A reinterpretation of the cosmological vacuum

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In this paper, we make a proposal for addressing the cosmological constant problem. Our approach will be based on a reinterpretation of two non-standard de Sitter solutions given by the Einstein vacuum equations with $\Lambda > 0$. As a first result, we derive an uncertainty principle for both variants of the de Sitter space (Theorem). Subsequently, a decomposition of the cosmological constant in a pair of time-dependent pieces is introduced (Corollary). The time-dependence of the corresponding energy and dark energy density is discussed and especially matched at the Planck scale. Furthermore, we show that for every instant of cosmic time this approach can be revealed in terms of a Schwarzschild-Anti-de Sitter cosmology with $\Lambda > 0$. The corresponding field equations are provided.

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I. INTRODUCTION

At present, one of the deepest problems in theoretical physics is harmonizing the theory of general relativity (GR), which describes gravitation, and applications to large-scale structures (stars, planets and galaxies), with quantum mechanics, which describes the other three fundamental forces acting on the atomic scale.

General relativity models gravity as curvature of spacetime. On the other hand, quantum field theory (QFT) is typically formulated in the flat spacetime used in special relativity. No theory has yet proven successful in describing the general situation where the dynamics of matter, modelled with quantum mechanics, affect the curvature of spacetime. Even in the simpler case where the curvature of spacetime is fixed a priori, developing QFT becomes more mathematically challenging, and many ideas physicists use in QFT on flat spacetime are no longer applicable [1]. A conceptual difficulty in combining quantum mechanics with GR also arises from the contrasting role of time within these two frameworks. In quantum theories time acts as an independent background through which states evolve, with the Hamiltonian operator acting as the generator of infinitesimal translations of quantum states through time. In contrast, GR treats time as a dynamical variable which interacts directly with matter and moreover requires the Hamiltonian constraint to vanish [2], removing any possibility of employing a notion of time similar to that in quantum theory.

One of the biggest confrontations between both theories is the Cosmological Constant Problem [3]. Quantum field theory predicts a huge vacuum energy density from various sources. On the other hand, GR requires that every form of energy gravitates in the same way. When combining these concepts together, it is widely supposed that the vacuum energy gravitates as a cosmological constant. However, depending on the Planck energy cutoff and other factors, the discrepancy between the observed cosmological constant and the prediction of QFT is as high as 50-120 orders of magnitude.

In the ΛCDM approach, the universe is approximated at late times by two fluids: pressureless matter and a cosmological constant Λ. Both baryonic matter and cold dark matter are unable to push the universe to accelerate [4]. Thus, besides dust-like fluids, one needs to include Λ to account for the observed speedup. However, the magnitude of Λ predicted by quantum fluctuations of flat spacetimes leads to a severe fine-tuning problem with the observed value of Λ. Even considering a curved spacetime one cannot remove the problem [3]. Further, both matter and Λ magnitudes are extremely close today, leading to the well-known coincidence problem [5][6][7]. Under these aspects the ΛCDM model seems to be incomplete, whereas from the observational point of view it adapts well to data.

In this paper, we make a proposal for addressing the cosmological constant problem. Similar to the ΛCDM model, we will consider two fluids in our description. However, in contrast to the ΛCDM approach, these fluids will be based on a reinterpretation of two time-dependent de Sitter solutions of the vacuum equations with $\Lambda > 0$ and no additional source term will be included. The solutions are different insofar that they correspond to spatial sectional curvatures of different sign and will be associated with the energy and dark energy density of the universe. As a first step, in Section II we derive an uncertainty principle for both of these solutions (Theorem). Subsequently, a decomposition of the curvature $K = \Lambda/3$ in a pair of time-dependent pieces is introduced (Corollary); in Section III, the energy and the dark energy density are discussed at different times and the cosmological constant problem is addressed; in Section IV, our approach is applied to obtain a Schwarzschild-Anti-de Sitter cosmology with $\Lambda > 0$ and the corresponding field equations are introduced; a summary is given in Section V.

II. THE UNCERTAINTY PRINCIPLE IN DE SITTER SPACE

As known from the history of Riemannian geometry and general relativity, the property of diffeomorphism invariance is one of the most important features for the generalization of physical laws to curved spaces.

For uncertainty principles given in 3-dimensional space this means that the applied measures of uncertainty should be chosen with caution. When the standard deviation of the momentum is based on the Laplace-Beltrami operator, then one can be sure that invariance under change of coordinates is satisfied. On the other hand, a proper choice for the measure of position uncertainty is hard to obtain if one is only concerned with applying the concept of standard deviation. As recently shown [8], fortunately the choice of a standard deviation in position space is not really necessary or even appropriate. Especially from the concept of projection-valued measures it becomes clear, alternatively, to consider suitable spatial domains for the representation of position uncertainty. Moreover, from the theory of spectral analysis on manifolds, we know that geodesic balls play an important role because these are the distinguished domains in many variational approaches. Since geodesic balls are uniquely classified by their geodesic radius (or diameter) it becomes self-evident that the geodesic radius is the appropriate measure for the representation of position uncertainty in curved spaces. For that reason it becomes clear why the requirement of coordinate invariance is hard to obtain by the known GUP and EUP in literature.

