

JUXTAPOSITION: A NEW WAY TO COMBINE LOGICS

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Abstract. This paper develops a new framework for combining propositional logics, called “juxtaposition”. Several general metalogical theorems are proved concerning the combination of logics by juxtaposition. In particular, it is shown that under reasonable conditions, juxtaposition preserves strong soundness. Under reasonable conditions, the juxtaposition of two consequence relations is a conservative extension of each of them. A general strong completeness result is proved. The paper then examines the philosophically important case of the combination of classical and intuitionist logics. Particular attention is paid to the phenomenon of collapse. It is shown that there are logics with two stocks of classical or intuitionist connectives that do not collapse. Finally, the paper briefly investigates the question of which rules, when added to these logics, lead to collapse.

§1. Introduction. Methods of combining logics are of great interest.¹ Formal systems that result from the combination of multiple logical systems into a single system have applications in mathematics, linguistics, and computer science. For example, there are many applications for logics with multiple kinds of modal operators – epistemic, temporal, and deontic.

There are also purely philosophical reasons to be interested in the combination of logics. One illustration of this comes from so-called collapse theorems.

Suppose there is a language with two stocks of the usual logical connectives (for propositional or for first-order logic) – \wedge_1 and \wedge_2 , \vee_1 and \vee_2 , and so on. There is a well-known result which states that given any logic for this language such that each logical constant obeys the usual natural deduction rules (for classical or even for intuitionist logic), sentences that differ only in some or all of their subscripts are intersubstitutable.² Corresponding connectives, such as \wedge_1 and \wedge_2 , behave as mere notational variants. In such a logic, if one stock of constants obeys the classical natural deduction rules, so does the other.

This result has been used to argue for several striking philosophical theses. For instance, it has been used to argue that the logical constants in our language

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¹Methods of combining logics include fibring and its variants, as developed by Gabbay, as well as algebraic fibring and its variants, as developed by A. Sernadas and his collaborators. See Gabbay (1998) and Carnielli et al. (2008), respectively, for comprehensive overviews of this work. See Caleiro et al. (2005) for a summary of the central results concerning algebraic fibring.

²See McGee (2000) for proof of this result. See Harris (1982) for a closely related result. The first collapse results of which I am aware appear in Carnap (1943), sections A.7–A.9. Also see Popper (1948) for the collapse of classical and intuitionist logic.

have unique and determinate extensions. The argument goes as follows: Suppose that one of our logical constants had an indeterminate extension. Then we could precisify the term. We could introduce two (or more) precise terms into our language with distinct extensions. These terms would presumably obey the usual natural deduction rules (understood to apply to the expanded language). But since the terms would have distinct extensions, they would not be intersubstitutable. By the collapse result, however, this is impossible. Therefore, the logical constants in our language must not be indeterminate.

Vann McGee makes use of a related line of thought to argue for striking claims about quantification.³ He argues that there is a unique determinate domain for the unrestricted universal quantifier. On his view, when metaphysicians intend to be talking about absolutely everything, they succeed in so doing. Disputes in ontology are genuine disputes; when metaphysicians disagree about what exists, they are not simply talking past each other by using “exists” in different ways.

Timothy Williamson uses the collapse result to argue for a claim about the relationship between classical and intuitionist logic.⁴ He claims that the collapse result demonstrates that there cannot be two stocks of logical constants – one classical and one intuitionist – with distinct extensions. The classicist and the intuitionist cannot both be right. Either intuitionism is correct and classical logic is incoherent, or (as Williamson suggests) classical logic is correct and the intuitionists fail to recognize certain genuinely valid entailments.⁵

There are several ways in which the philosophical arguments for these conclusions may be challenged, and there has been some discussion of these issues in the literature.⁶ What has not received sufficient attention, however, is the technical situation. It turns out that the technical situation is not as straightforward as has usually been supposed. For example, there are logics with two stocks of connectives that each obey the usual classical theorems and entailments (for the entire language), but in which corresponding constants are not intersubstitutable. There are logics with two stocks of connectives, one obeying the classical theorems and entailments and the other obeying the intuitionist theorems and entailments, but in which corresponding constants are not intersubstitutable.⁷ The caveat to these results is that the connectives do not obey the usual natural

³See McGee (2000).

⁴See Williamson (1987). A similar line of thought is present in Harris (1982). See Hand (1993) for relevant discussion.

⁵Analogous results have also been used to argue for striking philosophical claims about mathematical concepts. For instance, the categoricity of second-order arithmetic – a sort of collapse theorem – has been used to argue that “finite” has a unique determinate extension. The quasi-categoricity of second-order ZF set theory has been used to argue for the claim that we have a unique determinate conception of the sets, at least up to height. See McGee (1997), Parsons (1990), and Shapiro (1991) for discussion.

⁶See, for example, the postscript to Field (2001).

⁷This result can also be proved using the technique of crypto-fibring. In particular, see Caleiro & Ramos (2007) for proof of this result restricted to the implicational fragments of classical and intuitionist logic. The technique of modulated fibring, as described in Sernadas et al. (2002), also provides a way to combine classical and intuitionist logics without collapse. However, the resulting logic is not substitution invariant. Also see del Cerro & Herzig (1996) for a non-substitution invariant logic that combines classical and intuitionist logics.

deduction meta-rules in their full generality.⁸ The natural deduction rules encode more logical strength than one might have expected. Perhaps surprisingly, the collapse results rely on this extra strength. Thus, the collapse results turn out to be very fragile. Indeed, the issue of when exactly logics collapse turns out to be rather delicate.

The purpose of this paper is two-fold. First, I develop a new framework for combining logical systems, called “juxtaposition”. I prove general metalogical results concerning the combination of logics by juxtaposition. Second, I examine the particular case of combining classical and intuitionist logics. I show how the general results can be applied to shed light on the phenomenon of collapse. I demonstrate that the collapse results are much more limited than one might have expected.

The paper will proceed as follows. In the next section, I introduce the general logical apparatus that will be employed. In this paper, I focus on the case of propositional logics. The approach to semantics employed here is broadly algebraic. I consider two semantic frameworks. The first involves sets of logical matrices, algebras with an arbitrary set of designated values. The second, more interesting, framework involves sets of unital matrices, algebras with a single designated value.⁹ In section three, I present the main constructions for combining (“juxtaposing”) logical systems. Juxtaposing consequence relations is straightforward – the juxtaposition of two consequence relations is the least consequence relation that extends the original consequence relations. (In this paper, consequence relations are required to obey the usual structural rules and to be substitution invariant.) The juxtaposition of two algebraic structures is only slightly more complicated. A juxtaposed model is an ordered pair of models, each of which is based on the respective algebraic structure. There are two modifications to this basic idea that are needed to get the semantics to work. First, each of the two models must provide semantic values for sentences of the entire language. Therefore, each model treats sentences with main connectives governed by the other logic as though they were additional sentence symbols. Second, the two models must agree on which sentences get assigned designated values. In this way, juxtaposed models must be “coherent”.

In section four, I compare juxtaposition to two other methods of combining logics – algebraic fibring and modulated fibring. Section five is devoted to presenting basic metalogical results concerning juxtaposition. In particular, I show that under reasonable conditions, juxtaposition preserves strong soundness. I show that under reasonable conditions, juxtaposition preserves consistency. I also show that under reasonable conditions, the juxtaposition of two consequence relations is a strong conservative extension of the original relations. In section six, I turn to strong completeness. In this section, I present direct proofs of

⁸A meta-rule is a rule that governs relations among entailments but do not themselves state entailments. Conditional Introduction, Reasoning by Cases, and Classical Reductio are meta-rules. Conjunction Introduction and Elimination, Disjunction Elimination, Modus Ponens, and Double Negation Elimination are not meta-rules in this sense.

⁹There are other natural approaches to algebraic semantics. For example, we could make use of classes of logical matrices where the designated values can be characterized by a set of equations. See Blok & Pigozzi (1989).

strong completeness that apply in a wide range of cases. Indeed, I provide necessary and sufficient conditions for the case where the two stocks of connectives are disjoint. Finally, in section seven, I turn to the philosophically important case of combining classical and intuitionist logics. Applying the general metalogical results, I show that a logic with two stocks of classical connectives is consistent, conservative, and strongly sound and strongly complete with respect to a particular class of juxtaposed structures – the “bi-Boolean” structures. I show that a logic with two stocks of intuitionist connectives is consistent, conservative, and strongly sound and strongly complete with respect to the class of “bi-Heyting” structures.¹⁰ A logic with one stock of intuitionist connectives and one stock of classical connectives is consistent, conservative, and strongly sound and strongly complete with respect to the class of “Heyting-Boolean” structures. I prove that none of these logics collapse. I also investigate the question of which rules (and meta-rules) lead to collapse when added to these logics.

§2. Basic Notions.

2.1. Syntax. A language for propositional logic can be specified by a signature and a set of sentence symbols. A *signature* $C = \{C^n\}_{n \in \mathbb{N}}$ is an indexed family of sets over the natural numbers.¹¹ For each $n \in \mathbb{N}$, C^n is the (possibly empty) set of *connectives* of arity n . A set of *sentence symbols*, P , is a non-empty set. For convenience, we only work with infinite sets of sentence symbols. To avoid ambiguity, we assume that the elements of each C^n and P are not themselves sequences. We also assume that C^m and C^n are disjoint if $m \neq n$ and that each C^n is disjoint with P .

Suppose C and C' are two signatures. We say that C and C' are *disjoint* just in case for each $n \in \mathbb{N}$, C^n and C'^n are disjoint. Otherwise, we say that C and C' *overlap*. We say that C' is a *sub-signature* of C just in case for each $n \in \mathbb{N}$, $C'^n \subseteq C^n$.

Given a signature C and a set of sentence symbols P , the set of *sentences* generated by C and P , $\text{Sent}(C, P)$, is inductively defined to be the least set such that:

- If $\alpha \in P$ then $\alpha \in \text{Sent}(C, P)$;
- If $c \in C^n$ and $\alpha^1, \dots, \alpha^n \in \text{Sent}(C, P)$ then $c\alpha^1 \dots \alpha^n \in \text{Sent}(C, P)$.

We write α, β, γ , and δ (sometimes with superscripts) to stand for sentences.¹² We write Γ and Δ to stand for sets of sentences. We write p, q , and r to stand for sentence symbols.

We write $\alpha[\beta/p]$ to stand for the result of uniformly substituting each occurrence of p in α with β . We write $\Gamma[\beta/p]$ to stand for the set $\{\gamma[\beta/p] \mid \gamma \in \Gamma\}$. Let σ be any function from a subset of P to some set. We write α^σ to stand for the result of uniformly substituting each occurrence of any p in the domain of σ in α with $\sigma(p)$. We write Γ^σ to stand for the set $\{\gamma^\sigma \mid \gamma \in \Gamma\}$.

¹⁰This is a different usage of “bi-Heyting” than the one familiar from Skolem.

¹¹In this paper, superscripts are used as indices and not as exponents.

¹²Following standard practice, we don’t conform to strict conventions governing use and mention when there is little danger of confusion.

For ease of comprehension, when displaying sentences in the language of classical propositional logic, we use infix rather than prefix notation.

2.2. Consequence Relations. A *consequence relation*, \vdash , for a set of sentences $\text{Sent}(C, P)$ is a relation holding between subsets of $\text{Sent}(C, P)$ and elements of $\text{Sent}(C, P)$ such that the following conditions obtain for every $\alpha, \beta, \Gamma, \Delta$, and p :

Identity. $\{\alpha\} \vdash \alpha$;

Weakening. If $\Gamma \vdash \alpha$ then $\Gamma \cup \Delta \vdash \alpha$;

Cut. If $\Gamma \vdash \alpha$ and $\Delta \cup \{\alpha\} \vdash \beta$ then $\Gamma \cup \Delta \vdash \beta$;

Uniform Substitution. If $\Gamma \vdash \alpha$ then $\Gamma[\beta/p] \vdash \alpha[\beta/p]$.¹³

In this paper, we do not require that consequence relations be compact. That is, it need not be the case that if $\Gamma \vdash \alpha$ then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash \alpha$.

Suppose C^- is a sub-signature of C and P^- is a subset of P . Suppose \vdash^- is a consequence relation for $\text{Sent}(C^-, P^-)$ and \vdash is a consequence relation for $\text{Sent}(C, P)$. We say that \vdash *extends* \vdash^- just in case for every $\Gamma \subseteq \text{Sent}(C^-, P^-)$ and $\alpha \in \text{Sent}(C^-, P^-)$, if $\Gamma \vdash^- \alpha$ then $\Gamma \vdash \alpha$. We say that \vdash is a *strong conservative extension* of \vdash^- just in case for every $\Gamma \subseteq \text{Sent}(C^-, P^-)$ and $\alpha \in \text{Sent}(C^-, P^-)$, $\Gamma \vdash^- \alpha$ just in case $\Gamma \vdash \alpha$.

Let \vdash be a consequence relation for $\text{Sent}(C, P)$. We say that α is *deducible* from Γ just in case $\Gamma \vdash \alpha$. We say that α is a *theorem* of \vdash just in case $\emptyset \vdash \alpha$. As usual, we write $\vdash \alpha$ for $\emptyset \vdash \alpha$.

We say that $\Gamma \subseteq \text{Sent}(C, P)$ is *consistent* with respect to \vdash just in case there is an $\alpha \in \text{Sent}(C, P)$ such that $\Gamma \not\vdash \alpha$.¹⁴ We say that \vdash is *consistent* just in case there is an $\alpha \in \text{Sent}(C, P)$ such that $\not\vdash \alpha$. We say that \vdash is *non-trivial* just in case there is a non-empty $\Gamma \subseteq \text{Sent}(C, P)$ and an $\alpha \in \text{Sent}(C, P)$ such that $\Gamma \not\vdash \alpha$. We say that \vdash *has no mere followers* just in case $\vdash \alpha$ whenever $\Gamma \vdash \alpha$ for every non-empty $\Gamma \subseteq \text{Sent}(C, P)$.¹⁵ We say that \vdash *has theorems* just in case there is at least one theorem of \vdash . These notions are related as follows: Any consequence relation that has theorems has no mere followers. Any consistent consequence relation that has no mere followers is non-trivial. Any non-trivial consequence relation is consistent.

2.3. Semantics. The approach to semantics we employ is broadly algebraic. A *structure* over a signature C is an ordered triple $\mathcal{B} = \langle B, D, \Phi \rangle$ such that B is the carrier set of the structure, D is a non-empty proper subset of B , and for every $c \in C^n$, $\Phi(c)$ is a function from the n -th Cartesian power of B to B . B is the set of *semantic values* of \mathcal{B} . D is the set of *designated values* of \mathcal{B} . Φ is the *denotation function* of \mathcal{B} . Since D is a non-empty proper subset of B , B must have at least two elements. If D has only a single element, we say that \mathcal{B} is a *unital structure* and we write 1 to stand for the single designated value.

¹³A consequence relation is thus required to be structural in the sense of Łoś & Suszko (1958).

¹⁴Consistency is defined here as non-explosion. In a paraconsistent logic, a consistent set may contain both a sentence and its negation.

¹⁵I borrow the term “no mere followers” from Humberstone (2011), pages 459–460.

Given a structure $\mathcal{B} = \langle B, D, \Phi \rangle$ and a set of sentence symbols P , a *valuation* for \mathcal{B} and P is a function from P to B .

A *model* over a signature C and a set of sentence symbols P is an ordered pair $\mathcal{M} = \langle \mathcal{B}, V \rangle$ such that \mathcal{B} is a structure over C , and V is a valuation for \mathcal{B} and P . We say that the model \mathcal{M} is *based on* the structure \mathcal{B} . Given a class of structures \mathbb{B} over C , we say that \mathcal{M} is *based on* \mathbb{B} just in case \mathcal{M} is based on some element of \mathbb{B} . If \mathcal{M} is based on a unital structure, we call \mathcal{M} a *unital model*.

Suppose $\mathcal{M} = \langle \mathcal{B}, V \rangle$ is a model over C and P based on $\mathcal{B} = \langle B, D, \Phi \rangle$. For any $\alpha \in \text{Sent}(C, P)$, the *value* of α in \mathcal{M} , $\|\alpha\|^{\mathcal{M}}$, is recursively defined as follows:

- $\|\alpha\|^{\mathcal{M}} = V(\alpha)$ if $\alpha \in P$;
- $\|c\alpha^1 \dots \alpha^n\|^{\mathcal{M}} = \Phi(c)(\|\alpha^1\|^{\mathcal{M}}, \dots, \|\alpha^n\|^{\mathcal{M}})$ if $c \in C^n$ and $\alpha^1, \dots, \alpha^n \in \text{Sent}(C, P)$.

Given a model \mathcal{M} , we write $\mathcal{M} \models \alpha$ to mean that α has a designated value in \mathcal{M} . For short, we say that \mathcal{M} *designates* the sentence α . This obtains just in case $\|\alpha\|^{\mathcal{M}} \in D$. We write $\mathcal{M} \models \Gamma$ to mean that \mathcal{M} *designates* each of the sentences in Γ . This obtains just in case for every $\gamma \in \Gamma$, $\mathcal{M} \models \gamma$.

If a model \mathcal{M} over C and P either designates every element of $\text{Sent}(C, P)$ or designates no element of $\text{Sent}(C, P)$, we say that \mathcal{M} is *trivial*. Otherwise, we say that \mathcal{M} is *non-trivial*. For any structure \mathcal{B} over C and any set of sentence symbols P , there is a non-trivial model over C and P based on \mathcal{B} .

Given a class of structures \mathbb{B} over C , we write $\models^{\mathbb{B}} \alpha$ to mean that α is *valid* in \mathbb{B} . This obtains just in case for every model \mathcal{M} over C and P based on \mathbb{B} , $\mathcal{M} \models \alpha$. We write $\Gamma \models^{\mathbb{B}} \alpha$ to mean that Γ *entails* α in \mathbb{B} . This obtains just in case, for every model \mathcal{M} over C and P based on \mathbb{B} , if $\mathcal{M} \models \Gamma$ then $\mathcal{M} \models \alpha$.¹⁶

2.4. Soundness and Completeness. Given a consequence relation \vdash for the set of sentences $\text{Sent}(C, P)$ and a class of structures \mathbb{B} over C , we say that \vdash is *strongly sound* with respect to \mathbb{B} just in case for every $\Gamma \subseteq \text{Sent}(C, P)$ and $\alpha \in \text{Sent}(C, P)$, if $\Gamma \vdash \alpha$ then $\Gamma \models^{\mathbb{B}} \alpha$. We say that \vdash is *strongly complete* with respect to \mathbb{B} just in case for every $\Gamma \subseteq \text{Sent}(C, P)$ and $\alpha \in \text{Sent}(C, P)$, if $\Gamma \models^{\mathbb{B}} \alpha$ then $\Gamma \vdash \alpha$. We say that \vdash is *strongly determined* with respect to \mathbb{B} just in case for every $\Gamma \subseteq \text{Sent}(C, P)$ and $\alpha \in \text{Sent}(C, P)$, $\Gamma \models^{\mathbb{B}} \alpha$ just in case $\Gamma \vdash \alpha$.

We say that \vdash is *strongly sound*, *strongly complete*, or *strongly determined* (simpliciter) just in case \vdash is strongly sound, strongly complete, or strongly determined (respectively) with respect to some non-empty class of structures. We say that \vdash is *strongly unital sound*, *strongly unital complete*, or *strongly unital determined* (simpliciter) just in case \vdash is strongly sound, strongly complete, or strongly determined (respectively) with respect to some non-empty class of unital structures.

¹⁶Strictly speaking, these definitions require P to be fixed by context. However, whether α is valid in \mathbb{B} does not depend on P so long as P contains all of the sentence symbols that occur in α . Similarly, whether Γ entails α in \mathbb{B} does not depend on P so long as P contains all of the sentence symbols that occur in Γ and α .

2.5. Known Results. Let \vdash be a consequence relation for $\text{Sent}(C, P)$. We say that \vdash is *left-extensional* just in case for every $\alpha, \beta, \delta \in \text{Sent}(C, P)$ and p occurring in δ , $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\beta/p]$.¹⁷ The following results are known:

THEOREM 2.1. \vdash is strongly determined just in case \vdash is consistent and has no mere followers.¹⁸

THEOREM 2.2. \vdash is strongly unital determined just in case \vdash is consistent, has no mere followers, and is left-extensional.¹⁹

§3. Juxtaposition. We consider here the case of combining two logics. It is straightforward to extend the construction to combinations of arbitrarily many logics (indexed by some set). In what follows, we sometimes use i to range over $\{1, 2\}$.

3.1. Syntax. Let C_1 and C_2 be two signatures and P_1 and P_2 be two sets of sentence symbols. (Recall that P_1 and P_2 are required to be infinite.) We allow P_1 and P_2 to overlap. We also allow C_1 and C_2 to overlap.²⁰ For simplicity, we assume that C_1^m and C_2^n are disjoint if $m \neq n$. We also assume that each C_i^n is disjoint with each of P_1 and P_2 .

The *juxtaposition of the sets of sentence symbols* P_1 and P_2 , P_{12} , is $P_1 \cup P_2$. The *juxtaposition of the signatures* C_1 and C_2 , C_{12} , is $\{C_1^n \cup C_2^n\}_{n \in \mathbb{N}}$. That is, C_{12} is the signature that results from taking the union of each of the sets of connectives of arity n .²¹ The set of sentences generated by C_{12} and P_{12} , $\text{Sent}(C_{12}, P_{12})$, is defined just as above.²²

3.2. Consequence Relations. Let \vdash_1 be a consequence relation for $\text{Sent}(C_1, P_1)$ and let \vdash_2 be a consequence relation for $\text{Sent}(C_2, P_2)$. A *juxtaposed consequence relation* over \vdash_1 and \vdash_2 is a consequence relation for $\text{Sent}(C_{12}, P_{12})$ that extends both \vdash_1 and \vdash_2 .

It is immediate that if \vdash_1 or \vdash_2 is non-empty, so is any juxtaposed consequence relation over \vdash_1 and \vdash_2 . If \vdash_1 or \vdash_2 has theorems, so does any juxtaposed consequence relation over \vdash_1 and \vdash_2 . Using Uniform Substitution it is easy to show that if \vdash_1 or \vdash_2 is inconsistent, so is any juxtaposed consequence relation over \vdash_1 and \vdash_2 . Similarly, if \vdash_1 or \vdash_2 is trivial, so is any juxtaposed consequence relation over \vdash_1 and \vdash_2 .

¹⁷Wójcicki (1988), page 236, calls this property “strong replacement”. I borrow the term “left-extensional” from Humberstone (2011), section 3.23, who uses it for a slightly different property.

¹⁸See, for example, Observation 3.23.13 in Humberstone (2011). The core of this result is Theorem 3.1.5 in Wójcicki (1988). It is originally due to Lindenbaum.

