South American Journal of Logic
Vol. 4, n. 2, pp. 1–21, 2018
ISSN: 2446-6719

End of the square?

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Abstract

It has been recently argued that the well-known square of opposition is a gathering that can be reduced to a one-dimensional figure, an ordered line segment of positive and negative integers [3]. However, one-dimensionality leads to some difficulties once the structure of opposed terms extends to more complex sets. An alternative algebraic semantics is proposed to solve the problem of dimensionality in a systematic way, namely: partition (or bitstring) semantics. Finally, an alternative geometry yields a new and unique pattern of oppositions that proceeds with colored diagrams and an increasing set of bitstrings.

Keywords: Bitstring, implication, opposite-forming operators, opposition, structural semantics.

Introduction: Oppositions

A sample of preliminary questions should be asked about oppositions, before tackling our central issue of the square.
First, what are logical 'oppositions' for?
Whilst appearing as an old-fashioned concept from the Aristotelian tradition of term logic, opposition still remains a basic concept of logic and philosophy: it seems difficult to deprive from it for semantic, ontological and metaphysical issues; it has even been reintroduced as a core concept of paraconsistency, due to the central role of the Principle of Contradiction and the question of its validity. Notably, Slater made an insightful (albeit not conclusive) objection to Graham Priest's dialetheism by claiming that there cannot be 'true contradictions' semantically speaking [12]. In addition, Priest acknowledged himself that any pair of true sentences, albeit being called 'contradictories' in his dialethist theory, should be called 'subcontraries' properly speaking [9]. Thus the Aristotelian square of opposition played an important role in the area of

metalogic, as witnessed by both Priest's own theory of negation[1] and Slater's reply. It can also be used to deal with many-valued semantics and cases of philosophical logics like epistemic logic, thereby contributing to the debate about how two arbitrary agents may disagree with each other [11].
Second, how do they matter with respect to the relation of consequence?
Opposition is a metalogical concept just like consequence. Now the former is not to be viewed as a mere complementary of the latter, and one aim of the present paper is to show how any logical relations can be formed in terms of opposition whilst including entailment (or valid implication) as a particular case. However, we insist that truth-values appear as a channel concept connecting both metalogical relations of opposition and consequence: the latter aims at preserving truth-values from premises to consequence, whilst the former aims at separating them in some way to be explained further on.
Third, can there be a semantics for oppositions?
A central concern for logical oppositions is their failure of *truth-functionality*: apart from contradictoriness, it is not possible to determine the value of one formula that stands into opposition with a first one. This is because more than one term may be related to any other formula by contrariety, subcontrariety, of subalternation. In the face of such a non-deterministic situation, the present paper wants to endorse an alternative semantics where opposite-forming operators proceed as interpretation functions ranging over a finite domain.

Opposition is normally depicted as a relation of *incompatibility* between truth-values: for any formula φ, this cannot be true at the same time as its opposite ψ. This means that the well-known Aristotelian square includes some relations that are not of the same sort as opposition, i.e. subcontrariety and subalternation. From this observation, a new way of depicting logical relations has been proposed in several works by Demey & Smessaert. In [4], the two authors assume two main kinds of logical relation among the four Aristotelian ones, namely: *oppositional* relations, of the logical form $\neg(\varphi \wedge \psi)$ and $\neg(\neg\varphi \wedge \neg\psi)$; *implicational* relations, of the logical form $\varphi \to \psi$ and $\psi \to \varphi$. Note that the above expressions assume a deflationary translation of φ and $\neg\varphi$ as meaning φ's being true and φ's being false, respectively. Then for any logical system $\mathcal{S}$, the formulas φ, ψ stand into the following sorts of logical relations. First, oppositional relations:

Contrariety	$\models_{\mathcal{S}} \neg(\varphi \wedge \psi)$, $\not\models_{\mathcal{S}} \neg(\neg\varphi \wedge \neg\psi)$
Contradiction	$\models_{\mathcal{S}} \neg(\varphi \wedge \psi)$, $\models_{\mathcal{S}} \neg(\neg\varphi \wedge \neg\psi)$
Subcontrariety	$\not\models_{\mathcal{S}} \neg(\varphi \wedge \psi)$, $\models_{\mathcal{S}} \neg(\neg\varphi \wedge \neg\psi)$
Non-contradiction	$\not\models_{\mathcal{S}} \neg(\varphi \wedge \psi)$, $\not\models_{\mathcal{S}} \neg(\neg\varphi \wedge \neg\psi)$

[1] See [9], especially Chapter 4.

It clearly appears that the Aristotelian relations of subalternation do not occur any more among the above oppositional relations. This is because, according to Demey & Smessaert, the latter proceeds as a hybrid relation combining both properties of opposition and implication. At the same time, subcontrariety still appears in the class of oppositional relations by virtue of its common *logical form* with the other formulas. Thus, subcontrariety is a pair of (valid or invalid) negated conjunctions; at the same time, subalternation and superalternation will occur in the next class of (valid or invalid) entailment relations under the heading of 'left-implication' and 'right implication':

Bi-implication	$\models_{\mathcal{S}} \varphi \rightarrow \psi$, $\models_{\mathcal{S}} \psi \rightarrow \varphi$
Left implication	$\models_{\mathcal{S}} \varphi \rightarrow \psi$, $\not\models_{\mathcal{S}} \psi \rightarrow \varphi$
Right implication	$\not\models_{\mathcal{S}} \varphi \rightarrow \psi$, $\models_{\mathcal{S}} \psi \rightarrow \varphi$
Non-implication	$\not\models_{\mathcal{S}} \varphi \rightarrow \psi$, $\not\models_{\mathcal{S}} \psi \rightarrow \varphi$

Given that the latter sorts of formula have in common the crucial use of the logical constant of conditional, oppositional and implicational relations are distinguished from each other by the occurrence of one characteristic logical constant in them, rather than by a metalogical criterion of (in)compatibility.