More precisely, in order to measure the momentum one needs to consider a measure of position uncertainty. This is given by a domain D (typically the geodesic ball B_r) with boundary ∂D characterized by its geodesic radius r or diameter d and Dirichlet boundary conditions such that the wave function of the particle is confined in D. The method then reduces to the solution of an eigenvalue problem for the wave function ψ

$$\Delta\psi + \lambda\psi = 0 \tag{1}$$

inside D with the requirement that $\psi = 0$ on the boundary ∂D, while λ denotes the eigenvalue and Δ is the Laplace-Beltrami operator of the corresponding manifold. Then, one can write the following general inequality [8]

$$\sigma_p \geq \hbar\sqrt{\lambda_1}, \tag{2}$$

where λ_1 denotes the first Dirichlet eigenvalue of the problem. For the general class of 3-dimensional Riemannian manifolds of constant curvature k, there is a closed form solution and it was found that [8]

$$\sigma_p\, r \geq \pi\hbar\sqrt{1 - \frac{k}{\pi^2}\, r^2}, \tag{3}$$

where the corresponding position uncertainty of the particle is represented by the radius r of the associated geodesic ball. The underlying metric in spherical and hyperbolic coordinates can be written as

$$ds^2 = dr^2 + \frac{\sin(\sqrt{k}\, r)^2}{k}\, d\Omega^2, \tag{4}$$

with the 2-dimensional measure

$$d\Omega^2 = d\theta^2 + \sin(\theta)^2\, d\varphi^2, \tag{5}$$

for $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. Note that we use the formal identity $\sin(ix) \equiv i\sinh(x)$, such that this representation can be used for positive and negative k. For $k \geq 0$, we have the domain $0 \leq r < \pi/\sqrt{k}$. For $k < 0$, we have $r > 0$. It should also be mentioned that (3) is independent of the coordinate system (diffeomorphism invariance) and not of the same kind as the ordinary EUP or GUP in literature because it features the characteristic length of the confinement corresponding to r. Thus, r should be interpreted rather as uncertainty and does not describe the standard deviation of position [8][9].

Now, let us turn to the Einstein equations. Every n-dimensional space of constant curvature K is also an Einstein space defined by the standard condition

$$R_{ij} = (n-1)K g_{ij} \tag{6}$$

with Ricci tensor R_{ij} and metric g_{ij}, where $i, j = 1, 2, ..., n$. This is also the case for de Sitter spacetimes which are defined as the solution of Einstein's vacuum field equations with cosmological constant Λ,

$$R_{\mu\nu} = \Lambda\, g_{\mu\nu}, \tag{7}$$

for $\mu, \nu = 0, 1, 2, 3$. In four dimensions we have $\Lambda = 3K$. It has been shown in [8] that inequality (3) only holds in spaces of dimension three. That is why k in (3) cannot simply be replaced by K.

Since quantum fluctuations are expected to be relevant only on very small time scales compared to the cosmological circumstances, it is obvious to consider 3-dimensional foliations at a temporal vicinity of a fixed instance of cosmological time. Now, let us proceed under the assumption that the universe is homogeneous and isotropic. Then, there exists a one-parameter family of spacelike hypersurfaces Σ_τ, foliating the spacetime into pieces labelled by the proper time, τ, of a clock carried by any isotropic observer. In these coordinates the spacetime metrics can be written as

$$ds^2 = -c^2 d\tau^2 + a_i^2(\tau) \begin{cases} d\chi^2 + \sin^2\chi\, d\Omega^2, & 0 \leq \chi \leq \pi \\ d\chi^2 + \chi^2\, d\Omega^2, & \chi \geq 0 \\ d\chi^2 + \sinh^2\chi\, d\Omega^2, & \chi \geq 0 \end{cases} \tag{8}$$

where the three possibilities beside the bracket correspond to the three possible spatial geometries in this context [10]. The spatial sections of constant τ are 3-dimensional subspaces of constant curvature. The general form of (8) is called the Robertson-Walker cosmological model. In the case of the vacuum field equations, it is sufficient to consider only one differential equation corresponding to the first component of the Einstein tensor, $G_\mu^{\ \nu} = -3K\delta_\mu^{\ \nu}$, which is explicitly given by

$$\frac{\dot{a}(\tau)^2}{c^2} = K a_i(\tau)^2 - \epsilon, \tag{9}$$

with the scale function $a_i(\tau)$ of (8), and $\epsilon = 0, \pm 1$ corresponding to the three possible spatial geometries in (8). The textbook solutions $a_i(\tau)$ of this equation are as follows [10][11]:

$\underline{K > 0:}$ (De Sitter space)

$$a_1(\tau) = \frac{1}{\sqrt{K}} \cosh(\sqrt{K}\, c\tau) \qquad \epsilon = +1 \tag{10}$$

$$a_2(\tau) = \frac{1}{\sqrt{K}} \sinh(\sqrt{K}\, c\tau) \qquad \epsilon = -1 \tag{11}$$

$$a_3(\tau) = \frac{1}{2\sqrt{K}} e^{\sqrt{K}\, c\tau} \qquad \epsilon = 0 \tag{12}$$

$\underline{K < 0:}$ (Anti-de Sitter space)

$$a_4(\tau) = \frac{1}{\sqrt{|K|}} \cos(\sqrt{|K|}\, c\tau) \qquad \epsilon = -1 \tag{13}$$

$$a_5(\tau) = \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}\, c\tau) \qquad \epsilon = -1 \tag{14}$$

Because of the non-linear nature of the field equation, the pre-factors in $a_i(\tau)$, $i = 1, ..., 5$, cannot arbitrarily be chosen, but are uniquely determined. For this reason, the corresponding curvature is well defined.

The periodic solutions corresponding to the case of $K < 0$ are interrelated by time translation. This might be the reason why in literature, either (13) or (14) is expressed. Here, we consider both of them because the corresponding curvatures will be related to the curvatures corresponding to (10) and (11).

In the following theorem, we apply the inequality (3) to the spatial part of the de Sitter solutions. To do so, we first identify the sectional curvatures corresponding to the spatial part of the metric at a fixed time.

