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SOME RESULTS ON ORDERED STRUCTURES IN TOPOSES

A b s t r a c t. A topos version of Cantor's back and forth theorem is established and used to prove that the ordered structure of the rational numbers $\langle \mathbb{Q}, < \rangle$ is homogeneous in any topos with natural numbers object. The notion of effective homogeneity is introduced, and it is shown that $\langle \mathbb{Q}, < \rangle$ is a minimal effectively homogeneous structure, that is, it can be embedded in every other effectively homogeneous ordered structure.

1. Introduction

In [6] several interesting results on intuitionistic model theory concerning ordered structures were obtained. Inspired on this, we study in this paper ordered structures defined in an arbitrary topos with natural numbers object.

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The notion of homogeneity of a structure was introduced, in the context of Model Theory, by B. Jónsson in 1960. In general, a set-based structure $\mathfrak{A}$ is said to be κ-homogeneous (κ an infinite cardinal) if, for every partial isomorphism of $\mathfrak{A}$ of cardinal $\kappa' < \kappa$, there exists an automorphism of $\mathfrak{A}$ which extends it (see [2] for details). In this paper we deal only with finite partial isomorphisms, which means that our homogeneous ordered structures are the $\aleph_0$-homogeneous ones.

We introduce the notion of effectively homogeneous ordered structure, for which there is an effective procedure which extends every finite partial isomorphism to an automorphism. Our aim is to prove that the ordered structure $\langle \mathbb{Q}, < \rangle$ of the rational numbers object $\mathbb{Q}$ is itself effectively homogeneous in any topos with natural numbers object (Theorem 6.1) and, moreover, it is minimal, that is: If $\langle A, < \rangle$ is another effectively homogeneous ordered structure, then there exists a monomorphism $\langle \mathbb{Q}, < \rangle \hookrightarrow \langle A, < \rangle$ (Theorem 6.3).

For the definitions and proofs we use the logical tool of local set theory, which deserves some explanation.

Local set theory is a typed set theory whose underlying logic is higher-order intuitionistic logic. In this sense, it is a generalization of classical set theory, in which the primitive notion of set is replaced by that of type (accurately, sets are replaced by terms of certain types). The category $\mathbf{C}(S)$ built out of a local theory S – whose objects are the local sets (or S-sets) and whose arrows are the local maps (or S-maps) – can be shown to be a topos, called a linguistic topos. It is then possible to show that every topos $\mathbf{E}$ is equivalent to a linguistic topos, namely $\mathbf{C}(\mathrm{T}(\mathbf{E}))$, where $\mathrm{T}(\mathbf{E})$ is the local theory whose axioms are those which are valid by the canonical interpretation of the internal language of $\mathbf{E}$ in $\mathbf{E}$ itself. So, the categorial apparatus of a topos can be translated to a logical one and develop all the technical machinery in the environment of local set theory. For a detailed development of the subject, we address the reader to [1]. In the following paragraphs a few notational aspects are pointed out.

The structure of the paper is the following: In the first section we recall some elementary definitions and easy results from local set theory, ordered structures and natural numbers object. In Section 3 we prove the minimum principle for decidable properties on the natural numbers (Proposition 3.2). Using the results about finite sequences stated in Section 4, a topos version of Cantor's back and forth theorem is obtained in Section 5 (Theorem 5.2), as well as a more general result (Corollary 5.3). The latter is applied to the proof of the (effective) homogeneity of the ordered structure $\langle\mathbb{Q},<\rangle$ of the rational numbers (Theorem 6.1), subject of Section 6. Finally, it is proved that $\langle\mathbb{Q},<\rangle$ is minimal with respect to effective homogeneity (Theorem 6.3).

2. Preliminaries

A *local language* $\mathcal{L}$ is a higher-order language consisting of types, variables and terms defined as usual such that: If $\mathsf{A}_1,\ldots,\mathsf{A}_n$ and A are types then $\mathsf{A}_1\times\cdots\times\mathsf{A}_n$ and $P\mathsf{A}$ are types (the product and the power types, respectively). If $n=0$ then the empty product $\mathsf{A}_1\times\cdots\times\mathsf{A}_n$ is denoted by $\mathbf{1}$ (the *unity type*). There is another distinguished type denoted by Ω (the *truth-value type*). For every type A we settle a denumerable collection $x^1_{\mathsf{A}},x^2_{\mathsf{A}},\ldots$ of *variables of type* A. The collection of *terms* of a given type A is defined recursively over the collection of variables (the details of the construction can be found in [1]). Terms of type Ω are called *formulas* and denoted by α, β etc. A formula *in context* is an expression $\vec{x}.\alpha$, where $\vec{x}$ is a list $x_0,\ldots,x_{n-1}$ of distinct variables and α is a formula such that all its free variables are in $\vec{x}$.

Following [1], a *local set theory* is a sequent calculus S over a local language $\mathcal{L}$ satisfying specific rules for higher-order intuitionistic logic. Most of the notation and terminology we will use in the sequel is standard. In particular, we will write $\Gamma\mid\!\sim_S\alpha$ when the sequent $\Gamma\Rightarrow\alpha$ is derived from the collection of sequents S. In a local language $\mathcal{L}$, all the customary logical symbols can be *defined*. We assume the reader is acquainted with the basic axioms and inference rules of higher-order intuitionistic logic (cf. [1]).

