Verifiability: An Essay on an Empirical Account of Meaning

Paulo Guilherme Santos

Inscription

To my Wife (Pi): the sun of my planet, the moon of my tides.

To my Parents (Isabel and José Paulo): the eternal harbor to my sailing ship.

Prologue

The book you are about to read is a humble attempt to give new life to the (*Logical*) *Positivist* ideas from the beginning of the XX$^{\text{th}}$ century. We felt the urge to write our ideas because their formulation is found nowhere in the current literature; moreover, Logical Positivism is considered, for the most part, unfeasible; we hope to convince the reader of the opposite and to give new life to the movement.

Our approach differs from the original Positivism: we include logic and mathematics in the empirical realm, and we develop a finite way to implement the Verification Principle; we, consequently, do not resort to the common analytic/synthetic distinction. Although the principles we use date back to the original formulation of Logical Positivism, our approach is, in the technical sense, very distinct.

Contents

Chapter 1

The Verification Principle

Languages are the ways of expression: it is through languages that we express our thoughts, that we interact with our fellow human beings, that we formulate questions, that we seek answers, that we express happiness and sadness, and that we communicate some particular state of affairs about our perception of the world. They do not need to be symbolic, they might be spoken; they do not need to be written, they might be gesticulated—the fundamental aspect of languages is that they are the vehicle of information. There are forms of expression whose primary goal is not to express some state of affairs about the world, like music; we exclude those forms from our analysis of meaning, since they do not aim at making statements about any concrete reality, they rather aim at creating an impression; we will not call them 'languages', but rather 'forms of expression'.[1] We are going to focus our attention on common written (English) language and its symbolic representations. As the reader might recognize from her[2] own experience, there is no loss of generality in this: languages can be translated; the written language can be read, and the speech can be written.

A key feature of languages is the existence of grammar rules that exclude immediately forms of nonsensical speech. It is not late in our use of any form of language that we realize that some expressions are nonsensical, despite being grammatically correct. For example, the sentence 'The car inside the atom is powerful in its humanized development of the round-square.' is a grammatically sound sentence; however, we believe, for the sake of sanity, that most readers will not attribute to it any concrete meaning. This book seeks to clarify this aspect, namely to develop a method to decide the meaning of a sentence using only empirical grounds: we will analyze everyday language and scientific discourse, as well as other forms of discourse, like meta-

[1] We use the quotations '···' to refer to a particular array of symbols, and the quotations "···" to emphasize an idiomatic expression.

[2] For the sake of variety, inclusion, and politeness, for us, the reader is referred as 'she'. Why not? For a change... In addition, the natural scientist is also a 'she'. There are two other reasons for this choice. Firstly, my Wife—a natural scientist—was the first person to read and comment this work; this book was, in a sense, written for *her*. Secondly, in older forms of English, 'she' was used to evoke a sense of respect and admiration: for example, countries used to be called 'she', as well as the moon, nature, the Earth, and even cities; boats still are commonly referred as 'she'.

physics. Our approach does not rely on the analytic/synthetic distinction, namely the distinction of the propositions that are true by definition or by virtue of their relation to the world; this distinction was a defining factor of the original Logical Positivism, but not for our approach.

The main feature of our approach is that all meaningful discourse is going to be framed in empirical terms (even logic and mathematics are going to be analyzed in this fashion). The driving force of this world-view is the *Verification Principle* (VP). This principle claims that the meaningful sentences are the ones that are empirically verifiable, *id est* the ones that can be effectively tested in the world. The VP comes hand-in-hand with what we call the *Schlick Principle* (SP) that claims that the meaning of a proposition is its method of verification (see, for example, [11]).

This book should be read as an instruction manual where we describe the meaning of propositions using the SP and describe procedures to use the VP to clarify the scientific and the everyday languages. Just like the reader does not expect metaphysical considerations from an instruction manual of a fridge, the reader should also not expect them in this context; we want to take a pragmatic view by developing a theory of meaning and verifiability from the SP and the VP, we do not want to take the impossible endeavour of justifying the SP and the VP: in the case of the fridge, the realization that it exists and performs the desired tasks is something assumed by the instruction manual.

As the reader might already suspect, we are going to exclude metaphysics from the meaningful realm, but more on that later (see Chapter 4). In the current chapter, we will proceed as follows: firstly, we are going to see that the VP is meaningful; secondly, we are going to realize that what is not covered by the VP is nonsensical; and thirdly, this will give rise to a separation of the sentences of our language into two distinct types, to wit the '(actual) propositions' and the 'pseudo-propositions'. The VP can be viewed as the conjunction of the two following statements:

Soundness of the VP: The empirically verifiable propositions are meaningful.

Completeness of the VP: Only the empirically verifiable propositions are meaningful.

In the next chapters, we are going to analyze logic, science, and metaphysics using the VP. As we already mentioned, our approach differs from the original Logical Positivism in several aspects, in particular because we also reduce logic and mathematics to either empirical grounds or to conventions of language. Clearly, conventions are an extreme form of empirically verifiability: they can be viewed as a sort of verifiability that is test-free, to which any test will yield a positive result.

1.1 Soundness and completeness of the VP

The fact that the empirically verifiable propositions are meaningful comes, for most people, as a self-evident fact; the converse claim that only the empirical propositions are meaningful is the one that gives rise to more discussion. We do not contest the self-evidence of that fact, but as we are building a system of meaningful propositions based on the VP, we need, in the first place, to guarantee that the VP is sound, *scilicet* that it is adequate to use.

No proof of soundness can be given, the soundness of the VP can only be realized and agreed by the reader. We need to start our analysis from concrete grounds, it is hopeless to assume that a system of knowledge can be built from no assumptions whatsoever; as Parmenides cleverly said *ex nihilo nihil fit* (*nothing comes from nothing*). This does not mean that we should religiously take the soundness of the VP as a dogma, quite the opposite.

The soundness of the VP is used in every single day of our lives, even if unconsciously so. We use it to build bridges, to fly airplanes; to create cars, cellphones, and fountain pens. We unconsciously use it to avoid danger, to distance ourselves from a burning flame, to use our arms to protect our body when we are falling, to move the steering wheel of a car to the right when a right turn is to be performed, to firmly hold an object to avoid it falling on the ground, and so on. All these actions come from experience and can be expressed by meaningful propositions; moreover, if at any given moment we want to actually test them again we certainly can: if we are in doubt that we should firmly hold an object to avoid its fall, at any given moment we can release the object and confirm that it actually falls. These facts that we described are especially useful when they are written down for other fellow human beings to read and use.

In the previous paragraph, we are not using the soundness of the VP to justify its very soundness, we are appealing to a form of meta-soundness of the VP to create in the reader the realization that the VP is, in fact, sound. As we mentioned, we do not seek an (impossible) prove of soundness, we simply ask the reader to recall her own experiences of the world and to realize that the soundness of the VP is something that is an integral part of our animal nature. The examples we gave are not directly concerned with the expression of the empirical facts using language, they do not constitute propositions *per se*, but they surely can be expressed by useful propositions. Again, the very fact that we can make empirical propositions is something that needs to be recognized, not proved. Other animals, allegedly, do not make propositions—let us not jump to conclusions on this because we cannot access completely to their minds and their language—, but they do act in accordance with experience. This fact is linked to the soundness of the VP, it is, very naïvely, its "unspoken version".

Not subscribing to the soundness of the VP is a form of denial of either the way we, humans, as animals, act, or the denial that languages can meaningfully capture these actions. Both forms of denial are so fundamental that, unfortunately, give rise to an unsurpassable restriction of our discourse about reality. We move forward embracing our nature and the fact that this very simple correspondence of actions and empirical facts into our language is possible: empirical propositions are meaningful.

Despite not being able to prove the soundness of VP, we can do more than simply recognize it; we can recognize the general structure of an empirical proposition. When we are given a proposition with empirical content, by its very nature, it comes associated with a form of its verification. An empirical proposition is, consequently, nothing more than a pair constituted by a syntactical array of symbols (the actual proposition), and by a pair that, by its turn, is constituted by a description of the correspondence of those symbols to the world together with a method of testing. By a pair we simply mean the syntactical ability of keeping track of two distinct pieces of information; we use $\langle \cdot, \cdot \rangle$ to denote pairs; $\langle i_0, i_1 \rangle$ is the pair composed of information i_0 and information i_1 (these pieces of information are essentially arrays of symbols). We can, of course, have pairs inside pairs, such as in the previously described case $\langle i_0, \langle i_1, i_2 \rangle \rangle$. These pairs that we are using are simply a syntactical device to organize arrays of symbols, they do not assume the usual mathematical machinery to be used.

For example, the proposition 'There is a red duck in Lisbon.', when viewed as an empirical proposition, can be considered as the following pair:

$$\langle \text{'There is a red duck in Lisbon.'}, \\ \langle \texttt{Def}(\text{Duck}), \texttt{Method}(\text{Duck}, \text{Lisbon}, \text{Red}) \rangle \rangle.$$

`Def`(Duck) is some agreed definition of 'duck' and `Method`(Duck, Lisbon, Red) is an agreed method of finding red ducks in Lisbon. *Exempli gratia*, we can consider

$\texttt{Def}(\text{Duck}) =$ 'Member of the family *Anatidae*.';
$\texttt{Method}(\text{Duck}, \text{Lisbon}, \text{Red}) =$ 'Walking through Lisbon identifying members of the class *Aves* and testing if they belong to the family *Anatidae* and if the wave length of the visible radiation they emit is between 625 and 740 nanometers.'.

Of course, the definition at hand can at any moment be incorporated in the considered method of testing. This example might strike the reader as somehow comical and unpractical, but it highlights important aspects of the way empirically meaningful propositions work: we have a way to establish a correspondence between word-names and objects in the world, and a method to test the proposition. We do not claim to possess an unchanged definition of the concepts, such as 'duck', neither do we claim that we can persue this process in a finite reduction and reach primordial concepts; that is not our goal. The empirical way of analyzing propositions is more concerned with the method, rather than with establishing unchangeable atomic concepts: the important aspect is to find atomic concepts that, for the example at hand, the community agrees on the meaning; moreover, those very atomic concepts can be further scrutinized and reduced to possibly other concepts, and so on. Later in this book, we will give further details on the way language is used in empirical terms, including logic.

The empirical attitude is to find ways of verifying the propositions and, if needed, to proceed and reduce the verification to simpler verifications; no meaningful proposition can have meaning without a way to test it; this constitutes the SP, a principle related to the VP. Just like the soundness of the VP, the SP cannot be proved nor empirically tested. The SP can be viewed as a definition of meaning. We encourage the reader to think of possible propositions and to check if the ones that she attributes meaning to are exactly the ones that have a method of verification. Do propositions like 'The tears of the sea are signs of sorrow from the gods.' have meaning? For sure they can cause an impression or even an emotion in the reader, like music does; they are forms of expression; but outside poetry communities no one believes that those sorts of (pseudo-)propositions have meaning. Even the poet in a poetic context, when given a poem to read, does not believe that the poem has world-meaning, he might be touched by it, but he does not expect it to describe the world: the poem is a sketch of emotions. Let us proceed with this tentative definition of world-meaning of a proposition: a method to describe the world and not an impression nor an emotion. We are obviously excluding the use of the word 'meaning' as a synonym of something being treasured, like in 'That music has meaning to me.'.

A consequence of the SP is the completeness of the VP: the SP claims that a proposition has meaning if it has a method of being verified, this is a restatement of the completeness of the VP, because such a method always needs to be empirical (the methods are either forms of definition or of testing).

At this stage, the reader might be feeling that these views are, for sure, too restrictive: what about ethics, what about metaphysics? Is there no meaning in the proposition 'You shall not kill.'? These subjects, namely ethics and metaphysics, are going to be further investigated in our book, but for the moment let us directly answer that question by pointing-out that 'You shall not kill.' is an order, rather than an actual proposition. The statements of ethics are either commandments—that by their very nature are not descriptions of the world—, or facts that can be empirically analyzed as soon as the suitable context is given. The commandment 'You shall not kill.' becomes an empirical proposition if 'shall' is intended as 'not producing harm to the community' and the universe of discourse is centred in the community, and not in the individual. In that setting, the order turns into the proposition 'If you kill an individual of the community you produce harm to the community.', where 'harm' is obviously 'causing suffering', *et cetera*.

Clearly, other analyses can be carried out, for example by centring the discourse in the individual: this emphasizes the importance of using clear and unambiguous language. An array of symbols—'You shall not kill.'—can be meaningless when no context to attribute empirical meaning is given; it can be just viewed as a command; or it can even be given a concrete empirical meaning if further information is specified. This is not a restrictive view, this is rather a clarifying analysis of language. If someone claims that vague statements as 'You shall not kill.' are meaningful by their own, he is the one who should provide the account of meaning, the burden of explanation is not on the empiricist that patiently awaits for an explanation, but on the person that claims that the statement is a proposition: a generalized misuse of language does not create meaningful discourse; the fact that a group of individuals believe that 'You shall not kill.' is a proposition is not enough for it to actually be a proposition, a method of meaning needs to be provided. Evidently, this does not mean that the reader should perform unethical actions, like killing; nor does it mean that the reader does not have ethical standards! It simply means that, to speak meaningfully about those actions, a context needs to be given: this constitutes no problem whatsoever because, fortunately, we do not need to make propositions in order to avoid killing someone, or to act in general, only sporadic commands.

It is important to emphasize that propositions that deal with conventions and manipulation of symbols might also be empirical. Pure manipulation of symbols—like in a game, in a derivability system of logic, or like in a computer—does not constitute, by itself, a form of proposition; but one can make meaningful propositions about a system of symbolic manipulation; usually, in that context, a justification of a fact is a list of manipulations to obtain the desired final result, just like a mathematical proof. In the case of a first-order theory—a formal theory[3] in classical logic with quantifiers $\forall$ (for all) and $\exists$ (exists)—, we can make two sorts of propositions: we can make propositions about the very first-order theory, in that case we are making empirical claims about arrays of symbols that we can always test, assuming the right amount of resources and assuming that when one speaks about the theory one is mentioning the possible ways of actually writing it down; but we can also attribute a concrete meaning to the symbols of the theory and

[3] A theory is a collection of formal sentences obtained from a fixed set of axioms and using derivation rules.

then the theory itself has empirical meaning. We will develop this last idea later on.

1.2 Propositions and pseudo-propositions

The analysis we have been carrying out gives rise to a now necessary classification of propositions: the ones that have (empirical) meaning, that we will continue to call ‘proposition’; and the ones that (still?) do not have a meaning, that we now call ‘pseudo-propositions’. There are clearly two sorts of pseudo-propositions, to wit, the ones that can never have an actual empirical meaning and the ones that potentially, after some clarifications are made, can have a meaning. ‘You shall not kill.’ is clearly an example of the latter. An example of the former is ‘The round-square has area 17.’: it has a “self-contradictory” term, namely ‘round-square’ and we are, consequently, unable to attribute a concrete meaning to it, so it fails our meaning attribution criterion. All propositions start as pseudo-propositions, the defining difference being that a method of testing was, at a certain moment, developed for the former, while the latter is still lacking one (it might even be impossible to find such a method to certain pseudo-propositions).

The pseudo-proposition is neither true nor false, it is simply meaningless. The pseudo-propositions that cannot have an empirical meaning do not exclusively arise from the use of “self-contradictory” terms, they can also be a consequence of pseudopredicates. 'Existence' is an example of a non-predicate that is commonly used as a predicate: people often attribute the "property" of existence to objects in statements of the form 'Unicorns exist.'. Of course, the reader can believe whatever she wants, we simply argue that only a limited part of our discourse is (empirically) meaningful; moreover, beliefs are outside the realm of rational discourse, since the latter is only concerned with facts. We will analyze in full syntactic detail why existence is not a predicate, but for now we can give an intuitive justification.

A verification of an existential claim about a property $P(x)$ can be of two sorts:

Existence 1: Either a method of presenting an object c satisfying the desired property, i.e. $P(c)$; or

Existence 2: A method that guarantees that no method of establishing all non-occurrences of $P(x)$ (for every tested x) can be constructed.

Let us give two examples of these. Consider the properties $\texttt{Dog}(x)$ stating 'x is a dog.' and $\texttt{Blackhole}(x)$ stating 'x is a black hole.'. A possible method of verification of the statement 'There exists a dog.', more symbolically $\exists x.\texttt{Dog}(x)$, is having a textbook definition of 'dog' and explore the world testing if any of the animals we encounter satisfies the definition (in principle, this is a feasible method); if there are dogs, and if enough time is given to the person testing, at a certain moment she will find one or run out of animals to test. If she has a way to know that she has tested all animals (using satellite images, for example) she might be able not only to positively test the statement—*scilicet* to find a witness of dogness—, but she might even be able to scientifically decide on the existence of dogs. This is a clear example of the first form of verification of an existential claim (**Existence 1**). Let us now give an example of the second way of verifying existence, **Existence 2**. Let us consider the statement 'There exists a black hole.', i.e. $\exists x.\texttt{Blackhole}(x)$. We might not have a direct way of guaranteeing the existence of black holes, so a direct use of **Existence 1** might fail: possibly, a definition of such a concept is not sound or the method of exploring the universe looking everywhere for black holes is not feasible, *videlicet* it is an "infinite" search. A possible method of verifying this statement is to develop consequences of the non-occurrence of black holes in the universe and obtaining contradicting information.

Existence 1 is clearly preferable to **Existence 2**. Unlike **Existence 2**, **Existence 1** gives a direct method of verification; additionally, if an agreement of the used concepts is reached, it gives an unambiguous guarantee of existence. Unfortunately, in most situations, we are not able to guarantee the strong existential claim for a given property $P(x)$, only a consistency claim of the sort 'It is consistent with our data to assume that no general method to obtain non-occurrences of the property $P(x)$ can be constructed.' (**Existence 2**).

It might be the case that an application of **Existence 2** is not enough to guarantee that the statement at hand is not a pseudo-proposition. If we have no empirical way of testing the terms at hand, for example 'black holes'[4], these still belong to

[4] This serves as just an example, we are not making the claim that there

the realm of pseudo-terms; this mentioned way of testing might not be real, it might be a potential test, in the sense that we might conceive the use of some device that we still are unable to construct, but we can clearly state what needs to be done to test. The 'gravitational waves' are an excellent example of this: even before an actual way of testing them was built, they were a meaningful empirical concept, because we could clearly state the testing devices needed to experimentally test them, we simply did not have the know-how and the technology to do so.