¹⁹See, for example, Theorem 3.23.9 in Humberstone (2011). The core of this result is Theorem II.1.2 in Czelakowski (1981). It is originally due to Suszko.

²⁰In the literature on fibring, the combination of logics with overlapping signatures is referred to as “constrained” and the combination of logics with disjoint signatures is referred to as “unconstrained”.

²¹This is essentially the same construction as the categorial fibring of propositional signatures, as described in Sernadas et al. (1999), page 152.

²²There is an alternative choice we could have made here. We could have defined $\text{Sent}(C_{12}, P_{12})$ to disallow sentences with connectives from both signatures. But that would have yielded a much less interesting language.

The *juxtaposition of the consequence relations* \vdash_1 and \vdash_2 is the intersection of all juxtaposed consequence relations over \vdash_1 and \vdash_2 . It is routine to show that the juxtaposition of \vdash_1 and \vdash_2 is a juxtaposed consequence relation over \vdash_1 and \vdash_2 .²³

3.3. Semantics. In developing a semantics for juxtaposed consequence relations, one might think that we should base our semantics on classes of unital structures (or classes of structures, more generally) just as above. However, this is not the best approach to take. By Theorem 2.2, a consequence relation is strongly determined with respect to a non-empty class of unital structures only if it is left-extensional. But the juxtaposition of two left-extensional consequence relations need not be left-extensional. Indeed, as we will see below, although the classical consequence relation is left-extensional, the juxtaposition of two copies of the classical consequence relation is not. So, if we would like a strong unital determination result for this consequence relation, we must rely upon a somewhat different semantics.

A *juxtaposed structure* over the signatures C_1 and C_2 is an ordered pair $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ such that each \mathcal{B}_i is a structure over C_i . If \mathcal{B}_i is a unital structure, we write 1_i to stand for the designated value of \mathcal{B}_i . If \mathcal{B}_1 and \mathcal{B}_2 are unital structures, we say that $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ is a *juxtaposed unital structure*.

Suppose \mathcal{B}_1 is a structure over C_1 and \mathcal{B}_2 is a structure over C_2 . Then the *juxtaposition of the structures* \mathcal{B}_1 and \mathcal{B}_2 is the ordered pair $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$. Suppose \mathbb{B}_1 is a class of structures over C_1 and \mathbb{B}_2 is a class of structures over C_2 . Then the *juxtaposition of the classes of structures* \mathbb{B}_1 and \mathbb{B}_2 is the Cartesian product $\mathbb{B}_1 \times \mathbb{B}_2$.

The set of *i-atoms* contains the elements of $\text{Sent}(C_{12}, P_{12})$ that do not have a main connective from C_i . For example, if C_1 and C_2 are disjoint then the 1-atoms include the elements of P_{12} as well as those sentences with a main connective from C_2 . If C_1 and C_2 overlap then the 1-atoms include the elements of P_{12} as well as those sentences with a main connective from C_2 but not from C_1 .

A *juxtaposed model* over the signatures C_1 and C_2 and the set of sentence symbols P_{12} is an ordered quadruple $\langle \mathcal{B}_1, V_1, \mathcal{B}_2, V_2 \rangle$ such that $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ is a juxtaposed structure over C_1 and C_2 , and each V_i is a function from the set of i-atoms of $\text{Sent}(C_{12}, P_{12})$ to B_i . In effect, a juxtaposed model is a pair of models, each of which treats sentences with main connectives not from its signature as if they were additional sentence symbols.

We say that the juxtaposed model $\langle \mathcal{B}_1, V_1, \mathcal{B}_2, V_2 \rangle$ is *based on* the juxtaposed structure $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$. More generally, we say that a juxtaposed model \mathcal{M}_{12} is *based on* the class of juxtaposed structures \mathbb{B}_{12} just in case \mathcal{M}_{12} is based on one of the elements of \mathbb{B}_{12} . If \mathcal{M}_{12} is based on a juxtaposed unital structure, we say that \mathcal{M}_{12} is a *juxtaposed unital model*.

²³Cruz-Filipe et al. (2007) defines a more general notion of the fibring of consequence relations that also applies in the case of non-substitution invariant consequence relations. By Propositions 2.18 and 2.19 in that paper, the fibring of two substitution invariant consequence relations coincides with their juxtaposition.

Suppose $\mathcal{M}_{12} = \langle \mathcal{B}_1, V_1, \mathcal{B}_2, V_2 \rangle$ is a juxtaposed model over C_1 , C_2 , and P_{12} , where each $\mathcal{B}_i = \langle B_i, D_i, \Phi_i \rangle$. For any $\alpha \in \text{Sent}(C_{12}, P_{12})$, the i -value of α in \mathcal{M}_{12} , $\|\alpha\|_i^{\mathcal{M}_{12}}$, is recursively defined as follows:

- $\|\alpha\|_i^{\mathcal{M}_{12}} = V_i(\alpha)$ if α is an i-atom;
- $\|c\alpha^1 \dots \alpha^n\|_i^{\mathcal{M}_{12}} = \Phi_i(c)(\|\alpha^1\|_i^{\mathcal{M}_{12}} \dots \|\alpha^n\|_i^{\mathcal{M}_{12}})$ if $c \in C_i^n$ and $\alpha^1, \dots, \alpha^n \in \text{Sent}(C_{12}, P_{12})$.

We sometimes omit the \mathcal{M}_{12} if the relevant juxtaposed model is clear from context.

We say that a juxtaposed model is *coherent* just in case for every sentence $\alpha \in \text{Sent}(C_{12}, P_{12})$, $\|\alpha\|_1^{\mathcal{M}_{12}} \in D_1$ just in case $\|\alpha\|_2^{\mathcal{M}_{12}} \in D_2$. In other words, α has a designated 1-value just in case it has a designated 2-value. The notion of coherence is needed to provide a sensible definition of designation for juxtaposed models.

Given a coherent juxtaposed model \mathcal{M}_{12} , we write $\mathcal{M}_{12} \models \alpha$ to mean that \mathcal{M}_{12} *designates* the sentence α . This obtains just in case $\|\alpha\|_1^{\mathcal{M}_{12}} \in D_1$ (or equivalently, $\|\alpha\|_2^{\mathcal{M}_{12}} \in D_2$). We write $\mathcal{M}_{12} \models \Gamma$ to mean that \mathcal{M}_{12} *designates* each of the sentences in Γ . This obtains just in case for every $\gamma \in \Gamma$, $\mathcal{M}_{12} \models \gamma$.

If a coherent juxtaposed model \mathcal{M}_{12} over C_1 , C_2 , and P_{12} either designates every element of $\text{Sent}(C_{12}, P_{12})$ or designates no element of $\text{Sent}(C_{12}, P_{12})$, we say that \mathcal{M}_{12} is *trivial*. Otherwise, we say that \mathcal{M}_{12} is *non-trivial*. We show below (Proposition 5.2) that if C_1 and C_2 are disjoint then for any structure \mathcal{B}_{12} over C_1 , C_2 , and any set of sentence symbols P_{12} , there is a coherent non-trivial juxtaposed model over C_1 , C_2 , and P_{12} based on \mathcal{B}_{12} .

Given a class of juxtaposed structures \mathbb{B}_{12} over C_1 and C_2 , we write $\models^{\mathbb{B}_{12}} \alpha$ to mean that α is *valid* in \mathbb{B}_{12} . This obtains just in case for every coherent juxtaposed model \mathcal{M}_{12} over C_1 , C_2 , and P_{12} based on \mathbb{B}_{12} , $\mathcal{M}_{12} \models \alpha$. We write $\Gamma \models^{\mathbb{B}_{12}} \alpha$ to mean that Γ *entails* α in \mathbb{B}_{12} . This obtains just in case, for every coherent juxtaposed model \mathcal{M}_{12} over C_1 , C_2 , and P_{12} based on \mathbb{B}_{12} , if $\mathcal{M}_{12} \models \Gamma$ then $\mathcal{M}_{12} \models \alpha$.²⁴

Notice that this semantics is philosophically very natural. The two models in a coherent juxtaposed model can be thought of as representing speakers of different languages who take on board one another's claims without interpreting them in their home languages.

3.4. Soundness and Completeness. Suppose \vdash_{12} is a consequence relation for $\text{Sent}(C_{12}, P_{12})$, and \mathbb{B}_{12} is a class of juxtaposed structures over C_1 and C_2 . We say that \vdash_{12} is *strongly sound* with respect to \mathbb{B}_{12} just in case for every $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ and $\alpha \in \text{Sent}(C_{12}, P_{12})$, if $\Gamma \vdash_{12} \alpha$ then $\Gamma \models^{\mathbb{B}_{12}} \alpha$. We say that \vdash_{12} is *strongly complete* with respect to \mathbb{B}_{12} just in case for every $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ and $\alpha \in \text{Sent}(C_{12}, P_{12})$, if $\Gamma \models^{\mathbb{B}_{12}} \alpha$ then $\Gamma \vdash_{12} \alpha$. We say that \vdash_{12} is *strongly determined* with respect to \mathbb{B}_{12} just in case for every $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ and $\alpha \in \text{Sent}(C_{12}, P_{12})$, $\Gamma \models^{\mathbb{B}_{12}} \alpha$ just in case $\Gamma \vdash_{12} \alpha$.

²⁴As before, the relevant set of sentence symbols is fixed by context.

We say that \vdash_{12} is $\langle C_1, C_2 \rangle$ -*strongly sound*, $\langle C_1, C_2 \rangle$ -*strongly complete*, or $\langle C_1, C_2 \rangle$ -*strongly determined* just in case \vdash_{12} is strongly sound, strongly complete, or strongly determined (respectively) with respect to some class of juxtaposed structures over C_1 and C_2 that has a coherent non-trivial juxtaposed model based on it. We say that \vdash_{12} is $\langle C_1, C_2 \rangle$ -*strongly unital sound*, $\langle C_1, C_2 \rangle$ -*strongly unital complete*, or $\langle C_1, C_2 \rangle$ -*strongly unital determined* just in case \vdash_{12} is strongly sound, strongly complete, or strongly determined (respectively) with respect to some class of juxtaposed unital structures over C_1 and C_2 that has a coherent non-trivial juxtaposed model based on it.²⁵

§4. Algebraic Fibring and Modulated Fibring. To better understand juxtaposition, it may be helpful to compare it with other methods of combining logics. In this section, I briefly compare juxtaposition with two other such methods – algebraic fibring and modulated fibring.²⁶

The algebraic fibring of two consequence relations is essentially the same as their juxtaposition. The sole difference is that algebraic fibring is applied to Hilbert calculi, which correspond to compact consequence relations. The important differences arise in the semantics.

The semantics for algebraic fibring makes use of partially ordered unital matrices. In the notation of this paper, a partially ordered unital structure is a quadruple $\langle B, \leq, \{1\}, \Phi \rangle$ where $\langle B, \{1\}, \Phi \rangle$ is a unital structure and \leq is a partial order on B with the top element 1. The purpose of the partial ordering is to enable one to define two notions of entailment. Given a class of partially ordered unital structures, \mathbb{B} , Γ *globally entails* α in \mathbb{B} just in case in every model based on \mathbb{B} , if each element of Γ is designated, then so is α . By contrast, Γ *locally entails* α just in case in every model based on \mathbb{B} , the value of each element of Γ is less than or equal to the value of α . For simplicity, however, in what follows, we'll focus on global entailment.

Suppose $\mathcal{B} = \langle B, \leq, \{1\}, \Phi \rangle$ is a partially ordered unital structure over C . Suppose C' is a sub-signature of C . The *reduct* of \mathcal{B} to C' is $\mathcal{B}|_{C'} = \langle B, \leq, \{1\}, \Phi|_{C'} \rangle$, where $\Phi|_{C'}$ is the restriction of Φ to the connectives from C' . Let \mathbb{B}_1 be a class of partially ordered unital structures over C_1 and let \mathbb{B}_2 be a class of partially ordered unital structures over C_2 . The *algebraic fibring* of \mathbb{B}_1 and \mathbb{B}_2 is $\mathbb{B}_{12} = \{\mathcal{B} \mid \mathcal{B}|_{C_1} \in \mathbb{B}_1 \text{ and } \mathcal{B}|_{C_2} \in \mathbb{B}_2\}$. This is a very natural construction. The algebraic fibring of two classes of partially ordered unital structures is simply a class of partially ordered unital structures. This is unlike the case of juxtaposition – the juxtaposition of two classes of unital structures is a class of pairs of unital structures.

One weakness of algebraic fibring compared to juxtaposition is that the strong determination theorem is weaker. Consider the following principle:

²⁵In the case where C_1 and C_2 are disjoint, Proposition 5.2 enables us to simplify these definitions: \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly sound, $\langle C_1, C_2 \rangle$ -strongly complete, or $\langle C_1, C_2 \rangle$ -strongly determined just in case \vdash_{12} is strongly sound, strongly complete, or strongly determined (respectively) with respect to some non-empty class of juxtaposed structures over C_1 and C_2 . (And similarly for the cases of $\langle C_1, C_2 \rangle$ -strong unital soundness, completeness, and determination.)

²⁶See Carnielli et al. (2008) for a comprehensive overview of these two methods.

Entailment Congruence. If $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\beta\} \vdash \alpha$ then for any $\delta \in \text{Sent}(C, P)$ and p occurring in δ , $\Gamma \cup \{\delta[\alpha/p]\} \vdash \delta[\beta/p]$.²⁷

(This is a significant strengthening of left-extensionality.) The strong determination result for algebraic fibring relies upon the following basic result: Any consequence relation that has theorems and obeys Entailment Congruence is strongly determined with respect to some class of partially ordered unital structures.²⁸

We say that a consequence relation \vdash over $\text{Sent}(C, P)$ has *implication* \rightarrow just in case $\rightarrow \in C^2$ and \vdash obeys Modus Ponens and Conditional Introduction with respect to \rightarrow . Given a consequence relation \vdash that has implication \rightarrow , we say that \vdash has *equivalence* \leftrightarrow just in case $\leftrightarrow \in C^2$ and \vdash obeys the following rules: $\{\alpha \leftrightarrow \beta\} \vdash \alpha \rightarrow \beta$; $\{\alpha \leftrightarrow \beta\} \vdash \beta \rightarrow \alpha$; $\{\alpha \rightarrow \beta, \beta \rightarrow \alpha\} \vdash \alpha \leftrightarrow \beta$; and $\{\alpha \leftrightarrow \beta\} \vdash \delta[\alpha/p] \leftrightarrow \delta[\beta/p]$. The point of these definitions is that any consequence relation that has implication and equivalence obeys Entailment Congruence. Moreover, if two compact consequence relations each have the very same implication and equivalence then the algebraic fibring of the two consequence relations will have implication and equivalence, too.

The strong determination result for algebraic fibring is as follows: Suppose \vdash_1 and \vdash_2 are compact consequence relations that have the very same implication and equivalence. Suppose \vdash_{12} is the algebraic fibring of \vdash_1 and \vdash_2 . Then \vdash_{12} is strongly determined with respect to some class of partially ordered unital structures. In particular, \vdash_{12} is strongly determined with respect to the algebraic fibring of the class of all partially ordered unital structures for \vdash_1 and the class of all partially ordered unital structures for \vdash_2 .²⁹

This result is significantly weaker than our strong unital determination result below. By Corollary 6.30, if two consequence relations are each consistent and left-extensional and at least one of them has theorems, then their juxtaposition is strongly unital determined. The two consequence relations need not be compact. They need not have overlapping signatures. They need not have implication or equivalence. Indeed, for the result to apply, the juxtaposition of the two consequence relations need not obey Entailment Congruence.

A related issue is that algebraic fibring is not suitable to use in studying the collapse of classical and intuitionist logics.³⁰ One way to see the problem is in terms of the semantics. The classical consequence relation is strongly determined with respect to the class of all partially ordered unital structures in which the partial ordering corresponds to a Boolean algebra. The intuitionist consequence relation is strongly determined with respect to the class of all partially ordered unital structures in which the partial ordering corresponds to a Heyting algebra. The algebraic fibring of these two classes of partially ordered unital structures is a class of partially ordered unital structures in which the relevant reducts are

²⁷This property is called “congruentiality” in Rautenberg (1981). Consequence relations obeying Entailment Congruence are called “Fregean” in Czelakowski & Pigozzi (2004) and “weakly extensional” in Humberstone (2011), pages 455-456.

²⁸See Theorem 5.7 in Zanardo et al. (2001) and Theorem 3.3.15 in Carnielli et al. (2008) for a closely related but somewhat more complicated result.

²⁹See Theorem 6.7 in Zanardo et al. (2001) and Theorem 3.3.18 in Carnielli et al. (2008).

³⁰This is well-known in the literature on fibring and algebraic fibring. See, for instance, Gabbay (1998), page 8.

both Boolean and Heyting algebras. But these must all be Boolean algebras. Thus, the resulting logic ends up behaving purely classically for both stocks of connectives. Indeed, we can show that corresponding connectives become intersubstitutable. The logic collapses.

There is a second way to see the problem. Suppose we have a language with two of each of the usual logical connectives. Suppose \vdash^{ii} is the consequence relation for this language that obeys all of the intuitionist theorems and entailments for each stock of connectives. As we will see below, any consequence relation that extends \vdash^{ii} and obeys Entailment Congruence collapses. But the proof of the strong determination result for algebraic fibring crucially depends on the combined logic obeying Entailment Congruence. So there is no way to avoid collapse.

There are methods of combining logics designed to avoid collapse.³¹ Perhaps the most well worked out is modulated fibring.³² The method of combining consequence relations and the method of combining classes of structures are significantly more complicated for modulated fibring than for algebraic fibring or for juxtaposition. But the basic idea is straightforward. A modulated structure is a quadruple $\langle B, \leq, \{1\}, \Phi \rangle$ such that $\langle B, \{1\}, \Phi \rangle$ is a structure and \leq is a pre-order with finite meets and top element 1. Each modulated structure \mathcal{B}_{12} in the modulated fibring \mathbb{B}_{12} of two classes of modulated structures \mathbb{B}_1 and \mathbb{B}_2 corresponds to a pair (or pairs) of modulated structures $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$, where $\mathcal{B}_1 \in \mathbb{B}_1$ and $\mathcal{B}_2 \in \mathbb{B}_2$. The semantic values of \mathcal{B}_{12} are the union of the semantic values of the original two structures. To get the construction to work, however, a restriction has to be imposed on which pairs of modulated structures yield a modulated structure in \mathbb{B}_{12} . In particular, there must be translations between the semantic values of \mathcal{B}_1 and the semantic values of \mathcal{B}_2 . These translations are established by something called a bridge, which is provided as an input to the modulated fibring procedure.

The proof theory for modulated fibring relies on a variant of Hilbert calculi called modulated Hilbert calculi. The strong determination result for algebraic fibring relies upon the fact that any modulated Hilbert calculus that has theorems and obeys Entailment Congruence is strongly determined with respect to some class of modulated structures.³³ The strong determination result for modulated fibring is as follows: The modulated fibring (by an adequate bridge) of two modulated Hilbert calculi that each have theorems and obey Entailment Congruence is strongly determined with respect to a class of modulated structures. In particular, it is strongly determined with respect to the modulated fibring (by the bridge) of the class of all modulated structures in which \vdash_1 is sound and the class of modulated structures in which \vdash_2 is sound.³⁴ Moreover, a sufficient condition for there being an adequate bridge is that the two consequence relations are for disjoint languages.³⁵ So there is a strong determination

³¹See, for instance, crypto-fibring as defined in Caleiro & Ramos (2007).

³²See Sernadas et al. (2002). Also see Carnielli et al. (2008), chapter 8.

³³See Theorem 5.6 in Sernadas et al. (2002) and Theorem 8.5.10 in Carnielli et al. (2008) for a closely related result.

³⁴See Theorem 5.12 in Sernadas et al. (2002) and Theorem 8.5.16 in Carnielli et al. (2008).

³⁵See Example 5.13 in Sernadas et al. (2002) and Example 8.5.17 in Carnielli et al. (2008).

result for any pair of modulated Hilbert calculi for disjoint languages which have theorems and obey Entailment Congruence.

As before, this result is weaker than the strong determination result for juxtaposition. However, it can be used to combine the classical and intuitionist consequence relations. Moreover, it can be shown that the resulting modulated Hilbert calculus does not collapse. So modulated fibring does provide a way to avoid the collapse of classical and intuitionist logics.

This may sound strange given our result that any consequence relation that extends \vdash^{ii} and obeys Entailment Congruence collapses. Modulated fibring is designed to preserve Entailment Congruence. So the modulated fibring of classical and intuitionist logic obeys Entailment Congruence. Why doesn't it collapse?

The answer is that the result of modulated fibring classical and intuitionist logic (over disjoint languages by an adequate bridge) is not a consequence relation in the sense defined above. The resulting relation obeys the usual structural rules, but it is not substitution invariant. A modulated Hilbert calculus comes with a set of "safe substitutions" and is only guaranteed to be substitution invariant with respect to those substitutions. For the particular case of combining classical and intuitionist logic, the relevant set of safe substitutions only permit us to uniformly substitute sentences with a intuitionist main connective (or an intuitionist sentence symbol) into the intuitionist axioms and rules.

Thus, while modulated fibring enables us to combine classical and intuitionist logic without collapse, it does not yield a substitution invariant consequence relation. As we will see below, juxtaposition enables us to combine classical and intuitionist logic in a way that preserves substitution invariance. The cost is that the resulting consequence relation does not obey Entailment Congruence. But that seems to be a smaller cost to bear.

§5. Preservation Theorems. In this section, we prove general results about the metalogical properties of juxtaposition. In particular, we show that if the two signatures are disjoint, juxtaposition preserves strong soundness and strong unital soundness. We show that under reasonable conditions, juxtaposition preserves consistency. We also show that under reasonable conditions, the juxtaposition of two consequence relations is a strong conservative extension of each of them.

In what follows, in this section and the next, we assume that C_1 and C_2 are two signatures and C_{12} is their juxtaposition. Unless otherwise specified, C_1 and C_2 may overlap. We assume that P_1 and P_2 are sets of sentence symbols and P_{12} is their juxtaposition. We assume that \vdash_1 is a consequence relation for $\text{Sent}(C_1, P_1)$, \vdash_2 is a consequence relation for $\text{Sent}(C_2, P_2)$, and \vdash_{12} is a consequence relation for $\text{Sent}(C_{12}, P_{12})$. Unless otherwise specified, \vdash_{12} need not be the juxtaposition of \vdash_1 and \vdash_2 , or even a juxtaposed consequence relation over \vdash_1 and \vdash_2 . We also assume that C is a signature, C^- is a sub-signature of C , P is a set of sentence symbols, and \vdash is a consequence relation for $\text{Sent}(C, P)$.