Theorem. The uncertainty principle of position and momentum, corresponding to the metric (8) with conformal factors (10)-(14) at fixed time τ, is given by the inequality

$$\sigma_p\, r \geq \pi\hbar \sqrt{1 - \frac{k_i(\tau)}{\pi^2}\, r^2}, \tag{15}$$

where, for $K \neq 0$, the spatial curvature $k_i(\tau)$, $i = 1, 2$, is given by

$$k_1(\tau) = \frac{K}{\cosh^2(\sqrt{K}\, c\tau)} \equiv \frac{1}{a_1^2} \tag{16}$$

$$k_2(\tau) = \frac{-K}{\sinh^2(\sqrt{K}\, c\tau)} \equiv -\frac{1}{a_2^2}. \tag{17}$$

For $K > 0$, there is a spatially flat case

$$k_3(\tau) = 0. \tag{18}$$

Remark: As will be shown in the proof, the remaining cases k_4 and k_5, corresponding to (13) and (14), are contained in k_1 and k_2 and will be obtained by analytic continuation.

Proof. See appendix.

The time-dependence of k_1 and k_2 for some values K is shown in Fig. 1 and Fig. 2. For $K > 0$, there is an asymptotic behaviour given by

$$|k_i| \sim 4K e^{-2\sqrt{K}\, c\tau}, \tag{19}$$

for $\tau \to \infty$, $i = 1, 2$. That is, in the long run the relation (15) is simplified to the case of $\sigma_p r \geq \pi\hbar$. The turning point of k_1, for $K > 0$, is given by the solution of $\cosh(2\sqrt{K}c\tau) = 2$. This condition can be solved numerically and gives $\tau^* = 0.65/\sqrt{K}c$.

Moreover, the time behaviour of k_1 remains finite for $\tau \to 0$ according to

$$k_1(\tau) = K + \mathcal{O}(\tau^2). \tag{20}$$

On the other hand, the leading term of the asymptotic expansion of k_2, for $\tau \to 0$, is independent of K and given by

$$k_2(\tau) = -\frac{1}{c^2\tau^2} + \mathcal{O}(1). \tag{21}$$

Actually, this independence will be an important point in the resolution of the cosmological constant problem discussed below. Although k_1 and k_2 have a very different time-dependence near the beginning of time, we have the following decomposition of K:

Corollary. For $K \neq 0$, there is the identity

$$K = \frac{k_1 k_2}{k_1 + k_2}. \tag{22}$$

Proof. By substitution of (16) and (17).

This decomposition of K will be the key identity for the description of the Schwarzschild-Anti-de Sitter cosmology in Section IV.

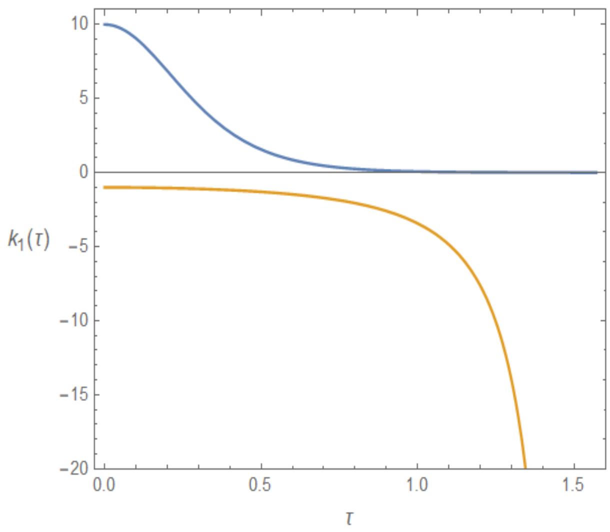

FIG. 1: Spatial curvature k_1, for $K = 10$ (blue) and $K = -1$ (orange) over time, in units of c. For positive K, there is a turning point at $\tau^* \approx 0.2$. For negative K, there is a singularity at $\pi/2$ (see text).

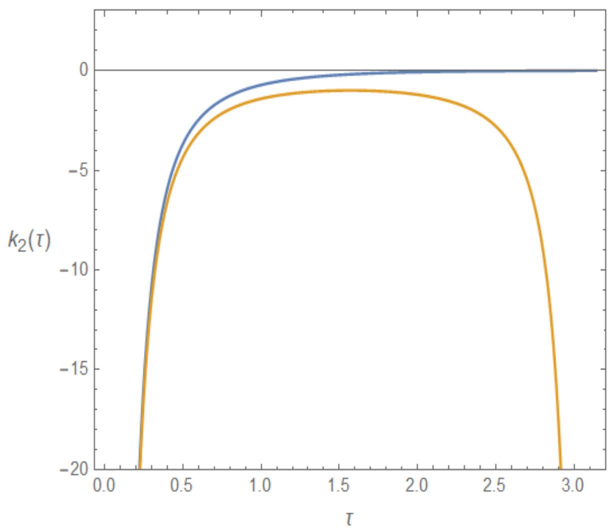

FIG. 2: The spatial curvature k_2, for $K = 1$ (blue) and $K = -1$ (orange) over time, in units of c. For negative K, there is an additional singularity π (see text).

III. DECOMPOSITION OF THE COSMOLOGICAL CONSTANT

In quantum field theories the notion of empty space has been replaced with that of a vacuum state, defined to be the ground state (lowest possible energy density) of a collection of quantum fields. A quantum mechanical feature of quantum fields is that they exhibit zero-point fluctuations everywhere in space, even in regions which are otherwise empty. These zero-point fluctuations give rise to a vacuum energy density u_{QFT}. This vacuum energy density is believed to act as a contribution to the cosmological constant [12].