Sometimes, in science, existence is used as an abbreviation of certain detection phenomena. That is to say, the application of **Existence 1** can be carried out via a complex defining method. The existence of electrons is of that sort; it is indirect in the sense that complex methods of detection are used, but it is *not* indirect in the sense of **Existence 2** because the detection of electrons does not rely on a method to guarantee that no method of identifying non-electrons exists. 'Electron' is, in a way, just an abbreviation for the positive detection cases of a device the scientists decided to create. This is very common in science: firstly to have a detection phenomena, and then to give a name to those positive occurrences. The majority of the scientific concepts are more fundamental in terms of verification when compared to the everyday-life-concepts, because the scientific concepts are agreed definitions of positive identifica-

---

is no positive way of identifying black holes; we leave those details for the physicist reader. In fact, very recently the scientific community was able to obtain a photo of a black hole: `https://www.nasa.gov/mission_pages/chandra/news/black-hole-image-makes-history`. So, in fact, black holes might have an **Existence 1**.

tion phenomena; in contrast, concepts like ‘car’ are useful social names one gives to simplify one’s everyday life, but they usually do not carry a straightforward identification method, they are not natural in the empirical sense—we could discuss what a chair is for hours and still find a counter-example to our definition of chair that some members of the community still identify as the object.

After this detour about existence, we believe that is now clearer to the reader that when one makes existential claims one is not actually predicating, one is abbreviating the possibility of a method, either of identification (**Existence 1**) or of non-constructiveness of a general non-identifier (**Existence 2**). Consequently, philosophically-relevant statements such as ‘God exists.’ are pseudo-propositions in two levels: firstly, because existence is not a predicate; secondly, because pseudo-terms occur in the statement, like ‘God’ that by “definition” is not empirically detectable.

We have so far described a way of distinguishing pseudo-propositions from propositions—the latter are pseudo-propositions equipped with a verification method. What about the VP itself? Is the VP a pseudo-proposition?

Asking whether the VP is a proposition or a pseudo-proposition is, in a sense, a superfluous question. The VP is like an ethical norm in its unwritten form: it is the recognition of a certain way of acting, not the actual written statement. It is not ethical in what the connotation of good and evil is concerned, nor in what some sort of obligation is concerned. It is a way of living, it is an attitude to knowledge; the VP is the never truly convinced attitude to knowledge, it is the everlasting seeking for evidence and clarification. As we previously saw, the ethical norm 'You shall not kill.' can be viewed as a pseudo-proposition or even as a proposition when the right set-up is devised: in the end, it does not matter because we are not describing the world, at that level we are simply recognizing it! We simply act in the world, we do not need an authority figure deciding to write down pseudo-propositions to know how to act, nor do we need myths passed down through generations to let us know how to act; as Plato mentioned, we have an inner *daemon*[5] that decides what to do. Like in the case of music, we can for sure try to make propositions about it, but that is totally missing the point of music. In order to act, we do not need to speak; to recognize the VP, we simply need to look at our experience of the world. As we previously described, the VP is something that one recognizes and does not argue about, like recognizing that killing is bad; one can (and should!) give as much evidence as one wants to support the claim that killing is bad, just like we did with the VP, but in the end it boils down to a (hopefully) reached recognition of it. These approaches—the ethical recognition and the VP recognition—clearly distance themselves from a sort of dogmatic one because these recognitions are based on *facts*.

After this clarification, we can give an answer to the question 'Is the VP a proposition?' similar to the one we gave to the statement 'You shall not kill.'. We can create an artificial setting that makes it a proposition, for example by considering a meta-version of the VP sustaining the VP itself, but as we mentioned,

[5]This pseudo-term gave rise to the pseudo-term 'demon', but should not be confused with it. *Daemon* can be roughly defined as a "supernatural spiritual being, not necessarily evil". Of course, these are all pseudo-terms.

that is not the point. The main purpose is to proceed from this recognition; to further develop clarifications of the propositions of science, to explain logic in definitional and empirical terms, and so on. Our goal in the following chapter is, consequently, not to further recognize the VP, but rather to apply it and to obtain a better understanding of reality through language.

There are a great variety of relevant references on Positivism, including: [13], [4], [3], and [11]. As general references for first-order logic, especially for the arithmetical part, we recommend [2] and [5]. The reason why we have so few references in the *corpus* of our book is because the matters, as we explain them, are: either original, or common knowledge, or fully defined by us.

Chapter 2

Logic and Mathematics

Logic and mathematics are the basis of rational discourse, so it is paramount to account for them in our empirical analysis. Throughout our book, we make the following fundamental and self-evident assumptions:

Finiteness 2.0.1. In an empirical endeavour, the reader might write down any symbol she wishes. The syntactic resources are *potentially* infinite, in the sense that the reader can write down more symbols than she has already written. The empirical endeavour is, consequently, like a laboratory book where the reader is writing down the propositions she has found and the methods of verifying them. At any given moment, the reader has a finite amount of propositions and methods of verifying them, and by using experience, she might expand the information on that book by writing down new discoveries: we call this the *record book*. In this hypothetical book that will be mentioned throughout our empirical analysis, the reader writes down every single *proposition*, *method of verification*, and the *actual experiments* that she has performed. We therefore exclude infinitary accounts of knowledge, since we cannot experience an infinite amount of information.

Testability 2.0.2. The meaningful propositions can, whenever the reader wishes, be actually tested, i.e. we can execute the empirical description of the method/test of the proposition at any given moment. When a test is performed, it might yield a *positive* result, a *negative* result, or it might even be *inconclusive*. It is not necessarily the case that when one has a way of testing a proposition one also has a method of identifying its non-occurrences; for example, we can easily test 'Some human beings die.', but we cannot develop a general method of identifying its non-occurrences, in our example they are instances of 'eternity', a pseudo-term that cannot be tested. We use the word 'method' and 'test' in an indistinguishable way; a 'method' is to be understood as some agreed procedure of testing/realizing statements.

Warning 2.0.3. The statements we make throughout this book are, in most cases, not propositions! We are describing methods of acting, like in ethics, but these descriptions might not belong to the realm of meaningful discourse via propositions; we aim at creating an impression on the reader in order to be able to follow these actions. In several cases, they are simply definitions and conventions that aim at capturing the everyday use of empirical concepts. In our justifications, we use a form of meta-principles to explain the very logic we are developing: these meta-principles are, *ad consequentiam*, used to give the desired impression on the reader to persuade into our described way of acting and the validity of our methods; these meta-principles are, essentially, simple facts about the use of language and its correspondence to the world, for example the ability to see if two methods, when actually performed, both test positive (this, of course, does not assume any logical rules). We use meta-syntactical variables to denote syntactical objects that the reader can, obviously, instantiate whenever she wishes to do so; for instance, A might denote a (generic) proposition.

Notation 2.0.4. As the reader might already suspect, we are going to deal with syntactical arrays of symbols. To be very strict, for an array of symbols $\xi_0 \cdots \xi_n$, we should always use the quotation '$\xi_0 \cdots \xi_n$' to mention it. For the sake of saving resources, and since this is a common practise in logic, we make the usual abuse of language and use $\xi_0 \cdots \xi_n$ to refer to '$\xi_0 \cdots \xi_n$'. In some circumstances where we want to emphasize that we are mentioning the actual symbols we explicitly write '$\xi_0 \cdots \xi_n$', otherwise we try not to bother the reader with overcrowded notation.

We will define in empirical terms the usual logical connectives and quantifiers with the usual meanings:

Symbols	Meaning
$A \wedge B$	‘A and B.’
$A \vee B$	‘A or B.’
$A \to B$	‘If A, then B.’
$\neg A$	‘Not A.’
$\exists x.A(x)$	‘There is x such that $A(x)$.’
$\forall x.A(x)$	‘For all x, $A(x)$.’

Firstly, we define the propositional part of logic in Section 2.1 where we account for the connectives; and secondly, we account for quantification in Section 2.2. As we did before with existential quantification (recall **Existence 1** and **Existence 2**), we separate the connectives and quantifiers into two sorts: type 1 denotes the actual verifiability, and type 2 denotes non-constructiveness of a non-occurrence. We write in subscript the respective information, for example $\wedge_1$ denotes the conjunction of the first sort, and $\forall_2$ the universal quantification of the second sort.

In this chapter, we give a possible reading of the connectives and quantifiers, it is not necessarily the unique way of framing them in empirical terms; moreover, the quantifiers and connectives we are considering do not have any special status when compared to other definable quantifiers and connectives (for example, exclusive disjunction ‘`XOR`’), we opted to consider these ones for their use in natural language. The fundamental aspect of this chapter is the way we frame logic using empirical testability: here the method of framing logic is the fundamental aspect, not the actual particular way we decided to follow. Mathematics is going to be analyzed after the logical Sections

2.1 and 2.2.

2.1 Empirical Propositional Logic

Following **Finiteness 2.0.1**, only a finite number of propositions is being considered at a given moment. We are going to denote these propositions using capital letters A, B, C, and so on. These letters stand for empirical propositions for which we have already given a concrete empirical meaning, more precisely a method of verification. To emphasize the method of verification of a proposition, we write $\mu : A$ to denote that μ is the method of verification of the proposition A. In the example from Chapter 1, this notation becomes

$$\texttt{Method}(\mathrm{Duck}, \mathrm{Lisbon}, \mathrm{Red}) : \text{'There is a red duck in Lisbon.'}.$$

As before, we assume that we are able to use a syntactical device to have two sorts of information considered together—pairing—that we denote using $\langle\cdot,\cdot\rangle$.

We say that a proposition $\mu : A$ is *true* (*for the method* μ), and we write $\mid\!\sim \mu : A$, or simply $\mid\!\sim A$, if, when the test/method μ is actually performed, A is verified: we recall that, in our context, a method/test is different from the actual action of performing it; the former is just a syntactical description of the latter. (When we write $\mid\!\sim A$, we actually mean that there is a suitable method for A, $\mu : A$, such that $\mid\!\sim \mu : A$.) When we say that a proposition is true, we assume that it was already written in the record book and that a test with positive result can, at any moment, be added to the book. We can enrich our symbolic language with the propositional connectives, but we still need to give them an empirical meaning. We do it as follows.[1]

Empirical Connectives 2.1.1. Let $\mu : A$ and $\nu : B$ be propositions.

Negation 1: If there is an empirical test $\mu_\neg$ associated with μ to identify the non-occurrences of A, then $\mu_\neg : (\neg_1 A)$ is a proposition. By 'associated' we mean that μ gives a way of testing $\mu_\neg$; here we do not allow inconclusive tests. When a test is performed, either μ or $\mu_\neg$ tests positive, and not both.

Conjunction 1: $\langle \mu, \nu \rangle : (A \wedge_1 B)$ is a proposition, where $\langle \mu, \nu \rangle$ denotes any test that, when actually implemented, tests positive exactly when both μ and ν do. For example, 'Test μ and test ν; output a positive result if, and only in that case, both test positive.'.

Disjunction 1: $\mathtt{disj}(\mu, \nu) : (A \vee_1 B)$ is a proposition, where $\mathtt{disj}(\mu, \nu)$ represents either the method μ or the method ν.[2]

Implication 1: $\mathtt{impl}_\mu^\nu : (A \rightarrow_1 B)$ is a proposition, where $\mathtt{impl}_\mu^\nu$ denotes any test that, when empirically implemented, tests positive when all positive occurrences of μ are

[1] Keep in mind that **Negation 1** mentions the type 1 negation, just like **Disjunction 2** is used to refer to the type 2 disjunction, *et cetera*.

[2] We could have presented a definition similar to **Conjunction 1**, for example by saying that '$\mathtt{disj}(\mu, \nu) : (A \vee_1 B)$ is a proposition, where $\mathtt{disj}(\mu, \nu)$ denotes any test that, when actually implemented, tests positive exactly when μ tests positive, or ν tests positive'.

necessarily accompanied by positive occurrences of ν. Here $\mathtt{impl}_{\mu}^{\nu}$ can be, for example, a test of the form 'Test μ and ν. Confirm that ν was positive when μ was positive. In the affirmative case, and only in that case, output positive result.'.[3]

Negation 2: Consider $\mathtt{neg}(A)$ the following method: 'The reader goes to her record book[4] and confirms that the tests performed on A were not positive; output positive in the affirmative case.': when we mention 'not positive', it might be the case that the reader has not yet performed an actual test, that the tests were not conclusive, that the tests were negative, or even that the proposition does not yet appear on the book. Then, $\mathtt{neg}(A) : (\neg_2 A)$ is a proposition. (More formally, we are, in fact considering, '$\mathtt{neg}(\mu)$' instead of '$\mathtt{neg}(A)$', but we opted for the latter notation for the sake of appealing to the reader's intuition.)

Conjunction 2: Suppose that $\mu_\neg : (\neg_1 A)$ and $\nu_\neg : (\neg_1 B)$ are propositions. Then, $\mathtt{neg}((\neg_1 A) \vee_1 (\neg_1 B)) : (A \wedge_2 B)$ is also a proposition, where $(A \wedge_2 B)$ abbreviates $(\neg_2((\neg_1 A) \vee_1 (\neg_1 B)))$.

Disjunction 2: Suppose that $\mu_\neg : (\neg_1 A)$ and $\nu_\neg : (\neg_1 B)$ are propositions. Then, $\mathtt{neg}((\neg_1 A) \wedge_1 (\neg_1 B)) : (A \vee_2 B)$ is also

[3] We could also add a condition stating, when $\neg_1 B$ is a proposition, that $\mathtt{impl}_{\mu}^{\nu}$ needs to test negative when A tests positive and $\neg_1 B$ also tests positive.

[4] This definition, to be implemented, depends on the considered moment in time, since more tests can be added to the record book.

a proposition, where $(A \vee_2 B)$ abbreviates $(\neg_2((\neg_1 A) \wedge_1 (\neg_1 B)))$.

Implication 2: $\texttt{disj}(\texttt{neg}(A), \nu) : (A \rightarrow_2 B)$ is a proposition, where $(A \rightarrow_2 B)$ abbreviates $((\neg_2 A) \vee_1 B)$.

As usual, we omit unnecessary parenthesis, for example: we write $\neg_1 A$ for $(\neg_1 A)$, and $\neg_2(\neg_1 A \vee_1 \neg_1 B)$ for $(\neg_2((\neg_1 A) \vee_1 (\neg_1 B)))$. We say that A is *false* if either $\mu_\neg : (\neg_1 A)$ is a proposition and $\mu_\neg$ was a successful test, or if $\texttt{neg}(A)$ was a successful test. It is important to emphasize that, for a generic proposition A, $\neg_1 A$ is not necessarily defined, but $\neg_2 A$ is. Moreover, we do not require any prior logical notions nor rules in our former definition, what we do require are very simple principles about testing the methods at hand: given two tests, we require the ability to see if both test positive, the ability to see if at least one of them tests positive, *et cetera*. It is important to emphasize that the methods we described can be instantiated by any method satisfying the described defining conditions; for example, $\texttt{impl}_\mu^\nu$ can denote and be instantiated by any method that satisfies the former definition, and not necessarily the one we exemplified.

In the common use of language, and also in the common way of framing propositional logic, the type 1 and type 2 connectives are not distinguished: *verbi gratia*, sometimes when we make a negative claim, we do not have at hand an actual way of testing it, like in **Negation 1**, but, nevertheless, we might still have good grounds to assert it, like in **Negation 2**. Additionally, we might not have a general method of constructing implications as in **Implication 1**, but it might still be meaningful to assert a form of **Implication 2**.

When the atomic propositions and concepts are agreed on, the use of type 1 connectives gives rise to propositions that maintain their empirical status, i.e. their truth value; on the contrary, the type 2 propositions might change their truth value when further empirical information is provided: in a given moment, it might be the case that $\mathrel{|\!\sim} \neg_2 A$, but, after more tests are performed, it might be the case that $\mathrel{|\!\sim} A$ in the next moment; all this depends on the content of the record book at a precise moment in time and on the performed tests. In the usual scientific discourse, both forms of connectives are used, especially the type 2. For the rest of this book, we will, in most situations, describe the content of the methods we wish to consider, instead of actually writing them down in full syntactical detail (we do this to save syntactical resources and for the sake of succinctness).

The facts we are about to mention throughout the rest of this book have a nature of meta-statements, *scilicet* statements about the system that we are describing of the use of the record book; they are not, strictly speaking, propositions, they are pseudo-meta-propositions that create an impression on the reader about their use, keep in mind what we said on Warning 2.0.3. To establish them, we of course use very simple meta-principles; these principles are useful to create an impression on the reader, but are not necessary in a strict sense, namely in the meaningful sense of the record book. These principles have a nature similar to 'A test cannot be both positive and negative.'; they are not truths in the actual sense, they are rather mere conventions or observations.

Fact 2.1.2. Let $\mu : A$, $\nu : B$, and $\chi : C$ be propositions. The following are true propositions[5]:

(i) $A \rightarrow_1 A$;

(ii) $A \rightarrow_1 (B \rightarrow_1 A)$;

(iii) $(A \rightarrow_1 (B \rightarrow_1 C)) \rightarrow_1 ((A \rightarrow_1 B) \rightarrow_1 (A \rightarrow_1 C))$.

Moreover, if $\neg_1 A$ and $\neg_1 B$ are propositions, then the following is a true proposition:

(iv) $(\neg_1 A \rightarrow_1 \neg_1 B) \rightarrow_1 (B \rightarrow_1 A)$.

Confirmation. [6]

(i) Obvious, because we can trivially create a method that makes all occurrences of μ be followed by occurrences of μ itself; this method will always yield a positive empirical application.

(ii) To see that $\mathtt{impl}_\mu^{\mathtt{impl}_\nu^\mu}$ tests positive it suffices to see that a positive occurrence of μ is accompanied by a positive occurrence of $\mathtt{impl}_\nu^\mu$. So, suppose that μ tests positive.

[5] They are (only) true when they are actually written down in the record book. Moreover, the claim that they are true is in fact a claim that there is a suitable method such that the considered proposition always tests positive.

[6] Confirmations have two distinct natures: they can either be viewed as an array of text that aims at producing in the reader a certain kind of impression, namely that the considered fact holds; or, more importantly, they can be viewed as descriptions of meta-procedures to justify the considered fact (in this view, the facts are interpreted as meta-propositions, that is to say propositions about other propositions).

By the definition, as μ tests positive, $\mathtt{impl}_{\nu}^{\mu}$ tests positive, since positive occurrences of ν yield the assumed positive occurrences of μ. This is guarantees that $\mathtt{impl}_{\mu}^{\mathtt{impl}_{\nu}^{\mu}}$ tests positive.