5.1. The Existence of Coherent Non-Trivial Models. We first provide a simple sufficient condition for when there is a coherent non-trivial juxtaposed model based on a juxtaposed structure. To prove this result, we introduce another semantic notion, that of a juxtaposition of two models.

Suppose $\mathcal{M}_1 = \langle \mathcal{B}_1, V_1 \rangle$ is a model over C_1 and P_1 and $\mathcal{M}_2 = \langle \mathcal{B}_2, V_2 \rangle$ is a model over C_2 and P_2 . A *juxtaposition of the models* \mathcal{M}_1 and \mathcal{M}_2 is a juxtaposed model, $\langle \mathcal{B}_1, V_1^+, \mathcal{B}_2, V_2^+ \rangle$, over C_1, C_2 , and P_{12} such that:

- If $p \in P_1$, $V_1^+(p) = V_1(p)$; and
- If $p \in P_2$, $V_2^+(p) = V_2(p)$.

Notice that a juxtaposition of two unital models is a juxtaposed unital model.

A juxtaposition of two models need not be coherent. Given two models, there need not be a coherent juxtaposition of them. If there is a coherent juxtaposition, it need not be unique.

Suppose \mathcal{M}_{12} is a juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 . It is routine to show that for any $\alpha \in \text{Sent}(C_i, P_i)$, $\|\alpha\|_i^{\mathcal{M}_{12}} = \|\alpha\|^{\mathcal{M}_i}$. Therefore, if \mathcal{M}_{12} is coherent then for any $\alpha \in \text{Sent}(C_i, P_i)$, $\mathcal{M}_{12} \models \alpha$ just in case $\mathcal{M}_i \models \alpha$.

The following lemma provides a simple sufficient condition on when two models have a coherent juxtaposition.

LEMMA 5.1. *Suppose C_1 and C_2 are disjoint signatures. Suppose $\mathcal{M}_1 = \langle \mathcal{B}_1, V_1 \rangle$ is a model over C_1 and P_1 and $\mathcal{M}_2 = \langle \mathcal{B}_2, V_2 \rangle$ is a model over C_2 and P_2 . Then there is a coherent juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 just in case for every $p \in P_1 \cap P_2$, $\mathcal{M}_1 \models p$ just in case $\mathcal{M}_2 \models p$.*

PROOF. Suppose there is some $p \in P_1 \cap P_2$ such that $\mathcal{M}_1 \models p$ and $\mathcal{M}_2 \not\models p$. $V_1(p) \in D_1$ and $V_2(p) \notin D_2$. So there is no coherent juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 . Similarly, if $\mathcal{M}_1 \not\models p$ and $\mathcal{M}_2 \models p$, there is no coherent juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 .

Now suppose for every $p \in P_1 \cap P_2$, $\mathcal{M}_1 \models p$ just in case $\mathcal{M}_2 \models p$. We show that there is a coherent juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 .

Let d_i be an element of D_i and let a_i be an element of $B_i - D_i$. Let $[\]_i$ be the function from $\text{Sent}(C_{12}, P_{12})$ to B_i inductively defined as follows:

- If $p \in P_i$, $[p]_i = V_i(p)$;
- If $p \in P_1 - P_2$, $[p]_2 = d_2$ if $V_1(p) \in D_1$ and $[p]_2 = a_2$ otherwise;
- If $p \in P_2 - P_1$, $[p]_1 = d_1$ if $V_2(p) \in D_2$ and $[p]_1 = a_1$ otherwise;
- If $c \in C_i^n$, $[c\alpha^1 \dots \alpha^n]_i = \Phi_i(c)([\alpha^1]_i \dots [\alpha^n]_i)$;
- If $c \in C_1^n$, $[c\alpha^1 \dots \alpha^n]_2 = d_2$ if $\Phi_1(c)([\alpha^1]_1 \dots [\alpha^n]_1) \in D_1$ and $[c\alpha^1 \dots \alpha^n]_2 = a_2$ otherwise;
- If $c \in C_2^n$, $[c\alpha^1 \dots \alpha^n]_1 = d_1$ if $\Phi_2(c)([\alpha^1]_2 \dots [\alpha^n]_2) \in D_2$ and $[c\alpha^1 \dots \alpha^n]_1 = a_1$ otherwise.

If α is an i -atom, let $V_i^+(\alpha) = [\alpha]_i$. Let $\mathcal{M}_{12} = \langle \mathcal{B}_1, V_1^+, \mathcal{B}_2, V_2^+ \rangle$. Clearly, \mathcal{M}_{12} is a juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 . We show that for each $\alpha \in \text{Sent}(C_{12}, P_{12})$, $\|\alpha\|_1^{\mathcal{M}_{12}} \in D_1$ just in case $\|\alpha\|_2^{\mathcal{M}_{12}} \in D_2$. Notice that $\|\alpha\|_i^{\mathcal{M}_{12}} = [\alpha]_i$. So we need only to show that $[\alpha]_1 \in D_1$ just in case $[\alpha]_2 \in D_2$.

If $p \in P_1 \cap P_2$, $[p]_1 \in D_1$ just in case $V_1(p) \in D_1$ just in case $\mathcal{M}_1 \models p$ just in case $\mathcal{M}_2 \models p$ just in case $V_2(p) \in D_2$ just in case $[p]_2 \in D_2$. If $p \in P_1 - P_2$, $[p]_1 \in D_1$ just in case $V_1(p) \in D_1$ just in case $[p]_2 = d_2$ just in case $[p]_2 \in D_2$. If $p \in P_2 - P_1$, $[p]_2 \in D_2$ just in case $V_2(p) \in D_2$ just in case $[p]_1 = d_1$ just in case $[p]_1 \in D_1$. If $c \in C_1^n$ and $\alpha^1 \dots \alpha^n \in \text{Sent}(C_{12}, P_{12})$, $[c\alpha^1 \dots \alpha^n]_1 \in D_1$ just in case $\Phi_1(c)([\alpha^1]_1 \dots [\alpha^n]_1) \in D_1$ just in case $[c\alpha^1 \dots \alpha^n]_2 = d_2$ (since C_1 and C_2 are disjoint) just in case $[c\alpha^1 \dots \alpha^n]_2 \in D_2$. If $c \in C_2^n$ and $\alpha^1 \dots \alpha^n \in$

$\text{Sent}(C_{12}, P_{12})$, $[\alpha^1 \dots \alpha^n]_2 \in D_2$ just in case $\Phi_2(c)([\alpha^1]_2 \dots [\alpha^n]_2) \in D_2$ just in case $[\alpha^1 \dots \alpha^n]_1 = d_1$ (since C_1 and C_2 are disjoint) just in case $[\alpha^1 \dots \alpha^n]_1 \in D_1$. \square

Applying this lemma, we have the following result:

PROPOSITION 5.2 (Existence of Non-Trivial Models). *Suppose C_1 and C_2 are disjoint signatures. Suppose $\mathcal{B}_{12} = \langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ is a juxtaposed structure over C_1 and C_2 . Then there is a coherent non-trivial juxtaposed model over C_1 , C_2 , and P_{12} based on \mathcal{B}_{12} .*

PROOF. Let V_1 be a function from P_1 to B_1 that maps some element of P_1 to an element of D_1 and some element of P_1 to an element of $B_1 - D_1$. Let V_2 be a function from P_2 to B_2 such that for every $p \in P_1 \cap P_2$, $V_2(p) \in D_2$ just in case $V_1(p) \in D_1$. It is easy to see that such functions exist. Then $\mathcal{M}_i = \langle \mathcal{B}_i, V_i \rangle$ is a model over C_i and P_i . For every $p \in P_1 \cap P_2$, $\mathcal{M}_1 \models p$ just in case $\mathcal{M}_2 \models p$. By Lemma 5.1, there is a coherent juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 . Any juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 is non-trivial and is based on \mathcal{B}_{12} . So there is a coherent non-trivial juxtaposed model over C_1 , C_2 , and P_{12} based on \mathcal{B}_{12} . \square

The restriction to disjoint signatures is important. There are juxtaposed structures that do not have coherent juxtaposed models based on them. For example, let C be a signature with a single unary connective, c . Let $B = \{0, 1\}$ and $D = \{1\}$. Let $\Phi_1(c)$ be the identity mapping on B and let $\Phi_2(c)$ be the function that maps 0 to 1 and 1 to 0. Let $\mathcal{B}_1 = \langle B, D, \Phi_1 \rangle$ and let $\mathcal{B}_2 = \langle B, D, \Phi_2 \rangle$. It is easy to see that there is no coherent juxtaposed model based on $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$. There are also juxtaposed structures that have only trivial coherent juxtaposed models based on them. For example, let $\Phi'_2(c)$ be the function that maps both 0 and 1 to 1. Let $\mathcal{B}'_2 = \langle B, D, \Phi'_2 \rangle$. It is easy to see that the only coherent juxtaposed models based on $\langle \mathcal{B}_1, \mathcal{B}'_2 \rangle$ are trivial.

5.2. The Preservation of Strong Soundness. Our next general result concerns the preservation of strong soundness by juxtaposition. We show that if \vdash_1 is strongly sound with respect to \mathbb{B}_1 and \vdash_2 is strongly sound with respect to \mathbb{B}_2 then the juxtaposition of \vdash_1 and \vdash_2 is strongly sound with respect to the juxtaposition of \mathbb{B}_1 and \mathbb{B}_2 . This result does not require that C_1 and C_2 be disjoint. However, to show that juxtaposition preserves strong soundness or strong unital soundness (simpliciter), we must show that there is a coherent non-trivial juxtaposed model based on \mathbb{B}_{12} . In the case where C_1 and C_2 are disjoint, we can apply Proposition 5.2.

LEMMA 5.3. *Suppose \mathbb{B}_{12} is a class of juxtaposed structures over C_1 and C_2 . Then $\models^{\mathbb{B}_{12}}$ is a consequence relation for $\text{Sent}(C_{12}, P_{12})$.*

PROOF. $\models^{\mathbb{B}_{12}}$ is a relation between subsets of $\text{Sent}(C_{12}, P_{12})$ and individual elements of $\text{Sent}(C_{12}, P_{12})$. It suffices to show that $\models^{\mathbb{B}_{12}}$ satisfies Identity, Weakening, Cut, and Uniform Substitution.

Identity: Trivially, every coherent juxtaposed model based on \mathbb{B}_{12} that designates α designates α . So $\{\alpha\} \models^{\mathbb{B}_{12}} \alpha$.

Weakening: Suppose $\Gamma \models^{\mathbb{B}_{12}} \alpha$. So every coherent juxtaposed model based on \mathbb{B}_{12} that designates each element of Γ also designates α . In particular, every

coherent juxtaposed model based on \mathbb{B}_{12} that designates each element of $\Gamma \cup \Delta$ also designates α . So $\Gamma \cup \Delta \models^{\mathbb{B}_{12}} \alpha$.

Cut: Suppose $\Gamma \models^{\mathbb{B}_{12}} \alpha$ and $\Delta \cup \{\alpha\} \models^{\mathbb{B}_{12}} \beta$. Suppose \mathcal{M}_{12} is a coherent juxtaposed model based on \mathbb{B}_{12} that designates each element of $\Gamma \cup \Delta$. Since every coherent juxtaposed model based on \mathbb{B}_{12} that designates each element of Γ also designates α , \mathcal{M}_{12} designates α . Since every coherent juxtaposed model based on \mathbb{B}_{12} that designates each element of $\Delta \cup \{\alpha\}$ also designates β , \mathcal{M}_{12} designates β . So $\Gamma \cup \Delta \models^{\mathbb{B}_{12}} \beta$.

Uniform Substitution: Suppose $\Gamma[\beta/p] \not\models^{\mathbb{B}_{12}} \alpha[\beta/p]$. Since $\Gamma[\beta/p] \not\models^{\mathbb{B}_{12}} \alpha[\beta/p]$, there is a coherent juxtaposed model $\mathcal{M}_{12} = \langle \mathcal{B}_1, V_1, \mathcal{B}_2, V_2 \rangle$ based on \mathbb{B}_{12} that designates each element of $\Gamma[\beta/p]$ but does not designate $\alpha[\beta/p]$. Let $V'_i(\delta) = \|\delta[\beta/p]\|_i^{\mathcal{M}_{12}}$ if δ is an i -atom. Let $\mathcal{M}'_{12} = \langle \mathcal{B}_1, V'_1, \mathcal{B}_2, V'_2 \rangle$. Clearly, \mathcal{M}'_{12} is a juxtaposed model based on \mathbb{B}_{12} . By a simple induction, for any $\delta \in \text{Sent}(C_{12}, P_{12})$, $\|\delta\|_i^{\mathcal{M}'_{12}} = \|\delta[\beta/p]\|_i^{\mathcal{M}_{12}}$. Since \mathcal{M}_{12} is coherent, so is \mathcal{M}'_{12} . \mathcal{M}'_{12} designates each element of Γ but does not designate α . Therefore, $\Gamma \not\models^{\mathbb{B}_{12}} \alpha$. \square

LEMMA 5.4. *Suppose \mathbb{B}_{12} is the juxtaposition of \mathbb{B}_1 and \mathbb{B}_2 . If $\Gamma \models^{\mathbb{B}_1} \alpha$ or $\Gamma \models^{\mathbb{B}_2} \alpha$, then $\Gamma \models^{\mathbb{B}_{12}} \alpha$.*

PROOF. Suppose $\Gamma \models^{\mathbb{B}_i} \alpha$ for some $i \in \{1, 2\}$, $\Gamma \subseteq \text{Sent}(C_i, P_i)$, and $\alpha \in \text{Sent}(C_i, P_i)$. Suppose $\mathcal{M}_{12} = \langle \mathcal{B}_1, V_1, \mathcal{B}_2, V_2 \rangle$ is a coherent juxtaposed model based on \mathbb{B}_{12} such that $\mathcal{M}_{12} \models \Gamma$. Let $V_i|_{P_i}$ be the restriction of V_i to P_i . Let $\mathcal{M}_i|_{P_i} = \langle \mathcal{B}_i, V_i|_{P_i} \rangle$. $\mathcal{M}_i|_{P_i}$ is a model over C_i and P_i . For every $\beta \in \text{Sent}(C_i, P_i)$, $\|\beta\|_i^{\mathcal{M}_i|_{P_i}} = \|\beta\|_i^{\mathcal{M}_{12}}$. So $\mathcal{M}_i|_{P_i} \models \Gamma$. Since $\mathcal{M}_i|_{P_i}$ is based on \mathbb{B}_i , $\mathcal{M}_i|_{P_i} \models \alpha$. So $\mathcal{M}_{12} \models \alpha$. Therefore, $\Gamma \models^{\mathbb{B}_{12}} \alpha$. \square

THEOREM 5.5 (Preservation of Strong Soundness). *Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 and \mathbb{B}_{12} is the juxtaposition of \mathbb{B}_1 and \mathbb{B}_2 . If \vdash_1 is strongly sound with respect to \mathbb{B}_1 and \vdash_2 is strongly sound with respect to \mathbb{B}_2 , then \vdash_{12} is strongly sound with respect to \mathbb{B}_{12} .*

PROOF. By the definition of \vdash_{12} , it suffices to show that $\models^{\mathbb{B}_{12}}$ is a consequence relation for $\text{Sent}(C_{12}, P_{12})$ such that if $\Gamma \vdash_1 \alpha$ or $\Gamma \vdash_2 \alpha$ then $\Gamma \models^{\mathbb{B}_{12}} \alpha$. By Lemma 5.3, $\models^{\mathbb{B}_{12}}$ is a consequence relation for $\text{Sent}(C_{12}, P_{12})$. By strong soundness, if $\Gamma \vdash_1 \alpha$ then $\Gamma \models^{\mathbb{B}_1} \alpha$ and if $\Gamma \vdash_2 \alpha$ then $\Gamma \models^{\mathbb{B}_2} \alpha$. By Lemma 5.4, if $\Gamma \models^{\mathbb{B}_1} \alpha$ or $\Gamma \models^{\mathbb{B}_2} \alpha$ then $\Gamma \models^{\mathbb{B}_{12}} \alpha$. Therefore, if $\Gamma \vdash_1 \alpha$ or $\Gamma \vdash_2 \alpha$ then $\Gamma \models^{\mathbb{B}_{12}} \alpha$. \square

This result is fully general – in particular, it does not require that C_1 and C_2 be disjoint. However, to show that juxtaposition preserves strong soundness or strong unital soundness (simpliciter), we must show that there is a coherent non-trivial juxtaposed model based on \mathbb{B}_{12} . If C_1 and C_2 are disjoint, we can apply Proposition 5.2. Indeed, if C_1 and C_2 are disjoint, we can show that the juxtaposition of two consequence relations is strongly sound just in case both of them are. To prove this stronger result, we first need to prove a simple lemma.

LEMMA 5.6. *Suppose C_1 and C_2 are disjoint signatures. Suppose \mathbb{B}_{12} is the juxtaposition of \mathbb{B}_1 and \mathbb{B}_2 , each of which is a non-empty class of structures. For any $i \in \{1, 2\}$, $\Gamma \subseteq \text{Sent}(C_i, P_i)$, and $\alpha \in \text{Sent}(C_i, P_i)$, if $\Gamma \models^{\mathbb{B}_{12}} \alpha$, then $\Gamma \models^{\mathbb{B}_i} \alpha$.*

PROOF. Suppose $\Gamma \subseteq \text{Sent}(C_1, P_1)$ and $\alpha \in \text{Sent}(C_1, P_1)$. Suppose $\Gamma \not\equiv^{\mathbb{B}_1} \alpha$. So there is a model \mathcal{M}_1 based on \mathbb{B}_1 such that $\mathcal{M}_1 \models \Gamma$ and $\mathcal{M}_1 \not\models \alpha$. Since \mathbb{B}_2 is non-empty, there are models based on it. By Lemma 5.1, there is a coherent juxtaposed model \mathcal{M}_{12} based on \mathbb{B}_{12} that is the juxtaposition of \mathcal{M}_1 with some \mathcal{M}_2 based on \mathbb{B}_2 . (We must pick \mathcal{M}_2 so that it designates the same elements of $P_1 \cap P_2$ as \mathcal{M}_1 .) So $\mathcal{M}_{12} \models \Gamma$ and $\mathcal{M}_{12} \not\models \alpha$. Therefore, $\Gamma \not\equiv^{\mathbb{B}_{12}} \alpha$.

The case where $\Gamma \subseteq \text{Sent}(C_2, P_2)$ and $\alpha \in \text{Sent}(C_2, P_2)$ is analogous. \square

COROLLARY 5.7. *Suppose C_1 and C_2 are disjoint signatures. Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . Then:*

1. \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly sound just in case \vdash_1 and \vdash_2 are each strongly sound;
2. \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly unital sound just in case \vdash_1 and \vdash_2 are each strongly unital sound.

PROOF. Suppose \vdash_1 is strongly sound with respect to \mathbb{B}_1 and \vdash_2 is strongly sound with respect to \mathbb{B}_2 , where \mathbb{B}_1 and \mathbb{B}_2 are non-empty classes of structures. Let \mathbb{B}_{12} be the juxtaposition of \mathbb{B}_1 and \mathbb{B}_2 . By Theorem 5.5, \vdash_{12} is strongly sound with respect to \mathbb{B}_{12} . Since \mathbb{B}_1 and \mathbb{B}_2 are non-empty, so is \mathbb{B}_{12} . By Proposition 5.2, there is a coherent non-trivial juxtaposed model based on \mathbb{B}_{12} . Therefore, \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly sound.

Now suppose \vdash_{12} is strongly sound with respect to \mathbb{B}_{12} , a non-empty class of structures. Suppose $\Gamma \subseteq \text{Sent}(C_i, P_i)$ and $\alpha \in \text{Sent}(C_i, P_i)$. Suppose $\Gamma \not\equiv^{\mathbb{B}_i} \alpha$. Since \mathbb{B}_{12} is non-empty, so are \mathbb{B}_1 and \mathbb{B}_2 . By Lemma 5.6, $\Gamma \not\equiv^{\mathbb{B}_{12}} \alpha$. By strong soundness, $\Gamma \not\models \alpha$. So $\Gamma \not\models_i \alpha$. So \vdash_i is strongly sound with respect to \mathbb{B}_i , a non-empty class of structures. Therefore, \vdash_1 and \vdash_2 are each strongly sound.

\mathbb{B}_{12} is a class of juxtaposed unital structures just in case \mathbb{B}_1 and \mathbb{B}_2 are each classes of unital structures. Therefore, \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly unital sound just in case \vdash_1 and \vdash_2 are each strongly unital sound. \square

5.3. Conservativeness and Consistency. Our next general results concern conservativeness and the preservation of consistency.

We first provide a sufficient condition on when the juxtaposition of two consequence relations is a strong conservative extension of the original relations.

PROPOSITION 5.8. *Suppose C_1 and C_2 are disjoint signatures. Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . If \vdash_1 and \vdash_2 are each strongly determined, then \vdash_{12} is a strong conservative extension of each of \vdash_1 and \vdash_2 .*

PROOF. Suppose $\Gamma \subseteq \text{Sent}(C_i, P_i)$ and $\alpha \in \text{Sent}(C_i, P_i)$.

Suppose $\Gamma \vdash_i \alpha$. By the definition of \vdash_{12} , $\Gamma \vdash_{12} \alpha$.

Now suppose $\Gamma \vdash_{12} \alpha$. Suppose \vdash_1 is strongly determined with respect to \mathbb{B}_1 and \vdash_2 is strongly determined with respect to \mathbb{B}_2 , where \mathbb{B}_1 and \mathbb{B}_2 are non-empty classes of structures. Let \mathbb{B}_{12} be the juxtaposition of \mathbb{B}_1 and \mathbb{B}_2 . By Theorem 5.5, \vdash_{12} is strongly sound with respect to \mathbb{B}_{12} . So $\Gamma \models^{\mathbb{B}_{12}} \alpha$. By Lemma 5.6, $\Gamma \models^{\mathbb{B}_i} \alpha$. Therefore, by strong completeness, $\Gamma \vdash_i \alpha$. \square

Conservativeness is closely tied to consistency. If Γ is consistent with respect to some consequence relation, Γ is consistent with respect to any strong conservative extension of it. Thus, we have the following:

PROPOSITION 5.9. *Suppose C_1 and C_2 are disjoint signatures. Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . Suppose \vdash_1 and \vdash_2 are each strongly determined. If (for $i = 1$ or 2) $\Gamma \subseteq \text{Sent}(C_i, P_i)$ is consistent with respect to \vdash_i , then Γ is consistent with respect to \vdash_{12} .*

Combining these results with Theorem 2.1, the following results are immediate:

THEOREM 5.10 (Preservation of Consistency). *Suppose C_1 and C_2 are disjoint signatures. Suppose each of \vdash_1 and \vdash_2 is consistent and has no mere followers. Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . Then \vdash_{12} is consistent. If $\Gamma \subseteq \text{Sent}(C_i, P_i)$ is consistent with respect to \vdash_i , Γ is consistent with respect to \vdash_{12} .*

THEOREM 5.11 (Strong Conservativeness). *Suppose C_1 and C_2 are disjoint signatures. Suppose each of \vdash_1 and \vdash_2 is consistent and has no mere followers. Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . Then \vdash_{12} is a strong conservative extension of each of \vdash_1 and \vdash_2 .³⁶*

§6. Strong Completeness and Strong Determination. As usual, it is somewhat more difficult to prove completeness results. In this section, we present direct proofs of strong completeness and strong unital completeness that apply in a wide range of cases.³⁷ The general strategy of proof relies on a modification of the familiar Lindenbaum-Tarski constructions.³⁸ We first prove a very abstract completeness result that requires there to be suitable equivalence relations with which to build our Lindenbaum-Tarski models. We then investigate when such relations exist.