Local sets, which are closed terms of power type, are denoted by capital letters A, B, X, Y etc. Recall from [1] that, if x is a variable of type A and t is a term of type $P\mathsf{A}$ then $x\in t$ is a formula; and if α is a formula then $\{x\,:\,\alpha\}$ is a term of type $P\mathsf{A}$ in which x does not occur free. We stipulate to represent elements with the same letter of the local set to which they belong, though in low case letters, possibly indexed. For example: $a,a',a_0\in A$; $b',b'',b_1\in B$; $x,x_1,x_2\in X$ etc. Special local sets are $U_{\mathsf{A}}=\{x_{\mathsf{A}}\,:\,\top\}$ and $\emptyset_{\mathsf{A}}=\{x_{\mathsf{A}}\,:\,\bot\}$.

We now recall some basic facts about any local set theory S.

An S-set X is said to be *inhabited* if $\mid\!\sim_S \mathrm{Inh}(X)$, where $\mathrm{Inh}(X):\Leftrightarrow$

$\exists x.x \in X$. It is certainly true that $\mid\!\sim_S \mathrm{Inh}(X) \rightarrow X \neq \emptyset$, but the converse does not necessarily hold.

We say that a formula in context $\vec{z}.\alpha$ is *decidable in* the S-set $\prod_{i<n} X_i$, $X_i \subseteq U_{\mathsf{A}_i}$ $(i = 0, \ldots, n-1)$ if

$$\mid\!\sim_S \forall x_0 \in X_0, \ldots, x_{n-1} \in X_{n-1}[\alpha \vee \neg\alpha].$$

This is equivalent to saying that the S-set

$$\left\{ \langle x_0, \ldots, x_{n-1} \rangle \in \prod_{i<n} X_i \ : \ \alpha \right\}$$

is a complemented element in the Heyting algebra of the S-subsets of $\prod_{i<n} U_{\mathsf{A}_i}$. In particular, $z.\alpha$ is decidable in $X \subseteq U_{\mathsf{A}}$ if $\mid\!\sim_S \forall x \in X[\alpha \vee \neg\alpha]$.

We say that $\vec{z}.\alpha$ is *decidable* if it is decidable in $\prod_{i<n} U_{\mathsf{A}_i}$.

Lemma 2.1. *If $\vec{z}.\alpha$ and $\vec{z}.\beta$ are decidable in $\prod_{i<n} X_i$, then so are $\vec{z}.(\alpha \wedge \beta)$, $\vec{z}.(\alpha \vee \beta)$ and $\vec{z}.(\alpha \rightarrow \beta)$.*

Proof. If x and y are complemented elements in a Heyting algebra, with complements, say, $\sim\!x$ and $\sim\!y$, respectively, it is easy to verify that $\sim\!x \vee \sim\!y$, $\sim\!x \wedge \sim\!y$ and $x \wedge \sim\!y$ are the respective complements of $x \wedge y$, $x \vee y$ and $x \rightarrow y$ (recall that, if both x, y are complemented, then $x \rightarrow y = \sim\!x \vee y$). □

We next exhibit some elementary definitions concerning the ordered structures in a local set theory. Recall that $\mathbf{C}(S)$ is the topos constructed out of the local set theory S.

A (*partially*) *ordered* $\mathbf{C}(S)$-*structure*, or simply "an order", is a pair $\langle A, < \rangle$, where A is an S-set and $< \subseteq A \times A$ is a relation satisfying $\mid\!\sim_S \neg(a < a)$ and $a' < a'', a'' < a''' \mid\!\sim_S a' < a'''$.

A *homomorphism* f from $\langle A, < \rangle$ into $\langle B, < \rangle$ is an S-map $f : A \rightarrow B$ which preserves the order, that is, $a' < a'' \mid\!\sim_S f(a') < f(a'')$. Ordered $\mathbf{C}(S)$-structures and homomorphisms form a category $\mathrm{Ord}[\mathbf{C}(S)]$.

An order $\langle A, < \rangle$ is *linear* if it satisfies $\mid\!\sim_S a' < a'' \vee a' = a'' \vee a'' < a'$.

The $\mathbf{C}(S)$-structure $\langle A, < \rangle$ is *dense in* $\langle B, < \rangle$ if there exists a monomorphism $i : \langle A, < \rangle \rightarrow \langle B, < \rangle$ in $\mathrm{Ord}[\mathbf{C}(S)]$ such that $b' < b'' \mid\!\sim_S \exists a.b' < i(a) < b''$. Usually we consider a canonical monomorphism i clear from the context. For example, in **Set** this monomorphism is mostly represented by the inclusion $A \subseteq B$. $\langle A, < \rangle$ is *dense* if it is dense in itself by means of id_A.

A $\mathbf{C}(S)$-structure $\langle A, <\rangle$ is *persistent in* $\langle B, <\rangle$ if it is dense in $\langle B, <\rangle$ and $\mid\!\sim_S \forall b \exists a', a''.i(a') < b < i(a'')$, where i is the canonical monomorphism which must be previously defined. $\langle A, <\rangle$ is *persistent* if it is persistent in itself. Intuitively, $\langle A, <\rangle$ is persistent if it is dense and does not have endpoints.

Finally we remind some elementary results concerning the natural numbers in a local set theory.

A local set theory N is said to be *naturalized* if its language has a type N, a closed term 0 (zero) of type N and an N-function s (successor) of type $\mathsf{N} \times \mathsf{N}$ satisfying the *Peano axioms.* The N-set $U_{\mathsf{N}} = \{x_{\mathsf{N}} \ : \ \top\}$ is then denoted by $\mathbb{N}$ and called N-set of natural numbers. Elements of $\mathbb{N}$ are denoted by m, n, n' etc.