In the following, however, logical relations are defined without the logical constants $\neg$, $\wedge$, $\vee$, and $\rightarrow$. This is a way to recall that such relations stand at the *metalogical* level of discourse and are used to define these logical relations instead of being defined by themselves. We are going to see how a sample of binary logical connectives may be characterized by the Aristotelian relations of the square, whether these be expressions of incompatibility or not. Above all, the present paper wants to insist that, unlike the above distinction between two independent kinds of logical relations, there is only one basic kind of logical relation: opposition, in the sense that even the compatible and implicational relations may be rephrased in terms of, and thus reduced to, the two basic oppositional relations of contrariety and contradiction. These will be used as irreducible and sufficient notions for constructing all the other ones depicted in terms of oppositional and implicational relations.

For this purpose, we will need a special semantic framework in order to analyze the basic properties of opposition. This will be introduced in Section 3.

What is logical opposition, beyond the restricted area of formulas ? For any arbitrary items x, y, a logical opposition is a relation $Op(x, y)$ that reads 'x and y are opposed to each other' and is such that $Op(x, y) = Op(x, op(x))$, where $op(x)$ reads as 'opposite to x'. More especially, a peculiar feature of the next opposite-forming operators is that most of them are not proper functions since the antecedent $op(x)$ may have less than or more than one image y.[2]

[2]Actually, opposition-forming operators mostly proceed as a unary function like 'is the

Generally speaking, any logical relation will be explained throughout the paper as a basic relation of opposition,

$$Op(\varphi, \psi) = Op(\varphi, op(\varphi))$$

where *op* is an opposite-forming function that yields the formula ψ whenever applied to φ. Now Demey & Smessaert [4] rightly noted that the Aristotelian square includes two kinds of compatible relations, namely: subcontrariety, and subalternation. If so, why should the Aristotelian square be called a 'square of oppositions' given the normal sense of incompatibility associated to the concept of opposition? Our answer comes from the possibility of reducing any kind of logical relations in terms of two basic cases of unanimous opposition. For example, subalternation can be explained as a composed or iterated function of other oppositional functions. The unifying key to do so is dealing with logical relations as mappings on truth-values. Now these mappings may be total or partial. When *total*, mapping applies to both values of truth and falsity; when partial, it applies to only one of these truth-values. Therefore, an investigation into the logical relations of oppositions requires a preliminary investigation into the relation Op and the operator op: What are the properties of Op? What are the properties of op?

The paper will be organized as follows, around the common notion of square and its various issues. In the first section, the main properties of an Aristotelian 'square' and its geometrical extensions will be recalled and exemplified through several families of sentential or conceptual logical relations. In the second section, we will consider a recent proposal to reduce every logical relation throughout a common pattern of segment. In the third section, we will introduce a special semantics provided to account for the meaning of oppositional relations through opposite-forming operators: a *bitstring* semantics, where the basicality of opposition stems from a common analysis of logical space in terms of *partition*. By doing so, we will complete the preceding proposal by generalizing the Aristotelian square within one common gathering.

1 Oppositions with a square

A square of oppositions is a logical structure including a number of logical relations between its elements. Such a structure has been scrutinized in various works from the middle of the 20th century, and it is still the central issue of some contemporary research programs.

Our coming point is about whether such a structure is a relevant object, in the sense of being a convenient trade-off between explanatory power and theo-

father of': y may be the father of none, one, or more than one antecedent x.

retical simplicity. For some ones [1,2,4,6], the Aristotelian square is too a simple structure to be really explanatory; its extension has been famously illustrated as a logical hexagon [2], but it also turns out that this extended structure does not include further kinds of logical opposition than those of the square: contrariety, contradiction, subcontrariety, and subalternation. Moretti [6] showed that the hexagon is nothing but a first extension of the square towards an indefinite series of increasingly complex or many-dimensional structures. For some other one [3], the logical square is too a complex structure to be really simple; according to the latter, all the four logical relations of the square can be related to each other into a simpler gathering which will be illustrated in the following section. We want to defend a middle position between a simple structure and a richer account of logical relations. For this purpose, we introduce later a square that does not increase the number of its vertices whilst augmenting its logical relations increasingly. But before that, let us consider the first option of a weakest structure of logical relations.

The history of logical oppositions goes on a par with the enrichment of its gatherings. The pioneer application was Aristotle's square of categorical statements. In this primary square, the four vertices correspond to four kinds of quantified propositions: affirmative universals, expressed by the typical sentence 'Every S is P' and symbolized by the Roman letter **A**; negative universals, expressed as 'No S is P' and symbolized as **E**; affirmative particulars, expressed as 'Some S is P' and symbolized as **I**; and negative particulars, expressed by 'Some S is not P' and symbolized as **O**. Then further additions occurred throughout the history of logic and philosophy, due to a richer structure of the related formulas. A medieval case in point was Buridan, who introduced in his *Summulae de dialectica* an octagon of quantified modal oppositions such that 'Every S is necessarily P' or 'Necessarily, every S is P'. It clearly appears that the addition of one more modal component into the quantified formulas thereby multiplies the number of related formulas. Both the structure and the content of the formulas considered in their logical relations may vary and go beyond the historical case of quantified statements. But a common precondition for all of these is to share a common logical structure, that is, a common set of components: any two formulas can be defined structurally if and only if their logical relations essentially depend upon their common structure, so that the logical relation cannot be established otherwise. Returning to the above case of Aristotle's square, **A** and **E** are logically related to each other by their structure –and not by axioms, in that their logical forms essentially rely upon a quality and a quantity. The general structure of the logical square of oppositions can be depicted in the following figures (see Figures 1, 2), where the kinds of logical opposition are depicted by functional expressions *ct* (for contrariety), *cd* (for contradictoriness), *sct* (for subcontrariety), *sb* (for subalternation), and *sp* (for superalternation).