On the other hand, when $\kappa = 8\pi G/c^4$ is Einstein's gravitational constant, the vacuum energy and the cosmological constant have identical behaviour in general relativity, as long as the vacuum energy density is identified with

$$u_\Lambda = \frac{\Lambda}{\kappa} \approx 5.3 \times 10^{-10} \frac{\text{J}}{\text{m}^3}, \tag{23}$$

where $\Lambda = 1.1056 \times 10^{-52} m^{-2}$ is the empirical estimate of the cosmological constant. Since the cosmological constant corresponds to the curvature K of the 4-dimensional spacetime, an experimental setup to measure it needs to include the dimension of time for its determination. Measurements of Λ in astrophysics are typically performed by (indirectly) comparing huge spatial distances or velocities of objects corresponding to different instances of cosmological time associated with signals coming from very far away.

However, the outstanding problem is that most quantum field theories predict a huge value of u_{QFT} for the quantum vacuum. A (simplified) standard argument for the determination of vacuum energy densities corresponding to vacuum fluctuations is given by summation of zero-point energies according to

$$u_{\text{QFT}} = \frac{1}{(2\pi\hbar)^3} \int_{|\mathbf{p}|\leq\Gamma} d^3p \, \frac{1}{2} \, E_p, \tag{24}$$

with cutoff parameter $\Gamma > 0$, and the energy spectrum $E_p^2 = (\mathbf{p}c)^2 + (mc^2)^2$. In contrast to the measurement of Λ from the astrophysical point of view, in the quantum field theoretic approach the measurement of vacuum energy is restricted to 3-dimensional (spatial) domains corresponding to the vicinity of only one specific instant of world time. No observed data at very different cosmological times are taken into account. Therefore, we ask for a decomposition of Λ into components which are associated with spatial foliations of spacetime and which are corresponding to the measurements of quantum field theory in the vicinity of a given world time. Here, we propose to consider the energy densities corresponding to k_1 and k_2. According to (23), it is obvious to introduce the notation

$$\Lambda_i = 3k_i, \tag{25}$$

for $i = 1, 2$. As a consequence from the corollary, we obtain the following decomposition of the cosmological constant

$$\Lambda = \frac{\Lambda_1 \Lambda_2}{\Lambda_1 + \Lambda_2}. \tag{26}$$

The left-hand side is proportional to the curvature in 4-dimensional spacetime. The right-hand side is composed by components of curvatures in 3-dimensional space at a given instant of world time. Obviously, we can identify the corresponding spatial vacuum energies u_i, $i = 1, 2$, at a given proper time by

$$u_i = \frac{\Lambda_i}{\kappa}. \tag{27}$$

After a few algebraic manipulations, the energy density corresponding to Λ is given in terms of the composition law

$$u_\Lambda = \frac{u_1 u_2}{u_1 + u_2}. \tag{28}$$

For $\Lambda > 0$, it follows that $u_1 > 0$, and $u_2 < 0$, for all $\tau \geq 0$. Thus, it is obvious to consider u_1 to be the positive energy density, and u_2 to be the contribution of the dark energy density. Now, it is interesting to consider the time evolution of the relative fractions with respect to the total amount of energy density by

$$q_1(\tau) = \frac{u_1}{u_1 + |u_2|} = \frac{1}{2}\left[1 - \operatorname{sech}\left(2c\sqrt{K}\tau\right)\right], \tag{29}$$

$$q_2(\tau) = \frac{|u_2|}{u_1 + |u_2|} = \frac{1}{2}\left[1 + \operatorname{sech}\left(2c\sqrt{K}\tau\right)\right], \tag{30}$$

where $\operatorname{sech}(x) = 1/\cosh(x)$ is the hyperbolic secant of x. According to these fractions, there is a high relative density of dark energy at $\tau = 0$, followed by a continuous decrease to approach 50 % in the long run for $\tau \to \infty$. After 7.68 billion years there is a turning point. The present fraction of about 70 % dark energy density is reached at an age of

$$\tau_0 = 13.64 \quad \text{billion years.} \tag{31}$$

This fits well to the best known estimate of 13.77 billion years obtained by the Planck Collaboration in [13]. We expressed the time evolution in Fig. 3.

Since we have obtained a time evolution of vacuum energy densities, let us also verify the situation at the Planck scale. For the moment at one Planck time $\tau_\mathrm{P} = \sqrt{\hbar G/c^5}$ after the initial state, we can restrict our considerations to the dark energy density u_2, because the relative fraction q_1 of positive energy density gives nearly zero at this time (see data below). According to (25) and (27), the absolute value of the dark energy density at the Planck time τ_P after the initial state is given by

$$|u_2(\tau_\mathrm{P})| = 5.3 \times 10^{112}\,\frac{\mathrm{J}}{\mathrm{m}^3}. \tag{32}$$

On the other hand, the textbook expression of the Planck energy density is given by

$$\frac{E_\mathrm{P}}{l_\mathrm{P}^3} = \frac{c^7}{\hbar G^2} = 4.6 \times 10^{113}\,\frac{\mathrm{J}}{\mathrm{m}^3}, \tag{33}$$

which is only about one order of magnitude larger than (32). Nevertheless, we would like to refine this comparison as follows: First, we replace the cubic Planck volume of edge length l_P by the spherical domain of volume

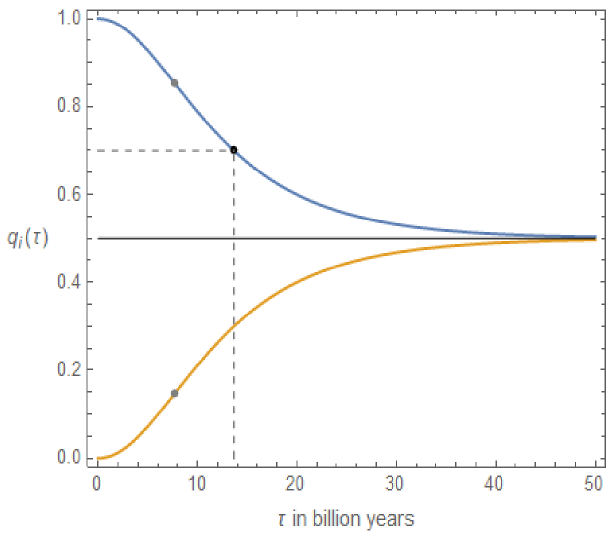

FIG. 3: Relative fraction of energy density (orange) and dark energy density (blue) over the age τ of the universe. The fraction of 70 % dark energy (black dot) is reached at the age of 13.64 billion years. There are also turning points at 7.68 billion years after the initial state (gray dots). In the long run, both energy densities approach an equal fraction (see text).