In a naturalized local theory it is possible to prove the *primitive recursion principle* (PRP) (see [1] for details):

$$x \in X, g \in X^{X \times \mathbb{N}} \mid\!\sim_N \exists! f \in X^{\mathbb{N}}[f(0) = x \wedge \forall n.f \circ s(n) = g(f(n), n)].$$

We define an N-function $[\cdot] : \mathbb{N} \to P\mathbb{N}$ which intuitively collect all the natural numbers less than a given n. By PRP:

$$\mid\!\sim_N \exists! [\cdot] \in (P\mathbb{N})^{\mathbb{N}}[[0] = \emptyset \wedge [s(n)] = [n] \cup \{n\}].$$

We can now define a relation of strict order in $\mathbb{N}$ by $m < n :\Leftrightarrow m \in [n]$. Therefore $\mid\!\sim_N [n] = \{m \ : \ m < n\}$. The following lemma is a selection of basic facts about the strict order and the N-set $\mathbb{N}$. The proofs can be verified in [1].

Lemma 2.2.

(a)	$\mid\!\sim_N \neg(n < 0)$	(minimal element);
(b)	$\mid\!\sim_N m < s(n) \leftrightarrow m = n \vee m < n$	(discretion);
(c)	$\mid\!\sim_N \neg(n < n)$	(irreflexivity);
(d)	$m' < n, n < m'' \mid\!\sim_N m' < m''$	(transitivity);
(e)	$\mid\!\sim_N m < n \vee m = n \vee n < m$	(linearity);
(f)	$\mid\!\sim_N m = n \vee \neg(m = n)$	(decidability).

The decidability of the equality in $\mathbb{N}$ allows us to define the linear partial order $\leq$ by $m \leq n :\Leftrightarrow m = n \vee m < n$. Addition, product and exponentiation are defined straightforwardly via PRP. For example, once we have addition and product, exponentiation $(\cdot)^{(\cdot)} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is obtained by $\mid\!\sim_N m^0 = 1 \wedge \ m^{s(n)} = m^n.m$.

The integers object $\mathbb{Z}$ is defined as usual, as being the coproduct of $\mathbb{N}$ with its image by s; that is, $\mathbb{Z} := \mathbb{N} + s(\mathbb{N})$. Extending strict order, addition and product from the system $\mathbb{N}$, the resulting $\mathbf{C}(N)$-structure satisfies the conditions of a linearly ordered commutative ring. Now the rational numbers object $\mathbb{Q}$ can be defined by imitating the quotient classical construction. The details can be appreciate in [4]. Extending again strict order, addition and product, this time from $\mathbb{Z}$, the resulting $\mathbf{C}(N)$-structure is also a linearly ordered commutative ring. Furthermore, the linearly ordered $\mathbf{C}(N)$-structure $\langle \mathbb{Q}, < \rangle$ is persistent. Since the N-set (not the corresponding ordered $\mathbf{C}(N)$-structure) $\mathbb{Q}$ can be shown to be isomorphic to the N-set $\mathbb{N}$, we shall infer from Theorem 5.2 that $\langle \mathbb{Q}, < \rangle$ is, up to isomorphism, the unique inhabited, linearly ordered, persistent and denumerable $\mathbf{C}(N)$-structure.

3. The minimum principle

Every primitive recursive function is representable in a topos with natural numbers object (see [3]). For example, the usual bijections between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ can be performed in a naturalized local set theory: $\langle m, n \rangle \mapsto 2^m(1 + 2n) - 1$ and $\langle m, n \rangle \mapsto \frac{1}{2}((m+n)^2 + 3m + n)$ are primitive recursive.

We prove in this section the minimum principle, which can be described intuitively as: if some natural number has a given *decidable* property, then there is a least natural number having that property. It is important to emphasize that we are dealing with decidable properties (here "property" stands for formula in the local language). In [5] there is a constructive proof that the unrestricted minimum principle implies the law of excluded middle.

Before the proof of the minimum principle, we introduce the following notation: the finite conjunction of a formula α is the N-function $\bigwedge_{i<(\cdot)} \alpha : \mathbb{N} \to U_\Omega$ defined by PRP:

$$\mathrel{|\!\!\sim}_N \bigwedge_{i<0} \alpha = \top \wedge \forall n \left[\bigwedge_{i<n+1} \alpha = \bigwedge_{i<n} \alpha \wedge \alpha(n) \right].$$

Finite disjunction $\bigvee_{i<(\cdot)} \alpha : \mathbb{N} \to U_\Omega$ is defined in the same way.

By induction it can be proved that

$$n \leq m \mathrel{|\!\sim}_N \bigwedge_{i<m} \alpha \to \bigwedge_{i<n} \alpha; \tag{1}$$

$$n \leq m \mathrel{|\!\sim}_N \bigvee_{i<n} \alpha \to \bigvee_{i<m} \alpha. \tag{2}$$

The following result can be easily proved:

$$\mathrel{|\!\sim}_N \left[\bigvee_{i<n+1} \alpha \wedge \neg \bigvee_{i<n} \alpha\right] \to \alpha(n). \tag{3}$$

The next lemma generalizes Lemma 2.1.

Lemma 3.1. *If the formula α is decidable in $\mathbb{N}$, then so are $\bigwedge_{i<n} \alpha$ and $\bigvee_{i<n} \alpha$ for each n.*

Proof. Using induction, the case $n = 0$ is immediate. For the remainder it suffices to apply Lemma 2.1. □

Finally, the minimum principle can be obtained.