6.1. The Lindenbaum-Tarski Construction. Let \sim be an equivalence relation on $\text{Sent}(C, P)$. We say that \sim is a *congruence* over C^- just in case:

- For every $c^- \in C^{-n}$, $\alpha^1, \dots, \alpha^n, \beta \in \text{Sent}(C, P)$, and $k \in \{1, \dots, n\}$, if $\alpha^k \sim \beta$ then $c^- \alpha^1 \dots \alpha^k \dots \alpha^n \sim c^- \alpha^1 \dots \beta \dots \alpha^n$.

We say that \sim is *compatible* with \vdash and $\Gamma \subseteq \text{Sent}(C, P)$ just in case:

- For every $\alpha, \beta \in \text{Sent}(C, P)$, if $\alpha \sim \beta$ then $\Gamma \vdash \alpha$ just in case $\Gamma \vdash \beta$.

We say that \sim is *strongly compatible* with \vdash and $\Gamma \subseteq \text{Sent}(C, P)$ just in case \sim is compatible with \vdash and Γ and:

- For every $\alpha, \beta \in \text{Sent}(C, P)$, if both $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$ then $\alpha \sim \beta$.

We say that \sim is *suitable* for C^- , \vdash , and Γ just in case \sim is a congruence over C^- compatible with \vdash and Γ . We say that \sim is *unital suitable* for C^- , \vdash , and Γ just in case \sim is a congruence over C^- strongly compatible with \vdash and Γ . We

³⁶See Cruz-Filipe et al. (2007), Proposition 2.17, for a slightly stronger result. In particular, the juxtaposition of two non-trivial consequence relations over disjoint signatures is a strong conservative extension of each of them. Their proof of this result is purely proof-theoretic, relying on a fixed-point argument.

³⁷See Zanardo et al. (2001) and Sernadas et al. (2002) for completeness results concerning algebraic fibring and modulated fibring, respectively.

³⁸See Rasiowa (1974) and Rasiowa & Sikorski (1970) for the application of the Lindenbaum-Tarski method to logics that contain conditionals. See Blok & Pigozzi (1989) for the application of this method in a more general setting.

make use of suitable and unital suitable equivalence relations in constructing our Lindenbaum-Tarski models.

Suppose Γ is a non-empty subset of $\text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} . Suppose for each $i \in \{1, 2\}$, \sim_i^Γ is an equivalence relation on $\text{Sent}(C_{12}, P_{12})$ suitable for C_i, \vdash_{12} , and Γ . We make the following definitions:

$$\begin{aligned} |\alpha|_i^\Gamma &= \{\beta \mid \alpha \sim_i^\Gamma \beta\} \\ \mathbf{B}_i^\Gamma &= \{|\alpha|_i^\Gamma \mid \alpha \in \text{Sent}(C_{12}, P_{12})\} \\ \mathbf{D}_i^\Gamma &= \{|\alpha|_i^\Gamma \mid \Gamma \vdash_{12} \alpha\} \\ \text{If } c_i \in C_i^n, \Phi_i^\Gamma(c_i)(|\alpha^1|_i^\Gamma, \dots, |\alpha^n|_i^\Gamma) &= |c_i \alpha^1 \dots \alpha^n|_i^\Gamma \\ \mathbf{B}_i^\Gamma &= \langle \mathbf{B}_i^\Gamma, \mathbf{D}_i^\Gamma, \Phi_i^\Gamma \rangle \\ \mathbf{B}_{12}^\Gamma &= \langle \mathbf{B}_1^\Gamma, \mathbf{B}_2^\Gamma \rangle \\ \text{If } \alpha \text{ is an i-atom, } \mathbf{V}_i^\Gamma(\alpha) &= |\alpha|_i^\Gamma \\ \mathbf{M}_{12}^\Gamma &= \langle \mathbf{B}_1^\Gamma, \mathbf{V}_1^\Gamma, \mathbf{B}_2^\Gamma, \mathbf{V}_2^\Gamma \rangle \end{aligned}$$

\mathbf{B}_{12}^Γ is the *Lindenbaum-Tarski juxtaposed structure* for C_1, C_2, \vdash_{12} , and Γ built with \sim_1^Γ and \sim_2^Γ . \mathbf{M}_{12}^Γ is the *Lindenbaum-Tarski juxtaposed model* for C_1, C_2, \vdash_{12} , and Γ built with \sim_1^Γ and \sim_2^Γ .

Notice that if $\sim_1^\Gamma = \sim_2^\Gamma$, then $\mathbf{B}_1^\Gamma = \mathbf{B}_2^\Gamma$, $\mathbf{D}_1^\Gamma = \mathbf{D}_2^\Gamma$, $\mathbf{V}_1^\Gamma(p) = \mathbf{V}_2^\Gamma(p)$ for $p \in P_{12}$, and $|\alpha|_1 = |\alpha|_2$ for $\alpha \in \text{Sent}(C_{12}, P_{12})$.

LEMMA 6.1. *Suppose Γ is a non-empty subset of $\text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} . Suppose for each $i \in \{1, 2\}$, \sim_i^Γ is an equivalence relation on $\text{Sent}(C_{12}, P_{12})$ suitable for C_i, \vdash_{12} , and Γ . Then \mathbf{M}_{12}^Γ is a juxtaposed model over C_1, C_2 , and P_{12} based on \mathbf{B}_{12}^Γ .*

PROOF. $|\alpha|_i^\Gamma$ is well defined: \sim_i^Γ is an equivalence relation.

\mathbf{D}_i^Γ is well-defined: Suppose $\alpha \sim_i^\Gamma \beta$. By compatibility with \vdash and Γ , $\Gamma \vdash_{12} \alpha$ just in case $\Gamma \vdash_{12} \beta$.

\mathbf{D}_i^Γ is a proper subset of \mathbf{B}_i^Γ : Clearly, \mathbf{D}_i^Γ is a subset of \mathbf{B}_i^Γ . Since Γ is consistent with respect to \vdash_{12} , there is some $\alpha \in \text{Sent}(C_{12}, P_{12})$ such that $\Gamma \not\vdash_{12} \alpha$. So $|\alpha|_i^\Gamma \notin \mathbf{D}_i^\Gamma$.

\mathbf{D}_i^Γ is non-empty: Since Γ is non-empty, there is some $\gamma \in \Gamma$. By Identity and Weakening, $\Gamma \vdash_{12} \gamma$. So $|\gamma|_i^\Gamma \in \mathbf{D}_i^\Gamma$.

For every $c_i \in C_i^n$, $\Phi_i^\Gamma(c_i)$ is well-defined: Suppose for some $k \in \{1, \dots, n\}$, $|\alpha^k|_i^\Gamma = |\beta|_i^\Gamma$. Since \sim_i^Γ is a congruence over C_i , $|c_i \alpha^1 \dots \alpha^k \dots \alpha^n|_i^\Gamma = |c_i \alpha^1 \dots \beta \dots \alpha^n|_i^\Gamma$.

Clearly, for every $c_i \in C_i^n$, $\Phi_i^\Gamma(c_i)$ is a function from the n -th Cartesian power of \mathbf{B}_i^Γ to \mathbf{B}_i^Γ . So \mathbf{B}_{12}^Γ is a juxtaposed structure over C_1 and C_2 . \mathbf{V}_i^Γ is a function from the set of i-atoms to \mathbf{B}_i^Γ . Therefore, \mathbf{M}_{12}^Γ is a juxtaposed model over C_1, C_2 , and P_{12} based on \mathbf{B}_{12}^Γ . \square

LEMMA 6.2. *Suppose Γ is a non-empty subset of $\text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} . Suppose for each $i \in \{1, 2\}$, \sim_i^Γ is an equivalence relation on $\text{Sent}(C_{12}, P_{12})$ suitable for C_i, \vdash_{12} , and Γ . Then:*

1. $\|\alpha\|_i^{\mathbf{M}_{12}^\Gamma} = |\alpha|_i^\Gamma$;
2. \mathbf{M}_{12}^Γ is coherent;
3. $\mathbf{M}_{12}^\Gamma \models \alpha$ just in case $\Gamma \vdash_{12} \alpha$;
4. \mathbf{M}_{12}^Γ is non-trivial.

PROOF. By Lemma 6.1, \mathbf{M}_{12}^Γ is a juxtaposed model over C_1 , C_2 , and P_{12} .

For every i -atom α , $\|\alpha\|_i^{\mathbf{M}_{12}^\Gamma} = \mathbf{V}_i^\Gamma(\alpha) = |\alpha|_i^\Gamma$. For every $c_i \in C_i^n$ and $\alpha^1, \dots, \alpha^n \in \text{Sent}(C_{12}, P_{12})$, $\|c_i \alpha^1 \dots \alpha^n\|_i^{\mathbf{M}_{12}^\Gamma} = \Phi_i^\Gamma(c_i)(|\alpha^1|_i^\Gamma \dots |\alpha^n|_i^\Gamma) = |c_i \alpha^1 \dots \alpha^n|_i^\Gamma$. Thus, by a simple induction, for every $\alpha \in \text{Sent}(C_{12}, P_{12})$, $\|\alpha\|_i^{\mathbf{M}_{12}^\Gamma} = |\alpha|_i^\Gamma$.

$\|\alpha\|_i^{\mathbf{M}_{12}^\Gamma} \in D_i^\Gamma$ just in case $|\alpha|_i^\Gamma \in D_i^\Gamma$ just in case $\Gamma \vdash_{12} \alpha$. So $\|\alpha\|_1^{\mathbf{M}_{12}^\Gamma} \in D_1^\Gamma$ just in case $\|\alpha\|_2^{\mathbf{M}_{12}^\Gamma} \in D_2^\Gamma$. Therefore, \mathbf{M}_{12}^Γ is coherent.

$\mathbf{M}_{12}^\Gamma \models \alpha$ just in case $\|\alpha\|_i^{\mathbf{M}_{12}^\Gamma} \in D_i^\Gamma$ just in case $\Gamma \vdash_{12} \alpha$.

Since Γ is non-empty, there is a $\gamma \in \text{Sent}(C_{12}, P_{12})$ such that $\gamma \in \Gamma$. By Identity and Weakening, $\Gamma \vdash_{12} \gamma$. So $\mathbf{M}_{12}^\Gamma \models \gamma$. Since Γ is consistent with respect to \vdash_{12} , there is an $\alpha \in \text{Sent}(C_{12}, P_{12})$ such that $\Gamma \not\vdash_{12} \alpha$. So $\mathbf{M}_{12}^\Gamma \not\models \alpha$. Therefore, \mathbf{M}_{12}^Γ is non-trivial. \square

The next result tells us when \mathbf{B}_{12}^Γ is a juxtaposed unital structure:

PROPOSITION 6.3. *Suppose Γ is a non-empty subset of $\text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} . Suppose for each $i \in \{1, 2\}$, \sim_i^Γ is an equivalence relation on $\text{Sent}(C_{12}, P_{12})$ suitable for C_i , \vdash_{12} , and Γ . Then \mathbf{B}_{12}^Γ is a juxtaposed unital structure just in case for each $i \in \{1, 2\}$, \sim_i^Γ is unital suitable for C_i , \vdash_{12} , and Γ .*

PROOF. Suppose for each $i \in \{1, 2\}$, \sim_i^Γ is unital suitable for C_i , \vdash_{12} , and Γ . By Lemma 6.1, \mathbf{B}_{12}^Γ is a juxtaposed structure. Suppose $|\alpha|_i^\Gamma \in D_i^\Gamma$ and $|\beta|_i^\Gamma \in D_i^\Gamma$. So $\Gamma \vdash_{12} \alpha$ and $\Gamma \vdash_{12} \beta$. By strong compatibility with \vdash and Γ , $\alpha \sim_i^\Gamma \beta$. So $|\alpha|_i^\Gamma = |\beta|_i^\Gamma$. Therefore, \mathbf{B}_{12}^Γ is a juxtaposed unital structure.

Now suppose \mathbf{B}_{12}^Γ is a juxtaposed unital structure. Suppose $\Gamma \vdash_{12} \alpha$ and $\Gamma \vdash_{12} \beta$. So $|\alpha|_i^\Gamma \in D_i^\Gamma$ and $|\beta|_i^\Gamma \in D_i^\Gamma$. Since \mathbf{B}_{12}^Γ is unital, $|\alpha|_i^\Gamma = |\beta|_i^\Gamma$. So $\alpha \sim_i^\Gamma \beta$. Therefore, \sim_i^Γ is unital suitable. \square

Suppose for every $i \in \{1, 2\}$ and every non-empty $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ consistent with \vdash_{12} , \sim_i^Γ is an equivalence relation on $\text{Sent}(C_{12}, P_{12})$ suitable for C_i , \vdash_{12} , and Γ . We define the following class of juxtaposed structures:

$$\mathbb{B}_{12}^\sim = \{\mathbf{B}_{12}^\Gamma \mid \Gamma \subseteq \text{Sent}(C_{12}, P_{12}) \text{ is non-empty and consistent with respect to } \vdash_{12} \\ \text{and } \mathbf{B}_{12}^\Gamma \text{ is built with } \sim_1^\Gamma \text{ and } \sim_2^\Gamma\}$$

\mathbb{B}_{12}^\sim is the Lindenbaum-Tarski class of juxtaposed structures for C_1 , C_2 , and \vdash_{12} built with the \sim_i^Γ relations.

THEOREM 6.4 (Strong Completeness). *Suppose \vdash_{12} has no mere followers. Suppose for every $i \in \{1, 2\}$ and non-empty $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} , \sim_i^Γ is an equivalence relation on $\text{Sent}(C_{12}, P_{12})$ suitable for C_i , \vdash_{12} , and Γ . Then \vdash_{12} is strongly complete with respect to \mathbb{B}_{12}^\sim .*

PROOF. Suppose $\Gamma \not\vdash_{12} \alpha$. We show $\Gamma \not\vdash^{\mathbb{B}_{12}^\sim} \alpha$. Since \vdash_{12} has no mere followers, without loss of generality, we can take Γ to be non-empty. (If $\Gamma = \emptyset$, there is a non-empty $\Delta \subseteq \text{Sent}(C_{12}, P_{12})$ such that $\Delta \not\vdash_{12} \alpha$. We can take Δ to be our non-empty set.) By Lemma 6.1, \mathbf{M}_{12}^Γ is a juxtaposed model over C_1 , C_2 , and P_{12} . By Lemma 6.2, \mathbf{M}_{12}^Γ is a coherent non-trivial juxtaposed model such that $\mathbf{M}_{12}^\Gamma \models \Gamma$ and $\mathbf{M}_{12}^\Gamma \not\models \alpha$. Since \mathbf{M}_{12}^Γ is based on \mathbb{B}_{12}^\sim , $\Gamma \not\vdash^{\mathbb{B}_{12}^\sim} \alpha$. \square

To show that \vdash_{12} is strongly determined with respect to \mathbb{B}_{12}^\sim , we also need to show that \vdash_{12} is strongly sound with respect to \mathbb{B}_{12}^\sim . If \vdash_{12} is *the* juxtaposition of \vdash_1 and \vdash_2 , we can make use of Theorem 5.5.

THEOREM 6.5 (Strong Soundness). *Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . Suppose for every $i \in \{1, 2\}$ and non-empty $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} , \sim_i^Γ is an equivalence relation on $\text{Sent}(C_{12}, P_{12})$ suitable for C_i , \vdash_i , and Γ . Then \vdash_{12} is strongly sound with respect to \mathbb{B}_{12}^\sim .*

PROOF. For $i = 1$ and 2 , let $\mathbb{B}_i^\sim = \{\mathbf{B}_i^\Gamma \mid \Gamma \subseteq \text{Sent}(C_{12}, P_{12}) \text{ is non-empty and consistent with respect to } \vdash_{12} \text{ and } \mathbf{B}_i^\Gamma \text{ is built with } \sim_i^\Gamma\}$. We first show that \vdash_i is strongly sound with respect to \mathbb{B}_i^\sim .

Suppose $\Gamma \not\vdash^{\mathbb{B}_i^\sim} \alpha$ for $\Gamma \subseteq \text{Sent}(C_i, P_i)$ and $\alpha \in \text{Sent}(C_i, P_i)$. Then there is a model \mathcal{M}_i over C_i and P_i based on \mathbb{B}_i^\sim such that $\mathcal{M}_i \models \Gamma$ and $\mathcal{M}_i \not\models \alpha$. \mathcal{M}_i is based on $\mathbf{B}_i^\Delta \in \mathbb{B}_i^\sim$, for some non-empty $\Delta \subseteq \text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} . By Lemmas 6.1 and 6.2, \mathbf{M}_{12}^Δ is a coherent juxtaposed model over C_1 , C_2 , and P_{12} . For each $p \in P_i$, let $\sigma(p)$ be a sentence in $\text{Sent}(C_{12}, P_{12})$ such that $\|\sigma(p)\|_i^{\mathbf{M}_{12}^\Delta} = \|p\|^{\mathcal{M}_i}$. Such a sentence exists by the construction of \mathbf{M}_{12}^Δ . By a simple induction, for any $\beta \in \text{Sent}(C_i, P_i)$, $\|\beta^\sigma\|_i^{\mathbf{M}_{12}^\Delta} = \|\beta\|^{\mathcal{M}_i}$. So $\mathbf{M}_{12}^\Delta \models \Gamma^\sigma$ and $\mathbf{M}_{12}^\Delta \not\models \alpha^\sigma$. By Lemma 6.2, for each $\gamma \in \Gamma$, $\Delta \vdash_{12} \gamma^\sigma$ and $\Delta \not\vdash_{12} \alpha^\sigma$. By Cut, $\Gamma^\sigma \not\vdash_{12} \alpha^\sigma$. By Uniform Substitution, $\Gamma \not\vdash_{12} \alpha$. Since \vdash_{12} extends \vdash_i , $\Gamma \not\vdash_i \alpha$. Thus, \vdash_i is strongly sound with respect to \mathbb{B}_i^\sim .

By Theorem 5.5, \vdash_{12} is strongly sound with respect to $\mathbb{B}_1^\sim \times \mathbb{B}_2^\sim$. Since $\mathbb{B}_{12}^\sim \subseteq \mathbb{B}_1^\sim \times \mathbb{B}_2^\sim$, \vdash_{12} is strongly sound with respect to \mathbb{B}_{12}^\sim .³⁹ \square

To show that \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly determined or $\langle C_1, C_2 \rangle$ -strongly unital determined, we need to show that there is a coherent non-trivial juxtaposed model based on \mathbb{B}_{12}^\sim . We can make use of the following simple result:

PROPOSITION 6.6. *Suppose \vdash_{12} has no mere followers. Suppose for every $i \in \{1, 2\}$ and non-empty $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} , \sim_i^Γ is an equivalence relation on $\text{Sent}(C_{12}, P_{12})$ suitable for C_i , \vdash_i , and Γ . Then if \vdash_{12} is consistent, there is a coherent non-trivial juxtaposed model based on \mathbb{B}_{12}^\sim .*

PROOF. Since \vdash_{12} is consistent, for some $\alpha \in \text{Sent}(C_{12}, P_{12})$, $\not\vdash_{12} \alpha$. Since \vdash_{12} has no mere followers, for some non-empty $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$, $\Gamma \not\vdash_{12} \alpha$. By Lemmas 6.1 and 6.2, \mathbf{M}_{12}^Γ is a coherent non-trivial juxtaposed model based on \mathbb{B}_{12}^\sim . \square

³⁹We could make use of $\mathbb{B}_1^\sim \times \mathbb{B}_2^\sim$ instead of \mathbb{B}_{12}^\sim in our results in this section. We focus on \mathbb{B}_{12}^\sim because it is a slightly more elegant construction.

Notice that in the case that \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly sound with respect to \mathbb{B}_{12}^\sim , we also get the converse of this result: Suppose there is a coherent non-trivial juxtaposed model \mathcal{M}_{12} based on \mathbb{B}_{12}^\sim . Since \mathcal{M}_{12} is non-trivial, for some $\alpha \in \text{Sent}(C_{12}, P_{12})$, $\mathcal{M}_{12} \not\models \alpha$. By strong soundness, $\not\vdash_{12} \alpha$. So, \vdash_{12} is consistent.

Summarizing our results, we have the following:

THEOREM 6.7. *Suppose \vdash_{12} has no mere followers. Suppose for every $i \in \{1, 2\}$ and non-empty $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} , \sim_i^Γ is an equivalence relation on $\text{Sent}(C_{12}, P_{12})$ suitable for C_i , \vdash_{12} , and Γ . Then:*

1. \vdash_{12} is strongly complete with respect to \mathbb{B}_{12}^\sim ;
2. If \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , then \vdash_{12} is strongly sound with respect to \mathbb{B}_{12}^\sim ;
3. If \vdash_{12} is consistent, then there is a coherent non-trivial model based on \mathbb{B}_{12}^\sim ;
4. \mathbb{B}_{12}^\sim is a class of juxtaposed unital structures just in case for every $i \in \{1, 2\}$ and non-empty $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} , \sim_i^Γ is unital suitable for C_i , \vdash_{12} , and Γ .

6.2. Strong Determination. Theorem 6.7 raises the question of when there is an equivalence relation on $\text{Sent}(C_{12}, P_{12})$ suitable or unital suitable for C_i , \vdash_{12} , and Γ . The case of suitability is straightforward. The following result is easy to prove:

LEMMA 6.8. *Suppose $\Gamma \subseteq \text{Sent}(C, P)$. Then the identity relation on $\text{Sent}(C, P)$ is an equivalence relation on $\text{Sent}(C, P)$ suitable for C^- , \vdash , and Γ .*

Let \mathbb{B}_{12}^\equiv be the Lindenbaum-Tarski class of juxtaposed structures for C_1, C_2 , and \vdash_{12} , using the identity relation on $\text{Sent}(C_{12}, P_{12})$ for the suitable equivalence relations. Combining Lemma 6.8 with Theorem 6.7, we arrive at the following result:

PROPOSITION 6.9. *Suppose \vdash_{12} has no mere followers. Then:*

1. \vdash_{12} is strongly complete with respect to \mathbb{B}_{12}^\equiv ;
2. If \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , then \vdash_{12} is strongly sound with respect to \mathbb{B}_{12}^\equiv ;
3. If \vdash_{12} is consistent, then there is a coherent non-trivial model based on \mathbb{B}_{12}^\equiv .