$V_\mathrm{P} = 4\pi l_\mathrm{P}^3/3$, of Planck radius l_P and define the following adjusted Planck energy density

$$u_\mathrm{P} \equiv \frac{E_\mathrm{P}}{V_\mathrm{P}} = \frac{3}{4\pi}\frac{E_\mathrm{P}}{l_\mathrm{P}^3}, \tag{34}$$

which is slightly different to (33) by the factor $3/4\pi$. Furthermore, we have to complete the Planck energy by regarding the factor 1/2 corresponding to the zero-point modes of the vacuum energy (24). Thus we have to equate

$$|u_2(\tau_\mathrm{P})| = \frac{u_\mathrm{P}}{2}. \tag{35}$$

Since the Planck time is very small, we can properly apply the asymptotic representation

$$u_2 = -\frac{3}{\kappa\, c^2 \tau^2} + \frac{\Lambda}{3\kappa} + \mathcal{O}(\tau^2), \tag{36}$$

which is independent of the cosmological constant in the leading term. The second term is about 1.77×10^{-10} and therefore negligible right now. After substitution of the leading term into (35), we obtain the equivalent expression

$$\kappa = \frac{8\pi G}{c^4}, \tag{37}$$

which is identically satisfied by the definition of Einstein's constant κ. Thus, we see that the vacuum energy density $u_2(\tau)$, at Planck time τ_P, is identical to the Planck energy density. This is a remarkable result, because it can be considered as a calibration of general relativity (κ) to the Planck scale of QFT.

Up to this point there is no indication in our approach that gravity breaks down even for $\tau \to 0$. Thus, one could suppose that something in quantum field theory has to be completed. One known way to do that is the introduction of a cutoff Γ to fix the ultra-high energy density scale to the amount of energy density given by the theory of gravity. Therefore, we consider the zero-point energy expression (24). For zero mass, its exact value is given by the closed form expression

$$u_{\mathrm{QFT}} = \frac{c\,\Gamma^4}{16\pi^2\hbar^3}. \tag{38}$$

The corresponding cutoff Γ can be fixed by equating it at one Planck time after the initial state according to

$$|u_2(\tau_{\mathrm{P}})| = u_{\mathrm{QFT}}, \tag{39}$$

and is given by

$$\Gamma = (6\pi)^{\frac{1}{4}}\, m_{\mathrm{P}} c, \tag{40}$$

with Planck momentum $m_{\mathrm{P}}c = \hbar/l_{\mathrm{P}}$. Although the cutoff is obtained purely from the vacuum field equations of general relativity, it is as far as possible independent of the cosmological constant itself. This seems to be necessary to overcome so many orders of magnitude to reach the Planck scale at all. Actually, such results are what one might require to overcome the cosmological constant problem.

There have been assumptions in literature that also non-zero masses might be necessary to fit the Planck scale. This is not confirmed in our approach because a suitable cutoff can already be reached for zero mass. However, we cannot exclude that there are also mass contributions in the cutoff, but we think that they are of minor effect in the very early Planck stage after the beginning of time.

Let us now consider some observable quantities at later ages of the universe. In Table I, we have summarized some steps in the time evolution of the cosmological vacuum energy densities. At the initial state the positive energy contribution is finite and given by u_Λ. The latter is our main interpretation of the quantity u_Λ. Its dark counterpart is characterized by an infinite density of negative sign.

Cosmological event	τ	u_1/u_Λ	u_2/u_Λ
Initial singularity	0	1	$-\infty$
Planck time	τ_{P}	≈ 1	-1.03×10^{122}
Turning point	7.67	0.83	-4.82
Present age ($q_2 = 0.7$)	13.64	0.57	-1.33
Asymptotic future	$+\infty$	0	-0

TABLE I: Positive and negative energy densities of the cosmological vacuum in units of u_Λ ($\neq u_1 + |u_2|$). The time is measured in billions of years from the initial state. For the numerical value of the Planck energy density see (35).

At one Planck time τ_{P} right after the initial state, the positive energy density is nearly unchanged, but the negative contribution has become finite and is equal to the huge energy density corresponding to the Planck scale (34).

After about 7 billion years later, there is a turning point when the fractions (q_1 and q_2) of both energy densities stop decreasing progressively in time, but begin to relax exponentially slower (Fig. 3). Subsequently, when the fraction of dark energy density has decreased to 70 %, then the present age has been reached. Later on, both energy densities asymptotically tend to zero. On the other hand, there is an exponential growth rate of space which is asymptotically preserved because the negative fraction of energy always dominates its positive counterpart. This phenomenon is one of the main advantages in our approach because the *horizon problem* does not come into the picture.

Let us also compare our results with the mass density given by the Wilkinson Microwave Anisotropy Probe (WMAP). The WMAP determined that the universe is flat, from which it follows that the mean energy density in the universe is equal to the critical density. This is equivalent to a mass density of 9.9×10^{-30} g/cm^3. Of this total density, we know (as of January 2013) the breakdown to be 71.4 % dark energy and 28.6 % of atoms and cold dark matter. For this fraction of dark energy, our estimated age of the universe is about 13 billion years. According to our approach, the corresponding mass densities at this age are given in the first line of Table II.