Proposition 3.2 (Minimum principle). *If the formula α is decidable in $\mathbb{N}$, then*

$$\exists n.\alpha(n) \mathrel{|\!\sim}_N \exists n[\alpha(n) \wedge \forall m(\alpha(m) \to n \leq m)].$$

Proof.
Define a formula β by:

$$\beta := \exists n[\alpha(n) \wedge \forall m(\alpha(m) \to n \leq m)].$$

We shall prove by induction that:

$$\mathrel{|\!\sim}_N \forall n \left[0 < n \to \left(\bigvee_{i<n} \alpha \to \beta\right)\right].$$

For $n = 1$, the statement is equivalent to $\mathrel{|\!\sim}_N \alpha(0) \to \beta$, which clearly holds. Suppose now that the statement holds for n and consider the case $n+1$: by Lema 3.1, the formula $\bigvee_{i<n} \alpha$ is decidable; thus we may consider two cases:

$$\mathrel{|\!\sim}_N \bigvee_{i<n} \alpha \vee \neg \bigvee_{i<n} \alpha.$$

If $\mid\!\sim_N \bigvee_{i<n} \alpha$, we deduce $\mid\!\sim_N \beta$ by induction hypothesis and so

$$\mid\!\sim_N \bigvee_{i<n+1} \alpha \to \beta;$$

if $\mid\!\sim_N \neg \bigvee_{i<n} \alpha$, we derive by intuitionistic rules that $\mid\!\sim_N \bigwedge_{i<n} \neg\alpha$; hence:

$$\begin{aligned} \textstyle\bigvee_{i<n+1} \alpha \quad & \mid\!\sim_N \alpha(n+1) \wedge \textstyle\bigwedge_{i<n} \neg\alpha. \\ & \mid\!\sim_N \alpha(n+1) \wedge \forall m[\alpha(m) \to n < m] \\ & \mid\!\sim_N \beta, \end{aligned}$$

completing the induction argument. Thus we have established the initial statement, which by intuitionistic rules implies that:

$$\exists n \left[0 < n \wedge \bigvee_{i<n} \alpha \right] \mid\!\sim_N \beta.$$

Therefore the minimum principle follows:

$$\exists n.\alpha(n) \mid\!\sim_N \exists n[\alpha(n) \wedge \forall m(\alpha(m) \to n \leq m)].$$

□

If $\alpha(n)$ is decidable and $\mid\!\sim_N \exists n.\alpha(n)$, then, since $\leq$ is antisymmetric, the minimum will be unique, that is:

$$\mid\!\sim_N \exists! n[\alpha(n) \wedge \forall m(\alpha(m) \to n \leq m)].$$

This unique minimum will be denoted by $\mu n.\alpha(n)$.

4. Finite sequences

A given N-set X is said to be *denumerable* if it satisfies the axiom $\mid\!\sim_N \mathrm{Den}(X)$, where

$$\mathrm{Den}(X) :\Leftrightarrow X = \emptyset \vee \exists g \in X^{\mathbb{N}}.\mathrm{Sur}(g).$$

We also say that each N-map g (which we represent mostly by g_X) satisfying the above condition is an *enumeration* of X.

If X satisfies the stronger condition $\mid\!\sim_N X \simeq \mathbb{N}$, where $X \simeq Y$ stands for the existence of an isomorphism in $\mathbf{C}(N)$ between X and Y, we say that it is *completely denumerable* or simply *countable*.

Let X and Y be denumerable N-sets and $g_{X\times Y}$ be the product $g_X \times g_Y : X \times Y \to \mathbb{N} \times \mathbb{N}$. Since the product $\mathbb{N} \times \mathbb{N}$ is countable, the composition $g_{\mathbb{N}\times\mathbb{N}} \circ g_{X\times Y}$ is clearly an enumeration of $X \times Y$. If X and Y are countable, so is $X \times Y$.

An N-set X is *finite* if $\mid\!\sim_N \mathrm{Fin}(X)$, where $\mathrm{Fin}(X) :\Leftrightarrow \exists n.[X \simeq [n]]$.

We define a *sequence on* X as an N-map $f : \mathbb{N} \to X + \{\sharp\}$, where $+$ denotes the coproduct in $\mathbf{C}(N)$ and $\sharp$ is any element, say 0. Since $X + \{\sharp\}$ represents a disjoint union in N, we shall suppose that $\mid\!\sim_N \sharp \notin X$.

A *finite sequence on* X is a sequence on X satisfying the axiom $\mid\!\sim_N \mathrm{Fsq}_X(f)$, where

$$\mathrm{Fsq}_X(f) :\Leftrightarrow \exists n[f([n]) \subseteq X \wedge f(\mathbb{N} - [n]) = \{\sharp\}].$$

We observe that the natural number n in the above definition is unique.

We collect all the finite sequences of X through the N-set:

$$X^* := \{f \in (X + \{\sharp\})^{\mathbb{N}} \ : \ \mathrm{Fsq}_X(f)\}.$$

The elements of X^* will be denoted by $\vec{x}$, $\vec{x}_0$, $\vec{x}_1$ etc. Although the same notation has already been used for contexts, this will not lead to ambiguity. Among the elements of X^* we find the constant N-map defined by $n \mapsto \sharp$, which will be also denoted by $\sharp$ (an abuse of notation).