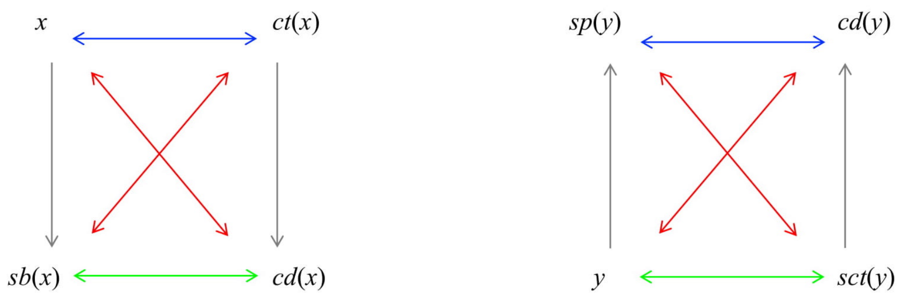


Figure 1: The logical square from two centered perspectives: x and y.

Here is a set of four illustrations of the Aristotelian 'oppositions', recalling that these include some compatible relations that Demey & Smessaert do not consider as cases of proper 'opposition'.

A first example is the set of categorical statements. Letting x for $\mathbf{A} = \forall x(Fx \rightarrow Gx)$, then $ct(x) = \mathbf{E} = \forall x(Fx \rightarrow \neg Gx)$, $cd(x) = \mathbf{O} = \neg\forall x(Fx \rightarrow Gx)$, and $sb(x) = \mathbf{I} = \neg\forall x(Fx \rightarrow \neg Gx)$. Letting y for $\mathbf{I}$, then $sct(y) = \mathbf{O} = \neg\forall x(Fx \rightarrow Gx)$, $cd(y) = \mathbf{E} = \forall x(Fx \rightarrow \neg Gx)$, and $sp(y) = \mathbf{A} = \forall x(Fx \rightarrow Gx)$.

A second example is the set of modal sentences [1].[3] Letting x for $\Box p$, then $ct(x) = \Box\neg p$, $cd(x) = \neg\Box p$, and $sb(x) = \neg\Box\neg p$. Letting y for $\neg\Box\neg p$, then $sct(y) = \neg\Box p$, $cd(y) = \Box\neg p$, and $sp(y) = \Box p$.

A third example is the set of binary sentences [7]. Letting x for $p \wedge q$, then $ct(x) = \neg p \wedge \neg q$, $cd(x) = \neg(p \wedge q)$, and $sb(x) = \neg(\neg p \wedge \neg q)$. Letting y for $\neg(\neg p \wedge \neg q)$, then $sct(y) = \neg(p \wedge q)$, $cd(y) = \neg p \wedge \neg q$, and $sp(y) = p \wedge q$.

A fourth example is the set of propositions in term logic [5]. Letting x for 'S is P', then $ct(x)$ = 'S is not-P', $cd(x)$ = 'S is not P', and $sb(x)$ = 'S is not not-P'. Letting y for 'S is not not-P', then $sct(y)$ = 'S is not P', $cd(y)$ = 'S is not-P',

[3] Modalities are taken here in the abstract or neutral sense of strong operators of necessity $\Box$ and weak operators of possibility $\Diamond$, such that both are structurally related to each other by the equivalence $\Box p \leftrightarrow \neg\Diamond\neg p$. This means that the above logical relations between modal sentences hold for every usual class of modal (alethic, temporal, epistemic, deontic, ...) structures in modal logic: K, T, D, S4, S5, and the like.

and $sp(y)$ = 'S is P'.

A first extension of the square has been introduced by some philosophers including Blanché [2].[4] This extension augments the initial square with two additional vertices. In the context of categorical statements, these are $\mathbf{U} = \mathbf{A} \vee \mathbf{E}$, and $\mathbf{Y} = \mathbf{I} \wedge \mathbf{O}$.

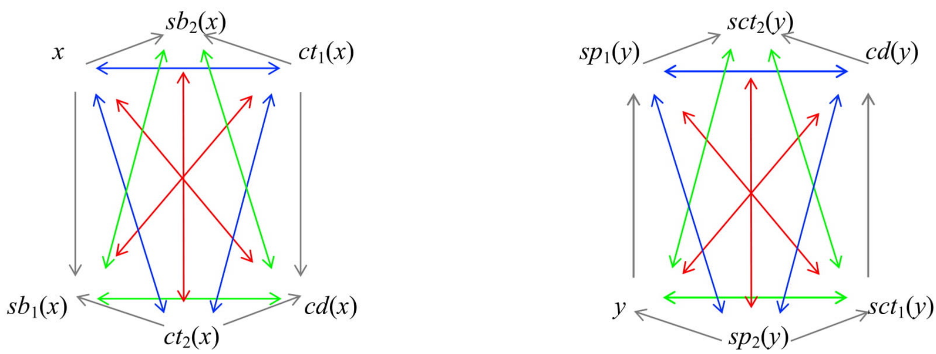


Figure 2: The logical hexagon from two centered perspectives: x and y.

It strikingly appears here above that the variously interpreted relations crucially relies on an additional parameter, viz. *negation*. A structural semantics will be endorsed in the third section, in order to show that the meaning of formulas does depend on their structure without the latter being of a syntactic order.

2 Oppositions without a square

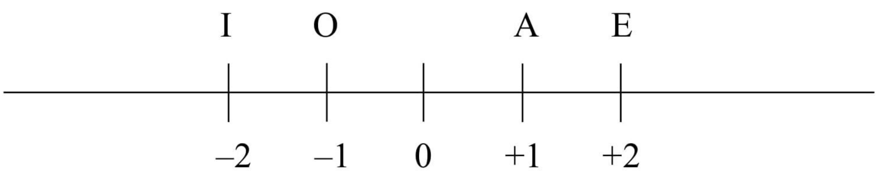
Costa-Leite [3:2] recently claimed that the square of oppositions is an old dogma that must be relativized:

[4]Another kind of hexagonal extension came from the Polish logician Czeżowski, in which the new vertices referred to the additional category of singulars ('This S is P', 'This S is not P') into the set of categorical statements.

Consider a question: is there a way to represent oppositions without two-dimensional objects such as squares or objects of higher dimensions? The answer is yes.