(iii) Let us consider general methods of testing $A \rightarrow_1 (B \rightarrow_1 C)$ and $(A \rightarrow_1 B) \rightarrow_1 (A \rightarrow_1 C)$, respectively of the form $\mathtt{impl}_{\mu}^{\mathtt{impl}_{\nu}^{\chi}}$ and $\mathtt{impl}_{\mathtt{impl}_{\mu}^{\nu}}^{\mathtt{impl}_{\mu}^{\chi}}$. Let us now see that positive occurrences of $\mathtt{impl}_{\mu}^{\mathtt{impl}_{\nu}^{\chi}}$ necessarily yield positive occurrences of $\mathtt{impl}_{\mathtt{impl}_{\mu}^{\nu}}^{\mathtt{impl}_{\mu}^{\chi}}$, and so we can very simply write-down a method for this (keep in mind that, for the sake of succinctness, we do not write-down every single method).

Assume that $\mathtt{impl}_{\mu}^{\mathtt{impl}_{\nu}^{\chi}}$ tested positively: it means that all positive occurrences of μ yield positive occurrences of $\mathtt{impl}_{\nu}^{\chi}$; the latter positive occurrence, by its turn, means that positive occurrences of ν entail positive occurrences of χ. Let us confirm that $\mathtt{impl}_{\mathtt{impl}_{\mu}^{\nu}}^{\mathtt{impl}_{\mu}^{\chi}}$ tests positively under the assumption that $\mathtt{impl}_{\mu}^{\mathtt{impl}_{\nu}^{\chi}}$ also does. For that, let us assume that $\mathtt{impl}_{\mu}^{\nu}$ tested positive. Let us confirm that $\mathtt{impl}_{\mu}^{\chi}$ when performed is also positive. If μ tests positive, as $\mathtt{impl}_{\mu}^{\mathtt{impl}_{\nu}^{\chi}}$ is positive, then $\mathtt{impl}_{\nu}^{\chi}$ is positive. From a previous assumption, $\mathtt{impl}_{\mu}^{\nu}$ is positive, and consequently ν is also positive. Consequently, ν is positive, which implies, when one has in mind that $\mathtt{impl}_{\nu}^{\chi}$ is positive, that χ is positive. This confirms that $\mathtt{impl}_{\mu}^{\chi}$ is positive, as desired.

(iv) Consider general methods of testing $\neg_1 A \rightarrow_1 \neg_1 B$ and

$B \to_1 A$, namely $\mathtt{impl}^{\nu_\neg}_{\mu_\neg}$ and $\mathtt{impl}^{\mu}_{\nu}$. Consider the method that describes the following way of obtaining positive tests of $\mathtt{impl}^{\mu}_{\nu}$ from $\mathtt{impl}^{\nu_\neg}_{\mu_\neg}$. Assume we are given a positive test of $\mathtt{impl}^{\nu_\neg}_{\mu_\neg}$ and we want to develop an also positive method for $\mathtt{impl}^{\mu}_{\nu}$, namely a method that obtains a positive occurrence of μ from a positive occurrence of ν. If ν tests positively, then $\nu_\neg$ cannot test positive (by definition); and so a positive test of $\mu_\neg$ cannot be the case, and so, by definition (since from **Negation 1** μ or $\mu_\neg$ need to test positive, and not both), a positive case of μ occurs. This confirms that $\mathtt{impl}^{\mu}_{\nu}$ is positive.

$\sim\!\!|$

As we already mentioned, we use the symbol '$\mid\!\sim$' to assert that a proposition is true; we sometimes omit the test/method. For example, from Fact 2.1.2, $\mid\!\sim A \to_1 (B \to_1 A)$. We recall that, in general, $\mid\!\sim \mu : A$ means that when the test μ was performed, it validated A. Further true propositions can be constructed from the usual classical propositional logic axioms, for example $\mid\!\sim A \to_1 A \vee_1 B$, $\mid\!\sim A \wedge_1 B \to_1 B$, and so on.

Fact 2.1.3. The record book is *incomplete*, i.e. at any given moment, there is always a true proposition that the reader can write on the record book that is missing from it.

Confirmation. This fact is a very intuitive one: the reader is only writing a finite number of propositions, so it is not a big surprise that she might always write down more true propositions. The more formal justification for this fact is the following.

Consider a fixed moment in time and consider the propositions the reader has written. Using Fact 2.1.2 she can build a new proposition missing from the list that, according to our explanations, will for sure test positive. $\dashv$

Fact 2.1.4. Let $\mu : A$ and $\mathtt{impl}_\mu^\nu : A \rightarrow_1 B$ be propositions. If $\mid\!\sim \mu : A$ and $\mid\!\sim \mathtt{impl}_\mu^\nu : A \rightarrow_1 B$, then $\mid\!\sim \nu : B$. This should be read as: if $\mu : A$ and $\mathtt{impl}_\mu^\nu : A \rightarrow_1 B$ are in the record book and a positive test was obtained for both of them (and written down in the book), then $\nu : B$ will test positive when the actual test is performed: it is important to keep in mind that the mentioned information should be added to the book. In a sense, this means that $\nu : B$ can, in the previous conditions, immediately be added to the record book, accompanied with the previous justification, without the need for a test on $\nu : B$, because such a test is contained in a test of $\mu : A$ and $\mathtt{impl}_\mu^\nu : A \rightarrow_1 B$.

Confirmation. Immediate from the definition of the methods of the form $\mathtt{impl}_\mu^\nu$. $\dashv$

We call the previous derivation (from Fact 2.1.4) *modus ponens* (MP), also known as *modus ponendo ponens* , literally "method of affirming". We can write the previous information in a more schematic way, namely

$$\frac{\mid\!\sim \mu : A \qquad \mid\!\sim \mathtt{impl}_\mu^\nu : A \rightarrow_1 B}{\mid\!\sim \nu : B}\,(\mathrm{MP}) .$$

Warning 2.1.5. Mind the reader: we do not actually have (full) classical propositional logic (the usual colloquial logic of everyday life) for the type 1 connectives! Despite the fact that (i)–(iv) constitute a possible axiomatization of classical propositional logic, and despite the fact that we have MP, there is one essential feature that we do not have in our analysis: not every proposition allows negation $\neg_1$. In addition, at a given moment, the reader only has a finite number of propositions, not an actual infinite number of propositions, just a *potential* infinite construction. For the propositions that allow the negation $\neg_1$, a form of classical propositional logic is allowed (when the tests are implemented), in the sense of Facts 2.1.2 and 2.1.4.

A *finite collection* (*or list*) *of propositions* is an array of symbols $\langle \mu_0 : A_0, \ldots, \mu_n : A_n \rangle$, where $\mu_0 : A_0, \ldots, \mu_n : A_n$ are all propositions. The "numbers" used in $\mu_i : A_i$ are just a shorthand notation, they obviously do not assume the actual numbers, they can be substituted for symbols of the form $\overbrace{||\cdots|}^{n \text{ times}}$; furthermore, $\mu_i : A_i$ are variables in the sense that they ought to be substituted by actual methods and actual propositions. In this context, we say that each $\mu_i : A_i$ is an element of the collection (list).

Now we can give a concrete meaning to the concept of a scientific theory.

Theory 2.1.6. A (*scientific*) *theory* T is a finite collection of propositions, for a given moment in the record book (this depends on the information on the record book). We define the *proofs in* T, also called the *justifications in* T or the *arguments in* T, using the following construction:

Proof 1: If $\tau_0 : T_0$ is a proposition in the collection T, then $\langle \tau_0 : T_0 \rangle$ is a proof.

Proof 2: If $\langle \mu_0 : A_0, \ldots, \mu_n : A_n \rangle$ is a proof and if $\mu : A$ and $\mathtt{impl}_\mu^\nu : A \rightarrow_1 B$ are elements of the considered proof, then $\langle \mu_0 : A_0, \ldots, \mu_n : A_n, \nu : B \rangle$ is a proof.

Proof 3: If $\langle \mu_0 : A_0, \ldots, \mu_n : A_n \rangle$ is a proof and $\nu : B$ is in the collection T, then $\langle \mu_0 : A_0, \ldots, \mu_n : A_n, \nu : B \rangle$ is a proof.

(a) We say that π is a *weak proof in* T if we allow the occurrence of implications $\rightarrow_2$ in **Proof 2**, instead of uses of $\rightarrow_1$.

(b) If $\langle \mu_0 : A_0, \ldots, \mu_n : A_n \rangle$ is a proof in T, then we say that $\mu_n : A_n$ is *provable in* T; we use the notation $T \vdash \mu_n : A_n$, or simply $T \vdash A_n$, to denote that fact.

(c) If $\langle \mu_0 : A_0, \ldots, \mu_n : A_n \rangle$ is a weak proof in T, then we say that $\mu_n : A_n$ is *weakly provable in* T; we use the notation $T \Vdash \mu_n : A_n$, or simply $T \Vdash A_n$, to denote that fact.

(d) We write $\mid\!\sim \langle \tau_0 : T_0, \ldots, \tau_n : T_n \rangle$ to denote that $\mid\!\sim \tau_0 : T_0, \ldots, \mid\!\sim \tau_n : T_n$.

When we are considering a particular proof, we assume that it was already written in the record book; that is to say, we do not consider the proofs "all-together", but we analyze particular individual proofs and, to do so in a meaningful way, the considered proofs have to be written in the record book. As the content of the record book, although it might change over time, is always finite, we can only analyze, at a given moment, a finite number of proofs.

Fact 2.1.7. Suppose that T is a theory and that $T \vdash \mu : A$. If $\mid\!\sim T$, then $\mid\!\sim \mu : A$.

Confirmation. We confirm this fact using the clauses of the definition of proof:

Proof 1: Immediate, by definition.

Proof 2: Suppose that $\langle \mu_0 : A_0, \ldots, \mu_n : A_n \rangle$ is a proof in T, that $\mu : A$ and $\mathtt{impl}_{\mu}^{\nu} : A \to_1 B$ are elements of that proof, and that it was already confirmed that $\mid\!\sim \mu_0 : A_0, \ldots, \mid\!\sim \mu_n : A_n$. In particular, $\mid\!\sim \mu : A$ and $\mid\!\sim \mathtt{impl}_{\mu}^{\nu} : A \to_1 B$. By definition $T \vdash \nu : B$. By Fact 2.1.4, we know that $\mid\!\sim \nu : B$.

Proof 3: Effortless, by definition.

The desired fact follows from the construction of proofs. $\dashv$

Theories, as the reader already knows, are very useful, especially in science. There, scientist use both proofs in T and, more often, weak proofs in T. Fact 2.1.7 confirms that provable propositions in theories that are verifiably confirmed (that each of the propositions of the theory tests positive) are also positively tested. This means that one can extract from a proof a method of testing the desired proposition.

2.2 Empirical First-Order Logic

We continue our clarification of language by accounting for the quantifiers. So far, we described the use of the connectives, but if the reader wants to deepen the empirical structure of the propositions she is testing, she needs to be able to speak about objects, properties of objects, and to quantify over them.

Objects 2.2.1. The objects the reader tests for the record book do not need to have a nature similar to 'cat', 'dog', 'door', and so on; they might be procedures in the world, they might be things we do not see, but that we can detect, *et cetera*.

First-Order Language 2.2.2. The term 'first-order' is used to mention that the quantifiers do not quantify meta-variables. In this setting, we allow the record book to have:

Constants: They are (a finite number of) symbols, for example `c`, that have a clear intended interpretation in the world, $\mathscr{I}(\texttt{c})$. We can, for instance, have a constant denoting a particular dog named 'Alpha', namely a constant `alphathedog`.

Variables: Symbols like $x_0, x_1, y_1, x, y, \ldots$ are used to denote arbitrary objects in the world, *variables*. They can, in a concrete proposition, be substituted by concrete objects; we use parenthesis to denote substitution of a variable by the content of the parenthesis. For example, 'x is a dog.'(`alphathedog`) is '`alphathedog` is a dog.'; more generally, the parenthesis in a proposition A, namely $A(x)$, are used to denote the substitutions of the variable x; in the previous example, A is 'x is a dog.' and $A(\texttt{alphathedog})$ is '`alphathedog` is a dog.'. $A(x)$ is clearly the same as A, in our example.

Methods: In this context, the methods might depend on variables, we emphasize this by writing $\mu(x_0, \ldots, x_m)$ for the possibility of substituting the variables $x_0, \ldots, x_m$ in the method μ, just like we previously described for propositions. For example, in the previous case where A is 'x is a dog.', if μ is a general method of identifying dogs, i.e. $\mu : A$ is a proposition, then $\mu(\texttt{alphathedog}) : A(\texttt{alphathedog})$ is also a proposition, where $\mu(\texttt{alphathedog})$ gives a concrete justification of '`alphathedog` is a dog.'.

Atomic concepts: We assume that the reader, in her record book, is dealing with a finite number of atomic concepts $\mathtt{R}^n(x_0, \ldots, x_{m_n})$; the n-th concept, $\mathtt{R}^n$, depends on the variables $x_0, \ldots, x_{m_n}$. For example, 'x is a dog.' might be one of the atomic concepts the reader is considering.

Function-symbols: These symbols, usually denoted by $\mathtt{f}, \mathtt{g}$, $\mathtt{f}_0, \mathtt{f}_1, \ldots$, represent (descriptions of) functions, i.e. real world procedures. To emphasize that they depend on certain variables, we write $\mathtt{f}(x_0, \ldots, x_m)$. The function--symbols are related to methods, since when one is given $\mathtt{f}$, we demand to have an actual way of creating the procedure described by $\mathtt{f}$. An example of such an $\mathtt{f}(x)$ is 'Find the chair closest to x.'.

Terms: Terms are defined in the following way:

Term 1: Every constant $\mathtt{c}$ and every variable x is a term;

Term 2: If $t_0, \ldots, t_n$ are terms and $\mathtt{f}$ is any function-symbol, then $\mathtt{f}(t_0, \ldots, t_n)$ is also a term.

Notation 2.2.3. As we previously mentioned, letters like A, B, and so on are meta-variables denoting propositions; greek letters like μ and ν are used to denote (syntactical descriptions of) methods; letters like $x_0, x_1, y_1, x, y, \ldots$ denote actual (syntactic) variables; and calligraphic letters like $\mathscr{O}$ and $\mathscr{I}$ denote real world objects, entities, procedures, *et cetera*.

The atomic concepts are the basis to construct more complex propositions, they might describe phenomena like: 'x was detected after the experiment named `asTx` was performed on y.', 'x is a red flower.', 'x emits X-ray radiation in the presence of y when passing through z.', 'When x_0 is dissolved in x_1, using a flame of intensity x_2, x_0 emits radiation which wavelength is x_3.', and so on. It might be the case that the reader wants to further analyze and decompose the atomic concepts into other atomic concepts; that does not affect what we describe below.

First-Order Empirical Logic 2.2.4. We now describe how to attribute meaning to the quantifiers and to the connectives where we allow the occurrence of variables. This construction takes as a starting-point the *atomic concepts* the reader is considering; from them, we create more complex propositions about possible description of the world.

Atomic Propositions: The atomic concepts are propositions, more precisely, for each atomic concept $\mathtt{R}^n$, there is a method μ^n such that $\mu^n(x_0, \ldots, x_{m_n}) : \mathtt{R}^n(x_0, \ldots, x_{m_n})$ is a proposition. For example, $\mathtt{dogtest}(x)$:'x is a dog.' might be an atomic proposition, for a suitable $\mathtt{dogtest}(x)$ that tests dogginess.

Suppose that $\mu(x_0, \ldots, x_m) : A(x_0, \ldots, x_m)$ and $\nu(y_0, \ldots, y_\ell) : B(y_0, \ldots, y_\ell)$ are propositions.

Constants: Consider constants $\mathtt{c}_0, \ldots, \mathtt{c}_m$ from the record book representing objects in the world. To confirm the proposition $\mu(\mathtt{c}_0, \ldots, \mathtt{c}_m) : A(\mathtt{c}_0, \ldots, \mathtt{c}_m)$ is to perform the test $\mu(\mathtt{c}_0, \ldots, \mathtt{c}_m)$ with the intended objects represented by the considered constants, namely $\mathscr{I}(\mathtt{c}_0), \ldots, \mathscr{I}(\mathtt{c}_m)$.

Function-symbols: Given a function-symbol $\mathtt{f}$ in the record book, a test of $A(\mathtt{f}_0(y^0_0, \ldots, y^0_{\ell_0}), \ldots, \mathtt{f}_m(y^m_0, \ldots, y^m_{\ell_m}))$ comprises a method to create each of the $\mathtt{f}_0(y^0_0, \ldots, y^0_{\ell_0}), \ldots, \mathtt{f}_m(y^m_0, \ldots, y^m_{\ell_m})$ by the described procedures $\mathtt{f}_0, \ldots, \mathtt{f}_m$, and a confirmation, using μ, that A holds for them. In more detail, the reader applies, for each i, $\mathtt{f}_i$ to the given $y^i_0, \ldots, y^i_{\ell_i}$ and obtains a_i; then tests $A(a_0, \ldots, a_m)$.

Universal 1: If there is a general method $\mathtt{uni}_1$ to guarantee that, for all (possibly infinite) ways to assign objects in the world to the variables, $A(x_0, \ldots, x_m)$ tests positive for each one of those assignments, then $\mathtt{uni}_1(A) : \forall_1 x_0. \cdots \forall_1 x_m.A(x_0, \ldots, x_m)$ is a proposition. We have no definitive answer to what such a method should look like; on the contrary, it depends heavily on the framework of the record book and on the methods the reader is considering in a specific moment in time. A $\mathtt{uni}_1(A)$ method can, after further information is provided, change its nature and no longer be considered a general method to attest all instances. In a given moment in time, the reader and the community settle down what methods are considered as sound $\texttt{uni}_1(A)$ methods. One should be especially careful with this: the type of science one is developing might considerably change when one mischaracterizes a $\texttt{uni}_1(A)$ method.

Universal 2: Consider the following method: 'Considering the (finite) set of objects from the record book, doing all possible ways to associate to each variable x_i a concrete object $\texttt{obj}(x_i)$ (from the record book), see if in the record book all the possible tests for $A(\texttt{obj}(x_0), \ldots, \texttt{obj}(x_m))$ were positive. Output a positive result in an affirmative case.'. We call the previous method $\texttt{uni}_2(A)$. Then, $\texttt{uni}_2(A) : \forall_2 x_0. \cdots \forall_2 x_m.A(x_0, \ldots, x_m)$ is a proposition.

