On Syntax and Semantics of Propositional Logic

Latif Salum, Independent Researcher, Türkiye
latif.salum@gmail.com

February 25, 2025

Abstract

We introduce syntactic aspect of disjunction based on the prime truth assignments of a 3CNF formula. In this manner, we describe the *semantics* of the formula *syntactically*. We use this novel approach to determine a sufficient condition for unsatisfiability.

A Sufficient Condition for Unsatisfiability

As is well known, a Boolean formula can be converted to an equisatisfiable 3CNF formula via the Tseytin transformation. Let $\phi = \bigwedge C_k$ be a 3CNF formula, C_k being a disjunction of two or three literals ℓ_i, where $\ell_i \in \{x_i, \overline{x}_i\}$.

Definition (Conventional/semantic aspect of disjunction)**.** Let $D = \ell_1 \vee \ell_2 \vee \cdots \vee \ell_n$. D is *true* if at least one of the literals is *true*.

Definition (Syntactic aspect of disjunction: *prime truth assignments*)**.** Let $C_{k'} = (\ell_1 \vee \ell_2)$ and $C_k = (\ell_1 \vee \ell_2 \vee \ell_3)$.
$\delta_{k'} = (\psi^1_{k'} \vee \psi^2_{k'} \vee \psi^3_{k'})$ such that $\psi^1_{k'} = (\ell_1 \wedge \overline{\ell}_2), \psi^2_{k'} = (\overline{\ell}_1 \wedge \ell_2), \psi^3_{k'} = (\ell_1 \wedge \ell_2)$.
$\delta_k = (\psi^1_k \vee \cdots \vee \psi^7_k)$ such that $\psi^1_k = (\ell_1 \wedge \overline{\ell}_2 \wedge \overline{\ell}_3), \psi^2_k = (\overline{\ell}_1 \wedge \ell_2 \wedge \overline{\ell}_3), \ldots, \psi^6_k = (\overline{\ell}_1 \wedge \ell_2 \wedge \ell_3), \psi^7_k = (\ell_1 \wedge \ell_2 \wedge \ell_3)$.

Remark. The prime truth assignments $\psi^1_k, \psi^2_k, \ldots, \psi^s_k$ of the clause C_k are denoted by δ_k the DNF formula. Then, C_k is *satisfiable* iff δ_k is *reducible* to one of its prime truth assignments ψ^s_k, which leads to the following definition.

Definition (Syntactic definition of the satisfiability of C_k)**.** C_k is satisfiable iff $\delta_k \vdash \psi^s_k$ for some $s \in \{1, 2, \ldots, 7\}$.

Definition (Syntactic description of the semantics of ϕ)**.** $\Psi = \bigwedge \delta_k$, where $\delta_k = (\psi^1_k \vee \cdots \vee \psi^{s_k}_k)$ for $s_k \in \{3, 7\}$.

Remark. Ψ the syntax describes the semantics of ϕ the formula via δ_k the prime truth assignments of every C_k.

Theorem (Unsatisfiability)**.** *ϕ is unsatisfiable if $\Psi \vdash \psi$ such that ψ is inconsistent, viz., $\psi \vdash x_i \wedge \overline{x}_i$ for some i.*

Proof. The proof is obvious. It is also obvious that $\phi \equiv \Psi$. Note that $\psi = \ell_i \wedge \ell_j \wedge \cdots \wedge \ell_u$, where $\ell_i \in \{x_i, \overline{x}_i\}$. □

A polynomial time decision procedure to decide the unsatisfiability of ϕ is described briefly as follows. Consider $\psi^s_k \wedge \Psi$ to evaluate the incompatibility of a prime truth assignment ψ^s_k. If $\psi^s_k \wedge \Psi \vdash \psi$ such that ψ is inconsistent, then ψ^s_k is incompatible and removed from Ψ. Note that $\psi^s_k \wedge \delta_k \vdash \psi^s_k$. Thus, the incompatibility of every prime truth assignment is evaluated so that each ψ^s_k incompatible is removed from Ψ. Also, if any ψ^s_k has been removed, then the incompatibility of every ψ^s_k is re-evaluated over the current Ψ. When the incompatibility evaluations terminate, it is the case that $\Psi \vdash \psi \wedge \Psi'$ such that $\psi = \psi^{s_1}_{k_1} \wedge \psi^{s_2}_{k_2} \wedge \cdots \wedge \psi^{s_m}_{k_m}$, where $k_1 \neq k_2 \neq \cdots \neq k_m$. If ψ is inconsistent, then ϕ is unsatisfiable. The following examples illustrate this decision procedure.

Let $\phi = (x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_2 \vee \overline{x}_3)$. Then, $\Psi = \delta_1 \wedge \delta_2 \wedge \delta_3$, in which $\delta_1 = (x_1 \wedge x_2) \vee (x_1 \wedge \overline{x}_2) \vee (\overline{x}_1 \wedge x_2)$, $\delta_2 = (x_2 \wedge x_3) \vee (x_2 \wedge \overline{x}_3) \vee (\overline{x}_2 \wedge x_3)$ and $\delta_3 = (x_2 \wedge \overline{x}_3) \vee (x_2 \wedge x_3) \vee (\overline{x}_2 \wedge \overline{x}_3)$. We evaluate the incompatibility of $(x_1 \wedge \overline{x}_2)$ the prime truth assignment. Consider $(x_1 \wedge \overline{x}_2) \wedge \Psi$. As a result, $\delta_1 = (x_1 \wedge \overline{x}_2)$, $\delta_2 = (\overline{x}_2 \wedge x_3)$ and $\delta_3 = (\overline{x}_2 \wedge \overline{x}_3)$. Because $\delta_2 \wedge \delta_3$ is inconsistent, $(x_1 \wedge \overline{x}_2)$ is incompatible and removed from Ψ. Also, $(\overline{x}_2 \wedge \ell_3)$ is incompatible for each $\ell_3 \in \{x_3, \overline{x}_3\}$. Thus, $\Psi \vdash \Psi'$ and $\Psi' = \delta_1 \wedge \delta_2 \wedge \delta_3$, in which $\delta_1 = (x_1 \wedge x_2) \vee (\overline{x}_1 \wedge x_2)$ and $\delta_2 = \delta_3 = (x_2 \wedge x_3) \vee (x_2 \wedge \overline{x}_3)$.