To each finite sequent $\vec{x}$ of X^* is associated a natural number which represents intuitively the length or number of relevant elements of $\vec{x}$. So we define an N-map $\mathrm{lgh}_X : X^* \to \mathbb{N}$ by $\vec{x} \mapsto \mu n.[\vec{x}(n) = \sharp]$. The N-map lgh allows us to collect all the finite sequences of a given length:

$$X^n := \{\vec{x} \in X^* \ : \ \mathrm{lgh}_X(\vec{x}) = n\}.$$

We define the *effective image* of a finite sequence $\vec{x}$ by:

$$\mathrm{eim}_X(\vec{x}) := \vec{x}([\mathrm{lgh}_X(\vec{x})]) = \{\vec{x}(n) \ : \ n < \mathrm{lgh}_X(\vec{x})\}.$$

When no confusion arises, we omit the indexes which represent the N-sets, writing simply Fsq, lgh and eim instead of Fsq_X, lgh_X e eim_X, respectively.

We may add a new element of X to any sequence in X^* through the N-map $* : X^* \times X \to X^*$ defined by:

$$\mid\!\sim_N \quad \forall n[(n = \mathrm{lgh}(\vec{x}) \to (\vec{x} * x)(n) = x) \\ \wedge(n \neq \mathrm{lgh}(\vec{x}) \to (\vec{x} * x)(n) = \vec{x}(n))].$$

It is also possible to determine an strict order between elements of X^*:

$$\vec{x}_0 < \vec{x}_1 :\Leftrightarrow \mathrm{lgh}(\vec{x}_0) < \mathrm{lgh}(\vec{x}_1) \wedge \forall n[n < \mathrm{lgh}(\vec{x}_0) \to \vec{x}_0(n) = \vec{x}_1(n)];$$

then:

$$\mid\!\sim_N [\mathrm{lgh}(\vec{x}_1) = \mathrm{lgh}(\vec{x}_0) + 1 \ \wedge \ \vec{x}_0 < \vec{x}_1] \leftrightarrow \vec{x}_1 = \vec{x}_0 * \vec{x}_1(\mathrm{lgh}(\vec{x}_0))$$

and $\mid\!\sim_N \forall \vec{x} \in X^*[\vec{x} = \sharp \ \vee \ \sharp < \vec{x}]$.

Next we present a primitive recursive bijection $h : \mathbb{N}^* \to \mathbb{N}$ defined by PRP:

$$\mid\!\sim_N h(\sharp) = 0 \wedge h(\vec{n} * n) = 2^n(1 + 2h(\vec{n})).$$

Therefore the N-set $\mathbb{N}^*$ is countable. The following proposition generalizes this fact.

Proposition 4.1. *If the N-set X is denumerable, so is X^*. If X is countable, so is X^*.*

Proof. Let $g_X : \mathbb{N} \to X$ be an enumeration of X. It suffices to present a surjection $g : \mathbb{N}^* \to X^*$, since $\mathbb{N} \simeq \mathbb{N}^*$. This is obtained by PRP:

$$\mid\!\sim_N g(\sharp) = \sharp \ \wedge \ g(\vec{n} * n) = g(\vec{n}) * g_X(n).$$

The same argument works for countability. □

5. Cantor's back an forth theorem

Now we come back to ordered structures.

An ordered $\mathbf{C}(N)$-structure $\langle A, < \rangle$ is *denumerable* if A is denumerable. The same holds for finiteness, countability etc.

So let $\langle A, < \rangle$ be denumerable and linear. We say that $\vec{a} \in A^*$ is *increasing* if $\mid\!\sim_N \mathrm{Crs}(\vec{a})$, where

$$\mathrm{Crs}(\vec{a}) :\Leftrightarrow \forall n, m[n < m < \mathrm{lgh}(\vec{a}) \to \vec{a}(n) < \vec{a}(m)]. \tag{4}$$

The N-set of all increasing finite sequences of elements of A is denoted by A^*_{cr}. Then $A^*_{cr} := \{\vec{a} \in A^* \ : \ \mathrm{Crs}(\vec{a})\}$.

Proposition 5.1. *If $\langle A, < \rangle$ is a ordered $\mathbf{C}(N)$-structure which is linear and denumerable, then A^*_{cr} is denumerable.*

Proof. Let $g_A : \mathbb{N} \twoheadrightarrow A$, as always, be an enumeration of A. Since $\mathbb{N} \simeq \mathbb{N}^*$, it suffices to present a surjection $g : \mathbb{N}^* \twoheadrightarrow A^*_{cr}$. But by PRP:

$$\begin{aligned} \mathrel{|\!\sim}_N \quad & g(\sharp) = \sharp \\ & \wedge [0 < \mathrm{lgh}(g(\vec{n})) \\ & \qquad \rightarrow [[g(\vec{n})(\mathrm{lgh}(g(\vec{n})) - 1) < g_A(n) \rightarrow g(\vec{n} * n) = g(\vec{n}) * g_A(n)] \\ & \qquad\quad \wedge [g_A(n) \leq g(\vec{n})(\mathrm{lgh}(g(\vec{n})) - 1) \rightarrow g(\vec{n} * n) = \sharp]]]. \end{aligned}$$

Another way would be to note that the expression (4) may be write as a double finite conjunction of a decidable formula; hence $\vec{a} \in A^*_{cr}$ is also decidable. □