Instantiate 1: From a particular method[7] $\texttt{uni}_1(A)$, we can, by definition, obtain a test of A for each instance; so there is a method $\texttt{inst}_{\texttt{uni}_1(A)}$ such that $\texttt{inst}_{\texttt{uni}_1(A)}(x_0, \ldots, x_m)$ is a test of $A(x_0, \ldots, x_m)$, i.e. $\texttt{inst}_{\texttt{uni}_1(A)}(x_0, \ldots, x_m) : A(x_0, \ldots, x_m)$ is a proposition. Clearly, by definition, if $\forall_1 x_0. \cdots \forall_1 x_m.A(x_0, \ldots, x_m)$ is a true proposition, then, for all possible ways to associate to each variable x_i a concrete object $\texttt{obj}(x_i)$, $\mid\!\sim \texttt{inst}_{\texttt{uni}_1(A)}(\texttt{obj}(x_0), \ldots, \texttt{obj}(x_m)) : A(\texttt{obj}(x_0), \ldots, \texttt{obj}(x_m))$.

Existential 1: If there is a method $\texttt{exists}_1(A)$ of finding objects such that the method tests positive exactly when $A(\texttt{obj}(x_0), \ldots, \texttt{obj}(x_m))$ is true for some of the found objects denoted by $\texttt{obj}(x_0), \ldots, \texttt{obj}(x_m)$; then, $\texttt{exists}_1(A)$:

[7]Keep in mind that $\texttt{uni}_1(A)$ does not stand for a fixed method, it rather stands for any method satisfying the definition.

$\exists_1 x_0. \cdots \exists_1 x_m.A(x_0, \ldots, x_m)$ is a proposition. We require that the method $\texttt{exists}_1(A)$ terminates, i.e. that it does not proceed by testing indefinitely the whole universe (the reader needs to be aware of this fact to write down an $\texttt{exists}_1(A)$ method). While implementing the method $\texttt{exists}_1(A)$, the reader keeps adding the new objects she might be using to do the tests to the record book. For the methods $\texttt{exists}_1(A)$, we also demand the existence of a function-symbol $\texttt{witness}_1$ describing a process to obtain the objects that are given by the $\texttt{exists}_1(A)$ method in the positive cases; that is to say, if $\texttt{exists}_1(A)$ tests positive, then $\texttt{witness}_1(\texttt{exists}_1(A))$, when implemented, gives a list $\langle \texttt{obj}_0, \ldots, \texttt{obj}_n \rangle$ of (names of) objects in the record book such that $\mathrel{|\!\!\sim} A(\texttt{obj}_0, \ldots, \texttt{obj}_n)$.

Existential 2: Consider the following method $\texttt{exists}_2(A)$: 'Go to the record book and confirm that no positive test was obtained for the proposition $\forall_2 x_0. \cdots \forall_2 x_m.\neg_2 A(x_0, \ldots, x_m)$; output positive in the affirmative case.'. By definition, $\texttt{exists}_2(A) : \exists_2 x_0. \cdots \exists_2 x_m.A(x_0, \ldots, x_m)$ is a proposition.

Closure: If in $A(x_0, \ldots, x_m)$ the variables are not bounded by quantifiers (in that context they are called *free variables*), then we define that $\mathrel{|\!\!\sim} A(x_0, \ldots, x_m)$ means that $\forall_1 x_0. \cdots \forall_1 x_m.A(x_0, \ldots, x_m)$ is a true proposition. We need to make this definition because, strictly speaking, a proposition of the form 'x is a dog.' only has concrete meaning when we substitute x for a concrete term without variables; without this definition, it is meaningless to say that propositions of the form 'x is a dog.' are true.

The definition of the connectives is very similar to the one we previously presented (see section 2.1), let us just give the example of the definition of conjunction.

Conjunction 1: Consider any method $\mathtt{conj}_A^B(x_0, \ldots, x_m, y_0, \ldots, y_\ell)$ that tests positive, for concrete choices of $x_0, \ldots, x_m, y_0, \ldots, y_\ell$, whenever $\mu(x_0, \ldots, x_m)$ and $\nu(y_0, \ldots, y_\ell)$, when performed, give a positive test. In this context, $\mathtt{conj}_A^B(x_0, \ldots, x_m, y_0, \ldots, y_\ell) : A(x_0, \ldots, x_m) \wedge_1 B(y_0, \ldots, y_\ell)$ is a proposition. Using our propositional notation, and using paring, $\mathtt{conj}_A^B(x_0, \ldots, x_m, y_0, \ldots, y_\ell)$ can be viewed as $\langle \mu(x_0, \ldots, x_m), \nu(y_0, \ldots, y_\ell) \rangle$.

Just like the case of $\neg_1$, not all propositions A allow the occurrence of the $\forall_1$ universal quantifier, one needs to have a general method for it. Let us give a toy example of a situation where one has such a method for $\forall_1$. Imagine that we are considering a room $\mathscr{R}$ with 15 chairs on it. Consider $A(x)$ as being the proposition that expresses 'x is a chair in the room.' $\to_1$ 'x is made of wood.'. It is easy to develop a general method to confirm $A(x)$, for example by confirming that each one of the 15 chairs in $\mathscr{R}$ is made of wood. In that case, we can claim that $\forall_1 x.A(x)$ is a proposition. If, in the previous example, the reader is not aware that the total number of chairs is 15, it might be the case that she is totally unaware of a general way of testing all the chairs, in that case only $\forall_2 x.A(x)$ is a proposition. An occurrence of $\forall_1$ does not need necessarily to quantify over a "finite" domain, it might be the case that we can have a general method for domains that are not necessarily known to be "finite".

Without loss of generality, we make the following convention.

Convention 2.2.5. When we consider an object denoted by `obj`, we assume, when tests are being considered in the record book, that `obj` was tested for all the finite number of propositions in the book.

Formulas *versus* Propositions 2.2.6. The propositions we are describing do not coincide with the logical concept of a *first-order formula*: the latter belongs to a (logical) formal theory, does not make the distinction about the quantifiers, and allows to add universal and existential quantifiers indefinitely, without the need of an actual way of testing. *Exempli gratia*, $\forall x.\exists y.\forall z.(x = z \to S(x, y, z))$ is a first-order formula. In the context of formulas, we do not need to attribute an empirical meaning, they are syntactic entities that, after a careful analysis, might have such a meaning, but we do not need to assume it to use the usual methods of Proof Theory.

Finite Universe 2.2.7. The statement 'The universe is finite.' is, *without further information*, a pseudo-proposition, so this remark should be read with the pseudo-eye-glasses. If the universe if finite, eventually the quantifier $\forall_2$ becomes the quantifier $\forall_1$: in that case, the $\forall_2$-method of testing all the (finite number of) objects in the universe gives a method to claim the $\forall_1$-quantifier.

Theory 2.2.8. A (*scientific*) *first-order theory* T is a finite collection of first-order propositions for a given moment in the record book: it can be though as the pair $\langle T, \mathscr{R} \rangle$, where T is the finite collection and $\mathscr{R}$ is all the (finite) content of the record book at a given moment. We define the *proofs in* T and *weak proofs in* T in the same way as we did before. We say that a theory T is a *tested theory* if $\mathrel{|\!\sim} T$: this definition depends on the content of $\mathscr{R}$ at the considered moment, that is why theories should, more formally, be considered as the previously mentioned pair $\langle T, \mathscr{R} \rangle$. Facts 2.1.2 and 2.1.4 still hold in this context. Some of the most relevant scientific theories are theories where the propositions are of type 2, for example the theory of evolution: we cannot test that every single animal, living or dead, obeys to the law of evolution, nor do we have a general method of guaranteeing that evolution holds for all animals; nevertheless, we have good grounds of type 2 to consider it a relevant sufficiently tested proposition!

Proofs and weak proofs correspond to the usual scientific argumentation, the only difference is that here we make explicit which arguments use type 1 and which arguments might use occurrences of type 2. There is no negative aspect in using, *in a conscious way*, type 2 propositions and arguments, the only danger is when type 2 propositions are interpreted as the stronger type 1 propositions. This is especially problematic when a negative conclusion of type 2 is empirically confirmed, but the community interprets it as a type 1 proposition instead of just a type 2 proposition: then, the erroneous feeling that science might be wrong appears.

Science is not wrong because Nature is not wrong; Nature simply *is*. Science is a syntactic description of Nature where certainty can be claimed, when the basis concepts are agreed on, for propositions of type 1; the plausibility realm is reserved for type 2 propositions, where the exhaustive testing part of science, namely the part where *consistency* claims are made, and not the actual claims *per se*: when we claim that 'All free objects fall to the ground.', we are actually claiming that we have not yet found a reason to deny it; moreover, by further testing this type 2 proposition we approximate certainty, but never actually reach it. The descriptions of science might differ from reality when one tries to frame type 2 propositions as type 1. Consequently, scientific revolutions might have two origins: either the concepts used in the type 1 propositions are no longer agreed and further explanation is needed, or some type 2 propositions, when further tested, turn out to not correspond to reality in the strict sense of type 1. (More on this in the next chapter.)

Fact 2.2.9. Let A and B be propositions such that y does not syntactically occur in A and such that $\forall_1 x.A(x)$, and $\forall_1 x.(A(x) \to_1 B(x))$ are propositions. Let t be any term without variables that corresponds to an object in the world[8]. The following are true propositions (assuming that they were written in the record book, *et cetera*):

(v) $(\forall_1 x.A(x)) \to_1 A(t)$;

(vi) $(\forall_1 x.(A(x) \to_1 B(x))) \to_1 ((\forall_1 x.A(x)) \to_1 (\forall_2 x.B(x)))$;

(vii) $A \to_1 \forall_1 y.A$;

[8] For example, if we consider `f`(x) as being 'Find the biggest petal in the flower x.', then `f(alphathedog)` is a term that has no correspondence in the world, since `alphathedog` has no petals.

(viii) $(\forall_1 x.A(x)) \to_1 (\forall_2 x.A(x))$.

Confirmation. The confirmation is similar to what was done in Fact 2.1.2. It is important to mention, regarding (vi), that in general we cannot substitute '$\forall_2 x.B(x)$' for '$\forall_1 x.B(x)$', since the latter requires that a strong type 1 positive test is at hand, which might not necessarily be the case; for instance, it can happen, under the assumption of $\forall_1 x.(A(x) \to_1 B(x))$ and $\forall_1 x.A(x)$ tested positive, that we have no general method to test $\forall_1 x.B(x)$, this is totally compatible with the way we framed things; only $\forall_2 x.B(x)$ needs to test positively. $\dashv$

Fact 2.2.10. Let A and B be propositions such that y does not syntactically occur in A, and `obj` denotes an object in the record book (`obj` can be regarded as a constant). The following are true propositions:

(ix) $(\forall_2 x.A(x)) \to_1 A(\texttt{obj})$;

(x) $(\forall_2 x.(A(x) \to_1 B(x))) \to_1 ((\forall_2 x.A(x)) \to_1 (\forall_2 x.B(x)))$;

(xi) $A \to_1 \forall_2 y.A$.

Confirmation. Similar to the previous Fact. (ix) and (x) rely on **Convention 2.2.5**. $\dashv$

Equality 2.2.11. So far, we have not yet attributed an empirical meaning to equality, i.e. to the symbol '='. This symbol is used to express that two syntactical symbols are names to the same object. Consider `presidentusa` a constant for the President of the United States of America and `whitehouseman` a constant for the man that lives in the White House. In this setting, to say that '`presidentusa` = `whitehouseman`' is the same as saying that the object that corresponds to `presidentusa` is the same as the object that corresponds to `whitehouseman`. In general, a method to confirm an equality '$x = y$' is simply going to the world and see that the object that x is denoting is the same object that y is denoting. We denote a method of testing equality by $\texttt{equa}(x, y)$; so, $\texttt{equa}(x, y) : x = y$ is a proposition[9]. Equality can be one of the atomic concepts that the reader is considering in the record book.

We have the following well-known facts about equality.

Fact 2.2.12. The following are true propositions:

(xii) $x = x$;

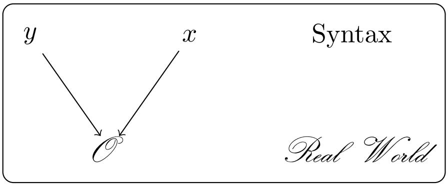
(xiii) $x = y \rightarrow_1 y = x$;

(xiv) $(x = y \wedge_1 y = z) \rightarrow_1 x = z$.

Confirmation. Immediate, by definition of equality, but let us give the example of (xiii) to emphasize an important aspect of equality. Any confirmation $\texttt{equa}(x, y)$ of $x = y$ attests that the object in correspondence with x is the same one as with y, but this also confirms that the object in correspondence with y is the same on as with x. So $x = y \rightarrow_1 y = x$ is always confirmed when tested.

[9]For historical reasons, we write '$x = y$' instead of '$= (x, y)$'.

What we described does not assume equality in the meta-language, it rather assumes that correspondences are unchanged, that is to say that we might see '$x = y$' as the following diagram



where '$\mathscr{O}$' denotes the common real world object; this means that equality is, *by definition*, not directional; we have the correspondences altogether, not one at a time. This confirms that the use of a form of meta-equality is not required. The same goes for the other justifications. $\mathrel{\sim\!|}$

Constants and Propositions 2.2.13. Let us consider a proposition $A(x)$ such that $\exists_1 x.A(x)$ is also a proposition. If $\mathrel{|\!\sim} \exists_1 x.A(x)$, then the reader might introduce a constant $\mathtt{c}_A$ denoting some arbitrarily fixed real wold object that satisfies the property A, i.e. such that $\mathrel{|\!\sim} A(\mathtt{c}_A)$. Conversely, if the reader is given a constant $\mathtt{c}$, if she is considering equality in the sense we previously described, she can create a proposition describing $\mathtt{c}$, namely $x = \mathtt{c}$.

Types 2.2.14. The type distinction we defined is not a fixed one over the passage of time, in the sense that a type 1 proposition can, after more information is provided to the reader about the methods she is using, be, afterall, considered instead a type

2 proposition: this can happen especially if methods the reader was previously considering became, for some empirical reason, not sound to use. Similarly, if the reader allows more methods into her framework, some type 2 propositions can now be considered type 1 if one of those methods allows to have a general way of testing the proposition at hand. Nevertheless, for a fix moment in time, and for a particular record book, the type 1 and type 2 distinction is a rigid one: a proposition is, by the way we defined them, of either type 1 (we only allow occurrences of type 1 quantifiers and connectives), or of type 2 (we allow occurrences of type 2 quantifiers and connectives); clearly, by definition, in a type 2 proposition it might occur type 1 connectives and quantifiers.

2.3 The Mathematical Meaning and Numbers

So far, we made an empirical account of first-order logic (the logic related to the colloquial use of quantifiers and connectives) and observed that this description is not the same as mathematical formal theories, since we have two distinct types of quantifiers and connectives, and since a method of confirmation needs to be provided at every single stage. This does not mean that, starting from a formal theory in the mathematical sense, we cannot attribute to it an empirical meaning; we, in fact, can do so for the most meaningful mathematical theories: theories about natural numbers, integers, rational numbers, real numbers, and many more.

Mathematical Meaning 2.3.1. A mathematical theory is a syntactical description of syntactic objects that, by its own, has no empirically meaningful content (besides the very symbols) in an *a priori* sense. Nevertheless, mathematical theories are extremely useful; in a sense, they are a warehouse of syntactic description ready to be used by the scientist whenever an empirical meaning is attributed to such a theory. To attribute an empirical meaning to a mathematical theory the scientist needs to:

Meaning attribution 1: Develop methods for the basis properties and objects described in the mathematical theory (number, set, and so on), and

Meaning attribution 2: Clarify which type of connectives and quantifiers in the mathematical theory, now in the empirical setting, are being considered.

In what follows, we give possible ways to interpret, in an empirical way, the natural numbers and the real numbers as world object.

Natural Numbers 2.3.2. The *natural numbers* are, intuitively, the numbers we use to count groups of objects that we tacitly decide to ignore the intricate inner-differences: we can count sheep (and not making an actual distinction between each individual sheep in the flock), dogs, bottles of water, and so on. They play an important role in our everyday life, since counting is one of the essential features of our way to communicate certain facts about the world. In what follows, we formalize the intuitive idea of counting.

Let $\overline{n}$ denote $\overbrace{||\cdots|}^{n \text{ times}}$, we call $\overline{n}$ the *numeral of* n; we assume that $\overline{0}$ is a blank space that, for the sake of facilitating reading it, we denote by $\texttt{blank}$. For example, $\overline{7}$ is |||||||.

There are several ways to empirically frame the natural numbers using numerals, we describe the one we consider the most intuitive. Let us fix a domain of discourse $\mathscr{D}$ whose syntactical representation is D; this domain can be the domain of chairs, books, real world objects, bottles of water, it does not matter for the description we are going to give, the essential feature is that we do not consider the differences between the objects, every object is the same, in what counting is regarded, as any other object; we just need the ability of having one object, and another, and another, and so on. The syntactical object $\texttt{number}_{\overline{n}}$ denotes any one-to-one correspondence between the dashes in $\overline{n}$ to different, but in our analysis indistinguishable, real objects from D. For example, if we have distinct elements of $\texttt{D}$ named $\texttt{d}_1, \texttt{d}_2$, and so on (keep in mind that the differences of the objects do not matter); $\texttt{number}_{|||||}$ can be the correspondence (following Notation 2.0.4, we use the quotations '$\cdots$' to emphasize that we are referring to a syntactical object):

$$`\begin{pmatrix} | \longrightarrow \texttt{d}_2 \\ | \longrightarrow \texttt{d}_5 \\ | \longrightarrow \texttt{d}_1 \\ | \longrightarrow \texttt{d}_3 \\ | \longrightarrow \texttt{d}_4 \end{pmatrix}' .$$

As before, the correspondence is considered in a joint way, that is to say that the previous syntactical object represents the same correspondences as

$$\text{`}\left(\begin{array}{l} |\longrightarrow \mathtt{d}_1 \\ |\longrightarrow \mathtt{d}_4 \\ |\longrightarrow \mathtt{d}_2 \\ |\longrightarrow \mathtt{d}_3 \\ |\longrightarrow \mathtt{d}_5 \end{array}\right)\text{'}.$$

Numbers are the interpretation of the correspondences described by `number`; for instance, $\mathtt{number}_{|||||}$ is the actual process of making corresponding each one of the 5 dashes in '|||||' to real world elements in $\mathscr{D}$. Just like the pastor used stones to make a one-to-one correspondence with each sheep (and for him the sheep, in what counting is regarded, are the same), here we use the dashes and `number` to have a syntactical representation of the action of making a correspondence. The numerals, as syntactical entities, are the names for the numbers, as long as the domain $\mathscr{D}$ is big enough to capture the amount of dashes we use. For example, for the number 5, the real world implementation of $\mathtt{number}_{|||||}$ for the domain $\mathscr{D}$, is named by '|||||'[10]. and is the actual real correspondence[11]

[10]Of course, we could also consider $\text{`}\left(\begin{array}{l} |\longrightarrow \mathtt{d}_1 \\ |\longrightarrow \mathtt{d}_4 \\ |\longrightarrow \mathtt{d}_2 \\ |\longrightarrow \mathtt{d}_3 \\ |\longrightarrow \mathtt{d}_5 \end{array}\right)\text{'}$ as being the name of $\mathtt{number}_{|||||}$; in a sense, '|||||' abbreviates the previous expression.