The identity relation on $\text{Sent}(C, P)$ is the most fine-grained equivalence relation suitable for C^- , \vdash , and Γ . We can also characterize the most coarse-grained such relation.

We say that p strictly C^- -occurs in δ just in case p does not occur within the scope of any connective not from C^- . Suppose $\Gamma \subseteq \text{Sent}(C, P)$. We define a binary relation on $\text{Sent}(C, P)$ as follows:

$$\langle \alpha, \beta \rangle \in \Omega_{C^-}^\Gamma \text{ just in case for every } \delta \in \text{Sent}(C, P) \text{ and } p \text{ that strictly } C^- \text{-occurs in } \delta, \\ \Gamma \vdash \delta[\alpha/p] \text{ just in case } \Gamma \vdash \delta[\beta/p].$$

This is a modification of the definition of the well-known Leibniz congruence.⁴⁰

We write $\alpha \equiv \beta \pmod{\Omega_{C^-}^\Gamma}$ to stand for the claim that $\langle \alpha, \beta \rangle \in \Omega_{C^-}^\Gamma$.

⁴⁰The Leibniz congruence may be defined as follows: $\alpha \equiv \beta \pmod{\Omega\Gamma}$ just in case for every $\delta \in \text{Sent}(C, P)$ and p occurring in δ , $\Gamma \vdash \delta[\alpha/p]$ just in case $\Gamma \vdash \delta[\beta/p]$. See Blok & Pigozzi (1989) for discussion of this congruence.

LEMMA 6.10. *Suppose $\Gamma \subseteq \text{Sent}(C, P)$. Then $\Omega_{C^-}^\Gamma$ is an equivalence relation on $\text{Sent}(C, P)$ suitable for C^- , \vdash , and Γ .*

PROOF. Suppose $\Gamma \subseteq \text{Sent}(C, P)$.

Clearly, $\Omega_{C^-}^\Gamma$ is an equivalence relation on $\text{Sent}(C, P)$.

Congruence over C^- : Suppose $c^- \in C^{-n}$; $\alpha^1, \dots, \alpha^n, \beta \in \text{Sent}(C, P)$; and $k \in \{1, \dots, n\}$. Suppose $\delta \in \text{Sent}(C, P)$ and p strictly C^- -occurs in δ . Suppose $\alpha^k \equiv \beta \pmod{\Omega_{C^-}^\Gamma}$. Let q be an element of P that does not occur in δ or in $\alpha^1, \dots, \alpha^n$. Then q strictly C^- -occurs in $\delta[c^- \alpha^1 \dots q \dots \alpha^n / p]$. So $\Gamma \vdash \delta[c^- \alpha^1 \dots \alpha^k \dots \alpha^n / p]$ just in case $\Gamma \vdash \delta[c^- \alpha^1 \dots q \dots \alpha^n / p][\alpha^k / q]$ just in case $\Gamma \vdash \delta[c^- \alpha^1 \dots q \dots \alpha^n / p][\beta / q]$ (by the definition of $\Omega_{C^-}^\Gamma$) just in case $\Gamma \vdash \delta[c^- \alpha^1 \dots \beta \dots \alpha^n / p]$. Hence, $c^- \alpha^1 \dots \alpha^k \dots \alpha^n \equiv c^- \alpha^1 \dots \beta \dots \alpha^n \pmod{\Omega_{C^-}^\Gamma}$.

Compatibility with \vdash and Γ : Suppose $\alpha \equiv \beta \pmod{\Omega_{C^-}^\Gamma}$. $\Gamma \vdash \alpha$ just in case $\Gamma \vdash p[\alpha / p]$ just in case $\Gamma \vdash p[\beta / p]$ (by the definition of $\Omega_{C^-}^\Gamma$) just in case $\Gamma \vdash \beta$. \square

It is straightforward to show that $\Omega_{C^-}^\Gamma$ is the most coarse-grained equivalence relation on $\text{Sent}(C, P)$ suitable for C^- , \vdash , and Γ :

PROPOSITION 6.11. *Suppose $\Gamma \subseteq \text{Sent}(C, P)$. Suppose \sim is an equivalence relation on $\text{Sent}(C, P)$ suitable for C^- , \vdash , and Γ . Then if $\alpha \sim \beta$, $\alpha \equiv \beta \pmod{\Omega_{C^-}^\Gamma}$.*

PROOF. Suppose $\alpha \sim \beta$. Suppose $\delta \in \text{Sent}(C, P)$ and p strictly C^- -occurs in δ . By induction on the complexity of δ , $\delta[\alpha / p] \sim \delta[\beta / p]$. By compatibility with \vdash and Γ for \sim , $\Gamma \vdash \delta[\alpha / p]$ just in case $\Gamma \vdash \delta[\beta / p]$. Therefore, $\alpha \equiv \beta \pmod{\Omega_{C^-}^\Gamma}$. \square

Let \mathbb{B}_{12}^Ω be the Lindenbaum-Tarski class of juxtaposed structures for C_1, C_2 , and \vdash_{12} , built using the $\Omega_{C_i}^\Gamma$ relations. Combining Lemma 6.10 with Theorem 6.7, we arrive at the following result:

PROPOSITION 6.12. *Suppose \vdash_{12} has no mere followers. Then:*

1. \vdash_{12} is strongly complete with respect to \mathbb{B}_{12}^Ω ;
2. If \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , then \vdash_{12} is strongly sound with respect to \mathbb{B}_{12}^Ω ;
3. If \vdash_{12} is consistent, then there is a coherent non-trivial model based on \mathbb{B}_{12}^Ω .

Making use of either Proposition 6.9 or Proposition 6.12, we have the following sufficient condition on $\langle C_1, C_2 \rangle$ -strong determination:

COROLLARY 6.13. *Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , is consistent, and has no mere followers. Then \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly determined.*

6.3. Strong Unital Determination. Let's now consider the case of unital suitability. We first provide a necessary and sufficient condition for $\Omega_{C^-}^\Gamma$ to be unital suitable for C^- , \vdash , and Γ .

We say that \vdash is C^- -left-extensional just in case for every $\alpha, \beta, \delta \in \text{Sent}(C, P)$ and p that strictly C^- -occurs in δ , $\{\alpha, \beta, \delta[\alpha / p]\} \vdash \delta[\beta / p]$. This is a generalization of the definition of left-extensionality presented above: \vdash is left-extensional (simpliciter) just in case \vdash is C^- -left-extensional.

LEMMA 6.14. \vdash is C^- -left-extensional just in case for every non-empty $\Gamma \subseteq \text{Sent}(C, P)$ consistent with respect to \vdash , $\Omega_{C^-}^\Gamma$ is unital suitable for C^- , \vdash , and Γ

PROOF. Suppose \vdash is C^- -left-extensional. Suppose $\Gamma \subseteq \text{Sent}(C, P)$. By Lemma 6.10, $\Omega_{C^-}^\Gamma$ is suitable for C^- , \vdash , and Γ . Suppose $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$. Suppose $\delta \in \text{Sent}(C, P)$ and p strictly C^- -occurs in δ . By C^- -left-extensionality, $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\beta/p]$. By Cut, if $\Gamma \vdash \delta[\alpha/p]$ then $\Gamma \vdash \delta[\beta/p]$. By analogous reasoning, if $\Gamma \vdash \delta[\beta/p]$ then $\Gamma \vdash \delta[\alpha/p]$. So $\alpha \equiv \beta \pmod{\Omega_{C^-}^\Gamma}$. Therefore, $\Omega_{C^-}^\Gamma$ is strongly compatible with \vdash and Γ .

Now suppose for every non-empty $\Gamma \subseteq \text{Sent}(C, P)$ consistent with respect to \vdash , $\Omega_{C^-}^\Gamma$ is unital suitable for C^- , \vdash , and Γ . Suppose $\alpha, \beta, \delta \in \text{Sent}(C, P)$ and p strictly C^- -occurs in δ . If $\{\alpha, \beta, \delta[\alpha/p]\}$ is inconsistent with respect to \vdash , then $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\beta/p]$. Suppose, then, that $\{\alpha, \beta, \delta[\alpha/p]\}$ is consistent with respect to \vdash . So $\Omega_{C^-}^{\{\alpha, \beta, \delta[\alpha/p]\}}$ is unital suitable for C^- , \vdash , and $\{\alpha, \beta, \delta[\alpha/p]\}$. By Identity and Weakening, $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \alpha$ and $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \beta$. By strong compatibility with \vdash and $\{\alpha, \beta, \delta[\alpha/p]\}$, $\alpha \equiv \beta \pmod{\Omega_{C^-}^{\{\alpha, \beta, \delta[\alpha/p]\}}}$. So $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\alpha/p]$ just in case $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\beta/p]$. By Identity and Weakening, $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\alpha/p]$. So, again, $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\beta/p]$. Therefore, \vdash is C^- -left-extensional. \square

Combining this lemma with Theorem 6.7, we arrive at the following result:

PROPOSITION 6.15. *Suppose \vdash_{12} has no mere followers. Then \mathbb{B}_{12}^Ω is a class of juxtaposed unital structures just in case \vdash_{12} is C_1 - and C_2 -left-extensional.*

Notice that the claim that \vdash_{12} is C_1 - and C_2 -left-extensional is weaker than the claim that \vdash_{12} is left-extensional (simpliciter).

Making use of Propositions 6.12 and 6.15, we have the following sufficient condition on $\langle C_1, C_2 \rangle$ -strong unital determination:

COROLLARY 6.16. *Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , is consistent, has no mere followers, and is C_1 - and C_2 -left-extensional. Then \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly unital determined.*

6.4. An Improvement. We can improve Corollary 6.16. Suppose $\Delta \subseteq \text{Sent}(C, P)$. We say that \vdash is *left-extensional* over Δ just in case for every $\alpha, \beta \in \text{Sent}(C, P)$, $\delta \in \Delta$, and p occurring in δ , $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\beta/p]$. This is a different generalization of the notion of left-extensionality than the one presented above. \vdash is left-extensional (simpliciter) just in case \vdash is left-extensional over $\text{Sent}(C, P)$. We can improve Corollary 6.16 by showing that if \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , is consistent, has no mere followers, and is left-extensional over $\text{Sent}(C_1, P_1)$ and over $\text{Sent}(C_2, P_2)$, then \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly unital determined.

The cost of this improvement is that we must move to a more complicated equivalence relation (for each C^- , \vdash , and $\Gamma \subseteq \text{Sent}(C, P)$). In particular, we make use of a modification of (one characterization of) the Suszko congruence.⁴¹

⁴¹The Suszko congruence may be defined as follows: $\alpha \equiv \beta \pmod{\Sigma\Gamma}$ just in case for every $\delta \in \text{Sent}(C, P)$ and p occurring in δ , $\Gamma \cup \{\delta[\alpha/p]\} \vdash \delta[\beta/p]$ and $\Gamma \cup \{\delta[\beta/p]\} \vdash \delta[\alpha/p]$. See Czelakowski (2003) for discussion of this congruence.

In defining this relation, it turns out to be helpful to work with an expansion of our language.

Let P^* be a countably infinite set of sentence symbols disjoint with P . (In what follows, we always choose P^* to be disjoint with whatever sets of sentence symbols we are working with.) Let \vdash^* be the least consequence relation for $\text{Sent}(C, P \cup P^*)$ that extends \vdash .

LEMMA 6.17. \vdash^* exists and is a strong conservative extension of \vdash .

PROOF. Let σ be any function from P^* to P . Let $\Gamma \vdash^\sigma \alpha$ just in case $\Gamma^\sigma \vdash \alpha^\sigma$. It is routine to show that \vdash^σ is a consequence relation for $\text{Sent}(C, P \cup P^*)$ that extends \vdash . It is also routine to show that the intersection of all consequence relations for $\text{Sent}(C, P \cup P^*)$ that extend \vdash is itself a consequence relation for $\text{Sent}(C, P \cup P^*)$. Therefore, \vdash^* exists.

Suppose $\Gamma \vdash \alpha$. Since \vdash^* extends \vdash , $\Gamma \vdash^* \alpha$. Now suppose $\Gamma \not\vdash \alpha$. Again let σ be any function from P^* to P . It is easy to see that $\Gamma \not\vdash^\sigma \alpha$. So $\Gamma \not\vdash^* \alpha$. Therefore, \vdash^* is a strong conservative extension of \vdash . \square

Suppose $\Gamma \subseteq \text{Sent}(C, P)$. We define a binary relation on $\text{Sent}(C, P)$ as follows:
 $\langle \alpha, \beta \rangle \in \Sigma_{C^-}^\Gamma$ just in case for every $\delta \in \text{Sent}(C^-, P \cup P^*)$ and p occurring in δ ,
 $\Gamma \cup \{\delta[\alpha/p]\} \vdash^* \delta[\beta/p]$ and $\Gamma \cup \{\delta[\beta/p]\} \vdash^* \delta[\alpha/p]$.

We write $\alpha \equiv \beta \pmod{\Sigma_{C^-}^\Gamma}$ to stand for the claim that $\langle \alpha, \beta \rangle \in \Sigma_{C^-}^\Gamma$.

In this definition, δ is restricted to sentences with connectives from C^- . This is what enables us to improve our strong unital determination result. However, this is also what motivates the use of an extra set of sentence symbols. By making use of the elements of P^* , we can apply Uniform Substitution to show that $\Sigma_{C^-}^\Gamma$ is a congruence over C^- even when Γ contains occurrences of every element in P .⁴²

LEMMA 6.18. Suppose $\Gamma \subseteq \text{Sent}(C, P)$. Then $\Sigma_{C^-}^\Gamma$ is an equivalence relation on $\text{Sent}(C, P)$ suitable for C^- , \vdash , and Γ .

PROOF. Suppose $\Gamma \subseteq \text{Sent}(C, P)$.

Clearly, $\Sigma_{C^-}^\Gamma$ is an equivalence relation on $\text{Sent}(C, P)$.

Congruence over C^- : Suppose $c^- \in C^-$; $\alpha^1, \dots, \alpha^n, \beta \in \text{Sent}(C, P)$; and $k \in \{1, \dots, n\}$. Suppose $\alpha^k \equiv \beta \pmod{\Sigma_{C^-}^\Gamma}$. Suppose $\delta \in \text{Sent}(C^-, P \cup P^*)$ and p occurs in δ . Let p^1, \dots, p^n be distinct elements of P^* that do not occur in δ . $\Gamma \cup \{\delta[c^- p^1 \dots p^k \dots p^n/p][\alpha^k/p^k]\} \vdash^* \delta[c^- p^1 \dots p^k \dots p^n/p][\beta/p^k]$. That is, $\Gamma \cup \{\delta[c^- p^1 \dots \alpha^k \dots p^n/p]\} \vdash^* \delta[c^- p^1 \dots \beta \dots p^n/p]$. By Uniform Substitution, $\Gamma \cup \{\delta[c^- \alpha^1 \dots \alpha^k \dots \alpha^n/p]\} \vdash^* \delta[c^- \alpha^1 \dots \beta \dots \alpha^n/p]$. By analogous reasoning, $\Gamma \cup$

⁴²There is an alternative approach that we could instead have adopted. The idea is to make use of the following equivalence relation: $\langle \alpha, \beta \rangle \in \Sigma_{C^-}^\Gamma$ just in case for every $\delta \in \text{Sent}(C^-, P)$ and p occurring in δ , $\Gamma \cup \{\delta[\alpha/p]\} \vdash \delta[\beta/p]$ and $\Gamma \cup \{\delta[\beta/p]\} \vdash \delta[\alpha/p]$. Using Uniform Substitution, we can show that if Γ lacks occurrences of infinitely many elements of P then $\Sigma_{C^-}^\Gamma$ is a congruence over C^- . If were to adopt this approach, we would have to modify our definition of the Lindenbaum-Tarski class of juxtaposed structures so that it contains a juxtaposed structure for every non-empty $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} that lacks occurrences of infinitely many elements of P_{12} . In effect, we would be working with restrictions of the language rather than an expansion of it.

$\{\delta[c^-\alpha^1 \dots \beta \dots \alpha^n/p]\} \vdash^* \delta[c^-\alpha^1 \dots \alpha^k \dots \alpha^n/p]$. Hence, $c^-\alpha^1 \dots \alpha^k \dots \alpha^n \equiv c^-\alpha^1 \dots \beta \dots \alpha^n \pmod{\Sigma_{C^-}^\Gamma}$.

Compatibility with \vdash and Γ : Suppose $\alpha \equiv \beta \pmod{\Sigma_{C^-}^\Gamma}$. Let $p \in P$. So $\Gamma \cup \{p[\alpha/p]\} \vdash^* p[\beta/p]$. That is, $\Gamma \cup \{\alpha\} \vdash^* \beta$. By Lemma 6.17, $\Gamma \cup \{\alpha\} \vdash \beta$. Similarly, $\Gamma \cup \{\beta\} \vdash \alpha$. By Cut, $\Gamma \vdash \alpha$ just in case $\Gamma \vdash \beta$. \square

Before we provide conditions for when $\Sigma_{C^-}^\Gamma$ is unital suitable for C^- , \vdash , and Γ , we first prove two simple results concerning left-extensionality over a set:

LEMMA 6.19. *Suppose $\Delta \subseteq \text{Sent}(C, P)$. Suppose \vdash is left-extensional over Δ . Then any consequence relation that extends \vdash is left-extensional over Δ .*

PROOF. Suppose C is a sub-signature of C^+ and P is a subset of P^+ . Suppose \vdash^+ is a consequence relation for $\text{Sent}(C^+, P^+)$ that extends \vdash . Suppose $\alpha, \beta \in \text{Sent}(C^+, P^+)$; $\delta \in \Delta$; and p occurs in δ . Let q and r be distinct elements of P that do not occur in δ . Since \vdash is left-extensional over Δ , $\{q, r, \delta[q/p]\} \vdash \delta[r/p]$. Since \vdash^+ extends \vdash , $\{q, r, \delta[q/p]\} \vdash^+ \delta[r/p]$. By Uniform Substitution, $\{\alpha, \beta, \delta[\alpha/p]\} \vdash^+ \delta[\beta/p]$. Therefore, \vdash^+ is left-extensional over Δ . \square

LEMMA 6.20. *Suppose P^- is an infinite subset of P . Then \vdash is left-extensional over $\text{Sent}(C^-, P)$ just in case \vdash is left-extensional over $\text{Sent}(C^-, P^-)$.*

PROOF. Suppose \vdash is left-extensional over $\text{Sent}(C^-, P)$. Since $\text{Sent}(C^-, P^-)$ is a subset of $\text{Sent}(C^-, P)$, \vdash is left-extensional over $\text{Sent}(C^-, P^-)$.

Now suppose \vdash is left-extensional over $\text{Sent}(C^-, P^-)$. Suppose that $\delta \in \text{Sent}(C^-, P)$; $\alpha, \beta \in \text{Sent}(C, P)$; and p occurs in δ . Let P^δ contain those elements of $P - P^-$ that occur in δ . Let P^* contain those elements of P^- that do not occur in α , β , or δ . Let σ be an injective function from P^δ to P^* . Such a function exists since P^δ is finite and P^* is infinite. By left-extensionality over $\text{Sent}(C^-, P^-)$, $\{\alpha^\sigma, \beta^\sigma, \delta^\sigma[\alpha^\sigma/p^\sigma]\} \vdash \delta^\sigma[\beta^\sigma/p^\sigma]$. By Uniform Substitution, $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\beta/p]$. Therefore, \vdash is left-extensional over $\text{Sent}(C^-, P)$. \square

Making use of these results, we can prove the following result:

LEMMA 6.21. *\vdash is left-extensional over $\text{Sent}(C^-, P)$ just in case for every non-empty $\Gamma \subseteq \text{Sent}(C, P)$ consistent with respect to \vdash , $\Sigma_{C^-}^\Gamma$ is unital suitable for C^- , \vdash , and Γ .*

PROOF. Suppose \vdash is left-extensional over $\text{Sent}(C^-, P)$. Suppose that $\Gamma \subseteq \text{Sent}(C, P)$. By Lemma 6.18, for every $\Gamma \subseteq \text{Sent}(C, P)$, $\Sigma_{C^-}^\Gamma$ is suitable for C^- , \vdash , and Γ . Suppose $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$. Suppose $\delta \in \text{Sent}(C^-, P \cup P^*)$ and p occurs in δ . By Lemma 6.19, \vdash^* is left-extensional over $\text{Sent}(C^-, P)$. By Lemma 6.20, \vdash^* is left-extensional over $\text{Sent}(C^-, P \cup P^*)$. So $\{\alpha, \beta, \delta[\alpha/p]\} \vdash^* \delta[\beta/p]$. Since \vdash^* extends \vdash , $\Gamma \vdash^* \alpha$ and $\Gamma \vdash^* \beta$. By Cut, $\Gamma \cup \{\delta[\alpha/p]\} \vdash^* \delta[\beta/p]$. Similarly, $\Gamma \cup \{\delta[\beta/p]\} \vdash^* \delta[\alpha/p]$. So $\alpha \equiv \beta \pmod{\Sigma_{C^-}^\Gamma}$. So $\Sigma_{C^-}^\Gamma$ is unital suitable for C^- , \vdash , and Γ .

Now suppose for every non-empty $\Gamma \subseteq \text{Sent}(C, P)$ consistent with respect to \vdash , $\Sigma_{C^-}^\Gamma$ is unital suitable for C^- , \vdash , and Γ . Suppose $\alpha, \beta \in \text{Sent}(C, P)$; $\delta \in \text{Sent}(C^-, P)$; and p occurs in δ . If $\{\alpha, \beta\}$ is inconsistent with respect to \vdash , then $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\beta/p]$. Suppose, then, that $\{\alpha, \beta\}$ is consistent with respect to \vdash . So $\Sigma_{C^-}^{\{\alpha, \beta\}}$ is unital suitable for C^- , \vdash , and $\{\alpha, \beta\}$. By Identity

and Weakening, $\{\alpha, \beta\} \vdash \alpha$ and $\{\alpha, \beta\} \vdash \beta$. By strong compatibility with \vdash and $\{\alpha, \beta\}$, $\alpha \equiv \beta \pmod{\Sigma_{C^-}^{\{\alpha, \beta\}}}$. So $\{\alpha, \beta, \delta[\alpha/p]\} \vdash^* \delta[\beta/p]$. By Lemma 6.17, $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\beta/p]$. Therefore, \vdash is left-extensional over $\text{Sent}(C^-, P)$. \square

Let \mathbb{B}_{12}^Σ be the Lindenbaum-Tarski class of juxtaposed structures for C_1, C_2 , and \vdash_{12} built using the $\Sigma_{C_i}^\Gamma$ relations. Combining Lemmas 6.18 and 6.21 with Theorem 6.7, we arrive at the following result:

PROPOSITION 6.22. *Suppose \vdash_{12} has no mere followers. Then:*

1. \vdash_{12} is strongly complete with respect to \mathbb{B}_{12}^Σ ;
2. If \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , then \vdash_{12} is strongly sound with respect to \mathbb{B}_{12}^Σ ;
3. If \vdash_{12} is consistent, then there is a coherent non-trivial model based on \mathbb{B}_{12}^Σ ;
4. \mathbb{B}_{12}^Σ is a class of juxtaposed unital structures just in case \vdash_{12} is left-extensional over $\text{Sent}(C_1, P_1)$ and over $\text{Sent}(C_2, P_2)$.