Now let $\langle A, < \rangle$ be a linearly ordered $\mathbf{C}(N)$-structure. We define N-maps $\curlywedge, \curlyvee : A \times A \rightarrow A$ by:

$$\mathrel{|\!\sim}_N (a' \curlywedge a'' = a') \leftrightarrow (a' \leq a'') \quad \text{and} \quad \mathrel{|\!\sim}_N (a'' \curlywedge a' = a') \leftrightarrow (a' \leq a'');$$

$$\mathrel{|\!\sim}_N (a' \curlyvee a'' = a'') \leftrightarrow (a' \leq a'') \quad \text{and} \quad \mathrel{|\!\sim}_N (a'' \curlyvee a' = a'') \leftrightarrow (a' \leq a'').$$

By convention, $\mathrel{|\!\sim}_N \sharp \curlywedge a = \sharp \curlyvee a = a \curlywedge \sharp = a \curlyvee \sharp = a$. We also define N-maps $\min, \max : A^* \rightarrow A + \{\sharp\}$ respectively by:

$$\mathrel{|\!\sim}_N \min(\sharp) = \sharp \ \wedge \ \min(\vec{a} * a) = \min(\vec{a}) \curlywedge a;$$

$$\mathrel{|\!\sim}_N \max(\sharp) = \sharp \ \wedge \ \max(\vec{a} * a) = \max(\vec{a}) \curlyvee a.$$

When there is a clear canonical way to convert sets to finite sequences, we shall apply the N-maps above also to N-sets.

We arrive at our first main result: the topos version of Cantor's back and forth theorem.

Theorem 5.2 (Cantor's back and forth theorem). *Let $\langle A, < \rangle$ and $\langle B, < \rangle$ be linearly ordered, inhabited, linear, persistent and denumerable $\mathbf{C}(N)$-structures. Then $\langle A, < \rangle \simeq \langle B, < \rangle$.*

Proof. The proof is divided into two parts. In the first we show that the structures $\langle A, <\rangle$ and $\langle B, <\rangle$ are partially isomorphic, that is, there is an N-set F of order-preserving finite sequences on $A \times B$ with the back and forth property. In the second part we construct the required isomorphism. Part 1. Let

$$\begin{aligned} F := \quad & \{f \in (A \times B)^* \ : \ \forall n, m[n < m < \mathrm{lgh}(f) \\ & \to (\pi' \circ f(n) < \pi' \circ f(m) \leftrightarrow \pi'' \circ f(n) < \pi'' \circ f(m))]\} \end{aligned} \tag{5}$$

be the N-set of all order-preserving finite sequences on $A \times B$. By simplicity we write simply $\pi' \circ f$ (and $\pi'' \circ f$) instead of $(\pi' + \mathrm{id}_{\{\sharp\}}) \circ f$ (and $(\pi'' + \mathrm{id}_{\{\sharp\}}) \circ f$, respectively). The product $A \times B$ of denumerable N-sets is denumerable; hence so is $(A \times B)^*$ by Proposition 4.1. Call β the formula

$$\pi' \circ f(n) < \pi' \circ f(m) \leftrightarrow \pi'' \circ f(n) < \pi'' \circ f(m).$$

Notice that we may write $\forall n, m[n < m < \mathrm{lgh}(f) \to \beta]$ as a double finite conjunction:

$$\bigwedge_{n<m} \left(\bigwedge_{m<\mathrm{lgh}(f)} \beta \right); \tag{6}$$

or shortly:

$$\bigwedge_{n<m<\mathrm{lgh}(f)} \beta.$$

Since β is decidable, so is the formula given by expression (6). Then the N-set F is characterized by a decidable property. Next we show that F possesses the back and forth property. We prove initially the expression:

$$\mid\!\sim_N \forall f \in (A \times B)^* \forall a \exists b.[f * \langle a, b\rangle \in F]. \tag{7}$$

For $\mathrm{lgh}(f) = 0$ it follows that $\mid\!\sim_N \sharp * \langle a, b\rangle \in F$. If $\mathrm{lgh}(f) > 0$, we use the abbreviations:

$$a_{\min} := \min(\pi' \circ f); \qquad a_{\max} := \max(\pi' \circ f);$$

$$b_{\min} := \min(\pi'' \circ f); \qquad b_{\max} := \max(\pi'' \circ f);$$

$$a^+ := \min\{a' \in \pi'(\mathrm{eim}(f)) \ : \ a < a'\};$$

$$a^- := \max\{a' \in \pi'(\mathrm{eim}(f)) \ : \ a' < a\}.$$

The following expression is a direct consequence of the persistence of $\langle B, <\rangle$:

$$\begin{aligned} \mid\!\sim_N \quad & a < a_{\min} \rightarrow \exists b.b < b_{\min} \\ & a_{\max} < a \rightarrow \exists b.b_{\max} < b \\ & a_{\min} < a < a_{\max} \rightarrow \exists b \exists n, m < \mathrm{lgh}(f).[f(n)(a^-) < b < f(m)(a^+)]. \end{aligned}$$

In any case we verify that $\mid\!\sim_N \exists b.[f * \langle a, b\rangle \in F]$. Similarly we can prove that:

$$\mid\!\sim_N \forall f, b \exists a.[f * \langle a, b\rangle \in F]. \tag{8}$$

So we can define the N-functions $\hat{f} : F \times A \rightarrow F$ and $\check{f} : F \times B \rightarrow F$, which satisfy explicitly the back and forth property (g_A and g_B are, as always, enumerations of A and B, respectively):

$$\langle f, a\rangle \mapsto f * \langle a, g_B(\mu n.f * \langle a, g_B(n)\rangle \in F)\rangle; \tag{9}$$

$$\langle f, b\rangle \mapsto f * \langle g_A(\mu n.f * \langle g_A(n), b\rangle \in F), b\rangle. \tag{10}$$