[11]The word 'correspondence' here might seem something very esoteric,

$$\left(\begin{array}{c} `|\text{'} \longrightarrow \mathscr{d}_1 \\ `|\text{'} \longrightarrow \mathscr{d}_4 \\ `|\text{'} \longrightarrow \mathscr{d}_2 \\ `|\text{'} \longrightarrow \mathscr{d}_3 \\ `|\text{'} \longrightarrow \mathscr{d}_5 \end{array}\right).$$

Numeral Operations 2.3.3. We can define operations, representable by function-symbols, in the numerals. Here we operate with '$\overline{n}$', since we want to use syntactical descriptions to describe actions on the syntactical entity $\overline{n}$.

Successor: Consider $\mathtt{S}$ the function-symbol 'Add another occurrence of '|' to x.'. For example, $\mid\!\sim \mathtt{S}(`||\text{'}) = `|||\text{'}$ since to find the object named by $\mathtt{S}(`||\text{'})$ we firstly need to implement it and obtain the array '|||', and then '|||' and '|||' of course correspond to, i.e. name, the same number (we remind that the number is a process of establishing correspondences in the previously described way).[12] Moreover, $\mid\!\sim \mathtt{S}(\mathtt{S}(`\overline{6}\text{'})) = \mathtt{S}(`\overline{7}\text{'})$. It is important to keep in mind that $\mathtt{S}$ operates on the representation the actual syntactical entity '$\overline{n}$', it is a syntactic description of a procedure on syntactical entities of the form $\overline{n}$.

Addition: We can define addition in two ways; a direct way, and a way through a schema.

---

but it can be viewed as one of two things: either a mental process, or an actual way of placing one thing close to another thing, or even a certain action like the pastor does when counts sheep using *correspondences* with stones.

[12] $\mid\!\sim \mathtt{S}(`||\text{'}) = `|||\text{'}$ holds because $\mathtt{S}(`||\text{'})$ and '|||' represent the same syntactical entity, namely three dashes |||.

Addition 1: + can be defined as the function-symbol 'Place the array x and y together to create a new array.'. Then, when actually carried through, '||'+'||' = '||||'.[13]

Addition 2: + can also be defined by the function-symbol 'For x and '|' it gives performs S(x), for x and S(y) it gives the application of S to the result of applying this very process to x and y.'. This is a rather funny description of a form of downward (recursion) procedure. Let us see that this makes sense. Let us compute '||' + '|||'. This is the same as '||' + S('||'), so it is S('||'+'||'). By its turn, it is S('||'+S('|')), and so is S(S('||' + '|')). Finally, S(S('||' + '|')) is S(S(S('||'))), i.e. '|||||'. This justifies '||' + '|||' = '|||||'.

It is a good exercise to justify that **Addition 1** and **Addition 2** give always the same result; more precisely, denoting the first one by $+_{(1)}$ and the second one by $+_{(2)}$, to find a general method to justify that $x +_{(1)} y = x +_{(2)} y$ is a true proposition.

Multiplication: $\times$ is the following function-symbol. 'Write down x in an emphasized position. For each occurrence of '|' in x write one occurrence of y. Join all the occurrences of y together.'. Let us also give an example of this. Let us implement (in the real world) '|||' $\times$ '|||||'. We start by writing '|||' in an emphasized position (we use $\underline{}$ to denote

[13] As usual, we write '$x + y$' instead of '$+ (x, y)$'.

the emphasized position):

$$\underline{\underline{|||}}$$

$$\texttt{blank}.$$

Then, we remove one occurrence of ‘|’ from ‘|||’ and write one occurrence of ‘|||||’, namely

$$\underline{\underline{||\ }}$$

$$|||||.$$

We repeat and obtain

$$\underline{\underline{|\ \ }}$$

$$|||||\ |||||,$$

and finally

$$\underline{\underline{\texttt{blank}}}$$

$$|||||\ |||||\ |||||.$$

Now we join everything and obtain |||||||||||||||. The intuitive idea is that we are doing

$$\underbrace{\underbrace{\underbrace{|||||}_{\text{one time}}\ |||||}_{\text{two times}}\ |||||}_{\text{three times}}\ .$$

At a syntactical level, this implementation confirms that ‘|||’ $\times$ ‘|||||’ $=$ ‘|||||||||||||||’.

Computable Functions: It is not hard to see that the usual computable function(-symbols) can also be considered in our setting.

It is important to emphasize that, in this framework, numbers are not groups of objects, they are rather a way of counting, namely a way of establishing a correspondence between the syntactical dashes '|' and each one of the objects in a given group. We defined the operations for the numerals, but these have a clear interpretation in terms of `number`; for instance, `S` denotes a way to add an extra correspondence to the original number, *et cetera*. We could have defined the operations for `number` and consider the latter as the names of the natural numbers, but we opted to consider the numerals—this confirms that a clear language is mandatory at every stage of a meaningful (empirically oriented) discourse.

Theories of Numbers 2.3.4. We described a way to make sense of the mathematical notion of 'natural number': firstly, we defined the numerals $\overline{n}$; secondly, we interpret numbers as a procedure of associating dashes '|' to objects; thirdly, we introduced the usual arithmetical operations for numerals. We draw the reader's attention to the fact that the function-symbols we described act on syntactical objects, numerals; they are syntactical entities that describe syntactical actions. But another view is possible: to interpret directly each string of dashes '|' as a group of objects by directly making the correspondence between the dashes and the objects, and not to interpret as the actual *way* of establishing a correspondence as we previously considered; in this view, we can also consider the operations on the groups of objects directly, without passing through the syntactic realm of the dashes '|'. Numbers are, consequently, now interpreted as actual groups of objects, and not as a way of counting them, i.e. not as a way of making a correspondence between dashes '|' and the objects as we did in **Natural Numbers 2.3.2**.[14]

Let us consider the first-order classical logical theory Q (called Robinson's Arithmetic) over the language $\langle 0, \mathtt{S}, +, \times, = \rangle$ with axioms:

Q1: The usual axioms of first-order classical logic and $=$;

Q2: $\neg\mathtt{S}(x) = 0$;

Q3: $\mathtt{S}(x) = \mathtt{S}(y) \rightarrow x = y$;

Q4: $y = 0 \vee \exists x.\mathtt{S}(x) = y$;

Q5: $x + 0 = x$;

Q6: $x + \mathtt{S}(y) = \mathtt{S}(x + y)$;

Q7: $x \times 0 = 0$;

Q8: $x \times \mathtt{S}(y) = (x \times y) + x$.

Let us use this theory to see how theories about numbers can be empirically interpreted (here, the symbols $\mathtt{S}, +,$ and $\times$ are used in a different way than the one we previously considered). Here, the variables, when interpreted, are assumed to represent

[14]This emphasizes the importance of giving a concrete meaning to the symbols in use: different empirical interpretations could be made.

groups (sets) of objects whose differences we opt to neglect and that belong to a certain domain $\mathscr{D}$; we call these objects $\mathscr{d}_1, \mathscr{d}_2$, and so on; here we assume that whenever the reader needs to consider more objects, she can do so. $\mathtt{S}$ is interpreted as the action of grouping one more element to an already considered group of elements, while 0 represents an "empty-table", i.e. not having any element in the space we are considering. $x + y$ is interpreted as joining two considered groups x and y. $x \times y$ is interpreted as the action of creating a new group by adding x-times (i.e. one time for each element of x) the elements of y. In this context, we also have a form of numerals that differs from our initial one: $\overbrace{\mathtt{S}(\mathtt{S}(\cdots \mathtt{S}(0)\cdots))}^{n \text{ times}}$.

We have already explained in a fairly amount of detail how to give empirical meaning to **Q1**, so we omit that part. In the rest of the axioms, the connectives and quantifiers should interpreted as being of type 1, so for instance **Q2** is $\neg_1 \mathtt{S}(x) = 0$: the reason for this is that we can give general explanations and methods of testing. Focusing on **Q2**, it is a proposition since we can easily create a method to justify that if we added an element to a group, then we can actually confirm that we are considering a "non-empty table", since it has at least the added element: the method of testing is taking a group that was obtained from an original group by adding one element and observing that the added element is still there, and consequently no occurrence of 0 is detected.[15] **Q3** is interpreted as $\mathtt{S}(x) = \mathtt{S}(y) \to_1 x = y$

[15] To consider this a type 1 proposition, namley $\forall_1 x.\neg_1 \mathtt{S}(x) = 0$, we, of course, are assuming some meta-assumptions in the record book; if the

and is also easily justified. The rest of the propositions have justifications similar to what we did on **Numeral Operations 2.3.3**. In general, it might happen that a formula of a theory of arithmetic might not have an interpretation as a proposition. It is important to mention that, in this context, equality is interpreted as having the same amount of objects; more precisely, $x = y$ means that x and y are names for the same amount of objects (keep in mind that the objects are different, but indistinguishable for our purpose). If the reader, in her record book, does not have sufficient meta-assumptions to consider the previously mentioned properties as type 1 propositions, then she ought to consider the type 2 quantifiers and connectives instead.

Induction 2.3.5. In the previous framework of **Theories of Numbers 2.3.4**, if $\mu : A(0)$ and $\nu : \forall_1 x.(A(x) \to_1 A(\mathtt{S}(x)))$ are propositions, we can easily create a method to justify that $\forall_1 x.A(x)$ is a proposition: the idea is that, for a given x, we construct a method for $A(x)$ by starting from μ and using ν "x-times"; in other words, μ gives a method for $A(0)$, from μ and one use of ν we get a method for $A(\mathtt{S}(0))$, and so on.[16] Moreover, the following form of induction holds: if $A(0)$ and $\forall_1 x.(A(x) \to_1 A(\mathtt{S}(x)))$ are propositions, then $(A(0) \wedge_1 \forall_1 x.(A(x) \to_1 A(\mathtt{S}(x)))) \to_1 \forall_1 x.A(x)$ is a true proposition in the domain of natural numbers[17]. Clearly, for having induction, the hard part is to justify that $\forall_1 x.(A(x) \to_1 A(\mathtt{S}(x)))$ is a proposition, a rather strong

reader is missing those assumptions from her book, the she should interpret the formula as $\forall_2 x.\neg_2 \mathtt{S}(x) = 0$.

[16] For this to make any sense, the reader needs, of course, to agree that this procedure is a valid method.

[17] More formally, when we mention that we quantify over the natural num-

assertion. Needless to say that not all forms of induction are allowed, since this requires that $\forall_1 x.(A(x) \rightarrow_1 A(\mathtt{S}(x)))$ is a proposition, which might easily fail to happen.

There is an important aspect that we should mention again:

Methods 2.3.6. The methods we describe in this book can, at any given stage, be subject to further analysis: what is considered as a method might, after a more careful analysis, be subdivided into other methods or might even cease to be considered a meaningful method. We do not claim to have at hand the actual perfect methods, what we do claim is that our way of acting is the way to achieve a meaningful discourse (recall the SP and the VP): the important aspect is not the particular details of the ways we frame logic, mathematics, and science, it is rather the overall mindset of looking for methods/tests for the propositions using the record book.

In particular, it might be the case that the reader can develop more fine-grained interpretations of logic and the natural numbers while still maintaining an empirically oriented mind—there is no issue with that; *on the contrary, that is the essence of our approach, namely the never-satisfied nature of science, where we are open to further clarifications of our propositions.* As an example to this, in **Theories of Numbers 2.3.4**, we presented methods that have implicit assumptions behind their meaning-

bers, we actually mean the following: when $A(0)$ and $\forall_1 x.(\mathtt{Natural}(x) \rightarrow_1 (A(x) \rightarrow_1 A(\mathtt{S}(x))))$ are propositions, then $(A(0) \wedge_1 (\forall_1 x.\mathtt{Natural}(x) \rightarrow_1 (A(x) \rightarrow_1 A(\mathtt{S}(x))))) \rightarrow_1 \forall_1 x.(\mathtt{Natural}(x) \rightarrow_1 A(x))$ is a true proposition, where $\mathtt{Natural}(x)$ represents a proposition that identifies the elements of the domain $\mathscr{D}$, namely 'x is a group of elements of $\mathtt{D}$.'.

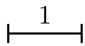
ful use; *verbi gratia*, in the case of **Q2**, we assumed, for the method we briefly described, that if an object is added to a given group, then in fact we do have an object in the group, in other words, the "table is no longer empty"; this "fact" can be further discussed and analyzed, it might even be doubted—if the reader is not happy proceeding with this "meta-fact", than she can, of course, find a more precise way of framing **Q2**; we, nevertheless, decided that the mentioned "fact" was a sufficiently firm basis to proceed and a good point to stop the concept-deconstruction process. Despite all this, the reader should keep in mind what we mentioned in Chapter 1: we always need some "meta-assumptions" that by themselves are not necessarily propositions and that underly our methods and the way they are constructed, but that have a good justification for being considered, like the SP, the VP, and the previously stated "fact". Of course, what in one context is considered a "meta-assumption", in a different context might have a concrete empirical justification (this is the case of the previously mentioned "fact").

Motto 2.3.7. The role of philosophy is to scrutinise the meta-assumptions of the natural scientist.

Real Numbers 2.3.8. The *real numbers* are a more complex mathematical entity when compared to the natural numbers; they are used, mostly in scientific contexts, to model continuous information, that is to say information that is not about group quantities, like the natural numbers, but is instead about indistinguishable streams of information, like a line. We can have several empirical interpretations of the real numbers. One possible way is to interpret them as time intervals, another possible way to give the real numbers an empirical meaning is to interpret them as line segments. We now give some details on the latter interpretation.

As we already explained the natural numbers in empirical terms with a fair amount of syntactic detail, for the real numbers we do not give that kind of detailed construction, we just give the essential ideas for the reader to implement similar procedures to the ones considered in **Natural Numbers 2.3.2** and **Numeral Operations 2.3.3**.[18] As we mentioned, we are interpreting the real numbers as actual line segments (drawn in the real world): these segments could be drawn in a piece of paper, could be drawn in the sand, and so on; the actual way of drawing does not matter, as long as the reader is able to perform basic geometric construction, like drawing lines, creating parallel lines to a given line, *et cetera*.[19] In our description, we do not account for *negative* real numbers; to fulfil that goal, one just needs to consider the line segments with certain orientations (we, again, omit the details).

The symbol '0' is interpreted in a similar way to the one we described for the natural numbers: it is an empty space. We fix some line segment to represent the number '1', say

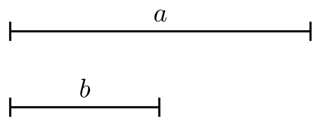


.

[18] We give intuitive description of the methods but, as the reader is aware, the methods always need to be syntactic descriptions, that is to say, for a rigorous account of the real numbers, a syntactical system expressing our intuitive ideas ought to be actually written down.

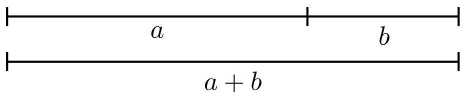
[19] Recall Methods 2.3.6, we need to have some basic meta-assumptions and constructions.

Given two line segments



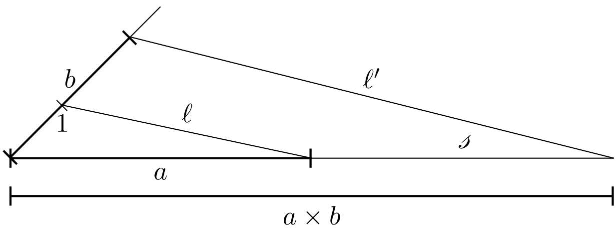
,

we can define their addition by joining them



;

and define their multiplication by the following process



,

where: firstly, we draw a and b in the shown way sharing one initial point and we superimpose to b the line segment 1; secondly, we draw a line ℓ that starts in the end point of the segment of size 1 (superimposed to b) and ends in the end part of a; thirdly, we draw a parallel line to ℓ that passes through the end of b, the line ℓ'; finally, $a \times b$ is the line segment that begins where a and b have a common point, and ends in the meeting point of ℓ' and the line $\mathscr{s}$, the latter is the extension of the segment a.

We can develop more complex procedures to represent other operations on the real numbers, for instance to compute '$\sqrt{x}$'. The essential aspect that one needs to have in mind when one speaks about a given "number" (and any object in general) is that one needs to have a procedure to construct it, otherwise no empirical meaning can be attributed, and thus making it just a syntactical mark without, at that moment, a representation in the world. Having this in mind, it comes as strikingly difficult task to give a meaning to a "non-computable real number" in certain contexts; without further information, they are, unfortunately, meaningless; this does not mean that there is no way of framing them empirically, it simply means that *so far* and *with our analysis of the real numbers* they do not have an attributed meaning: we do encourage the reader to consider other ways of empirically frame the real numbers.

Other Number Systems 2.3.9. We can give empirical meaning to other number systems: the integers modulo a certain natural number, the integers, the rational numbers, and many more. Usually, the way the reader was taught at school gives a good indication of how to frame the desired number system in empirical terms.

Mathematical Structures 2.3.10. The ways of giving empirical meaning to mathematical entities are certainly not confined to the number systems: we can give empirical meaning to group theory, semigroup theory, ring theory, topological domains, and many more. It is especially easy to give empirical meaning to finite mathematical structures: if the reader has enough resources, she might even built a real world machine for those finite structures! For example, if we are given a finite semigroup, we can use its *Cayley table*[20] to construct a machine. We believe that the reader now has the adequate tools and mindset to proceed in an autonomous way her own analysis of mathematical structures, so we move to a different subject.

We end this section with three important observations regarding our approach to Logical Positivism. These observations confirm that our approach is successful in view of the fact that it is not affected by and does not rely on the usual analytic/synthetic distinction of the propositions; of the fact that it does not have the usual problems identified by Popper regarding scientific induction[21] (see [10]); and of the fact that it is sensible

[20] A finite table that defines the operation of the considered semigroup, for example the operation $*$ might be defined by the following Cayley table:

$*$	1	2	3
1	1	1	1
2	1	2	1
3	1	1	3

.