Let $\phi = (x_1 \vee \overline{x}_2) \wedge (x_2 \vee \overline{x}_3) \wedge (x_3 \vee \overline{x}_1) \wedge (x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)$. Then, $\Psi = \delta_1 \wedge \delta_2 \wedge \delta_3 \wedge \delta_4 \wedge \delta_5$, where
$\delta_1 = (x_1 \wedge x_2) \vee (\overline{x}_1 \wedge \overline{x}_2) \vee (x_1 \wedge \overline{x}_2)$.
$\delta_2 = (x_2 \wedge x_3) \vee (\overline{x}_2 \wedge \overline{x}_3) \vee (x_2 \wedge \overline{x}_3)$.
$\delta_3 = (x_3 \wedge x_1) \vee (\overline{x}_3 \wedge \overline{x}_1) \vee (x_3 \wedge \overline{x}_1)$.
$\delta_4 = (x_1 \wedge \overline{x}_2 \wedge \overline{x}_3) \vee (x_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_1 \wedge x_2 \wedge x_3) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge x_3) \vee (x_1 \wedge \overline{x}_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$.
$\delta_5 = (x_1 \wedge \overline{x}_2 \wedge \overline{x}_3) \vee (x_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \vee (\overline{x}_1 \wedge x_2 \wedge x_3) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge x_3) \vee (x_1 \wedge \overline{x}_2 \wedge x_3) \vee (\overline{x}_1 \wedge \overline{x}_2 \wedge \overline{x}_3)$.
Consider $(x_1 \wedge \overline{x}_2 \wedge \overline{x}_3) \wedge \Psi$. Then, $(x_1 \wedge \overline{x}_2 \wedge \overline{x}_3) \wedge \delta_3 = (x_1 \wedge \overline{x}_2 \wedge \overline{x}_3) \wedge (x_3 \wedge \overline{x}_1)$, which is inconsistent. Hence, $(x_1 \wedge \overline{x}_2 \wedge \overline{x}_3)$ is removed.
Consider $(x_1 \wedge x_2 \wedge \overline{x}_3) \wedge \Psi$. Then, $(x_1 \wedge x_2 \wedge \overline{x}_3) \wedge \delta_3 = (x_1 \wedge x_2 \wedge \overline{x}_3) \wedge (x_3 \wedge \overline{x}_1)$ and $(x_1 \wedge x_2 \wedge \overline{x}_3)$ is removed.
Consider $(\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \wedge \Psi$. Then, $(\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \wedge \delta_1 = (\overline{x}_1 \wedge x_2 \wedge \overline{x}_3) \wedge (x_1 \wedge \overline{x}_2)$ and $(\overline{x}_1 \wedge x_2 \wedge \overline{x}_3)$ is removed.
Consider $(\overline{x}_1 \wedge x_2 \wedge x_3) \wedge \Psi$. Then, $(\overline{x}_1 \wedge x_2 \wedge x_3) \wedge \delta_1 = (\overline{x}_1 \wedge x_2 \wedge x_3) \wedge (x_1 \wedge \overline{x}_2)$ and $(\overline{x}_1 \wedge x_2 \wedge x_3)$ is removed.
Consider $(\overline{x}_1 \wedge \overline{x}_2 \wedge x_3) \wedge \Psi$. Then, $(\overline{x}_1 \wedge \overline{x}_2 \wedge x_3) \wedge \delta_2 = (\overline{x}_1 \wedge \overline{x}_2 \wedge x_3) \wedge (x_2 \wedge \overline{x}_3)$ and $(\overline{x}_1 \wedge \overline{x}_2 \wedge x_3)$ is removed.
Consider $(x_1 \wedge \overline{x}_2 \wedge x_3) \wedge \Psi$. Then, $(x_1 \wedge \overline{x}_2 \wedge x_3) \wedge \delta_2 = (x_1 \wedge \overline{x}_2 \wedge x_3) \wedge (x_2 \wedge \overline{x}_3)$ and $(x_1 \wedge \overline{x}_2 \wedge x_3)$ is removed.

Consequently, $\delta_4 = (x_1 \wedge x_2 \wedge x_3)$ and $\delta_5 = (\overline{x}_1 \wedge \overline{x}_2 \wedge \overline{x}_3)$. As a result, $\Psi \vdash \psi$ such that $\psi = \delta_4 \wedge \delta_5 = \psi^7_4 \wedge \psi^7_5$, where $\psi^7_4 = (x_1 \wedge x_2 \wedge x_3)$ and $\psi^7_5 = (\overline{x}_1 \wedge \overline{x}_2 \wedge \overline{x}_3)$. Because ψ is inconsistent, ϕ is unsatisfiable.

See also http://dx.doi.org/10.13140/RG.2.2.32590.78408.