Part 2. We construct now the isomorphism $h : \langle A, <\rangle \rightarrow \langle B, <\rangle$. First observe that we have the following property of the natural numbers:

$$\mid\!\sim_N \forall m \left[(m = 0) \vee \exists n(m = 2n + 1) \vee \exists n(m = 2n + 2)\right].$$

We can construct a unique N-map $\bar{f} : \mathbb{N} \rightarrow F$ that satisfies:

$$\begin{aligned} \mid\!\sim_N \quad & \bar{f}(0) = \sharp \ \wedge \ \bar{f}(2n + 1) = \hat{f}(\bar{f}(2n), g_A(n)) \\ & \wedge \ \bar{f}(2n + 2) = \check{f}(\bar{f}(2n + 1), g_B(n)). \end{aligned}$$

Finally we define the required isomorphism $h : \langle A, <\rangle \rightarrow \langle B, <\rangle$ by the image below:

$$h := \mathrm{img}\left(\bigcup_n \bar{f}(n)\right).$$

□

Corollary 5.3. *Suppose the following conditions:*

- $\langle A, <\rangle$ *and* $\langle B, <\rangle$ *are linearly ordered, inhabited, linear, persistent and denumerable* $\mathbf{C}(N)$*-structures;*
- $\alpha(x, y)$ *is a formula decidable in* $A \times B$*;*

- $\{a : \exists b.\alpha(a,b)\}$ *is persistent in* B *and* $\{b : \exists a.\alpha(a,b)\}$ *is persistent in* A.

Then there exists an isomorphism $h : \langle A,<\rangle \to \langle B,<\rangle$ *such that* $\mid\!\sim_N \forall a.\alpha(a,h(a))$.

Proof. We just need to add the new conditions to the proof of Theorem 5.2. Let F_α be an N-set defined by:

$$F_\alpha := \{f \in F \ : \ \forall n < \mathrm{lgh}(f).\alpha(f(n))\},$$

where F is defined by expression (5). Since α is decidable in $A \times B$ and the formula $\forall n < \mathrm{lgh}(f).\alpha(f(n))$ may be written in the form

$$\bigwedge_{n<\mathrm{lgh}(f)} \alpha,$$

and thus it is also decidable, we can assert that F_α is characterized by a decidable property. Moreover, since $\{a : \exists b.\alpha(a,b)\}$ is persistent in A, it is possible to adapt the expressions (7) and (8):

$$\mid\!\sim_N \forall f, b\exists a.[f * \langle a,b\rangle \in F_\alpha];$$

$$\mid\!\sim_N \forall f, a\exists b.[f * \langle a,b\rangle \in F_\alpha].$$

The rest of the proof is identical to that of Theorem 5.2, with F replaced by F_α. □

6. Homogeneous and effectively homogeneous structures

Let $\langle A,<\rangle$ be a (partially) order $\mathbf{C}(N)$-structure. We say that a finite sequence $f : \mathbb{N} \to A \times A$ *preserves the pairing order* if $\mid\!\sim_N \mathrm{Ppo}(f)$, where

$$\mathrm{Ppo}(f) :\Leftrightarrow \bigwedge_{m<n<\mathrm{lgh}(f)} [\pi' \circ f(m) < \pi' \circ f(n) \leftrightarrow \pi'' \circ f(m) < \pi'' \circ f(n)].$$

$\langle A,<\rangle$ is said to be *homogeneous* if, for every finite sequence $f : \mathbb{N} \to A \times A$ which preserves the pairing order, there exists an automorphism of $\langle A,<\rangle$ which extends $\mathrm{eim}(f)$. Intuitively, every finite partial isomorphism can be extended to an automorphism.

Now we arrive at our second main result: the homogeneity of $\langle \mathbb{Q},<\rangle$.

Theorem 6.1. *The* $\mathbf{C}(N)$*-structure* $\langle \mathbb{Q}, < \rangle$ *is homogeneous.*

Proof. Let $f : \mathbb{N} \to \mathbb{Q} \times \mathbb{Q}$ be a finite sequence which preserves the pairing order. We define a formula α by:

$$\alpha(q', q'') :\Leftrightarrow \bigwedge_{n<\mathrm{lgh}(f)} \left[\pi' \circ f(n) < q' \leftrightarrow \pi'' \circ f(n) < q''\right].$$

Then α is decidable in $\mathbb{Q} \times \mathbb{Q}$ and

$$\mid\!\sim_N \mathbb{Q} = \{q' \ : \ \exists q''.\alpha(q', q'')\} = \{q'' \ : \ \exists q'.\alpha(q', q'')\}.$$

Since α satisfies all the conditions of the Corollary 5.3, there exists an automorphism h of $\langle \mathbb{Q}, < \rangle$ such that $\mid\!\sim_N \forall q.\alpha(q, h(q))$, that is, h extends $\mathrm{eim}(f)$. □

We note that the automorphism h from the above proof was effectively constructed, that is, given a finite sequence preserving the pairing order, it was uniquely extended to an automorphism. This motivates the following definition.