[21] In [10, p. 70], Popper claims

> "Induction, i.e. inference based on many observations, is a myth. It is neither a psychological fact, nor a fact of ordinary life, nor one of scientific procedure."

to the nature of the evidence for the statements, avoiding the Raven's Paradox.

Analytic/Synthetic 2.3.11. Observe that in our approach we made no use of the usual analytic/synthetic distinction of Logical Positivism originated with E. Kant, i.e. the distinction of propositions that are true by their meaning or by virtue of their relation to the world; the way we framed propositions is independent of such a distinction; for us, only the propositions about the world (empirical ones) have meaning. We introduced logic in a syntactically defined way, but our system does not rely on being able to separate the propositions into to the two previously mentioned categories; for us, all propositions ought to have empirical meaning, even the ones that are constructed using conventions of methods (logic). Even the propositions that use logical quantifiers and connectives need to have an associate way of testing, this is a fundamental aspect of our approach.

Popper's Problem of Induction 2.3.12. Popper's problem of induction questions predictions about future observations based on past ones. It is considered a problem to several accounts of empirical meaning. Nevertheless, it does not constitute an obstacle to our approach, since we make a sharp distinction between the universal quantifications $\forall_1$ and $\forall_2$; Popper's problem of induction arises when one identifies $\forall_1$ with $\forall_2$. As an example, 'All man are mortal.' is a case of a $\forall_2$-proposition that cannot—at least to our knowledge about the nature of testability—be regarded as a $\forall_1$-proposition, since we can only confirm the particular instances of it, we have no general method to guarantee that 'All man are mortal.' in the sense of a $\forall_1$-quantifier.

Raven's Paradox 2.3.13. The Raven's Paradox (see [6] and [7]) states that confirmations of the statement 'All ravens are black.' coincide with confirmations of the statement 'Everything that is not a raven is not black.', and, from there, concludes that finding a green apple is evidence of the fact that 'All ravens are black.'. We believe that, after our analysis using the record book, the reader is totally aware that a test to the first $\forall_2$-quantification does not coincide with a test to the second statement, they have, by definition, very distinct ways of testing; moreover, it is by no means self-evident that the second statement is a proposition, since the reader might not have a general method of testing "non-raveness" and "non-blackness".

2.4 Existence is Not a Predicate

We have already explained why existence cannot be regarded as a predicate, nevertheless we are going to give a confirmation of this fact in a more symbolic way. We do this symbolic analysis to show the reader that our approach to meaning allows to clearly and undoubtably decide important philosophical questions. We start by symbolically define what a predicate is and then we show that existence cannot be a predicate. The analysis we carry out in this section complements the justifications we gave previously for existence not to be a predicate, this analysis is not strictly necessary to reach the conclusion that existence is not a predicate.

Firstly, let us say a few words about predicates. In coloquial terms, a predicate is simply a syntactical device to mention that certain objects do, or do not, have a certain property. For example, $\mathtt{Red}(x)$ defined by 'x is red.' is a predicate about the property of objects being 'red'; likewise 'x is the tallest man in the room.' is also a predicate. In our analysis, a predicate needs, obviously, to be a proposition; but we need some extra conditions to be satisfied. We need to be able to decide (in type 1 terms), for each object, if the predicate holds or not; moreover, we need to be able to know if there is any object satisfying the given property. Focusing in the previous example, for each object denoted by `obj`, we need to be able to decide if $\mathtt{Red}(\mathtt{obj})$ is a true proposition. We can consider predicates that we know not to have witnesses, for example $\mathtt{Roundstraight}(x)$ defined by 'x is a round and straight line.' that we know not to have witnesses, since there are no round and simultaneously straight lines[22].

In addition, for a predicate P, we need to know if there are any objects satisfying P; it might be the case that there are no objects whatsoever satisfying P, but we need to know that *a priori* for P to be considered a predicate: if we do not have deciding information about the existence of objects satisfying the considered P, then P is simply a general proposition. *Exempli gratia*, if we had no information about the existence or non-existence of red objects, than `Red` would be a proposition that thus far could not be considered a predicate; nevertheless, in that situation of

[22] This clearly relies on the meta-assumptions the reader is making. We could have considered, for example, the predicate $\mathtt{Red}(x) \wedge_1 \neg_1 \mathtt{Red}(x)$ and obtain the same conclusion.

not knowing about the existence or non-existence of red objects, clearly if we restrict the domain of discourse we get a predicate, namely $\texttt{Redearth}(x)$ defined by 'x is a red object in the planet Earth'. In addition, for a predicate P, we need to have a way to decide if there are witnesses to its non-occurrences. Having in mind what we mentioned, we can make the following definition.

Predicate 2.4.1. A proposition $P(x)$ (x the only free-variable, that is to say the only variable that is not under a quantifier) is a *predicate* if $\neg_1 P(x)$, $\exists_1 x.P(x)$, $\neg_1\exists_1 x.P(x)$, $\exists_1 x.\neg_1 P(x)$, $\neg_1\exists_1 x.\neg_1 P(x)$, and $P(x)\vee_1\neg_1 P(x)$ are propositions and, for any closed term a denoting an object, when the tests are actually performed, $\mathrel{|\!\!\sim} P(a) \vee_1 \neg_1 P(a)$.

For a given predicate P, we define $\texttt{witness}(P)$ to be[23] $\texttt{witness}_1(\texttt{exists}_1(P))$. Let us confirm that existence cannot be regarded as a predicate.

Confirmation that existence is not a predicate. Suppose, by way of contradiction, that there is a predicate $\texttt{E}$ representing existence. For $\texttt{E}$ to represent existence, we need the following properties to be satisfied:[24]

E1: For every predicate P, $\mathrel{|\!\!\sim} \texttt{E}(\texttt{witness}(P))$ if, and only if, $\mathrel{|\!\!\sim} \exists_1 x.P(x)$;

[23] This is well-defined because $\exists_1 x.P(x)$ is a proposition.

[24] $\leftrightarrow_1$ is the type 1 version of equivalence, it can easily be defined in the spirit of **First-Order Empirical Logic 2.2.4**. For example, we demand that if $\mathrel{|\!\!\sim} A \leftrightarrow_1 B$, then $\mathrel{|\!\!\sim} A$ holds exactly when $\mathrel{|\!\!\sim} B$ does, likewise for $\mathrel{|\!\!\sim} \neg_1 A$ and $\mathrel{|\!\!\sim} \neg_1 B$ for the case where $\neg_1 A$ and $\neg_1 B$ are also propositions.

E2: For every predicate P, when the suitable tests are performed, $\mathrel{|\!\sim} \mathtt{E}(\mathtt{witness}(P)) \leftrightarrow_1 \exists_1 x.P(x)$.

E1 expresses that $\mathtt{E}$ is capturing type 1 existence, that is to say the "*real*" existence. In addition, **E2** claims that one needs an actual method to see that $\mathtt{E}$ is representing type 1 existence. It is clear that **E1** follows from **E2**, but we decided to present the two conditions aiming at a greater clarity of the concepts in use.

By definition of predicate,

$$\mathrel{|\!\sim} \mathtt{E}(\mathtt{witness}(\neg_1\mathtt{E})) \vee_1 \neg_1\mathtt{E}(\mathtt{witness}(\neg_1\mathtt{E})).$$

So, either

A: $\mathrel{|\!\sim} \neg_1\mathtt{E}(\mathtt{witness}(\neg_1\mathtt{E}))$, or

B: $\mathrel{|\!\sim} \mathtt{E}(\mathtt{witness}(\neg_1\mathtt{E}))$.

Suppose, aiming at a contradiction, that **A** holds. Then, by **E2**, $\mathrel{|\!\sim} \neg_1\exists_1 x.\neg_1\mathtt{E}(x)$. In this case, it is not hard to justify—after the needed tests are performed in order to obtain no inconclusive conclusions—that[25] $\mathrel{|\!\sim} \forall_2 x.\mathtt{E}(x)$, which is obviously not the case, for example it fails for the predicate `Roundstraight`

[25] The justification for this is simple. If $\mathrel{|\!\sim} \forall_2 x.\mathtt{E}(x)$ were not the case when all the needed tests were performed to obtain no inconclusive answers, as, for each object $\mathtt{obj}$, $\mathrel{|\!\sim} \mathtt{E}(\mathtt{obj}) \vee_1 \neg_1\mathtt{E}(\mathtt{obj})$, then it would necessarily follow that for some object $\mathtt{obj}_0$, $\mathrel{|\!\sim} \neg_1\mathtt{E}(\mathtt{obj}_0)$; in that situation, we can obviously devise a type 1 existential test attesting $\mathrel{|\!\sim} \exists_1 x.\neg_1\mathtt{E}(x)$ (the test would just need to analyze the object $\mathtt{obj}_0$); this would clearly contradict the assumption that $\mathrel{|\!\sim} \neg_1\exists_1 x.\neg_1\mathtt{E}(x)$.

since[26] $\mathrel{|\!\sim} \neg_1 \exists_1 x.\texttt{Roundstraight}(x)$, that is the same as saying $\mathrel{|\!\sim} \neg_1 \texttt{E}(\texttt{witness}(\texttt{Roundstraight}))$. Thusly, **A** is not the case; therefore **B** holds. From what we have concluded in the preceding sections, it is not hard to see that, when P is a predicate, then

$$\mathrel{|\!\sim} \exists_1 x.P(x) \leftrightarrow_1 P(\texttt{witness}(P)).$$

B claims $\mathrel{|\!\sim} \texttt{E}(\texttt{witness}(\neg_1\texttt{E}))$, and so from **E1**, $\mathrel{|\!\sim} \exists_1 x.\neg_1 \texttt{E}(x)$. From the previous observation this entails $\mathrel{|\!\sim} \neg_1 \texttt{E}(\texttt{witness}(\neg_1\texttt{E}))$, which is impossible, since it contradicts the already establish **B**: keep in mind that by definition of $\neg_1$, A and $\neg_1 A$ cannot both hold. This concludes our confirmation. $\dashv$

[26] We can also consider, for a given predicate $P(x)$, the predicate $P(x) \wedge_1 \neg_1 P(x)$ and obtain the same conclusion as for the predicate `Roundstraight`. Let us summarily justify that $\mathrel{|\!\sim} \neg_1 \exists_1 x.P(x) \wedge_1 \neg_1 P(x)$. We start by recalling that, as $P(x)$ is a predicate, starting from $\mu(x) : P(x)$, by definition, there is an associated method $\mu_\neg(x) : \neg_1 P(x)$ that tests positive exactly when $\mu(x)$ tests negative, and *vice-versa*. A possible method for $\exists_1 x.P(x) \wedge_1 \neg_1 P(x)$ is simply a method that produces any object witness `obj` whatsoever and that test for `obj`, namely tests $P(\texttt{obj}) \wedge_1 \neg_1 P(\texttt{obj})$, outputting a positive result when a positive test was obtained, and a negative result otherwise (by definition either $\mu(\texttt{obj})$ or $\mu_\neg(\texttt{obj})$ tests positive). From our meta-assumptions, we know that the previously mentioned test will always yield a negative result; there is a very simple method to confirm this fact, namely by going to the instruction manual of the device that perform μ and confirm that $\mu(\texttt{obj})$ and $\mu_\neg(\texttt{obj})$ cannot simultaneously test positive; this former described method will always test positive and is a type 1 negative method associated with the method for $\exists_1 x.P(x) \wedge_1 \neg_1 P(x)$, hence it is a method for confirming that $\neg_1 \exists_1 x.P(x) \wedge_1 \neg_1 P(x)$ is true.

Chapter 3

Science

3.1 Natural Sciences

The natural sciences—that include (broadly speaking): physics, astronomy, chemistry, geology, and biology—are the fields were the most successful applications of the SP and the VP occur: we recall that the former claims that the meaning of a proposition is its method of verification, and the latter affirms that only the empirically verifiable propositions are meaningful. It is from these sciences that the idea of the VP arised, namely from observation that these sciences are especially successful in understanding and predicting the world.

One of the fundamental differences between natural sciences and other forms of science is the exhaustive use of quantitative methods, that is to say, methods to attribute concrete numerical aspects to the experiments at hand; in the natural sciences, we are not restricted to simple tests the propositions, we are also able to quantitatively assess them.

To Measure or Not To Measure 3.1.1. To apply the SP and the VP in a rigorous way, one does not necessarily need, in principle, to have quantitative methods. Nevertheless, it is a well-known fact that the more one is able to quantify, the more precise is the science one is developing. This is the case not in principle, but in practise: keep in mind that for the general framework we described, in the previous chapters, quantitative methods did not play a major role; in fact, we briefly explored the empirical meaning of some number systems. The reason for their practical utility is rather simple: when one has quantitative methods, one is able to make, in practical terms, sharper distinctions between the observed phenomena; non-quantitive observations that were considered to be the same might, after a quantitive analysis is performed, be distinguished. *Verbi gratia*, a pastor might be making claims about flocks of sheep—for example that his dog helps the flock in a certain way—, and the proposition he is testing might fail for a group of two sheep—the dog possibly is not interested in such a small group—; if the pastor is unable to quantify the flocks, he might conclude an apparently contradictory fact: that the phenomena both occurs and does not occur for flocks of sheep; possibly he will attribute it to the specific sheep, and not to the *quantity* of sheep at hand. If the pastor adds quantitative methods to his framework, he might be able to make sharper distinctions and to better understand the empirical aspects of its flock.

Vagueness 3.1.2. In the natural sciences, the use of vague terms is, for most cases, reduced to a minimum: everything can, in principle, be reduced to very fundamental observations, like atom detection, field detection, and so on. This is, in fact, one of the defining aspects of the natural sciences; if one considers, for the sake of giving an example, history, one certainly cannot reduce the fundamental observations, in a meaningful way, to atom detection and similar detections. Sure, one can view Gaius Julius Caesar as a particular array of atoms, in a particular interval of time, that were under certain force fields, but that gives no *meaningful* information: "Beware the Ides of March!", we do not frame historical events in those terms.

3.1.1 The Scientific Method

If the reader asks any natural scientist what is the one aspect that characterizes her way of creating knowledge, for sure the answer is going to be a resounding "*The scientific method!*". But why? What is so special about this method and why is it so good at making predictions? Moreover, where does it fit in our SP and VP analysis and distinction of propositions into type 1 and type 2? Firstly, let us state and analyze the scientific method (SM).

The Scientific Method 3.1.3. The SM can be viewed as the following process:

SM1. Ask a (meaningful) question or make an initial observation.

SM2. Do background tests to see if the asked question makes sense and if the initial observation is sound.

SM3. Construct an hypothesis to test. This hypothesis can be a proposition that we want to see that tests true, or a (still) pseudo-proposition that we want to give an empirical meaning.

SM4. Design a possible experiment to test the hypothesis; more precisely, if the hypothesis is a proposition, then test it, otherwise use your imagination to come up with a possible method to the (still) pseudo-proposition (making it an actual proposition).

SM5. Test the designed experiment.

SM6. If the test is not working then do one of the following moves:

- **SM6.1.** Ignore the question and move to a different one and move directly to **SM9**, or
- **SM6.2.** Redo the initial testing, or
- **SM6.3.** Repeat from **SM4**.

SM7. If the test was successful, analyze the acquired data.

SM8. If the results contradict the hypothesis, repeat from **SM3** by creating a new hypothesis and, in parallel, do also **SM9**.

SM9. If the results agree with the hypothesis, or you are redirected from **SM6.1**, or from **SM8**, then communicate the results.

Observe that the SM is closely related to the VP, since the scientific hypothesis is, before actually tested, meaningless: it is through thorough testing that the hypothesis becomes a tested proposition, or a pseudo-proposition that is promoted to a proposition and also positively tested; a positive result yields, in both cases, a true (scientifically confirmed) proposition. There is, in the implementation of SM, no meaning outside testability; the hypothesis before being tested has no scientific value whatsoever. Science lives from the interaction of these two entities: hypothesis creation, and hypothesis verification. One does not live without the other, but it is the latter the gives the scientific statue to an array of syntactical symbols (the hypothesis).

But science is not only about positively confirmed hypothesis, negative results ought to be communicated also (*conferatur* **SM8**): they are, by no means, *negative* in the pejorative sense, they are a simple non-verification of an hypothesis, but they are equally important. In fact, major scientific revolutions started with negative results, for example the theory of relativity gained traction with the Michelson-Morley experiment that ruled out the hypothesis of a luminiferous aether [9]. Unfortunately, there is, even in some scientific communities, the view that negative results ought not to be communicated, something that is totally against the scientific nature! One can have quality negative results much more superior and impactful than a variety of positive results: they allow the reader to pass from one positive type 2 proposition to a negative type 1 proposition: we used the term 'negative' in the sense that it claims the *intuitive opposite* of the type 2 proposition, for example the type 1 opposite of a $\exists_2 x.\forall_2 y.(\cdots)$-proposition, is a $\forall_1 x.\exists_1 y.\neg_1(\cdots)$-proposition.

Let us see an example of what we described, namely the Michelson-Morley experiment. The hypothesis at hand is 'There is a medium such that all light rays/particles propagate in that medium.'. For it to be a proposition, it must be a type 2 proposition, more precisely, the proposition[1]

$$\exists_2 medium.\forall_2 ray.\mathtt{Travelthrough}(ray, medium), \tag{Prop. Aether}$$

since we cannot, in principle and without any further information, positively detect the medium and test all light rays. Such a medium would impact on the way light is detected on Earth: since Earth is moving, and assuming such a medium, as light travels through that medium it must be detected differently at different instances. To better understand why, imagine that someone is producing waves in a swimming pool, if the reader is swimming she detects the waves in ways that depend on the direction she is moving; moreover, different parts of her body detect the waves differently to other parts.

The Michelson-Morley experiment was conducted and showed that there can be no such medium, by testing particular light rays and showing that their detection was not affected by the movement of the Earth. The former proposition (Prop. Aether)

[1] Of course, more details should be given here to define in a rigorous way the predicate `Travelthrough`.

that affirmed the luminiferous aether, now became the confirmed intuitive negated proposition, namely

$$\forall_1 medium.\exists_1 ray.\neg_1 \texttt{Travelthrough}(ray, medium),$$

that expresses 'For all possible media, there is a light ray that does not move according to that media.'. We have a detection device—the Michelson-Morley experiment—that, for any possible describable medium, shows that a particular light ray does not follow it; under the assumption that one can empirically define medium, this is the previous type 1 proposition, since we have a general construction to produce an existential witness—a light ray—that does not obey any possible medium.