According to the author, any of the Aristotelian relations can be represented in a mere line segment and thereby simplify the geometry of oppositions. For this purpose, an alternative model is proposed by translating logical relations into arithmetical operations.
Let $\mathbb{Z}$ be a set of integers, $\mathbb{Z}_+$ a set of positive integers, $\mathbb{Z}_-$ a set of negative integers, $\mathbb{Z}^*$ a set of non-nil integers, and $\mathbb{Z}' = \{-r, -q, q, r\}$. Let $\mathcal{C}$ be a set of a categorical statements $\{\mathbf{A},\mathbf{E},\mathbf{I},\mathbf{O}\}$ and i a function on $\mathcal{C}$ such that $i : \mathcal{C} \mapsto \mathbb{Z}'$. The set of Aristotelian oppositions between categoricals can be redefined as the following set of operations between integers: $j \in \mathbb{Z}^*_+$ iff $j \in \{\mathbf{A}, \mathbf{E}\}$ (universal sentences), and $j \in \mathbb{Z}^*_-$ iff $j \in \{\mathbf{I}, \mathbf{O}\}$ (particular sentences). Then for every $\alpha, \beta \in \mathcal{C}$:

α and β are contraries iff $i(\alpha)$, $i(b) \in \mathbb{Z}^*_+$;
α and β are contradictories iff $i(\alpha) + i(\beta) = 0$;
α and β are subcontraries iff $i(\alpha)$, $i(\beta) \in \mathbb{Z}^*_-$;
β is the subaltern of α iff $i(\alpha) \neq i(\beta)$ and $i(\beta) \in \mathbb{Z}^*_-$.



$+2 = ct(+1)$; $-1 = sct(-2)$; $-1 = cd(+1)$; $+2 = cd(-2)$; $-2 = sb(+1)$; $+1 = sp(-2)$

Figure 3: A segment line for the logical square.

Albeit correct for the square, Costa-Leite notes that the above definitions happen to fail with the hexagon of oppositions that augment the Aristotle's historical square of categorical statements with two further formulas: $\mathbf{U}$, whose arithmetical value Costa-Leite defines as $i(\mathbf{U}) = i(\mathbf{A}) + i(\mathbf{E})$; and $\mathbf{Y}$, to be defined by the same author as $i(\mathbf{Y}) = i(\mathbf{I}) + i(\mathbf{O})$. Let $\mathbb{Z}'' = \{-s, -r, -q, q, r, s\} \in \mathbb{Z}$

and $\mathcal{C} = \{\mathbf{A}, \mathbf{U}, \mathbf{E}, \mathbf{O}, \mathbf{Y}, \mathbf{I}\}$, such that $\mathbf{U} = \mathbf{A} \vee \mathbf{E}$ and $\mathbf{Y} = \mathbf{I} \wedge \mathbf{O}$. Now a sample of arbitrary integers can yield counter-examples to Costa-Leite's pattern.

Proof: Let $i(\mathbf{A}) = +1$, $i(\mathbf{U}) = +3$, $i(\mathbf{E}) = +2$, $i(\mathbf{O}) = -1$, $i(\mathbf{Y}) = r - 3$, $i(\mathbf{I}) = -2$.
$\mathbf{Y} = ct(\mathbf{A})$. Now $i(\mathbf{Y}) + i(\mathbf{A}) = -3 + 1 = -2$, therefore $i(\mathbf{Y}) + i(\mathbf{A}) \notin \mathbb{Z}^*_+$.
$\mathbf{U} = sct(\mathbf{I})$. Now $i(\mathbf{U}) + i(\mathbf{I}) = +3 - 2 = +1$, therefore $i(\mathbf{U}) + i(\mathbf{I}) \notin \mathbb{Z}^*_-$.
$\mathbf{U} = sb(\mathbf{A})$. Now $i(\mathbf{Y}) + i(\mathbf{A}) = +3$, therefore $i(\mathbf{Y}) + i(\mathbf{A}) \notin \mathbb{Z}^*_+$.

Costa-Leite was aware of this counter-result and noticed it from the beginning of his paper, actually. The troubles caused by the logical hexagon thus led the author to bring new arithmetic operations to preserve the line segment pattern:

For every $\varphi, \psi, \gamma \in \mathcal{C}$:
α, β, γ are *contraries* iff $i(\alpha) + i(\beta) + i(\gamma) = 0$ and $i(\gamma) \in \mathbb{Z}^*_-$;
α and β are *contradictories* iff $i(\alpha) + i(\beta) = 0$;
α, β, γ are *subcontraries* iff $i(\alpha) + i(\beta) + i(\gamma) = 0$ and $i(\gamma) \in \mathbb{Z}^*_+$;
β is the *subaltern* of α iff $i(\alpha) \neq i(\beta)$ and $i(\beta) \in \mathbb{Z}^*_-$
or $i(\alpha) \neq i(\beta)$ and either (a) $i(\beta) > i(\alpha)$ and $i(\beta)$, $i(\alpha) \in \mathbb{Z}^*_+$
or (b) $i(\beta) > i(\alpha)$ and $i(\beta), i(\alpha) \in \mathbb{Z}^*_-$.

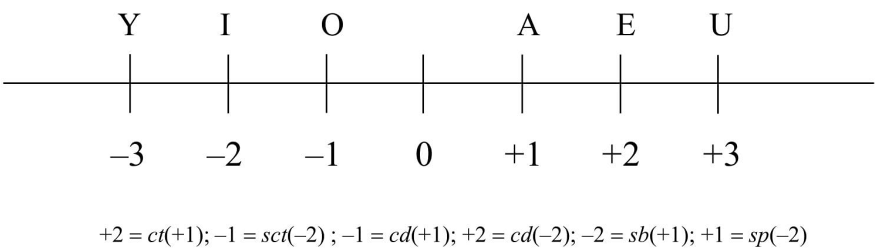


Figure 4: A segment line for the logical hexagon.

And yet, Costa-Leite [3:9] rightly acknowledges that these definitions might fail again with higher structures of oppositions:

> There are, notwithstanding, some problems which remain open: the question to determine whether the same procedure can also be applied to solids and higher dimensions, as well as to more than four oppositions, are very complicated and still have to investigated in detail.