An ordered $\mathbf{C}(N)$-structure $\langle A, < \rangle$ is *effectively homogeneous* if there exists an N-map

$$\hbar : \{f \in (A \times A)^{\mathbb{N}} \ : \ \mathrm{Fsq}(f) \wedge \mathrm{Ppo}(f)\} \to \{h \in A^A \ : \ \mathrm{Aut}(h)\}$$

such that

$$\mid\!\sim_N \forall f \in \mathrm{dom}(\hbar).\hbar(f) \circ \pi'(\mathrm{eim}(f)) = \pi''(\mathrm{eim}(f)),$$

where $\mid\!\sim_N \mathrm{Aut}(h)$ stands formally for "$h : \mathfrak{A} \to \mathfrak{A}$ is an automorphism". Intuitively, the definition says that, for every ordered $\mathbf{C}(N)$-structure, there is an effective procedure, represented by the N-map $\hbar$, which extends every finite partial isomorphism of that $\mathbf{C}(N)$-structure to an automorphism.

Corollary 6.2. *The* $\mathbf{C}(N)$*-structure* $\langle \mathbb{Q}, < \rangle$ *is effectively homogeneous.*

By Theorem 5.2, $\langle \mathbb{Q}, < \rangle$ is, up to isomorphism, the unique inhabited, linearly ordered, persistent and denumerable $\mathbf{C}(N)$-structure. We also showed that it is (effectively) homogeneous. A natural question is to determine if $\langle \mathbb{Q}, < \rangle$ is minimal as an effective homogeneous ordered $\mathbf{C}(N)$-structure. Next theorem, our third main result, gives a positive answer.

Theorem 6.3. *For every effective homogeneous* $\mathbf{C}(N)$*-structure of the form* $\langle A, <, a_0, a_1\rangle$*, where* $a_0 < a_1$*, there exists a monomorphism* $f : \langle \mathbb{Q}, <\rangle \to \langle A, <\rangle$.

Proof. Let $g : \mathbb{N} \to \mathbb{Q}$ a bijection. We shall define an injection $f' : \mathbb{N} \to A$ such that the composition $f' \circ g^{-1}$ will be a monomorphism $f' \circ g^{-1} : \langle \mathbb{Q}, <\rangle \hookrightarrow \langle A, <\rangle$. Before that we establish some conventions. When the N-map $\hbar$ is applied to a sequence of length 1, say $0 \mapsto \langle a', a''\rangle$, we write simply $\hbar(a' \mapsto a'')$ for the resulting automorphism; analogously, if the sequence has length 2, we write $\hbar(a' \mapsto a'', \dot{a} \mapsto \ddot{a})$. Now we define the N-maps $\hat{g}, \check{g} : s(\mathbb{N}) \to \mathbb{N}$ respectively by:

$$n \mapsto g^{-1}(\max\{g(m) \ : \ g(m) < g(n) \ \wedge \ m < n\});$$

$$n \mapsto g^{-1}(\min\{g(m) \ : \ g(n) < g(m) \ \wedge \ m < n\}).$$

Consider also the formulas:

$$\alpha^{<}(n) :\Leftrightarrow 1 < n \ \wedge \ \bigwedge_{m<n} g(m) < g(n);$$

$$\alpha^{>}(n) :\Leftrightarrow 1 < n \ \wedge \ \bigwedge_{m<n} g(n) < g(m);$$

$$\alpha(n) :\Leftrightarrow 1 < n \ \wedge \ \bigvee_{m',m''<n} g(m') < g(n) < g(m'').$$

Observe that:

$$\mathrel{|\!\sim}_N \forall n[1 < n \to [\alpha^{<}(n) \ \vee \ \alpha^{>}(n) \vee \alpha(n)]].$$

Thus the N-map f' can be defined by:

$$\begin{aligned} \mathrel{|\!\sim}_N \quad & f'(0) = a_0 \ \wedge \ f'(1) = a_1 \\ & \wedge [\alpha^{<}(n) \to f'(n) = \hbar(a_0 \mapsto f' \circ \hat{g}(n))(f' \circ \hat{g}(n))] \\ & \wedge [\alpha^{>}(n) \to f'(n) = \hbar(a_1 \mapsto f' \circ \check{g}(n))(f' \circ \check{g}(n))] \\ & \wedge [\alpha(n) \to f'(n) = \hbar(f' \circ \hat{g}(n) \mapsto f' \circ \hat{g}(n), \\ & \quad \hbar(f' \circ \hat{g}(n) \mapsto f' \circ \check{g}(n))(f' \circ \check{g}(n)) \mapsto f' \circ \check{g}(n))(f' \circ \check{g}(n))]. \end{aligned}$$

Intuitively:

$f'(0) := a_0$;

$f'(1) := a_1;$

$$f'(n) := \begin{cases} \hbar(a_0 \mapsto a)(a), & \text{where } a = f'(\hat{g}(n)), \quad \text{if } \alpha^{<}(n); \\ \hbar(a_1 \mapsto a)(a), & \text{where } a = f'(\check{g}(n)), \quad \text{if } \alpha^{>}(n); \\ \hbar(a' \mapsto a', \hbar(a' \mapsto a'')(a'') \mapsto a'')(a''), & \\ \quad \text{where } a' = f'(\hat{g}(n)) \text{ and } a'' = f'(\check{g}(n)), & \text{if } \alpha(n). \end{cases} \qquad \square$$

The theorem above says that the ordered $\mathbf{C}(N)$-structure of the rational numbers is a minimal effective homogeneous structure. Therefore, assuming that homogeneity is a fundamental aspect of the concept of real numbers, this result is the formal counterpart of the intuitive idea that the $\mathbf{C}(N)$-structure $\langle \mathbb{Q}, < \rangle$ is a lower bound for the ordered $\mathbf{C}(N)$-structures of the real numbers.

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