In the previously example, by

$$\exists_2 medium.\forall_2 ray.\texttt{Travelthrough}(ray, medium)$$

we actually mean that '*medium*' ranges over media and '*ray*' over light rays, i.e.

$$\exists_2 x.(\texttt{Medium}(x) \wedge_1 (\forall_2 y.\texttt{Ray}(y) \rightarrow_2 \texttt{Travelthrough}(y, x))),$$

where Medium defines media and Ray defines rays; and by

$$\forall_1 medium.\exists_1 ray.\neg_1 \texttt{Travelthrough}(ray, medium)$$

we do mean

$$\forall_1 x.\texttt{Medium}(x) \rightarrow_1 (\exists_1 y.\texttt{Ray}(y) \wedge_1 \neg_1 \texttt{Travelthrough}(y, x)). \tag{Michelson-Morley Exp.}$$

Of course, for all this to go through, the record book needs to have an empirical meaning to the previous statements and the reader needs to agree that the Michelson-Morley experiment gives a possible test to confirm the very last proposition we wrote. Of course, a definition of `Medium` needs to include that it interacts with light rays, that is why we can, in certain frameworks, consider (Michelson-Morley Exp.) as a $\forall_1$ proposition. The actual details of the previous example are not that important to the idea we want to express at this point, the main feature we want the reader to have in mind is that negative scientific results yield, in most cases, type 1 propositions, since they "transform" meaningful type 2 propositions into empirically verified type 1 propositions. Strictly speaking, one needs several meta-assumptions to agree that the Michelson-Morley experiment, an $\forall_1 medium.\exists_1 ray.\neg_1(...)$ statement, was empirically confirmed, otherwise we are only able to state the $\forall_2 medium.\exists_1 ray.\neg_1(...)$ statement: all this boils down to the fact that no detection of a change in the path of light, under a suitable definition of medium, is enough to confirm that the proposition (Michelson-Morley Exp.) holds. In fact, the Michelson-Morley experiment allows one to conclude the stronger proposition

$$\exists_1 y.\texttt{Ray}(y) \wedge_1 (\forall_1 x.\texttt{Medium}(x) \rightarrow_1 \neg_1 \texttt{Travelthrough}(y,x)) \tag{Strong Michelson-Morley Exp.}$$

The SM is also deeply connected with the SP, since the scientific hypothesis, before being given a method of testing (and actually tested), is, for the scientist, meaningless. The SM can be used to produce both propositions of type 1 and type 2. The previous example we saw, of the proposition (Michelson-Morley Exp.), is an example of a type 1 proposition that can be given by the SM: we emphasize once more, this particular characterization deeply depends on the record book and on the allowed methods; but it is for sure a type 2 proposition of the form $\forall_2 medium.\exists_1 ray.\neg_1(...)$. But of course, the SM mainly produces type 2 propositions, its very layout is similar to the type 2 definitions we gave for the quantifiers and connectives.

It is paramount to observe that when simple tests, namely a finite number of tests, are performed on a proposition $A(x)$, we might conclude from those tests $\vdash\!\sim \exists_1 x.A(x)$ and $\vdash\!\sim \forall_2 x.A(x)$, but we *cannot*, simply on the basis of the mentioned tests and without further information and assumptions, conclude that $\vdash\!\sim \forall_1 x.A(x)$. In general, a finite number of unrelated tests is not enough to have the strong statement $\forall_1 x.A(x)$, we might only claim $\forall_2 x.A(x)$ (the majority of scientific statements are of the previous form).

3.1.2 The Scientific Endeavour

Scientific Theories and Revolutions in Science 3.1.4. In **Theory 2.2.8**, and before in **Theory 2.1.6**, we defined formally what a scientific theory is: it is any finite collection of propositions that the reader decides to emphasize from her record book. These propositions can have, and commonly do have, a general nature (in the sense of being universal quantifications), either of type 1 or of type 2. The theories that interest the scientist the most are, without a doubt, the theories T that were actually tested and verified, i.e. $\vdash\!\sim T$. We recall that the verified nature of scientific theories depends on the actual moment one is reading the record book: from one moment to another, information might be added to the book and alter the status of some provable propositions in the considered theory. It is important to emphasize that, despite the fact that sometimes scientists claim that some theories that were tested turned out to be wrong (they mean something slightly different from what we are mentioning in this sharper analysis), the verified scientific theories, in a fixed moment in time, cannot be wrong, assuming the community agrees with the descriptions of the methods, since, by definition, they are verified methods that were actually carried through: only a misuse of the type 1 and type 2 propositions gives rise to such an impression. When we say that a theory is correct, we are saying that the (finite) content of the record book to which that theories reports to was confirmed.

It can happen that, for a verified theory T, in a certain moment, $T \vdash \neg_2 A$ (this means that it is provable in T that all the tests conducted on A were not positive, with the possibility of inconclusive methods and not having actually carried the experiment through), and after more information is added to the record book (for example by making more tests) $T^+ \vdash A$, where T^+ is the theory T adapted to the added information (for example, the additionally performed tests): more formally and following **Theory 2.2.8**, the new theory is, in fact, $\langle T^+, \mathscr{R}'\rangle$, where $\mathscr{R}'$ is the new content added to the original record book $\mathscr{R}$. Both theories are, for the considered descriptions, correct! In the first one, we had insufficient information regarding A, so $\neg_2 A$ was verified for $\mathscr{R}$, but more information was given, giving rise to $\mathscr{R}'$, and A was now verified: none of them is wrong (how can they be if, by definition, they are verified?).

We say that a *scientific revolution occurs for a theory T (for a given record book $\mathscr{R}$)*, if a proposition A changes its status[2] when more information is added to $\mathscr{R}$ (like in the previous example).

Our definition of scientific revolution captures "small revolutions" and not necessarily actual shifts in intellectual paradigms (like passing from classical mechanics to quantum mechanics). It is through scientific revolutions, in the sense we defined, that science moves forward, namely by writing more propositions in the record book and by conducting more tests: this can be done in a systematic way using the SM.

Actual shifts of paradigms play a more defining role in science and they are a particular kind of scientific revolution where the community decides to shift the way tests were conducted and the propositions that were considered: it might be the case that former proposition now become pseudo-propositions and *vice versa*, since the methods of testing are new. In our analysis using the record book, it corresponds to starting a new separate record book where we may copy some of the propositions of the old book, but in which we change our way of testing; in a sense, the scientific endeavour restarts in this new record book.

So far, we saw how transversally present the SP and the VP are in the scientific way of acting; as we are going to see, they are also present in most scientific revolutions.

[2]By this we mean that it changes from $\neg_2 A$ to A, from $\forall_2 x.A(x)$ to $\exists_1 x.\neg_1 A(x)$, and so on.

Motto 3.1.5. Clarification of language is behind every major scientific enterprise.

Let us give two examples of very successful uses of the SP and the VP: the special theory of relativity (SR), and quantum mechanics (QM).

Special Theory of Relativity 3.1.6. SR is a physical theory about space and time [14]. It irradiates from two fundamental (type 2 verified) postulates:

Postulate 1: The laws of physics are invariant for every inertial frame of reference[3].

Postulate 2: The speed of light is constant (in the vacuum) in all inertial frames of reference.

SR is a very relevant example of a theory where the clarification of language played an important role: it was—among other important factors such as the Michelson-Morley experiment—from the need of clarifying what an observer is and how information is perceived by different observers that this theory emerged. In this sense, it is an ultimate example of what we aim at. It is important to emphasize that it is, from the beginning, a meaningful theory: the concepts and propositions at hand were, in most cases, not possible (at that time) to test, but upon request a description of a possible way of testing could be given; as the reader is aware, only much latter in time was such actual testing possible.

[3] An 'inertial frame of reference' is a frame of reference that is not experiencing acceleration.

For the sake of giving a concrete example of what we have been describing, we provide a brief syntactic description of SR in a way that could, after some polishing is done, appear in the record book; this serves also as a form of exercise for the reader to recall the main concepts we have been using. The axiomatic approach we follow here is from [1] (see this reference for further details on why this axiomatization fully-captures the first-order part of special relativity; we also recommend [8] and [12]). In this setting, the following atomic propositions are considered in our basis:

- $\mathtt{B}(x)$ means 'x is a body.';
- $\mathtt{IOb}(x)$ represents 'x is an inertial observer.';
- $\mathtt{Ph}(x)$ means 'x is a photon.';
- $\mathtt{Quant}(x)$ denotes 'x is a space quantity.';
- $\mathtt{Time}(x)$ represents 'The moment x in time.';
- $\mathtt{W}(k, b, x_1, x_2, x_3, t)$ means 'Body k coordinatizes body b at coordinates x_1, x_2, x_3 and instant t.'.

We assume, in this setting, that those atomic properties are the starting point on the record book and that the reader has ways to give them an empirical meaning, namely the usual physical meaning; we give no further details on that, since it belongs to the *canonical corpus* of physics. Following the definitions, it is not hard to establish:

SR1: $\mathrel{|\!\!\sim} \mathtt{IOb}(x) \rightarrow_1 \mathtt{B}(x)$.

SR2: $\mathrel{|\!\!\sim} \texttt{Ph}(x) \rightarrow_1 \texttt{B}(x)$.

SR3: $\mathrel{|\!\!\sim} \texttt{W}(k, b, x_1, x_2, x_3, t) \rightarrow_1 (\texttt{B}(k) \wedge_1 \texttt{B}(b) \wedge_1 \texttt{Quant}(x_1) \wedge_1 \texttt{Quant}(x_2) \wedge_1 \texttt{Quant}(x_3) \wedge_1 \texttt{Time}(t))$.

We now consider the following symbols:

- 0_t denotes some "0-instant in time"[4];
- 1_t represents the "1-instant in time";
- 0_s denotes the "0-length";
- 1_s denotes the "1-length";
- $+_t, \times_t$ denote the usual addition and multiplication for time instants;
- $+_s, \times_s$ represent the common addition and multiplication in space;
- $-_t, -_s$ denote, respectively, the symmetric elements in time and space;
- $/_t, /_s$ represent, respectively, division in time and space;
- $\sqrt[t]{}, \sqrt[s]{}$ denote, respectively, the square roots in time and in space;
- $\leq_t, \leq_s$ denoting the usual smaller-or-equal relations for time and space, respectively;

[4]Please, do observe the change in quotations, here we mean the expression and not the syntactical array.

- $=_t$, $=_s$ denoting the usual equality relations for time and space;
- $=$ to compare time magnitudes with space magnitudes, for example $\mathrel{|\!\sim} 1_s = 1_t$ because they have the same magnitude.

We can give concrete empirical meanings to the previous symbols and the previous function-symbols; in **Real Numbers 2.3.8** we gave a possible interpretation using line segments, similar accounts can be given for time instants and for other ways to frame space. Moreover, it is not hard to empirically define square roots in both context (*exempli gratia*, the usual, actual and empirical, way of measuring and multiplying length is enough here).

We can empirically confirm the following propositions[5]:[6]

SR4: $\mathrel{|\!\sim} \forall_2 \mathtt{Time}(x).\forall_2 \mathtt{Time}(y).\forall_2 \mathtt{Time}(z).x +_t (y +_t z) =_t (x +_t y) +_t z.$

SR5: $\mathrel{|\!\sim} \forall_2 \mathtt{Time}(x).\forall_2 \mathtt{Time}(y).x +_t y =_t y +_t x.$

SR6: $\mathrel{|\!\sim} \forall_2 \mathtt{Time}(x).x +_t 0_t = x \wedge_1 x \times_t 1_t =_t x.$

SR7: $\mathrel{|\!\sim} \forall_2 x.(\mathtt{Time}(x) \wedge_1 (\neg_1 x =_t 0_t)) \rightarrow_1 x \times_t (1_t /_t x) =_t 1_t.$

SR8: $\mathrel{|\!\sim} \forall_2 \mathtt{Time}(x).\forall_2 \mathtt{Time}(y).\forall_2 \mathtt{Time}(z).x \times_t (y +_t z) =_t (x \times_t y) +_t (x \times_t z).$

---

[5] '$\forall_2 \mathtt{Time}(x).A(x)$' denotes '$\forall_2 x.(\mathtt{Time}(x) \rightarrow_1 A(x))$'; we use similar abbreviations throughout. Moreover, '$\forall_2 x_1, x_2, \ldots$' represents '$\forall_2 x_1.\forall_2 x_2.\cdots$'.

[6] $\leftrightarrow_1$ is the type 1 version of equivalence, it can easily be defined in the spirit of **First-Order Empirical Logic 2.2.4**.

SR9: $\mid\!\sim \neg_1 0_t =_t 1_t$.

SR10: $\mid\!\sim \forall_2 \mathtt{Time}(x).\forall_2 \mathtt{Time}(y).(x \leq_t y \leftrightarrow_1 \exists_1 z.x + z \times_t z =_t y)$.

SR11: $\mid\!\sim \forall_2 x.(\mathtt{Time}(x) \wedge_1 0_t \leq x) \rightarrow_1 0_t \leq_t \sqrt[t]{x}$.

SR12: $\mid\!\sim \forall_2 x.(\mathtt{Time}(x) \wedge_1 0_t \leq x) \rightarrow_1 (\sqrt[t]{x}) \times_t (\sqrt[t]{x}) =_t x$.

SR13: $\mid\!\sim$'**SR4-SR12** for `Quant` instead of `Time`, and space instead of time.'.

SR14: Axioms for $=$.

As usual, x^2 denotes $x \times_i x$, where i is t or s. These are the usual *mathematical* propositions used to measure time and space (the reader can measure time with a clock and define the time operations in terms of that clock). It might be the case that the reader has a general accepted way of confirming them, in that case they become $\forall_1$ propositions; in the worst case, they are testable true type 2 propositions. Now we present the (true) propositions that are specific of relativity.

AxPh: "For any inertial observer, the speed of light is the same in every direction everywhere, moreover, it is finite. In addition, it is possible to send out a light signal in any desired direction.”. In formal terms,

$$\begin{aligned}&\mathrel{|\!\!\sim} \forall_2 \mathtt{IOb}(m).\exists_1 c.\forall_1 x_1, x_2, x_3, y_1, y_2, y_3, t, r.\\ &\Big((\exists_1 p.\mathtt{Ph}(p) \wedge_1 \mathtt{W}(m,p,x_1,x_2,x_3,t) \wedge_1 \mathtt{W}(m,p,y_1,y_2,y_3,r))\\ &\leftrightarrow_1 \sqrt[s]{(y_1 -_s x_1)^2 +_s (y_2 -_s x_2)^2 +_s (y_3 -_s x_3)^2} =\\ &\qquad\qquad c \times_t \sqrt[t]{(t -_t r)^2}\Big).\end{aligned}$$

Let us analyze briefly the previous proposition. It is a $\forall_2 x.(\cdots)$-proposition because, without any further consideration on the methods at hand, the reader can only test a finite number of inertial observers. If a general method is developed, than it can be considered as a $\forall_1 x.(\cdots)$-proposition. The inside part of the previous type 2 universal quantification is, on the other hand, generally testable: the physicist is obviously the best person to explain in detail how such a general method works, but for our purposes it is enough to emphasize that the Michelson-Morley experiment can give such a general method for the inside part and that, for example, the ‘$\exists_1 c.$’ can actually be calculated (that is why a type 1 quantifier is used). If, for any reason, the reader doubts the general methods of the physicist or the record book does not have yet enough information to use them, she can substitute all the type 1 occurrences of quantifiers and connectives for the respective type 2 occurrences.

AxEv: “All inertial observers coordinatize the same events.”.

Formally,

$$\begin{aligned}&\mathrel{|\!\sim} \forall_2 \mathtt{IOb}(m).\forall_2 \mathtt{IOb}(k).\forall_1 x_1, x_2, x_3, t.\exists_1 y_1, y_2, y_3, r.\\ &\quad \forall_1 b.(\mathtt{W}(m, b, x_1, x_2, x_3, t) \leftrightarrow_1 \mathtt{W}(k, b, y_1, y_2, y_3, r)).\end{aligned}$$

The considerations we made on **AxPh** regarding the type 1 quantifiers and connectives apply here and throughout the rest of the propositions we are presenting concerning relativity.

AxSf: "Any inertial observer sees herself on the time axis.":

$$\begin{aligned}&\mathrel{|\!\sim} \forall_2 \mathtt{IOb}(m).\forall_1 x.\mathtt{W}(m, m, x_1, x_2, x_3, t_1) \leftrightarrow_1\\ &\qquad (x_1 =_s 0_s \wedge_1 x_2 =_s 0_s \wedge_1 x_3 =_s 0_s \wedge_1 t =_t 0_t).\end{aligned}$$

AxSm: "Any two inertial observers agree about the spatial distance between two events if these two events are simultaneous for both of them; moreover, the speed of light is 1."[7]:

[7] Obviously, the speed of light is 1 for an appropriate way of measuring time and distances, we believe the reader is aware of all those details.

$$\begin{aligned}
\mathrel{|\!\!\sim} \Bigg(&\forall_2 \mathtt{IOb}(m).\forall_2 \mathtt{IOb}(k).\forall_1 x_1^0, x_2^0, x_3^0, y_1^0, y_2^0, y_3^0, t, r.\\
&\forall_1 x_1^1, x_2^1, x_3^1, y_1^1, y_2^1, y_3^1.\\
&\Big(\Big((\forall_1 b.\mathtt{W}(m, b, x_1^0, x_2^0, x_3^0, t) \leftrightarrow_1 \mathtt{W}(k, b, x_1^1, x_2^1, x_3^1, r))\wedge_1\\
&(\forall_1 b.\mathtt{W}(m, b, y_1^0, y_2^0, y_3^0, t) \leftrightarrow_1 \mathtt{W}(k, b, y_1^1, y_2^1, y_3^1, r))\Big) \rightarrow_1\\
&\sqrt[s]{(y_1^0 -_s x_1^0)^2 +_s (y_2^0 -_s x_2^0)^2 +_s (y_3^0 -_s x_3^0)^2} =_s\\
&\sqrt[s]{(y_1^1 -_s x_1^1)^2 +_s (y_2^1 -_s x_2^1)^2 +_s (y_3^1 -_s x_3^1)^2}\Big)\Big)\wedge_1\\
&\Bigg(\forall_2 \mathtt{IOb}(m).\exists_1 p.\mathtt{Ph}(p) \wedge_1 \mathtt{W}(m, p, 0_s, 0_s, 0_s, 0_t)\wedge_1\\
&\mathtt{W}(m, p, 1_s, 0_s, 0_s, 1_t)\Bigg).
\end{aligned}$$

Special relativity is the (verified) theory

$$\langle \mathbf{SR1}\text{–}\mathbf{SR14}, \mathbf{AxPh}, \mathbf{AxEv}, \mathbf{AxSf}, \mathbf{AxSm} \rangle.$$

We suggest, as an exercise, that the reader does a similar analysis for her favourite scientific theory.