The following corollary is immediate:

COROLLARY 6.23. *Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , is consistent, has no mere followers, and is left-extensional over $\text{Sent}(C_1, P_1)$ and over $\text{Sent}(C_2, P_2)$. Then \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly unital determined.*

Notice that the claim that \vdash_{12} is left-extensional over $\text{Sent}(C_i, P_i)$ is weaker than the claim that \vdash_{12} is C_i -left-extensional. Thus, this is a genuine strengthening of Corollary 6.16.

6.5. Necessary Conditions. Corollaries 6.13 and 6.23 provide sufficient conditions on $\langle C_1, C_2 \rangle$ -strong determination and $\langle C_1, C_2 \rangle$ -strong unital determination, respectively. This raises the question of whether these conditions are necessary, too. We can show that they are almost necessary. In particular, non-triviality (and hence, consistency) is necessary for $\langle C_1, C_2 \rangle$ -strong determination. Left-extensionality over $\text{Sent}(C_1, P_1)$ and over $\text{Sent}(C_2, P_2)$ is necessary for $\langle C_1, C_2 \rangle$ -strong unital determination. However, we cannot show that having no mere followers is a necessary condition on $\langle C_1, C_2 \rangle$ -strong determination. What we can show is that if C_1 and C_2 are disjoint, having no mere followers is a necessary condition.⁴³

PROPOSITION 6.24. *\vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly sound only if \vdash_{12} is non-trivial.*

PROOF. Suppose \vdash_{12} is strongly sound with respect to \mathbb{B}_{12} , a class of juxtaposed structures over C_1 and C_2 with a coherent non-trivial juxtaposed model

⁴³We could modify our definitions so that having no mere followers is in general necessary for $\langle C_1, C_2 \rangle$ -strong determination. One way to do so would be to say that \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly determined just in case \vdash_{12} is strongly determined with respect to some class \mathbb{B}_{12} of juxtaposed structures over C_1 and C_2 such that (i) there is a non-trivial coherent juxtaposed model based on \mathbb{B}_{12} ; and (ii) if a sentence is invalid in \mathbb{B}_{12} , then there is a non-trivial coherent juxtaposed model based on \mathbb{B}_{12} that does not designate the sentence. (We would also have to redefine $\langle C_1, C_2 \rangle$ -strong unital determination in the corresponding way.) This is a natural proposal, since the analogue of condition (ii) is automatically satisfied in the case of ordinary (non-juxtaposed) strong determination. Moreover, since our Lindenbaum-Tarski juxtaposed models are non-trivial, the additional condition would not affect our proofs of $\langle C_1, C_2 \rangle$ -strong determination and $\langle C_1, C_2 \rangle$ -strong unital determination. However, since this alternative definition is somewhat more complicated, we will stick with our original definition.

\mathcal{M}_{12} based on it. Since \mathcal{M}_{12} is non-trivial, for some $\alpha, \beta \in \text{Sent}(C_{12}, P_{12})$, $\mathcal{M}_{12} \models \beta$ and $\mathcal{M}_{12} \not\models \alpha$. So $\{\beta\} \not\models^{\mathbb{B}_{12}} \alpha$. By soundness, $\{\beta\} \not\vdash_{12} \alpha$. So \vdash_{12} is non-trivial. \square

It immediately follows that \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly sound only if \vdash_{12} is consistent.

PROPOSITION 6.25. *\vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly complete with respect to a class of juxtaposed unital structures only if \vdash_{12} is left-extensional over $\text{Sent}(C_1, P_1)$ and over $\text{Sent}(C_2, P_2)$.*

PROOF. Suppose \vdash_{12} is strongly complete with respect to \mathbb{B}_{12} , a class of juxtaposed unital structures over C_1 and C_2 . Let $i = 1$ or 2 . Suppose $\delta \in \text{Sent}(C_i, P_i)$ and $\alpha, \beta \in \text{Sent}(C_{12}, P_{12})$. Suppose \mathcal{M}_{12} is a coherent juxtaposed unital model over C_1, C_2 , and P_{12} based on \mathbb{B}_{12} such that $\mathcal{M}_{12} \models \alpha$, $\mathcal{M}_{12} \models \beta$, and $\mathcal{M}_{12} \models \delta[\alpha/p]$. In \mathcal{M}_{12} , $\|\alpha\|_i = \|\beta\|_i = \|\delta[\alpha/p]\|_i = 1_i$. By induction on the complexity of δ , $\|\delta[\beta/p]\|_i = \|\delta[\alpha/p]\|_i$. So $\|\delta[\beta/p]\|_i = 1_i$. So $\mathcal{M}_{12} \models \delta[\beta/p]$. So $\{\alpha, \beta, \delta[\alpha/p]\} \models^{\mathbb{B}_{12}} \delta[\beta/p]$. By strong completeness, $\{\alpha, \beta, \delta[\alpha/p]\} \vdash_{12} \delta[\beta/p]$. Therefore, \vdash_{12} is left-extensional over $\text{Sent}(C_i, P_i)$. \square

PROPOSITION 6.26. *Suppose C_1 and C_2 are disjoint. \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly determined with respect to a class of juxtaposed structures only if \vdash_{12} has no mere followers.*

PROOF. Suppose \vdash_{12} is strongly determined with respect to \mathbb{B}_{12} , a class of juxtaposed structures over C_1 and C_2 . Suppose $\not\vdash_{12} \alpha$. Let p be an element of P_{12} that does not occur in α . We show that $\{p\} \not\vdash_{12} \alpha$.

By completeness, $\not\models^{\mathbb{B}_{12}} \alpha$. So there is a coherent juxtaposed model $\mathcal{M}_{12} = \langle \mathcal{B}_1, V_1, \mathcal{B}_2, V_2 \rangle$ over C_1, C_2 , and P_{12} based on \mathbb{B}_{12} such that $\mathcal{M}_{12} \not\models \alpha$. Let each $\mathcal{B}_i = \langle B_i, D_i, \Phi_i \rangle$.

Let d_i be an element of D_i and let a_i be an element of $B_i - D_i$. Let $[\]_i$ be the function from $\text{Sent}(C_{12}, P_{12})$ to B_i inductively defined as follows:

- If α has no occurrence of p , $[\alpha]_i = \|\alpha\|_i^{\mathcal{M}_{12}}$;
- $[p]_i = d_i$;
- If $c \in C_i^n$ and p occurs in at least one of $\alpha^1, \dots, \alpha^n$, $[c\alpha^1 \dots \alpha^n]_i = \Phi_i(c)([\alpha^1]_i \dots [\alpha^n]_i)$;
- If $c \in C_1^n$ and p occurs in at least one of $\alpha^1, \dots, \alpha^n$, $[c\alpha^1 \dots \alpha^n]_2 = d_2$ if $\Phi_1(c)([\alpha^1]_1 \dots [\alpha^n]_1) \in D_1$ and $[c\alpha^1 \dots \alpha^n]_2 = a_2$ otherwise;
- If $c \in C_2^n$ and p occurs in at least one of $\alpha^1, \dots, \alpha^n$, $[c\alpha^1 \dots \alpha^n]_1 = d_1$ if $\Phi_2(c)([\alpha^1]_2 \dots [\alpha^n]_2) \in D_2$ and $[c\alpha^1 \dots \alpha^n]_1 = a_1$ otherwise.

Since C_1 and C_2 are disjoint, $[\]_i$ is well-defined.

If α is an i -atom, let $V'_i(\alpha) = [\alpha]_i$. Let $\mathcal{M}'_{12} = \langle \mathcal{B}_1, V'_1, \mathcal{B}_2, V'_2 \rangle$.

Clearly, \mathcal{M}'_{12} is a juxtaposed model based on \mathbb{B}_{12} . It is straightforward to show that in \mathcal{M}'_{12} , $\|\alpha\|_i = [\alpha]_i$. It is also straightforward to show that \mathcal{M}'_{12} is coherent. $\mathcal{M}'_{12} \models p$ and $\mathcal{M}'_{12} \not\models \alpha$. So $\{p\} \not\models^{\mathbb{B}_{12}} \alpha$. By strong soundness, $\{p\} \not\vdash_{12} \alpha$. \square

Combining Corollaries 6.13 and 6.23 with Propositions 6.24, 6.25, and 6.26, we arrive at the following nice result:

COROLLARY 6.27. *Suppose C_1 and C_2 are disjoint. Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . Then:*

1. \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly determined just in case \vdash_{12} is consistent and has no mere followers;
2. \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly unital determined just in case \vdash_{12} is consistent, has no mere followers, and is left-extensional over $\text{Sent}(C_1, P_1)$ and over $\text{Sent}(C_2, P_2)$.

6.6. Juxtaposed Consequence Relations. Our main interest in this paper concerns the combination of logics. A natural question to ask is: What properties of \vdash_1 and \vdash_2 suffice for a juxtaposed consequence relation over \vdash_1 and \vdash_2 to be $\langle C_1, C_2 \rangle$ -strongly determined or $\langle C_1, C_2 \rangle$ -strongly unital determined? The results above enable us to provide a fairly comprehensive answer to this question.

We first define a new notion. Suppose $\alpha \in \text{Sent}(C, P)$. We say that an occurrence of β in α is a *maximal non- C^- -occurrence* just in case the main connective of β is not from C^- and the occurrence is not properly within the scope of any connective not from C^- .

LEMMA 6.28. *Suppose \vdash is C^- -left-extensional. Then any consequence relation that extends \vdash is C^- -left-extensional.*

PROOF. Suppose C is a sub-signature of C^+ and P is a subset of P^+ . Suppose \vdash^+ is a consequence relation for $\text{Sent}(C^+, P^+)$ that extends \vdash . Suppose $\alpha, \beta, \delta \in \text{Sent}(C^+, P^+)$ and p strictly C^- -occurs in δ . Without loss of generality, we can assume that $\delta \in \text{Sent}(C^+, P)$. (We can use Uniform Substitution to cover the case where it doesn't.) Let δ^- be the result of replacing the maximal non- C^- -occurrences in δ with distinct elements of P that do not occur in α, β , or δ . Let q and r be distinct elements of P that do not occur in δ^- . So $\delta^- \in \text{Sent}(C, P)$ and p strictly C^- -occurs in δ . Since \vdash is C^- -left-extensional, $\{q, r, \delta^-[q/p]\} \vdash \delta^-[r/p]$. Since \vdash^+ extends \vdash , $\{q, r, \delta^-[q/p]\} \vdash^+ \delta^-[r/p]$. Since p strictly C^- -occurs in δ , by Uniform Substitution $\{q, r, \delta[q/p]\} \vdash^+ \delta[r/p]$. Again using Uniform Substitution, $\{\alpha, \beta, \delta[\alpha/p]\} \vdash^+ \delta[\beta/p]$. Therefore, \vdash^+ is C^- -left-extensional. \square

PROPOSITION 6.29. *Suppose \vdash_{12} is a juxtaposed consequence relation over \vdash_1 and \vdash_2 . Suppose at least one of \vdash_1 or \vdash_2 has theorems. Then:*

1. \vdash_{12} is strongly complete with respect to each of $\mathbb{B}_{12}^=$, \mathbb{B}_{12}^Ω , and \mathbb{B}_{12}^Σ ;
2. If \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , then \vdash_{12} is strongly sound with respect to each of $\mathbb{B}_{12}^=$, \mathbb{B}_{12}^Ω , and \mathbb{B}_{12}^Σ ;
3. If \vdash_{12} is consistent, then there are coherent non-trivial models based on each of $\mathbb{B}_{12}^=$, \mathbb{B}_{12}^Ω , and \mathbb{B}_{12}^Σ ;
4. If \vdash_1 and \vdash_2 are each left-extensional, then \mathbb{B}_{12}^Ω and \mathbb{B}_{12}^Σ are classes of juxtaposed unital structures.

PROOF. If at least one of \vdash_1 and \vdash_2 has theorems, \vdash_{12} has theorems, and so \vdash_{12} has no mere followers. Claims 1–3 then follow from Propositions 6.9, 6.12, and 6.22. If \vdash_i is left-extensional, then by Lemma 6.28, \vdash_{12} is C_i -left-extensional (and thus, is left-extensional over $\text{Sent}(C_i, P_i)$). Claim 4 then follows from Propositions 6.15 and 6.22. \square

The following corollary is immediate:

COROLLARY 6.30. *Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . Suppose \vdash_{12} is consistent. Suppose at least one of \vdash_1 and \vdash_2 has theorems. Then:*

1. \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly determined;
2. If \vdash_1 and \vdash_2 are each left-extensional, then \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly unital determined.

In the case where C_1 and C_2 are disjoint, we have the following nice results:

PROPOSITION 6.31. *Suppose C_1 and C_2 are disjoint. Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . Suppose each of \vdash_1 and \vdash_2 is consistent and has no mere followers. Suppose at least one of \vdash_1 and \vdash_2 has theorems. Then:*

1. \vdash_{12} is strongly determined with respect to each of $\mathbb{B}_{12}^{\bar{}}$, \mathbb{B}_{12}^{Ω} , and \mathbb{B}_{12}^{Σ} ;
2. There is a coherent non-trivial model based on each of $\mathbb{B}_{12}^{\bar{}}$, \mathbb{B}_{12}^{Ω} , and \mathbb{B}_{12}^{Σ} ;
3. \mathbb{B}_{12}^{Ω} and \mathbb{B}_{12}^{Σ} are classes of juxtaposed unital structures just in case \vdash_1 and \vdash_2 are each left-extensional.

PROOF. By Theorem 5.10, \vdash_{12} is consistent. If at least one of \vdash_1 and \vdash_2 has theorems, \vdash_{12} has theorems, and so \vdash_{12} has no mere followers. Claims 1–3 then follow from Propositions 6.9, 6.12, and 6.22. If \vdash_i is left-extensional, then by Lemma 6.28, \vdash_{12} is C_i -left-extensional (and thus, is left-extensional over $\text{Sent}(C_i, P_i)$). By Theorem 5.11, if \vdash_{12} is left-extensional over $\text{Sent}(C_i, P_i)$, then \vdash_i is left-extensional. Claim 4 then follows from Propositions 6.15 and 6.22. \square

COROLLARY 6.32. *Suppose C_1 and C_2 are disjoint. Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . Suppose at least one of \vdash_1 and \vdash_2 has theorems. Then:*

1. \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly determined just in case each of \vdash_1 and \vdash_2 is consistent and has no mere followers;
2. \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly unital determined just in case each of \vdash_1 and \vdash_2 is consistent, has no mere followers, and is left-extensional.

PROOF. The right-to-left direction of each claim follows from Proposition 6.31. The left-to-right direction follows from Propositions 6.24, 6.25, and 6.26 combined with Theorem 5.11. \square

Using Theorems 2.1 and 2.2, the following result is immediate:

COROLLARY 6.33. *Suppose C_1 and C_2 are disjoint. Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 . Suppose at least one of \vdash_1 and \vdash_2 has theorems. Then:*

1. \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly determined just in case each of \vdash_1 and \vdash_2 is strongly determined;
2. \vdash_{12} is $\langle C_1, C_2 \rangle$ -strongly unital determined just in case each of \vdash_1 and \vdash_2 is strongly unital determined.

There is an additional result worth stating. We will make use of this result in our discussion of classical and intuitionist logic below. We say that a class of structures \mathbb{B} is *full* for the consequence relation \vdash just in case \vdash is strongly sound with respect to \mathbb{B} and \mathbb{B} contains at least one representative from every isomorphism class of structures for \vdash that has a set of semantic values with cardinality at most that of the language of \vdash .⁴⁴ We say that \mathbb{B} is *unital full* for

⁴⁴This is a weakening of Definition 4.4 in Zanardo et al. (2001) and Definition 3.3.11 in Carnielli et al. (2008).

the consequence relation \vdash just in case \vdash is strongly unital sound with respect to \mathbb{B} and \mathbb{B} contains at least one representative from every isomorphism class of unital structures for \vdash that has a set of semantic values with cardinality at most that of the language of \vdash .

PROPOSITION 6.34. *Suppose \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 and \mathbb{B}_{12} is the juxtaposition of \mathbb{B}_1 and \mathbb{B}_2 . Suppose at least one of \vdash_1 and \vdash_2 has theorems. Then:*

1. *If \mathbb{B}_1 is full for \vdash_1 and \mathbb{B}_2 is full for \vdash_2 , then \vdash_{12} is strongly determined with respect to \mathbb{B}_{12} ;*
2. *If \vdash_1 and \vdash_2 are each left-extensional, \mathbb{B}_1 is unital full for \vdash_1 , and \mathbb{B}_2 is unital full for \vdash_2 , then \vdash_{12} is strongly determined with respect to \mathbb{B}_{12} , a class of juxtaposed unital structures.*

PROOF. Since \vdash_1 is strongly sound with respect to \mathbb{B}_1 and \vdash_2 is strongly sound with respect to \mathbb{B}_2 , by Theorem 5.5, \vdash_{12} is strongly sound with respect to \mathbb{B}_{12} .

Suppose \mathbb{B}_1 is full for \vdash_1 and \mathbb{B}_2 is full for \vdash_2 . By Proposition 6.29, \vdash_{12} is strongly determined with respect to \mathbb{B}_{12}^Σ . Using fullness, it is routine to show that each element of \mathbb{B}_{12}^Σ is isomorphic to an element of $\mathbb{B}_1 \times \mathbb{B}_2 = \mathbb{B}_{12}$. Therefore, \vdash_{12} is strongly complete with respect to \mathbb{B}_{12} .

Suppose \mathbb{B}_1 is unital full for \vdash_1 and \mathbb{B}_2 is unital full for \vdash_2 . Since \vdash_1 and \vdash_2 are each left-extensional, by Proposition 6.29, \vdash_{12} is strongly determined with respect to \mathbb{B}_{12}^Σ , a class of juxtaposed unital structures. Using fullness, it is again routine to show that each element of \mathbb{B}_{12}^Σ is isomorphic to an element of \mathbb{B}_{12} . So \vdash_{12} is strongly complete with respect to \mathbb{B}_{12} . Since \mathbb{B}_{12} is the juxtaposition of two classes of unital structures, it is a class of juxtaposed unital structures. \square

6.7. Equivential Logics. There is an important special case worth discussing. It concerns logics that contain a set of formulas that behave like a generalized biconditional.

Let q and r be distinct elements of P . Suppose $\Theta \subseteq \text{Sent}(C, \{q, r\})$. For simplicity, if $\theta \in \Theta$, we write $\theta(\alpha, \beta)$ to stand for the result of uniformly substituting each occurrence of q in θ with α and each occurrence of r in θ with β . We write $\Theta(\alpha, \beta)$ to stand for the set $\{\theta(\alpha, \beta) \mid \theta \in \Theta\}$. We write $\vdash \Theta(\alpha, \beta)$ to abbreviate the claim that for every $\theta \in \Theta$, $\vdash \theta(\alpha, \beta)$. We write $\Gamma \vdash \Theta(\alpha, \beta)$ to abbreviate the claim that for every $\theta \in \Theta$, $\Gamma \vdash \theta(\alpha, \beta)$.

We say that Θ is an *equivalence set* over C^- for \vdash just in case Θ satisfies the following conditions for every $\alpha, \beta, \gamma \in \text{Sent}(C, P)$:

Reflexivity. $\vdash \Theta(\alpha, \alpha)$;

Symmetry. $\Theta(\alpha, \beta) \vdash \Theta(\beta, \alpha)$;

Transitivity. $\Theta(\alpha, \beta) \cup \Theta(\beta, \gamma) \vdash \Theta(\alpha, \gamma)$;

Modus Ponens. $\{\alpha\} \cup \Theta(\alpha, \beta) \vdash \beta$;

Congruence over C^- . For every $c^- \in C^{-n}$, $\alpha^1, \dots, \alpha^n, \beta \in \text{Sent}(C, P)$, and $k \in \{1, \dots, n\}$, $\Theta(\alpha^k, \beta) \vdash \Theta(c^- \alpha^1 \dots \alpha^k \dots \alpha^n, c^- \alpha^1 \dots \beta \dots \alpha^n)$.

There is an additional condition worth stating:

Substitution over C^- . For every $\delta \in \text{Sent}(C, P)$ and p that strictly C^- -occurs in δ , $\{\delta[\alpha/p]\} \cup \Theta(\alpha, \beta) \vdash \delta[\beta/p]$.

PROPOSITION 6.35. *Suppose Θ satisfies Reflexivity and Congruence over C^- . Then Θ satisfies Substitution over C^- just in case Θ satisfies Modus Ponens.*

PROOF. Suppose Θ satisfies Modus Ponens. Suppose $\delta \in \text{Sent}(C, P)$ and p strictly C^- -occurs in δ . Using Reflexivity and Congruence over C^- , by induction on the complexity of δ , $\Theta(\alpha, \beta) \vdash \Theta(\delta[\alpha/p], \delta[\beta/p])$. By Modus Ponens and Cut, $\{\delta[\alpha/p]\} \cup \Theta(\alpha, \beta) \vdash \delta[\beta/p]$.

Now suppose Θ satisfies Substitution over C^- . So $\{p[\alpha/p]\} \cup \Theta(\alpha, \beta) \vdash p[\beta/p]$. That is, $\{\alpha\} \cup \Theta(\alpha, \beta) \vdash \beta$. \square

We say that Θ is a *regular equivalence set* over C^- for \vdash just in case Θ is an equivalence set over C^- for \vdash that satisfies the following additional condition:

Regularity. $\{\alpha, \beta\} \vdash \Theta(\alpha, \beta)$.

We say that Θ is an *equivalence set* for \vdash just in case Θ is an equivalence set over C for \vdash . We say that Θ is a *regular equivalence set* for \vdash just in case Θ is a regular equivalence set over C for \vdash . In many familiar logics – including classical and intuitionist logic – the sets $\{q \leftrightarrow r\}$ and $\{q \rightarrow r, r \rightarrow q\}$ are regular equivalence sets.