In other words, the above new definitions seem to be *ad hoc* if they hold for $\mathbb{Z}''$ only. What of $\mathbb{Z}'^{\cdots\prime}$ for any corresponding set $\mathcal{C}'^{\cdots\prime}$, given that there is a maximal number of 2^n elements for these? We are in need of a semantics able to determine the meaning of expressions of arbitrary complexity. Actually, Costa-Leite's theory can be confirmed and simplified within an alternative semantics encompassing any number of related formulas. More especially, we want to show in the following that his aforementioned strengthened clauses for contrariety and subcontrariety work only when *all* the contraries and subcontraries are taken into account in the calculus. This is the case with, e.g., the triad **A**, **E**, **Y** in a Blanché-like hexagon; more generally, it turns out that every set $\mathbf{C}'^{\cdots\prime}$ requires an adaptation of these clauses by augmenting the number of contrary and subcontrary formulas. The same kind of generalization holds with his clause for subalternation: that any two formulas are not contradictories and have opposed sign does entail such a relation between them only within structurally limited sets of formulas but cannot be warranted further on.
In order to corroborate Costa-Leite's theory by means of a generalized pattern, let us introduce now our alternative formal semantics based on two basic and unanimous kinds of opposition.

3 Oppositions with another square

The following structural semantics helps to show why any extension of the initial logical square increases the instances of opposition without never increasing the number of oppositions themselves. As seen previously in Section 1, Blanché's hexagon of oppositions augments the initial square with two vertices by adding two cases of contrariety, two cases of subcontrariety, two cases of contradiction, and four cases of subalternation/superalternation. But no further kind of opposition occurs for all. Actually, the 'basic' square can be reduced to the simpler gathering of a line, by reducing the structure of formulas. All of this overtly follows from a special kind of semantic structure based on one essential parameter: a *bitstring*, which is an ordered set of *bits*.

With respect to the initial square of oppositions, the coming new square borrows from graph theory by dealing with operations between vertices [10].

In other words, any line between two vertices x,y is not considered from the perspective of a relation $Op(x, y)$ but, rather, from the perspective of a function like $op(x) = y$. The following figure compares an arbitrary graph with our alternative functional way of interpreting the Aristotelian square.

$$
\begin{aligned}
y &= f(x)\\
z &= g(y) = g(f(x))\\
x &= h(z) = h(g(f(x)))
\end{aligned}
$$

$$
\begin{tikzcd}
& x \arrow[d, "f"] \\
z \arrow[ur, "h"] & y \arrow[l, "g"]
\end{tikzcd}
$$

$$
\begin{aligned}
y &= sb(x))\\
z &= sct(y) = sct(sb(x))\\
x &= cd(z) = cd(sct(sb(x)))
\end{aligned}
$$

$$
\begin{tikzcd}
& x \arrow[d, "sb"] \\
z \arrow[ur, "cd"] & y \arrow[l, "sct"]
\end{tikzcd}
$$

Figure 5: The logical square as an oriented graph.

3.1 Bistring semantics

The following semantics is a special application of a broader semantic framework: *Question-Answer Semantics*, where the meaning of any meaningful item (individual, concept, or sentence) results from an ordered set of exhaustive predicates to characterize it.[5] More generally, these predicates may be ac-

[5]How the 'logical space' of a set of items is exhausted requires a general technique of *partition* [9,10,11]. This process will be considered in a later work about a proper 'partition

cepted or not as a mapping from the item x onto {1,0}. Let us consider again the preceding four illustrations of logical relations. The required number of questions to characterize a kind of formula determines its structural complexity, insofar as n predicates result in 2^n possible ordered answers and corresponding items. In the case of Aristotle's categorical statements, $n = 3$ predicates may characterize any item x belonging to this set of formulas: $\beta_1(x)$ is about whether P is true of every S, $\beta_2(x)$ about whether P is neither true of every S nor true of no S, and $\beta_3(x)$ about whether P is true of no S. Concerning (mono)modal sentences including only one operator, there are also $n = 3$ questions: $\beta_1(x)$ is about whether x is necessarily true; $\beta_2(x)$ is about whether x is contingently true; and $\beta_3(x)$ is about whether x is impossibly true. In the case of binary sentences, $n = 4$ predicates are required: $\beta_1(x)$ is about both p and q are true; $\beta_2(x)$ is about whether p is true whereas q is not; $\beta_3(x)$ is about whether p is not true whereas q is; and $\beta_4(x)$ is about whether neither p nor q are true. And $n = 3$ questions are required to characterize propositions of term logic: $\beta_1(x)$ is about whether S is P absolutely; $\beta_2(x)$ is about whether S is neither P absolutely nor not-P absolutely; and $\beta_3(x)$ is about whether S is not-P absolutely.

Let us consider the model of such a structural semantics. Every kind of formula φ is a partition of a logical space Σ, such that φ can be rephrased as a Disjunctive Normal Form $\beta(\varphi) = \beta_1(\varphi) \vee \ldots \vee \beta_n(\varphi)$ – where each disjunct $\beta_i(\varphi)$ is a *bit* mapping onto $\{1, 0\}$. The whole disjunction $\beta(\varphi)$ is called a *bitstring*. Let us take as an example the case of binary sentences $\varphi = f^2(p, q)$, where f^2 is an arbitrary binary connective characterized by a common disjunctive normal form including four possible disjuncts: $\beta_1(\varphi) = p \cap q, \beta_2(\varphi) = p \cap \overline{q}$, $\beta_3(\varphi) = \overline{p} \cap q, \beta_4(\varphi) = \overline{p} \cap \overline{q}$. Each of these $n = 4$ denotes one way of combining any two atomic sentences p and q, and the result is a logical constant that satisfies none, some, or all of the resulting $2^4 = 16$ items.[6] Thus:

$\beta(\bot) = 0000$ $\beta(p \wedge q) = 1000$ $\beta(\neg(p \to q)) = 0100$ $\beta(\neg(p \leftarrow q)) = 0010$
$\beta(\neg(p \vee q)) = 0001$ $\beta(p) = 1100$ $\beta(\neg(p \leftrightarrow q)) = 0110$ $\beta(\neg p) = 0011$
$\beta(p \to q) = 1001$ $\beta(\neg q) = 0101$ $\beta(q) = 1010$ $\beta(p \vee q) = 1110$
$\beta(\neg(p \wedge q)) = 0111$ $\beta(p \leftrightarrow q)) = 1011$ $\beta(p \leftarrow q)) = 1101$ $\beta(\top) = 1111$

Note that, in the above characterization, $\bot$ and $\top$ denote nothing and every-

semantics'.

