get:

$$\left(p^2+q^2-r\right)K(p,q,r)=U(p,q,r) \qquad (7)$$

Where

$$U(p,q,r)=A_yA_t\left(f(0,y,t)\right)+\frac{1}{p}A_yA_t\left(\frac{\partial f(0,y,t)}{\partial x}\right)$$
$$+A_xA_t\left(f(x,0,t)\right)+\frac{1}{q}A_yA_t\left(\frac{\partial f(x,0,t)}{\partial y}\right)-\frac{1}{r}A_xA_y\left(f(x,y,0)\right)$$
$$=\frac{p^2+q^2-r}{p(p-1)q(q-2)r(r-1)}$$

By substituting the value of $U(p,q,r)$, equation (7) becomes:

$$K(p,q,r)=\frac{1}{p(p-1)q(q-2)r(r-1)} \qquad (8)$$

Applying the triple inverse Aboodh transform on equation(8), we get :

$$f(x,y,t)=e^{x+2y+t}.$$

7. Conclusion

In this present work, triple Aboodh transform and its inverse are defined in order to solve partial differential equations and fractional differential equations, furthermore, we present several properties and theorems of triple Aboodh transform. To see the efficiency of triple Aboodh transform, we apply this transform on three different examples; the results show that the triple Aboodh transform method is an appropriate method for solving partial differential equations of both integer and fractional order.

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