	ρ_0 [g/cm^3]	ρ_1 [g/cm^3]	ρ_2 [g/cm^3]
From u_1, u_2 of (27)	$12.4\cdot 10^{-30}$	$3.55\cdot 10^{-30}$	$8.85\cdot 10^{-30}$
WMAP (2013)	$9.9\cdot 10^{-30}$	$2.83\cdot 10^{-30}$	$7.07\cdot 10^{-30}$

TABLE II: Total mass density ρ_0 and its fractions ρ_1, ρ_2 corresponding to 26.8 % of atoms/dark matter and 71.4 % of dark energy (WMAP). The densities of the first line are our estimations for the present age and $\rho_i(\tau) = q_i(\tau)\,(u_1 + |u_2|)/c^2$, for $i = 1, 2$ (see text).

Even though our values are 25 % above the estimations of the WMAP, they are still within the bounds of possibility. This is remarkable because the same formula also fits to the Planck scale shortly after the initial state. Since there is only one single parameter (Λ) in our approach, this description of the large scale in space and time encourages us to unify both a_1 and a_2 in a closed form static solution of the field equations.

IV. SCHWARZSCHILD-ANTI-DE SITTER COSMOLOGY

The results of the previous sections can now be applied to consider the corresponding Schwarzschild-Anti-de Sitter (SAdS) solution. But before we proceed, let us briefly describe the state-of-the-art situation of the ordinary Schwarzschild-de Sitter (SdS) approach. This metric describes a static spherically symmetric vacuum solution of the Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ and is given by

$$ds^2 = -f(r)\, d(ct)^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega^2, \tag{41}$$

with

$$f(r) = 1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2 \tag{42}$$

and Schwarzschild radius

$$r_s = \frac{2GM}{c^2}. \tag{43}$$

It describes one part of the maximal extension of the SdS spacetime. Cosmologically, this is a good model of an isolated region in an asymptotic background of constant positive curvature $\Lambda/3$. Negative values of Λ in (42) are also possible, but its order of magnitude would be much too small to solve the cosmological constant problem.

As opposed to this, our approach is leading to a static cosmology composed by a 3-sphere of constant curvature and energy u_1 in an asymptotic background of negative curvature corresponding to the energy u_2, but still recognizing $\Lambda > 0$. This picture fits well into a static SAdS solution of the field equation, where we have to consider both internal and external parts, which are continuously matched together at a suitable hypersurface of both spacetime. More formally, we know that for a fixed time slice τ, the curvatures of both regions are given by k_1 and k_2 of the theorem. For reasons of compatibility we apply the notation Λ_1 and Λ_2 of (25). Then, for the interior static solution, we consider the following field equation

$$G_{\mu\nu} + \Lambda_1\, g_{\mu\nu} = 0, \tag{44}$$

whereas for the exterior metric we have to solve

$$G_{\mu\nu} + \Lambda_2\, g_{\mu\nu} = 0. \tag{45}$$

We denote the associated solutions by $g^{(1)}_{\mu\nu}$ and $g^{(2)}_{\mu\nu}$. They have to match continuously at the boundary between their domains. The Birkhoff theorem states that the vacuum metric of a spherical symmetric distribution of mass (energy) is static and identical to the Schwarzschild metric of the enclosed total gravitational mass, while the vacuum region can either be outside all mass, or interior to some or all mass [14]. Therefore, the interior region is not affected by the homogeneous (dark) background of the exterior region and the solution of (44) can be considered as a special case of the interior Schwarzschild metric, when the domain inside is completely filled by the vacuum fluid u_1. In this case, we obtain

$$ds_1^2 = -g_1(r)\, d(ct)^2 + \frac{dr^2}{g_1(r)} + r^2\, d\Omega^2, \tag{46}$$

with

$$g_1(r) \;=\; 1 - \frac{\Lambda_1}{3}\, r^2. \tag{47}$$

Actually, this solution also solves (44) when there is a time-dependent factor in front of the first term of (46). For the original interior Schwarzschild solution this factor is a constant and given by 1/4. At this point, the common practice is to redefine the time coordinate to absorb the factor, so that $g_{00} \equiv -g_{11}^{-1}$. But such a choice makes g_{00} discontinuous across the boundary and can lead to observational effects such as light deflection or delay. However, we will match the interior and exterior solution without any change of the time coordinate to serve the continuity of time across the boundary.

For the exterior solution of the field equation (45), we obtain the static Schwarzschild-Anti-de Sitter metric

$$ds_2^2 = -g_2(r)\, d(ct)^2 + \frac{dr^2}{g_2(r)} + r^2\, d\Omega^2, \tag{48}$$

where

$$g_2(r) = 1 - \frac{r_0}{r} - \frac{\Lambda_2}{3}\, r^2. \tag{49}$$

It should be mentioned here that this metric is of type SAdS because according to our theorem we always have $\Lambda_2 < 0$. Moreover, this metric solves (45) for every constant value of r_0. This is an important degree of freedom to get a proper matching of both solutions. The trivial case $r_0 = 0$ is not appropriate to continuously fit the interior solution. We require connecting both solutions at the curvature radius a_1 of the interior solution, which is given in (16) of our theorem. Accordingly, we have to consider the equation $g_1(a_1) = g_2(a_1)$. From this condition it uniquely follows

$$r_0 \;=\; a_1 \left(1 - \frac{\Lambda_2}{\Lambda_1}\right). \tag{50}$$

We have chosen a mixed notation to emphasise the meaning of the pre-factor as the curvature radius of the interior domain. This matching implies that the only singularity of the exterior solution is given at $r = a_1$, and we have $g_2(r) \geq 0$, for all $r \geq a_1$. Moreover, there is an absolute minimum of r_0 when $\Lambda_1 \approx 0.44\,\Lambda$, so that r_0 remains strictly positive for every instant of cosmological time. However, the value of r_0 approaches to infinity if Λ_1 is chosen to be one of its extreme values 0 or Λ. The advantage of our solution (49) against the approach in (42) is threefold. On the one hand our solution is of type SAdS although Λ is positive. Furthermore, the value of Λ_2 can increase (negatively) beyond all bounds depending on the cosmological time which is considered. This property opens the possibility to reach even the Planck scale in the vicinity of the initial state. Finally, there is a natural relation between both terms r_0 and Λ_2, such that they are not different kinds of objects but are fundamentally related. This property is missing in the common solution (42). However, black hole's inside the interior region can be expressed by applying Λ_1 in (42), instead of Λ.