We say that \vdash is *equivalential* over C^- if there is an equivalence set over C^- for \vdash . We say that \vdash is *regularly equivalential* over C^- if there is a regular equivalence set over C^- for \vdash . We say that \vdash is *equivalential* (simpliciter) just in case \vdash is equivalential over C . We say that \vdash is *regularly equivalential* (simpliciter) just in case \vdash is regularly equivalential over C .⁴⁵

It is straightforward to show that if \vdash is regularly equivalential over C^- , then \vdash is C^- -left-extensional (and is therefore left-extensional over $\text{Sent}(C^-, P)$).

PROPOSITION 6.36. *If \vdash is regularly equivalential over C^- , then \vdash is C^- -left-extensional.*

PROOF. Suppose Θ is a regular equivalence set over C^- for \vdash . Suppose $\delta \in \text{Sent}(C, P)$ and p strictly C^- -occurs in δ . By Proposition 6.35, $\{\delta[\alpha/p]\} \cup \Theta(\alpha, \beta) \vdash \delta[\beta/p]$. By Regularity, $\{\alpha, \beta\} \vdash \Theta(\alpha, \beta)$. By Cut, $\{\alpha, \beta, \delta[\alpha/p]\} \vdash \delta[\beta/p]$. Therefore, \vdash is C^- -left-extensional. \square

Equivalence sets over C^- provide simple ways to construct equivalence relations suitable for C^- , \vdash , and Γ . Suppose $\Theta \subseteq \text{Sent}(C, P)$ is an equivalence set over C^- for \vdash . Suppose $\Gamma \subseteq \text{Sent}(C, P)$. We define a binary relation on $\text{Sent}(C, P)$ as follows:

$$\langle \alpha, \beta \rangle \in \Theta^\Gamma \text{ just in case } \Gamma \vdash \Theta(\alpha, \beta).$$

We write $\alpha \equiv \beta \pmod{\Theta^\Gamma}$ to stand for the claim that $\langle \alpha, \beta \rangle \in \Theta^\Gamma$.

LEMMA 6.37. *Suppose Θ is an equivalence set over C^- for \vdash . Then for every $\Gamma \subseteq \text{Sent}(C, P)$, Θ^Γ is an equivalence relation on $\text{Sent}(C, P)$ suitable for C^- , \vdash , and Γ .*

⁴⁵The notion of an equivalential logic (simpliciter) is originally due to Prucnal & Wroński (1974). See Czelakowski (1981) for discussion.

PROOF. Suppose $\Gamma \subseteq \text{Sent}(C, P)$. By Reflexivity, Symmetry, Transitivity, Weakening, and Cut, Θ^Γ is an equivalence relation on $\text{Sent}(C, P)$.

Congruence over C^- : Suppose $c^- \in C^-$; $\alpha^1, \dots, \alpha^n, \beta \in \text{Sent}(C, P)$; and $k \in \{1, \dots, n\}$. Suppose $\alpha^k \equiv \beta \pmod{\Theta^\Gamma}$. So $\Gamma \vdash \Theta(\alpha^k, \beta)$. Since Θ satisfies Congruence over C^- , $\Theta(\alpha^k, \beta) \vdash \Theta(c^- \alpha^1 \dots \alpha^k \dots \alpha^n, c^- \alpha^1 \dots \beta \dots \alpha^n)$. By Cut, $\Gamma \vdash \Theta(c^- \alpha^1 \dots \alpha^k \dots \alpha^n, c^- \alpha^1 \dots \beta \dots \alpha^n)$. Hence, $c^- \alpha^1 \dots \alpha^k \dots \alpha^n \equiv c^- \alpha^1 \dots \beta \dots \alpha^n \pmod{\Theta^\Gamma}$.

Compatibility with Γ and \vdash : Suppose $\alpha \equiv \beta \pmod{\Theta^\Gamma}$. So $\Gamma \vdash \Theta(\alpha, \beta)$. Suppose $\Gamma \vdash \alpha$. By Modus Ponens, $\{\alpha\} \cup \Theta(\alpha, \beta) \vdash \beta$. By Cut, $\Gamma \vdash \beta$. Similarly, if $\Gamma \vdash \beta$, then $\Gamma \vdash \alpha$. So $\Gamma \vdash \alpha$ just in case $\Gamma \vdash \beta$. \square

LEMMA 6.38. *Suppose Θ is an equivalence set over C^- for \vdash . Then Θ is a regular equivalence set over C^- for \vdash just in case for every non-empty $\Gamma \subseteq \text{Sent}(C, P)$ consistent with \vdash , Θ^Γ is unital suitable for C^- , \vdash , and Γ .*

PROOF. Suppose Θ is a regular equivalence set over C^- for \vdash . Suppose $\Gamma \subseteq \text{Sent}(C, P)$. Suppose $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$. By Regularity, $\{\alpha, \beta\} \vdash \Theta(\alpha, \beta)$. By Cut, $\Gamma \vdash \Theta(\alpha, \beta)$. So $\alpha \equiv \beta \pmod{\Theta^\Gamma}$. Therefore, Θ^Γ is unital suitable for C^- , \vdash , and Γ .

Now suppose for every non-empty $\Gamma \subseteq \text{Sent}(C, P)$ consistent with \vdash , Θ^Γ is unital suitable for C^- , \vdash , and Γ . Suppose $\alpha, \beta \in \text{Sent}(C, P)$. If $\{\alpha, \beta\}$ is inconsistent with respect to \vdash , $\{\alpha, \beta\} \vdash \Theta(\alpha, \beta)$. Suppose, then, that $\{\alpha, \beta\}$ is consistent with respect to \vdash . So $\Theta^{\{\alpha, \beta\}}$ is unital suitable for C^- , \vdash , and $\{\alpha, \beta\}$. By Identity and Weakening, $\{\alpha, \beta\} \vdash \alpha$ and $\{\alpha, \beta\} \vdash \beta$. By strong compatibility with \vdash and $\{\alpha, \beta\}$, $\alpha \equiv \beta \pmod{\Theta^{\{\alpha, \beta\}}}$. So, again, $\{\alpha, \beta\} \vdash \Theta(\alpha, \beta)$. Therefore, Θ is a regular equivalence set over C^- for \vdash . \square

Suppose Θ_1 is an equivalence set over C_1 for \vdash_{12} and Θ_2 is an equivalence set over C_2 for \vdash_{12} . Let $\mathbb{B}_{12}^{(\Theta_1, \Theta_2)}$ be the Lindenbaum-Tarski class of juxtaposed structures for C_1, C_2 , and \vdash_{12} built using the Θ_i^Γ relations. Any consequence relation that is equivalential over some signature has theorems. Combining this fact with Lemmas 6.37 and 6.38 and Theorem 6.7, we arrive at the following result:

PROPOSITION 6.39. *Suppose Θ_1 is an equivalence set over C_1 for \vdash_{12} and Θ_2 is an equivalence set over C_2 for \vdash_{12} . Then:*

1. \vdash_{12} is strongly complete with respect to $\mathbb{B}_{12}^{(\Theta_1, \Theta_2)}$;
2. If \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , then \vdash_{12} is strongly sound with respect to $\mathbb{B}_{12}^{(\Theta_1, \Theta_2)}$;
3. If \vdash_{12} is consistent, then there is a coherent non-trivial model based on $\mathbb{B}_{12}^{(\Theta_1, \Theta_2)}$;
4. $\mathbb{B}_{12}^{(\Theta_1, \Theta_2)}$ is a class of juxtaposed unital structures just in case Θ_1 is a regular equivalence set over C_1 for \vdash_{12} and Θ_2 is a regular equivalence set over C_2 for \vdash_{12} .

There is a particularly well-behaved kind of equivalence set over C^- . We say that Θ is an *internal equivalence set* over C^- for \vdash just in case Θ is an equivalence set over C^- for \vdash and $\Theta \subseteq \text{Sent}(C^-, \{q, r\})$. We say that Θ is an *regular internal equivalence set* if, in addition, Θ satisfies the Regularity condition. We say that

\vdash is *internally equivalential* over C^- if there is an internal equivalence set over C^- for \vdash . We say that \vdash is *regularly internally equivalential* over C^- if there is a regular internal equivalence set over C^- for \vdash . Notice that \vdash is equivalential (simpliciter) just in case \vdash is internally equivalential over C . \vdash is regularly equivalential (simpliciter) just in case \vdash is regularly internally equivalential over C .

We first put forward two alternative ways to characterize internal equivalence sets over C^- .

PROPOSITION 6.40. *Suppose $\Theta \subseteq \text{Sent}(C^-, \{q, r\})$. Θ is an internal equivalence set over C^- for \vdash just in case Θ satisfies Reflexivity, Modus Ponens, and Congruence over C^- .*

PROOF. Suppose Θ satisfies Reflexivity, Modus Ponens, and Congruence over C^- . Suppose $\theta \in \Theta$.

Symmetry: Since Θ is an internal equivalence set over C^- for \vdash , p strictly C^- -occurs in $\theta(p, \alpha)$. So by Proposition 6.35, $\{\theta(p, \alpha)[\alpha/p]\} \cup \Theta(\alpha, \beta) \vdash \theta(p, \alpha)[\beta/p]$. That is, $\{\theta(\alpha, \alpha)\} \cup \Theta(\alpha, \beta) \vdash \theta(\beta, \alpha)$. By Reflexivity, $\vdash \theta(\alpha, \alpha)$. By Cut, $\Theta(\alpha, \beta) \vdash \theta(\beta, \alpha)$.

Transitivity: Since Θ is an internal equivalence set over C^- for \vdash , p strictly C^- -occurs in $\theta(\alpha, p)$. By Proposition 6.35, $\{\theta(\alpha, p)[\beta/p]\} \cup \Theta(\beta, \gamma) \vdash \theta(\alpha, p)[\gamma/p]$. That is, $\theta(\alpha, \beta) \cup \Theta(\beta, \gamma) \vdash \theta(\alpha, \gamma)$. \square

PROPOSITION 6.41. *Suppose $\Theta \subseteq \text{Sent}(C^-, \{q, r\})$. Θ is an internal equivalence set over C^- for \vdash just in case Θ satisfies Reflexivity and Substitution over C^- .*

PROOF. Suppose Θ is an internal equivalence set over C^- for \vdash . Θ satisfies Reflexivity. By Proposition 6.35, Θ satisfies Substitution over C^- .

Now suppose Θ satisfies Reflexivity and Substitution over C^- . Suppose $c^- \in C^{-n}$; $\alpha^1, \dots, \alpha^n, \beta \in \text{Sent}(C, P)$; p is an element of P that does not occur in $\alpha^1, \dots, \alpha^n$; and $k \in \{1, \dots, n\}$. Suppose $\theta \in \Theta$. Since $\theta \in \text{Sent}(C^-, \{q, r\})$, p strictly C^- -occurs in $\theta(c^- \alpha^1 \dots \alpha^k \dots \alpha^n, c^- \alpha^1 \dots p \dots \alpha^n)$. By Substitution over C^- , Reflexivity, and Cut, $\Theta(\alpha^k, \beta) \vdash \theta(c^- \alpha^1 \dots \alpha^k \dots \alpha^n, c^- \alpha^1 \dots \beta \dots \alpha^n)$. So Θ satisfies Congruence over C^- . By Proposition 6.35, Θ satisfies Modus Ponens. By Proposition 6.40, Θ satisfies Symmetry and Transitivity. Therefore, Θ is an internal equivalence set over C^- for \vdash . \square

We next show that any two internal equivalence sets over C^- for \vdash are inter-derivable:

PROPOSITION 6.42. *Suppose Θ and Θ' are both internal equivalence sets over C^- for \vdash . Then $\Theta(\alpha, \beta) \vdash \Theta'(\alpha, \beta)$.*

PROOF. Suppose $\theta' \in \Theta'$. Since Θ' is an internal equivalence set over C^- for \vdash , p strictly C^- -occurs in $\theta'(\alpha, p)$. By Proposition 6.35, $\{\theta'(\alpha, p)[\alpha/p]\} \cup \Theta(\alpha, \beta) \vdash \theta'(\alpha, p)[\beta/p]$. That is, $\{\theta'(\alpha, \alpha)\} \cup \Theta(\alpha, \beta) \vdash \theta'(\alpha, \beta)$. By Reflexivity, $\vdash \theta'(\alpha, \alpha)$. By Cut, $\Theta(\alpha, \beta) \vdash \theta'(\alpha, \beta)$. \square

It follows from this result that if Θ and Θ' are both internal equivalence sets over C^- for \vdash , then $\Theta^\Gamma = \Theta'^\Gamma$. (It also follows that if Θ is regular, so is Θ' .)

Indeed, it is straightforward to show that if \vdash is internally equivalential and regularly equivalential, then every internal equivalence relation is regular.)

Suppose \vdash is internally equivalential over C^- . Let $\Theta_{C^-}^\Gamma$ stand for the unique equivalence relation defined on $\text{Sent}(C, P)$ generated by any of the internal equivalence sets over C^- for \vdash . We can show that $\Theta_{C^-}^\Gamma = \Omega_{C^-}^\Gamma = \Sigma_{C^-}^\Gamma$ is the most coarse-grained equivalence relation on $\text{Sent}(C, P)$ suitable for C^- , \vdash , and Γ .

LEMMA 6.43.

1. If Θ is an equivalence set over C^- for \vdash , then Θ is an equivalence set over C^- for any consequence relation that extends \vdash ;
2. If Θ is a regular equivalence set over C^- for \vdash , then Θ is a regular equivalence set over C^- for any consequence relation that extends \vdash .

PROOF. Suppose C is a sub-signature of C^+ and P is a subset of P^+ . Suppose \vdash^+ is a consequence relation for $\text{Sent}(C^+, P^+)$ that extends \vdash . Suppose p, q , and r are distinct elements of P . Suppose $\alpha, \beta, \gamma \in \text{Sent}(C^+, P^+)$.

Reflexivity: $\vdash \Theta(p, p)$. So $\vdash^+ \Theta(p, p)$. By Uniform Substitution, $\vdash^+ (\alpha, \alpha)$.

Symmetry: $\Theta(p, q) \vdash \Theta(q, p)$. So $\Theta(p, q) \vdash^+ \Theta(q, p)$. By Uniform Substitution, $\Theta(\alpha, \beta) \vdash^+ \Theta(\beta, \alpha)$.

Transitivity: $\Theta(p, q) \cup \Theta(q, r) \vdash \Theta(p, r)$. So $\Theta(p, q) \cup \Theta(q, r) \vdash^+ \Theta(p, r)$. By Uniform Substitution, $\Theta(\alpha, \beta) \cup \Theta(\beta, \gamma) \vdash^+ \Theta(\alpha, \gamma)$.

Modus Ponens: $\{p\} \cup \Theta(p, q) \vdash q$. So $\{p\} \cup \Theta(p, q) \vdash^+ q$. By Uniform Substitution, $\{\alpha\} \cup \Theta(\alpha, \beta) \vdash^+ \beta$.

Congruence over C^- : Let $c^- \in C^{-n}$. Let p^1, \dots, p^n, q be distinct elements of P . Let $\alpha^1, \dots, \alpha^n, \beta \in \text{Sent}(C^+, P^+)$ and $k \in \{1, \dots, n\}$. $\Theta(p^k, q) \vdash \Theta(c^- p^1 \dots p^k \dots p^n, c^- p^1 \dots q \dots p^n)$. So $\Theta(p^k, q) \vdash^+ \Theta(c^- p^1 \dots p^k \dots p^n, c^- p^1 \dots q \dots p^n)$. By Uniform Substitution, $\Theta(\alpha^k, \beta) \vdash^+ \Theta(c^- \alpha^1 \dots \alpha^k \dots \alpha^n, c^- \alpha^1 \dots \beta \dots \alpha^n)$.

Regularity: By Regularity, $\{p, q\} \vdash \Theta(p, q)$. So $\{p, q\} \vdash^+ \Theta(p, q)$. By Uniform Substitution, $\{\alpha, \beta\} \vdash \Theta(\alpha, \beta)$. \square

PROPOSITION 6.44. Suppose \vdash is internally equivalential over C^- . Suppose $\Gamma \subseteq \text{Sent}(C, P)$. Then $\Theta_{C^-}^\Gamma = \Omega_{C^-}^\Gamma = \Sigma_{C^-}^\Gamma$ is the most coarse-grained equivalence relation over $\text{Sent}(C, P)$ suitable for C^- , \vdash , and Γ .

PROOF. Suppose Θ is an internal equivalence set over C^- for \vdash .

We first show that $\alpha \equiv \beta \pmod{\Theta_{C^-}^\Gamma}$ just in case $\alpha \equiv \beta \pmod{\Omega_{C^-}^\Gamma}$.

Suppose $\alpha \equiv \beta \pmod{\Theta_{C^-}^\Gamma}$. By Proposition 6.11, $\alpha \equiv \beta \pmod{\Omega_{C^-}^\Gamma}$.

Now suppose $\alpha \equiv \beta \pmod{\Omega_{C^-}^\Gamma}$. Suppose $\theta \in \Theta$. Let p be an element of P that does not occur in α . Since Θ is an internal equivalence set over C^- for \vdash , p strictly C^- -occurs in $\theta(\alpha, p)$. So $\Gamma \vdash \Theta(\alpha, p)[\alpha/p]$ just in case $\Gamma \vdash \Theta(\alpha, p)[\beta/p]$. That is, $\Gamma \vdash \Theta(\alpha, \alpha)$ just in case $\Gamma \vdash \Theta(\alpha, \beta)$. By Reflexivity and Weakening, $\Gamma \vdash \Theta(\alpha, \alpha)$. So $\Gamma \vdash \Theta(\alpha, \beta)$. Therefore, $\alpha \equiv \beta \pmod{\Theta_{C^-}^\Gamma}$.

We next show that $\alpha \equiv \beta \pmod{\Theta_{C^-}^\Gamma}$ just in case $\alpha \equiv \beta \pmod{\Sigma_{C^-}^\Theta}$.

Suppose $\alpha \equiv \beta \pmod{\Theta_{C^-}^\Gamma}$. So $\Gamma \vdash \Theta(\alpha, \beta)$. Suppose $\delta \in \text{Sent}(C^-, P \cup P^*)$ and p occurs in δ . By Lemma 6.43, Θ is an equivalence set over C^- for \vdash^* . By Proposition 6.35, $\{\delta[\alpha/p]\} \cup \Theta(\alpha, \beta) \vdash^* \delta[\beta/p]$. Since \vdash^* extends \vdash , $\Gamma \vdash^* \Theta(\alpha, \beta)$. By Cut, $\Gamma \cup \{\delta[\alpha/p]\} \vdash^* \delta[\beta/p]$. Moreover, by Symmetry, $\Gamma \vdash^* \Theta(\beta, \alpha)$. So by analogous reasoning, $\Gamma \cup \{\delta[\beta/p]\} \vdash^* \delta[\alpha/p]$. Therefore, $\alpha \equiv \beta \pmod{\Sigma_{C^-}^\Gamma}$.

Now suppose $\alpha \equiv \beta \pmod{\Sigma_{C^-}^\Gamma}$. Suppose $\theta \in \Theta$ and $p \in P^*$. Since Θ is an internal equivalence set over C^- for \vdash , $\Gamma \cup \theta(p, \alpha) \vdash^* \theta(p, \beta)$. By Uniform Substitution, $\Gamma \cup \theta(\alpha, \alpha) \vdash^* \theta(\alpha, \beta)$. By Lemma 6.17, $\Gamma \cup \theta(\alpha, \alpha) \vdash \theta(\alpha, \beta)$. By Reflexivity, $\vdash \theta(\alpha, \alpha)$. By Cut, $\Gamma \vdash \theta(\alpha, \beta)$. Therefore, $\alpha \equiv \beta \pmod{\Theta_{C^-}^\Gamma}$.

By Proposition 6.11, $\Omega_{C^-}^\Gamma$ is the most coarse-grained equivalence relation on $\text{Sent}(C, P)$ suitable for C^- , \vdash , and Γ . \square

Suppose \vdash_{12} is internally equivalential over C_1 and over C_2 . Let \mathbb{B}_{12}^Θ be the Lindenbaum-Tarski class of juxtaposed structures for C_1 , C_2 , and \vdash_{12} , built using the $\Theta_{C_i}^\Gamma$ relations.

PROPOSITION 6.45. *Suppose \vdash_{12} is internally equivalential over C_1 and over C_2 . Then:*

1. $\mathbb{B}_{12}^\Theta = \mathbb{B}_{12}^\Omega = \mathbb{B}_{12}^\Sigma$;
2. \mathbb{B}_{12}^Θ is a class of juxtaposed unital structures just in case \vdash_{12} is regularly internally equivalential over C_1 and over C_2 just in case \vdash_{12} is left-extensional over $\text{Sent}(C_1, P_1)$ and over $\text{Sent}(C_2, P_2)$.

PROOF. Claim 1 follows from Proposition 6.44 Claim 2 follows from Proposition 6.44 and Lemmas 6.14 and 6.21. \square

Since $\mathbb{B}_{12}^\Theta = \mathbb{B}_{12}^\Omega = \mathbb{B}_{12}^\Sigma$, it is natural to think of this class as the *canonical* Lindenbaum-Tarski class of juxtaposed structures for C_1 , C_2 , and \vdash_{12} .

Suppose each of \vdash_1 and \vdash_2 is equivalential. When is \vdash_{12} $\langle C_1, C_2 \rangle$ -strongly determined or $\langle C_1, C_2 \rangle$ -strongly unital determined? Our results provide a fairly comprehensive answer to this question.

PROPOSITION 6.46. *Suppose \vdash_{12} is a juxtaposed consequence relation over \vdash_1 and \vdash_2 . Suppose each of \vdash_1 and \vdash_2 is equivalential. Then:*

1. \vdash_{12} is strongly complete with respect to \mathbb{B}_{12}^Θ ;
2. If \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , then \vdash_{12} is strongly sound with respect to \mathbb{B}_{12}^Θ ;
3. If \vdash_{12} is consistent, then there is a coherent non-trivial model based on \mathbb{B}_{12}^Θ ;
4. If each of \vdash_1 and \vdash_2 is regularly equivalential then \mathbb{B}_{12}^Θ is a class of juxtaposed unital structures.