Quantum Mechanics 3.1.7. It is extremely hard to give a summarized description of QM. Roughly speaking, QM is the branch of physics that is concerned with the behaviour of matter and light on the atomic and subatomic scales. Just like SR is an example of a scientific revolution that occurred, among other factors, due to clarification of several aspects of language, QM was also created to account for the precise concept of *measure*. QM can also be viewed as a physical theory of measuring, where the actual act of measuring interferes with the very system the scientist is studying.

Other Natural Sciences 3.1.8. We have, so far, analyzed physical theories, but similar accounts can be given for other natural sciences. Moving away from physics, it becomes harder to find suitable ways to frame the scientific theories in the terms we framed SR, since the concepts at hand become increasingly difficult to express and to detect; for example, it is harder to express the theory of evolution with all the modern day empirical tests that are conducted on genes, it would necessarily include an account of a great deal of physics. This is by no means a critic to the other natural sciences, what we mention is just an intrinsic aspect about their complex nature; in a similar way, the previous claim should not be read with disdain for physics. We mainly use examples from physics because they are easier to objectively state when compared to, say, biology.

Let us now focus on a completely different science to draw the reader's attention to an important aspect: psychology.

Freudian Psychology 3.1.9. It is sometimes claimed that psychology belongs to the realm of pseudo-science, but, as we are going to briefly observe, that is an unfair claim. The basis facts about psychology are observable. Human beings dream, so some form of being alive and not consciente must exist (we call it the *unconscious*; these brain processes are detectable when we sleep). We, as all animals, have desires and instincts; moreover, our parents play a major role in our minds when we grow. The key-concepts of psychology can, to a great deal, be either tested on humans, or observed in animals and extrapolated (type 2 move) to humans. In addition, meaningful biological tests can be conducted to study certain aspects of psychology. It is due to this connection to biology that we decided to include this remark on psychology in the natural sciences section.

The use of computers and Artificial Intelligence in the record book 3.1.10. The reader might use computers to analyze the data she collects in her experiments; but this is, by no means, an innocent and straightforward use; on the contrary, it assumes that the computer is trustworthy. To use computers in her record book, the reader needs to make explicit all the assumptions she is making in that use, namely that the computer behaves as expected, that it is constant and uniform with the data, that it does not produce new data out of the blue, that the programs she is using are sound, *et cetera*. Computers are, of course, very useful in everyday science, but one should always keep in mind all the background assumptions needed for its use. A similar sort of problem also occurs—but clearly in a different level—when the reader writes down her investigations: she also assumes that the medium she is using to write everything down does not magically change its content.

Artificial Intelligence (AI), especially in the form of neural networks, is very troublesome to account for in the record book, since it is generally a "black box" that somehow does what is expected from it, some sort of an oracle from Ancient Greece. This oracle can, obviously, be subject to scientific investigations by its own right; but its use as a basis for scientific investigations needs to include practical empirical information that necessarily needs to be more than simply "a black box that performs the desired tasks". Just like the case of the use of computers in general, also the use of AI should be explicitly mentioned in the record book.

3.2 Other Sciences

Scientific enterprises can also be carried out outside the realm of natural phenomena. Examples of such enterprises are sociology, anthropology, education, linguistics, and history; we call them *social sciences*. There is a clear transversal challenge in these fields: the concepts are even more complex than the ones used in the natural sciences; consequently, a conceptual analysis as the one we carried for SR is almost impossible to be achieved in practical terms. They are to be considered sciences because *in principle* (possibly, very hardly implemented) such an analysis can be done to a fragment of their discourse, but social sciences are not completely empirical in the sense of the natural sciences, since in several cases they deal with pseudo-propositions.

Unlike the natural sciences, where the reader can carry out a conceptual analysis—as the one we did for SR—without changing too much the practical way they are usually done, if one does such an analysis to a social science one either loses a great deal of the statements of that field, or the statements to be considered as propositions are utterly different from the original formulation.

Let us illustrate what we described by focusing, as an example, our attention to history. Historical statements are usually of the form ‘Such and such happened then to help Bob and attack Alice.’. For example, this is part of the Wikipedia on the crusades[8]:

> ‘The Crusades were a series of religious wars initiated, supported, and sometimes directed by the Latin Church in the medieval period. The best known of these Crusades are those to the Holy Land in the period between 1095 and 1291 that were intended to recover Jerusalem and its surrounding area from Islamic rule. Beginning with the First Crusade, which resulted in the recovery of Jerusalem in 1099, dozens of Crusades were fought, providing a focal point of European history for centuries.’

Is any of the previous statements empirically verifiable? For sure they can be *supported* by historical documents, like texts mentioning the crusades, but we do not have an actual way of testing their occurrence (we cannot time travel). More formally, the previous statements appear to have concrete type 1 quantification (mostly existential), but we do not have an actual way of

[8]We consulted `https://en.wikipedia.org/wiki/Crusades` on December 2022.

finding the existential witnesses, the best we can do is to frame them in terms of type 2 quantifiers; either that or consider them pseudo-propositions altogether. (They also use vague concepts, but let us not focus on that right now.) This is the point we were trying to rise: either the statements of the social sciences are considered meaningless, or they change in a considerable way their meaning. Sometimes the social scientist does not have a sharp distinction between type 1 and type 2 existence. We can find the dead body of a king (type 1 move), but we cannot be sure about his actions and his words based on accounts of that time (type 2 move). The historian is not commonly concerned about the scientific existence of Gaius Julius Caesar, he[9] wants to work from that basis and make propositions about his actions based on historical accounts. It is important to observe that this diverges a great deal from a similar science, namely palaeontology: in the latter, the scientist is mostly concerned with actual direct evidence, based on bones, for the existence of a certain animal (type 1 move), and sometimes concerned with extrapolation about other findings, like trying to figure out the way a certain animal walked and so on (type 2 move); in every moment the palaeontologist is fully aware of the nature of her propositions. Unfortunately, without time travel, this is the best we can do in history.

The social sciences are also characterized by the widespread use of vague and biased language. The following are examples of vague language from the mentioned Wikipedia page: 'religious wars', 'sometimes directed by', 'intended to recover Jerusalem',

[9] The social scientist is, for a change, a 'he'.

'Islamic rule', and 'providing a focal point of European history for centuries'. Some of the previous uses of vagueness can be substituted for concrete language, but sometimes that quest for concreteness is not possible without distorting completely the statement at hand: how does the reader define 'intended to recover Jerusalem', or 'providing a focal point of European history for centuries'?[10] As we mentioned, some social sciences are also *intentionally* biased, mostly in a political fashion. These two aspects increase the distance between empirical knowledge and the social sciences.

Not all is bad news. There are, of course, social scientists that strive to affirm type 1 propositions, just like there are natural scientists that are not so competent in the sharp distinction between type 1 and type 2 propositions. The special challenge the social scientist faces is to be informed about the nature of his propositions and to totally avoid vagueness and intentional bias: to achieve such a goal, the social scientist needs, regrettably, to distance himself from certain common practises in his field of study; just like the natural scientist, he needs to be fully aware of the verification methods of his propositions and needs to discharge any statement that lacks empirical meaning, he must therefore exclude pseudo-propositions.

[10] We, ourselves, do use vague language, but we have already warned the reader about this (see **Warning 2.0.3**), we are fully aware that a great deal of this book is, in a strict sense, meaningless in empirical terms.

3.3 Pseudo-sciences

Outside the scientific realm, there are activities that mimic some aspects of the scientific discourse, but without having any actual empirical testable grounds. These activities usually aim at persuading people for their own benefit—usually financial—, not for the creation of knowledge *per se*. Let us list a few of these activities: astrology, homeopathy, race theories, creationism, flat earth theory, feng shui, acupuncture, animal magnetism, ayurveda, crystal healing, quantum medicine, anti-vaccination, and, unfortunately, many more.[11] They usually employ scientific terms in a meaningless way, like 'healing energy', 'quantum crystals', and many more occurrences. Some of them are *flatly* and clearly wrong, like the *flat* earth theory: we do have type 1 evidence that the Earth is not flat; this "theory" is the simple denial of facts. But others, like quantum medicine, appear, to the untrained eye, to have some scientific content.

Let us analyze pseudo-sciences in the following steps: firstly, we mention why and how they strive; secondly, we mention the social impact they have; and thirdly, we explain how the reader can be safe and sound from all these nonsensical activities.

Pseudo-sciences take refuge in the lack of a strong scientific culture in our society and in the lack of preparation of our fellow citizens to the SP and the VP. In our society, crime is considered a problem, but lack of education is considered a feature of societies. We, as a society, ask for respect of unsound and nonempirical beliefs; we praise beliefs and traditions, and not facts

[11] In `https://en.wikipedia.org/wiki/List_of_topics_characterized_as_pseudoscience`, the reader can find an exhaustive list of pseudo-sciences.

and empirically guided change. We do not raise our children in an inquisitive spirit, prepared to put to the ground any form of nonsense, like political radicalism: that is simply too dangerous for the *status quo*. There are too many fellow citizens attached to pseudo-propositions, a change in this aspect of society is an ordeal.

We obviously have no problem whatsoever with personal beliefs; any fellow citizen can, and should, believe in what he wants to believe; the problem is that the citizen chooses to believe instead of looking for facts, the problem is that he wants to believe. We do not want to attack in any way the freedom to believe, we want to question the strive and the impulsive to believe. A society guided by facts is, inevitably, a healthier, wealthier, and more accepting society. It is when we are faced with the biological fact that we, humans, belong to the same species, and to the fact that the concept of subspecies (race) is inadequate to describe our biological existence, that we see how misleading racism is; it is when we study quantum mechanics that we understand that it is by no means connected to healing; it is when we biologically study women's brains and anatomy that we see that they are not inferior nor less intelligent than men; *et alii*. It is the concrete use of the SP and the VP that puts away all the previously mentioned harmful nonsense.

If, on the one hand, any individual is totally free to have any sort of beliefs, on the other hand, the society is also totally free to question those beliefs and to put them to the test. If an individual comes from a background that "taught" him to believe things without questioning, we should not be surprised when he totally denies facts; for him, beliefs have the status of facts. Are there some beliefs that should be accepted without evidence? How can we, as a society, choose the admissible beliefs not to be questioned? As, by definition, beliefs have no empirical grounds, if we allow one, we have to allow another; besides that, it is the whole attitude of belief that puts a halt to the SP and the VP use in society. The same goes for doing things for traditional grounds: 'We do this because our parents did so.', we are better than that; we, as a species, made admirable things, all of them either using the SP and the VP, in what facts are concerned, or using art and ethics, in what our human nature is concerned. There is no in-between, there is no meaningful ground between the sciences and the arts/ethics, there is no room, in an empirically guided society, for meaningless propositions that aim at having an impact in the world. It is important to say that religion, as a practise, just like what we mention in Chapter 1 about ethics, is not concerned with written propositions, but with actions; in that aspect, namely the religious practise, the religious person does not aim at making propositions. Only when a religious person makes claims about the world (maybe using their religious beliefs) does he need to put forward empirical evidence.

The arts, human love, and ethical impulses are totally outside the realm of belief: they do not aim at making propositions, they rather aim at acting, producing, and creating an impression, consequently they are free from the burden of verifiability. Pseudo-sciences and similar belief movements aim at describing, although totally failing to do so, the world; they are trying to step in the realm of meaningful discourse, a realm where only the SP and the VP are sources of meaning.

It is in a society full of beliefs and empty of facts that pseudo-sciences have a fertile soil to harvest. Any individual that has the misfortune of never having considered the SP and the VP is a possible victim to non-sensical speech. There is an obvious solution to all this: continue respecting beliefs, but strongly invest in scientific education, and all the rest will follow.

Pseudo-sciences and ungrounded beliefs are not mere beliefs, they have an impact in society. As a general starting point, they move the citizens away from reason and facts, and closer to superstition. But the harm goes deeper. "Healing practises" that are not based on science can cause harm to the individuals that decide to practise them: either they are not useful to solve the real medical problem, making the patient loose precious time; or, as they have no grounds on science and as they are not empirically tested, they might cause other problems besides the original problem the patient wanted to address. Race theory can lead to racism, we will not state the obvious harm racism causes. Creationism may appear to be harmless, but it deflects children from a scientific perspective of the world, it moves children away from the SP and the VP. Few of them are mere superstitions, like feng shui: worst-case scenario, if someone subscribes to feng shui, besides moving away from the SP and the VP, that person just looses time moving furniture around. The list goes on, but we stop it here.

There are several ways to identify pseudo-scientific discourse, but the fundamental aspect is to ask for the methods of testing the claims made by the suspected pseudo-science and the actual results of the performed tests. Moreover, ask for definitions of the basic concepts. In this aspect, the very implementation of the SP and the VP gives the reader a clear and direct way to identify pseudo-sciences: if the reader is unable to carry, even if in principle and not necessarily in practise, the analysis we proposed using a record book, then the reader can be sure that she is, in that context, dealing with pseudo-propositions, and consequently with a pseudo-science.

Chapter 4

Metaphysics

Metaphysics, as the name suggests, is concerned with the philosophical discourse of what, allegedly, is beyond physics, i.e. the tangible world. We believe that, at this point, it is clear to the reader that the SP and the VP totally discard such a discourse from the meaningful realm; more precisely, any metaphysical subject whose concepts cannot be expressed in empirical terms is, in the eyes of the SP and the VP, meaningless.

In Chapter 1, we saw that ethics is mostly concerned with actions, rather than with expressing those very actions; moreover, we also concluded that, when a concrete framework is given, the ethical statements can have an empirical meaning, thus they can have an actual meaning. The example we analyzed was the ethical norm 'You shall not kill.'. It can be viewed as an interjection that is not a proposition, it is, in that context, an order; and orders are not meaningful propositions, they are calls for action. But we also saw that we can precisely define the ethical terms in such a way that ‘You shall not kill.’ becomes a proposition; for example by framing the proposition in terms of harm to the society (this concept, by its turn, can have a similar empirical deconstruction).

Now that we have fully expressed our ideas about the use of the SP and the VP (with the attached distinction of propositions into type 1 and type 2), we can add an important aspect about ethics to what we have stated so far. The SP and the VP, mostly in the form of a concrete science, can be used to guide the ethical actions: again, here we are interested in actions, not in writing them down. When one makes a clarification about the world using the SP and the VP, one's ethical actions should be performed in accordance with that clarification. As an example, in Section 3.3, we emphasized that, using science, we are able to conclude that it makes no sense to be racist and to discriminate women. When one reaches the previously mentioned conclusions, one should act in accordance. There is yet a great aspect about ethical actions that are guided by the SP and the VP, like in the previous examples: when asked for a concrete reason for one's actions (when one is asked for a proposition), one can create an actual clear and meaningful proposition about the performed actions; *scilicet* one can give a real reason for the actions! Climate concerns are another important aspect of ethical norms drawn from science. Despite all this, there should be a meta-guiding line: when an empirically concluded ethical norm moves against one's ethical impulse—for instance by concluding that killing should be performed—one should double-check and parley with the community about this; as we mentioned in Chapter 1, ethics is to be performed, and the ethical drive of the individual should prevail.

Unfortunately, not all metaphysics is like ethics: some metaphysical discourses cannot have, by their very nature and way of design, a concrete empirical framework. Ontology is one of those fields. There are several reasons for why ontology is not empirically grounded, one of the most important ones is that existence is not a predicate; existence is, as we saw, a procedure, either of type 1 or of type 2: we do not predicate existence, we either give witnesses to propositions (type 1), or reasons to not conclude that a proposition is universally not the case (type 2).

For the reader's amusement, let us end this essay with a jolly simple way to create a meaningless metaphysical discourse. Pick a verb with a very widespread use in English: we pick the verb 'to be', we suggest that the reader picks her own verb. Create a noun from the verb, usually using the 'ing-form': in our case 'Being'. Use the noun in a generic and out of context way, especially by using it in grammatical sentences with the verb the reader started with accompanied by random pseudo-quantifications: 'Being is what is.', 'Being exists.', 'Nothing exists outside Being.', 'Being is the opposition of the existence of Nothing.', 'Being is to what is, like Nothing is for Being what is not.'.

Considering all that we have stated, there is no meaning outside verifiability.

Bibliography

[1] H. Andréka, I. Németi, J. Madarász, and G. Székely. On logical analysis of relativity theories. *arXiv preprint arXiv:1105.0885*, 2011.

[2] S. Buss. *Handbook of Proof Theory*. Elsevier Science, 1998.

[3] R. Carnap. *Meaning and Necessity: A Study in Semantics and Modal Logic*. Midway reprints. University of Chicago Press, 1988.

[4] R. Carnap. *The Logical Structure of the World: And, Pseudoproblems in Philosophy*. Open Court classics. Open Court, 2003.

[5] P. Hájek and P. Pudlák. *Metamathematics of First-Order Arithmetic*. Perspectives in Logic. Cambridge University Press, 1998.

[6] C. G. Hempel. Studies in the Logic of Confirmation (I). *Mind*, LIV(213):1–26, 01 1945.

[7] C. G. Hempel. Studies in the Logic of Confirmation (II). *Mind*, LIV(214):97–121, 04 1945.

[8] J. Madarász, I. Németi, and G. Székely. First-order logic foundation of relativity theories. In *Mathematical problems from applied logic II*, pages 217–252. Springer, 2007.

[9] A. A. Michelson and E. W. Morley. On the relative motion of the Earth and the luminiferous ether. *American Journal of Science*, s3-34(203):333–345, 11 1887.

[10] K. Popper. *Conjectures and Refutations: The Growth of Scientific Knowledge*. Routledge Classics. Taylor & Francis, 2014.

[11] M. Schlick. *General Theory of Knowledge*. Library of Exact Philosophy. Open Court Pub., 1985.

[12] M. Stannett and I. Németi. Using isabelle/hol to verify first-order relativity theory. *Journal of automated reasoning*, 52(4):361–378, 2014.

[13] L. Wittgenstein. *Tractatus Logico-philosophicus*. International library of psychology, philosophy, and scientific method. Routledge and Kegan Paul, 1974.

[14] N. M. J. Woodhouse. *Special Relativity*. Springer Undergraduate Mathematics Series. Springer London, 2012.