[6]Note that each such bit β_i corresponds to the four rows of a two-valued truth-table.

thing into a given set of expressions: the former includes no part of the logical space whilst the latter includes everything. These borderline cases will be let aside of our consideration about logical relations, especially in order to avoid complications with the definitions of the following opposite-forming operators.

An algebraic definition of the opposite-forming operators consists in defining these logical relations between items as transformations upon bitstrings. Here is such a calculus of logical relations $Op(x, y)$ based on the transformation of bits by opposite-forming operators $op(x) = y$.

Calculus of logical relations.
$\beta(x) = 1 \Rightarrow \beta(ct(x)) = 0$
$cd(\beta(x)) = 1 \Leftrightarrow \beta(x) = 0$, i.e. $cd(\beta(x)) = 1 \Rightarrow \beta(x) = 0$ and $cd(\beta(x)) = 0 \Rightarrow \beta(x) = 1$
$\beta(x) = 0 \Rightarrow \beta(sct(x)) = 1$
$\beta(x) = 1 \Rightarrow \beta(sb(x)) = 1$
$\beta(sp(x)) = 1 \Rightarrow \beta(x) = 1$

To argue for Demey & Smessaert [4], the above definitions show that not every such opposition-forming operator is defined in the same way; for example, contrariety and subcontrariety turn one bit into another one whereas subalternation and superalternation do not do so. This is a reason not to consider these unary operators as applying the same pattern, whereas the two authors take this to motivate a distinction between oppositional and implicational relations. To argue against them, our following point is that all of the logical relations can be defined with the same pattern by only two basic cases, namely: contradiction, and contrariety.

3.2 Logical connectives

Echoing with Costa-Leite's clauses, his introduction of sums between arithmetical functions of the logical hexagon amounts hereby to the set-theoretical operation of *union* $\cup$ and is on a par with the truth-functional connective of disjunction. The dual operation of conjunction can also be defined set-theoretically with the equally dual operation of intersection $\cap$. Both can be defined as follows: for any i-th element in a bistring of length n,

$\beta(\varphi \vee \psi) = \beta_i(\varphi) \cup \beta_i(\psi)$;
$\beta(\varphi \wedge \psi) = \beta_i(\varphi) \cap \beta_i(\psi)$.

Recalling that the categorical statement **U** is a disjunction of the affirmative universal **A** and the negative universal **E**, this entails that

$\beta(\mathbf{U}) = \beta(\mathbf{A}) \cup \beta(\mathbf{E}) = 100 \cup 001 = 101.$

The dual categorical formula **Y** can be characterized correspondingly, such as

$\beta(\mathbf{Y}) = \beta(\mathbf{I}) \cap \beta(\mathbf{O}) = 110 \cap 011 = 010.$

At the same time, our bitstring semantics helps to make a clear difference between the generally confused notions of conditional (or material implication) and consequence. A set-theoretical definition of conditional should yield a counterpart of its classical definition of conditional,

$$\varphi \to \psi =_{df} \neg\varphi \vee \psi.$$

Now assuming that classical negation is nothing but the extensional operator of contradictoriness, i.e.,

$$\neg\varphi =_{df} cd(\varphi),$$

this entails that every material implication is not a logical relation but a single formula like, e.g., **U**. Let us consider the formula $\mathbf{A} \to \mathbf{E}$, for example. According to the above definition, the value of the resulting formula is

$\beta(\mathbf{A} \to \mathbf{E}) = \beta(cd(\mathbf{A}) \vee \mathbf{E}) = 011 \cup 001 = 011.$

We are thus led in a semantically weird situation where **E** and $\mathbf{A} \to \mathbf{E}$ have the same value, i.e., mean the same thing. The point is that such a translation of material conditional is completely irrelevant in our bitstring semantics. At the same time, it could be used as a way to identify a consequence relation between any two formulas. Just as consequence (or formal implication) can be defined as a material implication that is *valid*, i.e., true for any assignment of truth-values to its components, we can also say that any formula ψ is a consequence of any other formula φ if, and only if, *always* φ implies ψ. That is,

$$\varphi \models_S \psi \text{ holds if, and only if, } \beta(\varphi \to \psi) = \top$$

Let us take **A**. The aforementioned explanation does entail both **I** and **U** in the sense that it fulfills our algebraic clause, namely,

$\beta(\mathbf{A} \to \mathbf{I}) = \beta(cd(\mathbf{A}) \vee \mathbf{I}) = 011 \cup 110 = \top$

$\beta(\mathbf{A} \to \mathbf{U}) = \beta(cd(\mathbf{A}) \vee \mathbf{U}) = 011 \cup 101 = \top$

This also reminds one that formulas may have more than one consequence, thus preventing from characterizing subalternation by a truth-functional operator.