Since the interior region is corresponding to a 3-sphere, we can compute the maximum possible (physical) distance of two points in this region. At the present age $t = \tau_0$, we obtain a value of $d_1(\tau_0) = \pi a_1 \approx 6.85 \times 10^{26}$m. For the diameter just at the initial state we obtain the finite value of $d_1(0) = \pi\sqrt{3/\Lambda} \approx 5.17 \times 10^{26}$m, which is not very far apart from the present value. On the other hand, the physical distance $d_{\rm obs}$ of two photons sent in opposite directions is twice the observable radius (or the particle horizon) and given by

$$\begin{aligned} d_{\rm obs}(t) &= 2\, a_1(t) \int_0^t \frac{c\, d\tau}{a_1(\tau)} \\ &= 2\, a_1(t)\, \mathrm{gd}(\sqrt{K} c\, t), \end{aligned} \tag{51}$$

where $\mathrm{gd}(\cdot)$ is the Gudermann function [15] defined by $\mathrm{gd}(x) = \int_0^x \mathrm{sech}\, \xi\, d\xi$. For the present age we obtain a value of $d_{\rm obs} \approx 3.11 \times 10^{26}$m, which might be compared with the diameter of the observable universe in literature of about $8.8 \times 10^{26} m$. For the exterior region no horizon is present.

All in all, at the initial state, the interior solution corresponds to a 3-sphere (closed universe) with finite curvature $\Lambda/3 > 0$ such that there is no curvature singularity in this region. At the same time, the curvature of the background space is divergent ($k_2 \to -\infty$). Later on ($\tau > 0$), the space corresponding to the interior solution remains a closed universe but approaches a flat geometry as time goes by. The exterior region is an open background space of finite (negative) curvature and also approaches a flat geometry at later times. In this limit the expansion rate of both curvature radii is given by an exponential scale factor $\propto e^{\sqrt{\Lambda/3}\, c\tau}$, which is consistent with the necessity of inflation. Therefore, the phenomenon of inflation is intrinsically contained in this approach, such that the *horizon problem* does not appear.

V. SUMMARY AND OUTLOOK

We have proposed a reinterpretation of two non-standard de Sitter solutions of Einstein's vacuum field equations with $\Lambda > 0$. This approach is new insofar that it takes into account that the spatial curvatures k_1 and k_2 of the associated hypersurfaces are non-zero and therefore explicitly dependent on the given time-slice. The vacuum energy densities corresponding to these curvatures have been discussed and properly matched at the Planck scale. For every instant of cosmological time, we also introduced the associated field equation and matched the corresponding interior and exterior solution. This spacetimes provide the possibility to implement quantum field theories at any instant of cosmic time and even near the initial state.

In the present day experiments we measure the shape of the radiation spectrum in the universe while the energy of the radiation is proportional to T^4. The derivation of this law uses the relation between radiation pressure and the internal energy density of a black body. In the cosmology described above, the history of the universe is mainly dominated by the dark energy contribution, which homogeneously surrounds the interior 3-sphere of positive vacuum energy. Photons are their own anti-particles. Therefore, the corresponding temperature distribution of the radiation given by our dark surrounding might be obtained by a generalization of Planck's law to the case of negative energy densities. Then, the relation between energy density and temperature depends on the cosmological time and might be given by

$$\Lambda_1 + \Lambda_2 \approx -\kappa\, \sigma\, T^4, \tag{52}$$

where σ is the Stefan-Boltzmann constant [16]. We already know from the previous sections that our dark energy approach fits well into the large scale and one can expect that the time-dependence of temperature will also be appropriate, especially at the Planck scale or at the early universe. When time goes to infinity the left-hand side approaches zero and we obtain $T \to 0$, which seems to be consistent too. However, at the present cosmological age this law cannot carry up because also other kinds of energy sources become more relevant. A (rough) estimation of today's temperature according to (52) gives 27 K, which is one order of magnitude above 2.73 K. At this point it might be appropriate also to include additional sources (e.g. cosmological dust) to complete the picture. This task is left for a future consideration.

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[1] Wald R. M., *Quantum field theory in curved spacetime and black hole thermodynamics.* University of Chicago Press. (1994).

[2] Novello, M. and Bergliaffa S. E., *Cosmology and Gravitation: Xth Brazilian School of Cosmology and Gravitation*; 25th Anniversary (1977–2002), Mangaratiba, Rio de Janeiro, Brazil. Springer Science & Business Media, (2003-06-11), p. 95.

[3] Weinberg S., *The cosmological constant problem,* Mod. Phys. **61** (1989) 1.

[4] Copeland E. J., Sami M. and Tsujikawa S. *Dynamics of dark energy,* International Journal of Modern Physics **D15** (2006) 1753.

[5] Wetterich C., *The cosmon model for an asymptotically vanishing time-dependent cosmological "constant"* Astronomy & Astrophysics, **301**, (1995) 321.