3.3 Iterated oppositions

Nevertheless, another parallel way to characterize subalternation is to define it by means of iterated functions. Here is the central point of the present paper: logical relations that are not taken to be oppositions by Demey & Smessaert [4] can be reduced after all to an iteration of basic oppositions. To begin with such a process, any *subaltern* of an arbitrary formula x is to be defined as the *contradictory of a contrary* of x:

(1) $$sb(\beta(x)) = cd(ct(\beta(x)))$$

For example, let $\beta(x) = 1000$. According to a series of proofs given previously [9], how many contraries there are for any given formula x depends upon the k number of 0-bits in x. Thus, $Card(ct(x)) = 2^n - 1$. However, a further constraint on the calculus of oppositions is that it exclude the extreme cases of tautology and antilogy, $\beta(x) = 0000 = \bot$ and $\beta(x) = 1111 = \top$: these two formulas mean nothing in particular, so that they should removed from the set of the relevant logical relations. Accordingly, $Card(ct(x)) = 2^n - 2$ by excluding the case of antilogy. As there is a number of $k = 3$ 0-bits in the above example, it follows that $Card(ct(x)) = 2^3 - 2 = 6$ contraries of x:

$ct((1000)) = \{0100, 0010, 0001, 0110, 0011, 0101\}$
$cd(ct((1000)) = \{1011, 1101, 1110, 1100, 1010\}$

The above law of cardinality explains why more than one kind of opposite occurs in such a gathering as the logical hexagon, whenever the structural complexity of bitstrings admits of several cases of contraries, subcontraries, and the like. It also helps so show that some formulas have no contrary according to their number of 0-bits: if, e,g, $\beta(x) = 1110$, then $Card(1110) = 2^1 - 2 = 0$. Finally, the characteristic bitstring of a set of items helps to determine when a logical structure is complete, or when it is a mere fragment of a larger structure. In the case of the Aristotelian square, the 3-bits structure of its items entails that this square is a mere fragment of a complete structure including $2^3 = 8$ items –including the two borderline cases of tautology and antilogy, so that a complete structure of bitstrings with 3 bits (that is, of length 3) is a hexagon including $8 - 2 = 6$ vertices. With respect to this criterion of structural completeness, we are now in position to make sense of Costa-Leite's strenghtened clauses for contrariety and subcontrariety (see Section 2). These can be generalized as follows.

For any set of formulas of length n in a given logical space Σ,

basic contraries of φ are the n formulas including only one 1-bit, such that their complete union exhausts the logical space: $ct_1(\varphi) \cup \ldots \cup ct_n(\varphi) = \top$;

basic subcontraries of φ are the n contradictories of basic contraries, such that their complete intersection exhausts the logical space: $ct_1(\varphi) \cap \ldots \cap ct_n(\varphi) = \bot$.

This helps to show that Costa-Leite's clauses should be adapted to the structural complexity or length of the related formulas, having in mind that the above definitions are set-theoretical counterparts of Costa-Leite's arithmetical account in terms of zero sum.[7]

Conversely to (1), any superlatern of x is to be defined as the *contrary of the contradictory* of x:

$$sp(\beta(x)) = ct(cd(\beta(x))) \tag{2}$$

Let $\beta(x) = 1110$. Then
$cd(\beta(1110)) = 0001$
$ct(0001) = \{1000, 0100, 0010, 1100, 0110, 1010\}$

A *subcontrary* of any x is *the contradictory of the superaltern* of x or, by substituting the latter relation for its iterative definition, the *contradictory of the contrary of the contradictory* of x:

$$sct(\beta(x)) = cd(sp(\beta(x))) = cd(ct(cd(\beta(x)))) \tag{3}$$

Thus let $\beta(x) = 1110$. Then
$cd(1110) = 0001$
$ct(0001) = \{1000, 0100, 0010, 1100, 0110, 1010\}$
$cd(ct(0001)) = \{0111, 1011, 1101, 0011, 1001, 0101\}$

This functional calculus structurally explains why some expressions as 'the contrary of the subcontrary of x' denote nothing, given that no instance of

[7] Referring to categorical statements again, this means that the three formulas **A**,**E**,**U** are all the contraries that there can be in a set of formulas of length $n = 3$ only; this is not so whenever $n > 3$, however.

subcontrariety can be an instance of contrariety as well (and conversely). Let $\beta(x) = 100$ as an example. Then $\beta(y) = 001$ is a contrary of x, and y cannot have any subcontrary by virtue of the above definitions: $\beta(sct(001)) = \emptyset$.[8] Accordingly, any computation including one empty result in its process always leads to a final empty result.

Our central thesis is thus that all the implicational relations can be defined iteratively, in the light of this calculus of logical relations. Thus right implication is subalternation and left implication is superalternation, whilst bi-implication is a combination of both, $sb(\beta(x)) \cap sp(\beta(x))$, non-implication being none of the previous relations. Consequently, the distinction made by Demey & Smessaert [4] between two main kinds of logical relations: oppositional and implicational, can be reduced to a unique set of logical relations based upon two basic genuine opposite-forming operators (contrary-forming and contradictory-forming) and iterations of these.

3.4 Universal quadrilateral of oppositions

In addition, our proposed semantic algebra can be completed by a common geometry of logical relations. This alternative geometry wants to combine the two virtues of a model: explanatory power, by applying to any set of structured items; and simplicity, by applying the same geometric pattern to any set of logical relations. It relies upon the following increasing pattern:

[8] Note that any such lack of corresponding bitstring, symbolized as $\emptyset$, differs from antilogy $\top$ in that the latter *is* a proper bitstring among all the other ones. Set-theoretically speaking, this means that $\top$ is a proper element of any set of bitstrings whereas $\emptyset$ is not.

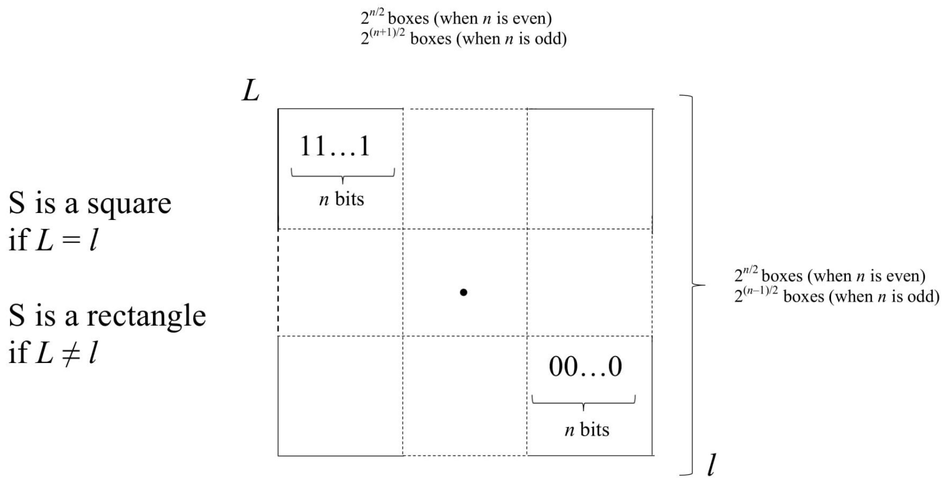


Figure 6: A pattern of the non-standard quadrilateral of oppositions.