[6] Amendola L., *Scaling solutions in general nonminimal coupling theories,* Phys. Rev. **D60** (1999) 043501.

[7] Chimento L. P., Jakubi A. S., Pavón, D. and Zimdahl W., *Interacting quintessence solution to the coincidence problem,* Phys. Rev. **D67** (2003) 083513.

[8] Schürmann T., *Uncertainty principle on 3-dimensional manifolds of constant curvature,* Found. Phys. **48** (2018) 716.

[9] Schürmann T. and Hoffmann I., *A closer look at the uncertainty relation of position and momentum,* Found. Phys. **39** (2009) 958.

[10] Wald R. M., *General Relativity.* University of Chicago Press. (1984).

[11] De Sitter W., *On the curvature of space,* KNAW, Proceedings, **20 I** (1918), Amsterdam, 1918, pp. 229.

[12] Rugh S. E. and Zinkernagel, H. *The vacuum and the cosmological constant problem,* Studies in History and Philosophy of Modern Physics, **33** (2002), 663.

[13] Planck Collaboration (2016). *Planck 2015 results. XIII. Cosmological parameters,* Astronomy and Astrophysics. **594** A13, Table 4, p. 32.

[14] Rindler W., *Thermodynamics of Black Holes in Anti-de Sitter Space,* Physics Letters **A 245** (1998) 363.

[15] [DLMF] *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/ [DLMF, Eq. 4.23.39].

[16] Stefan-Boltzmann constant: $\sigma = \pi^2 k_B^4/15\hbar^3 c^3$.

APPENDIX

Proof of the Theorem. For $K > 0$, we express the spatial part of the first solution corresponding to (10) by

$$ds_\tau^2 = \cosh^2(\sqrt{K} c\,\tau)\Big[d\tilde{r}^2 + \frac{\sin^2(\sqrt{K}\,\tilde{r})}{K}\,d\Omega^2\Big], \quad (53)$$

where we have introduced the notation ds_τ to indicate the restriction to the 3-dimensional space considered at fixed τ. Then, we apply the transformation $r = \tilde{r}\cosh(\sqrt{K} c\,\tau)$, to get

$$ds_\tau^2 = dr^2 + \frac{\sin^2(\sqrt{k_1}\,r)}{k_1}\,d\Omega^2, \quad (54)$$

with k_1 defined by (16) and the domain

$$0 \le r \le \frac{\pi}{\sqrt{k_1}}. \quad (55)$$

After applying (3) and (4), we obtain the first statement (16) of the theorem. Next, we consider the (spatially) hyperbolic representation corresponding to (11), by

$$ds_\tau^2 = \sinh^2(\sqrt{K} c\tau)\Big[d\tilde{r}^2 + \frac{\sinh^2(\sqrt{K}\,\tilde{r})}{K}\,d\Omega^2\Big], \quad (56)$$

and apply the transformation, $r = \tilde{r}\sinh(\sqrt{K} c\,\tau)$. Then, we obtain the representation

$$ds_\tau^2 = dr^2 + \frac{\sinh^2(\sqrt{|\tilde{k}_2|}\,r)}{|\tilde{k}_2|}\,d\Omega^2, \quad (57)$$

where the curvature $\tilde{k}_2$ is identified by

$$\tilde{k}_2(\tau) \;=\; \frac{-K}{\sinh^2(\sqrt{K}\,c\tau)} \equiv k_2. \quad (58)$$

The remaining two cases (13) and (14) correspond to $K < 0$. Both of them have only hyperbolic representations in the spatial domain. For the first of them, we write

$$ds_\tau^2 = \cos^2(\sqrt{|K|} c\tau)\Big[d\tilde{r}^2 + \frac{\sinh^2(\sqrt{|K|}\,\tilde{r})}{|K|}\,d\Omega^2\Big]. \quad (59)$$

This can be transformed by $r = \tilde{r}\cos(\sqrt{|K|} c\,\tau)$, and we get the representation

$$ds_\tau^2 = dr^2 + \frac{\sinh^2(\sqrt{|\tilde{k}_1|}\,r)}{|\tilde{k}_1|}\,d\Omega^2, \quad (60)$$

with curvature

$$\tilde{k}_1(\tau) \;=\; \frac{-|K|}{\cos^2(\sqrt{|K|}\,c\tau)} \quad (61)$$

$$\;=\; \frac{K}{\cosh^2(\sqrt{K}\,c\tau)} \equiv k_1. \quad (62)$$

The analytical continuation of the last line is performed according to the identity $\cos(ix) = \cosh(x)$.

The metric of the remaining hyperbolic case (14) is

$$ds_\tau^2 = \sin^2(\sqrt{|K|} c\tau)\Big[d\tilde{r}^2 + \frac{\sinh^2(\sqrt{|K|}\,\tilde{r})}{|K|}\,d\Omega^2\Big], \quad (63)$$

which is expressed by

$$ds_\tau^2 = dr^2 + \frac{\sinh^2(\sqrt{|\tilde{k}_2|}\,r)}{|\tilde{k}_2|}\,d\Omega^2, \quad (64)$$

with curvature

$$\tilde{k}_2(\tau) = \frac{-|K|}{\sin^2(\sqrt{|K|}\, c\tau)} \quad (65)$$

$$= \frac{-K}{\sinh^2(\sqrt{K}\, c\tau)} \equiv k_2. \quad (66)$$

Here, we have applied the identity $\sin(ix) = \sinh(x)$.

Finally, we consider the spatially flat solution corresponding to (12), which is easily expressed by

$$ds_\tau^2 = \frac{e^{2\sqrt{K}\, c\tau}}{4K} \Big[d\tilde{r}^2 + \tilde{r}^2 \, d\Omega^2 \Big]. \quad (67)$$

The spatial flatness of the corresponding subspace is equivalent to statement (18). □