The main difference with the standard geometry of oppositions is that the above pattern does not include vertices any more, insofar as any item of a set is an included box of a quadrilateral instead of a graph with points. The simplicity of such a geometry comes from its common process: either the characteristic items include an even number of bits, and the quadrilateral is a square; or it includes an odd number of bits, and the quadrilateral is a rectangle. Its explanatory virtue comes from its visual ability to show how any items are related to each other: by sustaining the central symmetry of contradictoriness –as shown in the figure here below, the way bitstrings are organized in the quadrilateral helps to determine any kind of logical relation by the spatial positions of items. In the following figure, black boxes symbolize a starting item x whose logical relations with the other ones are represented with the standard colors of Aristotelian oppositions (in addition with the orange color of mere compatibility between x and $op(x) = y$.

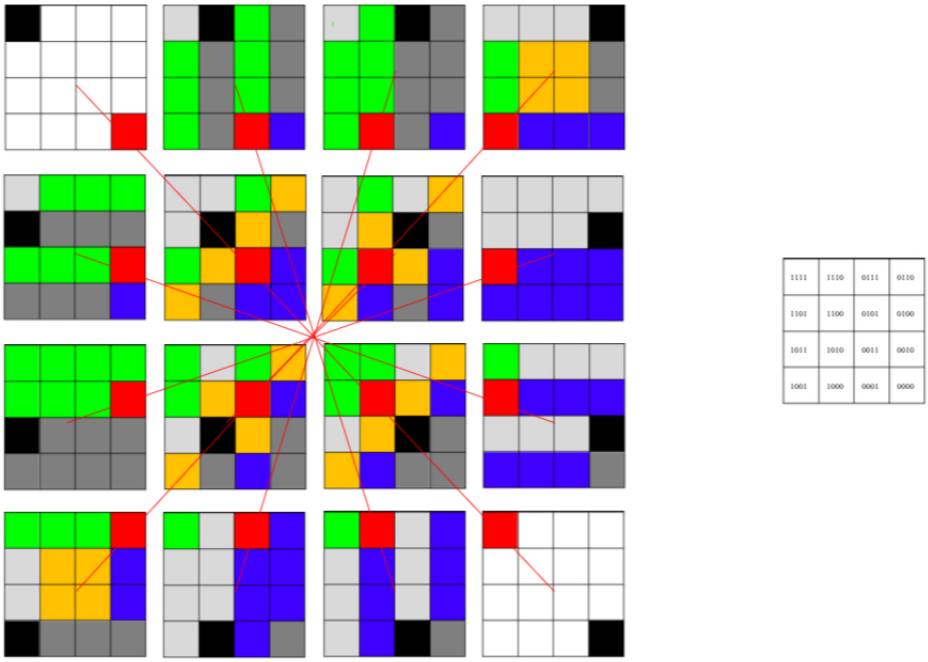


Figure 7: A colored square of oppositions.

We take our universal gathering to have a number of explanatory virtues, thereby generalizing both Costa-Leite's unidimensional segment and any other gathering of logical relations. Firstly, any such quadrilateral is a complete structure whilst some famous figures as the Aristotelian square or Blanché's hexagon are incomplete fragments. Secondly, the central symmetry of contradictories is maintained in our structure whilst including the borderline cases of tautology $\top$ and antilogy $\perp$ as proper formulas –at the top left and bottom right border sides of the given quadrilateral. Thirdly, it replaces Costa-Leite's complicated clauses by a unique set of definitions for any set of structured formulas. Fourthly, the previous distinction between material implication and logical consequence (see Section 3.2) completes the equivalence between classical negation and contradictoriness into a general, 'Slaterian' reflection about how logical connectives can be translated into combined oppositions. And fifthly, the above figure shows that each of the embedded quadrilaterals belongs to a higher structure that also satisfies the clause of central symmetry for contradictories; such a jump into abstraction could lead to interesting metalogical results, but this is to be done in another paper.

Conclusion: Beyond the old square

To recapitulate the present paper, the old Aristotelian square of oppositions can be replaced by some other clearer logical structures. A first reason is that this square is a set of logical relations mixing compatible and incompatible cases, and this conceptual unclarity has been emphasized by Demey & Smessaert [4] in order to motivate a general reconstruction of the geometry of logic. A second reason is that the Aristotelian square is a hidden graph that does not help to explain the interrelations between its vertices. Our above aim was to establish a systematic algebra and geometry of logical relations through two basic relations of opposition, viz. contradiction and contrariety.

Does this mean that opposition is a more basic property of logic than that of consequence, albeit conspicuous in the mainstream theory of logic? This may be the case, but provided that any item of a given formal language may be characterized by a bitstring. Moreover, a limitation of our bitstring semantics is that it applies only between items whose bitstrings are of the same length n or, equivalently, resort to the same characteristic predicates. Ideally, every domain of items relies upon a common bitstring of maximal length, such that 2^n exhausts the universe of discourse. But the discovery of such a bitstring appears to be the same ideal inquiry as the Leibnizian *calculus ratiocinator*, according to which any thought could be considered to a calculus of elementary relations. For want of such a theoretical possibility, our bitstring semantics wants to show that the main features of logical relations could be at least reduced to a binary calculus. The explanatory virtue of it in addition to our colored graphs aimed at fulfilling the expected relevance of a theory.

More is to be said about the underlying process of partition that characterizes any items inside a logical space of bitstrings. More especially, a central issue is the applicability of bitstrings to kinds of expression that behave in a non-truthfunctional or intentional way. This work needs to be done in a close future, in order to pursue our general theory of logical relations.

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