PROOF. By Lemma 6.43, if \vdash_i is equivalential, then \vdash_{12} is internally equivalential over C_i . By Proposition 6.42, \mathbb{B}_{12}^Θ is well-defined. Claims 1–4 then follow by Proposition 6.39. \square

PROPOSITION 6.47. *Suppose C_1 and C_2 are disjoint. Suppose \vdash_{12} is a juxtaposed consequence relation over \vdash_1 and \vdash_2 . Suppose each of \vdash_1 and \vdash_2 is consistent and equivalential. Then:*

1. \vdash_{12} is strongly complete with respect to \mathbb{B}_{12}^Θ ;
2. If \vdash_{12} is the juxtaposition of \vdash_1 and \vdash_2 , then \vdash_{12} is strongly sound with respect to \mathbb{B}_{12}^Θ ;
3. There is a coherent non-trivial model based on \mathbb{B}_{12}^Θ ;
4. \mathbb{B}_{12}^Θ is a class of juxtaposed unital structures just in case each of \vdash_1 and \vdash_2 is regularly equivalential.

PROOF. If \vdash_1 and \vdash_2 are consistent, then by Theorem 5.10, \vdash_{12} is consistent. Claims 1–3 then follow from Proposition 6.46. Claim 4 follows from Proposition 6.39 and Theorem 5.11. \square

§7. Classical and Intuitionist Logics. In this section, we make use of our general results about juxtaposition to investigate cases of particular interest – juxtapositions of classical and intuitionist logics.

Let $P_1 = P_2 = P_{12}$ be the countably infinite set $\{p^1, p^2, \dots\}$. For $i = 1$ and 2 , let the signature C_i contain the following sets of connectives:

- $C_i^1 = \{\neg_i\}$.
- $C_i^2 = \{\wedge_i, \vee_i, \rightarrow_i, \leftrightarrow_i\}$.

C_{12} thus contains two copies of each of the standard propositional connectives.

We say that a consequence relation, \vdash_{12} , for $\text{Sent}(C_{12}, P_{12})$ *collapses* just in case for every $\delta, \delta' \in \text{Sent}(C_{12}, P_{12})$ exactly alike except perhaps for some or all of their subscripts, $\{\delta\} \vdash_{12} \delta'$.⁴⁶

Let \vdash_1^c be the classical consequence relation for $\text{Sent}(C_1, P_1)$. Let \vdash_1^i be the intuitionist consequence relation for $\text{Sent}(C_1, P_1)$. Let \vdash_2^c and \vdash_2^i be defined similarly.

Let \vdash^{cc} be the juxtaposition of \vdash_1^c and \vdash_2^c . We call this the “bi-classical” consequence relation for $\text{Sent}(C_{12}, P_{12})$. Let \vdash^{ii} be the juxtaposition of \vdash_1^i and \vdash_2^i . We call this the “bi-intuitionist” consequence relation for $\text{Sent}(C_{12}, P_{12})$. Finally, let \vdash^{ic} be the juxtaposition of \vdash_1^i and \vdash_2^c . We call this the “intuitionist-classical” consequence relation for $\text{Sent}(C_{12}, P_{12})$.

7.1. Applying the Results. Let us first consider the case of bi-classical logic, the juxtaposition of two classical consequence relations. Applying the results above, we can derive several properties of \vdash^{cc} .

Given any non-trivial Boolean algebra, $\langle B, \leq \rangle$, there is a corresponding unital structure $\langle B, \{1\}, \Phi \rangle$ where B is the same set of semantic values, 1 is the greatest element of the Boolean algebra, and for every $a, b \in B$:

$$\begin{aligned} \Phi(\neg)(a) &= -a; \\ \Phi(\wedge)(a, b) &= a \sqcap b; \\ \Phi(\vee)(a, b) &= a \sqcup b; \\ \Phi(\rightarrow)(a, b) &= -a \sqcup b; \\ \Phi(\leftrightarrow)(a, b) &= (-a \sqcup b) \sqcap (-b \sqcup a). \end{aligned}$$

Here, $-$, \sqcap , and \sqcup are the complement, infimum, and supremum relations on the Boolean algebra, respectively.

Let us call such structures “Boolean structures”. Given a Boolean structure, the partial order of the corresponding Boolean algebra can be recovered: $a \leq b$ just in case $\Phi(\rightarrow)(a, b) = 1$.

⁴⁶In the literature on fibring, the term “collapse” is typically used for the property called “weak collapse” in section 7.4 below.

The classical consequence relation is strongly determined with respect to the class of Boolean structures.⁴⁷ It is consistent, has theorems, and is left-extensional. By Theorem 5.10, \vdash^{cc} is consistent. By Theorem 5.11, \vdash^{cc} is strongly conservative over \vdash_1^c and \vdash_2^c . It can be axiomatized using two copies of any Hilbert-style axiomatization for classical logic. (It can also be axiomatized using two copies of any natural deduction-style axiomatization for classical logic restricted so that the rules for one stock of connectives cannot be applied within any sub-derivation used in the application of a meta-rule governing a connective from the other stock.) By Corollary 6.32, \vdash^{cc} is $\langle C_1, C_2 \rangle$ -strongly unital determined.

We can extract additional information about \vdash^{cc} using Proposition 6.34. The class of all Boolean structures is unital full for the classical consequence relation. We say that a juxtaposed unital structure $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ is a *bi-Boolean structure* just in case both \mathcal{B}_1 and \mathcal{B}_2 are Boolean structures. By Proposition 6.34, \vdash^{cc} is strongly determined with respect to the class of all bi-Boolean structures.

The case of the bi-intuitionist consequence relation, \vdash^{ii} is analogous. Given any non-trivial Heyting algebra, $\langle B, \leq \rangle$, there is a corresponding unital structure $\langle B, \{1\}, \Phi \rangle$ where B is the same set of semantic values, 1 is the greatest element of the Heyting algebra, and for every $a, b \in B$:

$$\begin{aligned}\Phi(\neg)(a) &= a \Rightarrow 0; \\ \Phi(\wedge)(a, b) &= a \sqcap b; \\ \Phi(\vee)(a, b) &= a \sqcup b; \\ \Phi(\rightarrow)(a, b) &= a \Rightarrow b; \\ \Phi(\leftrightarrow)(a, b) &= (a \Rightarrow b) \sqcap (b \Rightarrow a).\end{aligned}$$

Here, \sqcap , \sqcup , and \Rightarrow , are the infimum, supremum, and implication relations on the Heyting algebra, and 0 is its least element.

Let us call such structures “Heyting structures”. Given a Heyting structure, the partial order of the corresponding Heyting algebra can be recovered: $a \leq b$ just in case $\Phi(\rightarrow)(a, b) = 1$.

The intuitionist consequence relation is strongly determined with respect to the class of Heyting structures.⁴⁸ It is consistent, has theorems, and is left-extensional. By Theorem 5.10, \vdash^{ii} is consistent. By Theorem 5.11, \vdash^{ii} is strongly conservative over \vdash_1^i and \vdash_2^i . It can be axiomatized using two copies of any Hilbert-style axiomatization for intuitionist logic. (It can also be axiomatized using two copies of any natural deduction-style axiomatization for intuitionist logic restricted so that the rules for one stock of connectives cannot be applied within any sub-derivation used in the application of a meta-rule governing a connective from the other stock.) By Corollary 6.32, \vdash^{ii} is $\langle C_1, C_2 \rangle$ -strongly unital determined.

⁴⁷The classical consequence relation is also strongly determined with respect to the class of structures defined as above except that the set of designated values is allowed to be any proper filter on the Boolean algebra.

⁴⁸The intuitionist consequence relation is also strongly determined with respect to the class of structures defined as above except that the set of designated values is allowed to be any proper filter on the Heyting algebra.

The class of all Heyting structures is unital full for the intuitionist consequence relation. We say that a juxtaposed unital structure $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ is a *bi-Heyting structure* just in case both \mathcal{B}_1 and \mathcal{B}_2 are Heyting structures. By Proposition 6.34, \vdash^{ii} is strongly determined with respect to the class of all bi-Heyting structures.

Finally, consider the intuitionist-classical consequence relation, \vdash^{ic} . By Theorem 5.10, \vdash^{ic} is consistent. By Theorem 5.11, \vdash^{ic} is strongly conservative over \vdash_1^i and \vdash_2^c . It can be axiomatized by combining a Hilbert-style axiomatization for classical logic and a Hilbert-style axiomatization for intuitionist logic. (It can also be axiomatized using a copy of any natural deduction-style axiomatization for intuitionist logic and a copy of any natural deduction-style axiomatization for classical logic, each restricted so that the rules for one stock of connectives cannot be applied within any sub-derivation used in the application of a meta-rule governing a connective from the other stock.) By Corollary 6.32, \vdash^{ic} is $\langle C_1, C_2 \rangle$ -strongly unital determined.

We say that a juxtaposed unital structure $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ is a *Heyting-Boolean structure* just in case \mathcal{B}_1 is a Heyting structure and \mathcal{B}_2 is a Boolean structure. Again using Proposition 6.34, \vdash^{ic} is strongly determined with respect to the class of all Heyting-Boolean structures.

7.2. Non-Collapse Results. These results already have implications for the collapse of classical and intuitionist logics. Consider the case of \vdash^{ic} . We have shown that this consequence relation is strongly conservative over \vdash_1^i and \vdash_2^c . This suffices to show that \vdash^{ic} does not collapse.⁴⁹ Since $p \vee \neg p$ is a theorem of classical logic but not a theorem of intuitionist logic, $\not\vdash^{ic} p \vee_1 \neg_1 p$ and $\vdash^{ic} p \vee_2 \neg_2 p$. Similarly, Peirce's law, $((p \rightarrow q) \rightarrow p) \rightarrow p$, can be used to show that \rightarrow_1 and \rightarrow_2 are not intersubstitutable. Double Negation Elimination can be used to show that \neg_1 and \neg_2 are not intersubstitutable, either.

The interest of these results should not be underemphasized. Juxtaposition is a natural way to combine classical and intuitionist logic. \vdash^{ic} has all of the entailments of intuitionist logic (for the 1-connectives) and all of the entailments of classical logic (for the 2-connectives). Indeed, it is a strong conservative extension of both intuitionist and classical logic. It obeys the usual structural rules, and is substitution invariant. Yet, the classical and intuitionist connectives are not intersubstitutable. There is no collapse.

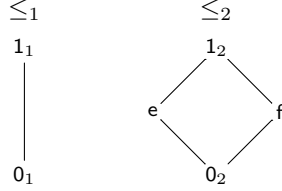
Notice that the non-collapse result also applies to \vdash^{ii} . This is a strictly weaker relation than \vdash^{ic} . Thus, if corresponding connectives are not intersubstitutable in \vdash^{ic} , they are not intersubstitutable in \vdash^{ii} , either.

What about bi-classical logic? Does this logic avoid collapse? The answer is yes, but proving this takes a bit more work. In particular, to show that \vdash^{cc} does not collapse, we make use of Lemma 5.1 to build a coherent bi-Boolean model.

A Boolean model can be specified by specifying a Boolean algebra and a valuation. Boolean algebras can be visually represented using *Hasse diagrams*. Such a diagram contains nodes – each representing a distinct element of the carrier set of the algebra – and line segments connecting pairs of nodes. Given two elements of the carrier set, a and b , $a \leq b$ just in case there is a path from a to

⁴⁹This can also be proved using the technique of cryptofibring. See Caleiro & Ramos (2007).

b monotonically increasing in height. For example, here are the Hasse diagrams for two Boolean algebras:



Let \mathcal{B}_1 be the Boolean structure corresponding to the first of these Boolean algebras. Let \mathcal{B}_2 be the Boolean structure corresponding to the second Boolean algebra. Let $V_1(p^1) = V_1(p^2) = V_1(p^3) = 0_1$. Let $V_2(p^1) = V_2(p^2) = e$ and $V_2(p^3) = f$. Let $V_i(p) = 1_i$ for every other $p \in P_{12}$. Let $\mathcal{M}_1 = \langle \mathcal{B}_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{B}_2, V_2 \rangle$. \mathcal{M}_1 and \mathcal{M}_2 designate the very same sentence symbols. By Lemma 5.1, there is a coherent juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 . We can single out a specific coherent juxtaposition of these models by specifying which non-designated elements to use in the construction relied upon in the proof of that lemma. (Since \mathcal{B}_1 and \mathcal{B}_2 are unital, there is no choice about which designated elements to use in the construction.) In particular, let $a_1 = 0_1$ and $a_2 = 0_2$. Let \mathcal{M}_{12}^* be the coherent bi-Boolean model that results from this choice in the construction.

Using this juxtaposed model, we can prove the non-collapse result. Indeed, we can prove the stronger result that no two corresponding connectives in C_{12} are intersubstitutable according to \vdash^{cc} :

PROPOSITION 7.1. *In \vdash^{cc} , no pair of corresponding connectives are intersubstitutable. In particular:*

- $\{\neg_1 p\} \not\vdash^{cc} \neg_2 p$;
- $\{p \vee_1 q\} \not\vdash^{cc} p \vee_2 q$;
- $\{p \rightarrow_1 q\} \not\vdash^{cc} p \rightarrow_2 q$;
- $\{p \leftrightarrow_1 q\} \not\vdash^{cc} p \leftrightarrow_2 q$;
- $\{\neg_2(p \wedge_1 q)\} \not\vdash^{cc} \neg_2(p \wedge_2 q)$.

Therefore, \vdash^{cc} does not collapse.

PROOF. Since \vdash^{cc} is strongly sound with respect to the class of bi-Boolean structures, all we have to do is to find a coherent bi-Boolean countermodel to each putative entailment claim. We make use of the model \mathcal{M}_{12}^* ,

In \mathcal{M}_{12}^* , $\|\neg_1 p^1\|_1 = 1_1$ and $\|\neg_2 p^1\|_2 = f$. So $\{\neg_1 p^1\} \not\vdash^{cc} \neg_2 p^1$. $\|p^1 \vee_2 p^3\|_2 = 1_2$ and $\|p^1 \vee_1 p^3\|_1 = 0_1$. So $\{p^1 \vee_2 p^3\} \not\vdash^{cc} p^1 \vee_1 p^3$. By symmetry, $\{p^1 \vee_1 p^3\} \not\vdash^{cc} p^1 \vee_2 p^3$. $\|p^1 \rightarrow_1 p^3\|_1 = 1_1$ and $\|p^1 \rightarrow_2 p^3\|_2 = f$. So $\{p^1 \rightarrow_1 p^3\} \not\vdash^{cc} p^1 \rightarrow_2 p^3$. $\|p^1 \leftrightarrow_1 p^3\|_1 = 1_1$. $\|p^1 \leftrightarrow_2 p^3\|_2 = 0_2$. So $\{p^1 \leftrightarrow_1 p^3\} \not\vdash^{cc} p^1 \leftrightarrow_2 p^3$.

The trickiest case is the final one, since it involves an embedded connective. This is what requires us to specify a particular coherent juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 . In \mathcal{M}_{12}^* , $\|\neg_2(p^1 \wedge_1 p^2)\|_2 = \neg_2\|p^1 \wedge_1 p^2\|_2 = \neg_2 0_2 = 1_2$. $\|\neg_2(p^1 \wedge_2 p^2)\|_2 = f$. So $\{\neg_2(p^1 \wedge_1 p^3)\} \not\vdash^{cc} \neg_2(p^1 \wedge_2 p^3)$. \square

Since \vdash^{ii} and \vdash^{ic} are weaker than \vdash^{cc} , the result applies to these logics, as well.

Notice that the two conjunction connectives, \wedge_1 and \wedge_2 , are intersubstitutable as *main connectives* in \vdash^{ii} . This is because $\{p^1 \wedge_1 p^2\} \vdash^i p^1$, $\{p^1 \wedge_1 p^2\} \vdash^i p^2$, and $\{p^1, p^2\} \vdash^i p^1 \wedge_2 p^2$. By the definition of juxtaposition, $\{p^1 \wedge_1 p^2\} \vdash^{ii} p^1 \wedge_2 p^2$. By Uniform Substitution, $\{\alpha \wedge_1 \beta\} \vdash^{ii} \alpha \wedge_2 \beta$. The other direction is analogous. It follows that \wedge_1 and \wedge_2 , are intersubstitutable as main connectives in stronger logics, including \vdash^{cc} and \vdash^{ic} .

It is also worth noting that, as advertised above, \vdash^{cc} is not left-extensional.

PROPOSITION 7.2. \vdash^{cc} is not left-extensional.

PROOF. Let \mathcal{B}_1 and \mathcal{B}_2 be the Boolean structures defined above. Let $V_1(p^1) = V_1(p^2) = 1_1$. Let $V_2(p^1) = V_2(p^2) = 1_2$. Let $V_i(p) = 0_i$ for every other $p \in P_{12}$. Let $\mathcal{M}_1 = \langle \mathcal{B}_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{B}_2, V_2 \rangle$. By Lemma 5.1, there is a coherent juxtaposition of \mathcal{M}_1 and \mathcal{M}_2 . Using a simple modification of the construction appearing in the proof of that lemma, we can ensure that $\|\neg_1 p^1\|_2 = 0_2$ and $\|\neg_1 p^2\|_2 = e$. (We let a_2 vary as needed in the construction.)

In the resulting model, $\|\neg_2 \neg_1 p^1\|_2 = 1_2$ and $\|\neg_2 \neg_1 p^2\|_2 = f$. By strong soundness, $\{p^1, p^2, \neg_2 \neg_1 p^1\} \not\vdash^{cc} \neg_2 \neg_1 p^2$. \square

It follows that \vdash^{ii} and \vdash^{ic} are not left-extensional, either.

7.3. Interaction Rules. Our non-collapse results raise a question: Which additional axioms and rules lead to collapse when added to \vdash^{ii} , \vdash^{ic} , or \vdash^{cc} ? In particular, which interaction rules – rules involving both stocks of connectives – lead to collapse?

Our general results about juxtaposition can help us to tackle this question. Many of our results do not merely to apply to *the* juxtaposition of \vdash_1 and \vdash_2 , but to *any* juxtaposed consequence relation over \vdash_1 and \vdash_2 (so long as the relevant constraints are met). Thus, we can make use of these results to help determine whether a juxtaposed consequence relation collapses. For example, we have the following nice result:

PROPOSITION 7.3. *Suppose \vdash_{12} is a consequence relation for $\text{Sent}(C_{12}, P_{12})$ that extends \vdash^{ii} . Suppose for any $\alpha, \beta \in \text{Sent}(C_{12}, P_{12})$, $\{\alpha \rightarrow_1 \beta\} \vdash_{12} \alpha \rightarrow_2 \beta$ and $\{\alpha \rightarrow_2 \beta\} \vdash_{12} \alpha \rightarrow_1 \beta$. Then \vdash_{12} collapses.*

PROOF. $\{q \rightarrow r, r \rightarrow q\}$ is a regular equivalence set for intuitionist logic. By Lemma 6.43, $\{q \rightarrow_i r, r \rightarrow_i q\}$ is a regular internal equivalence set over C_i for \vdash_{12} . For any $i \in \{1, 2\}$ and non-empty $\Gamma \subseteq \text{Sent}(C_{12}, P_{12})$ consistent with respect to \vdash_{12} , let $\langle \alpha, \beta \rangle \in \Theta_{C_1}^\Gamma$ just in case $\Gamma \vdash_{12} \alpha \rightarrow_i \beta$ and $\Gamma \vdash_{12} \beta \rightarrow_i \alpha$. By Lemma 6.38, $\Theta_{C_i}^\Gamma$ is unital suitable for C_i , \vdash_{12} and Γ . Let $\mathbf{M}_{12}^\Gamma = \langle \mathbf{B}_1^\Gamma, \mathbf{V}_1^\Gamma, \mathbf{B}_2^\Gamma, \mathbf{V}_2^\Gamma \rangle$ be the Lindenbaum-Tarski juxtaposed model for C_1, C_2, \vdash_{12} , and Γ built with $\Theta_{C_1}^\Gamma$ and $\Theta_{C_2}^\Gamma$. Since \rightarrow_1 and \rightarrow_2 are intersubstitutable as main connectives, $\Theta_{C_1}^\Gamma = \Theta_{C_2}^\Gamma$. Thus, $\mathbf{B}_1^\Gamma = \mathbf{B}_2^\Gamma$, $\mathbf{1}_1^\Gamma = \mathbf{1}_2^\Gamma$, $\mathbf{V}_1^\Gamma(p) = \mathbf{V}_2^\Gamma(p)$ for every $p \in P_{12}$, and $|\alpha|_1 = |\alpha|_2$ for every $\alpha \in \text{Sent}(C_{12}, P_{12})$. By Lemma 6.2, $\|\alpha\|_1 = \|\alpha\|_2$ for every $\alpha \in \text{Sent}(C_{12}, P_{12})$.

The partial order of the Heyting algebra corresponding to \mathbf{B}_i^Γ , \leq_i^Γ , can be defined as follows: $|\alpha|_i^\Gamma \leq_i^\Gamma |\beta|_i^\Gamma$ just in case $\Phi_i^\Gamma(\rightarrow_i)(|\alpha|_i^\Gamma, |\beta|_i^\Gamma) = \mathbf{1}_i^\Gamma$ just in case $\Gamma \vdash_{12} \alpha \rightarrow_i \beta$. By Cut, $\Gamma \vdash_{12} \alpha \rightarrow_1 \beta$ just in case $\Gamma \vdash_{12} \alpha \rightarrow_2 \beta$. Thus, $\leq_1^\Gamma = \leq_2^\Gamma$. As described above, Φ_i^Γ assigns to the connectives in C_i the functions induced by

\leq_i^Γ to make \mathbf{B}_{12}^Γ into a Heyting structure. Hence, Φ_1^Γ and Φ_2^Γ map corresponding connectives to the very same functions. It follows that any two sentences that are exactly alike except for their subscripts will have the same i -values in \mathbf{M}_{12}^Γ .

Let $\delta, \delta' \in \text{Sent}(C_{12}, P_{12})$ be sentences that are exactly alike except for some or all of their subscripts. If $\{\delta\}$ is inconsistent with respect to \vdash_{12} , $\{\delta\} \vdash_{12} \delta'$. Suppose, then, that $\{\delta\}$ is consistent with respect to \vdash_{12} . By Lemma 6.2, $\mathbf{M}_{12}^{\{\delta\}} \models \delta$. By the above reasoning, δ and δ' have the same i -values in $\mathbf{M}_{12}^{\{\delta\}}$. So $\mathbf{M}_{12}^{\{\delta\}} \models \delta'$. Again using Lemma 6.2, $\{\delta\} \vdash_{12} \delta'$. Therefore, \vdash_{12} collapses. \square

Thus, if we enrich \vdash^{ii} (or a stronger logic) with the rule that says that \rightarrow_1 and \rightarrow_2 are intersubstitutable as main connectives, the resulting logic collapses.

We can leverage this result to get further collapse results. In intuitionist logic, $\alpha \rightarrow \beta$ is equivalent to $\alpha \leftrightarrow (\alpha \wedge \beta)$. So it follows that if we enrich \vdash^{ii} (or a stronger logic) with (i) the rule that says that \leftrightarrow_1 and \leftrightarrow_2 are intersubstitutable as main connectives as well as (ii) the rule that says that \wedge_1 and \wedge_2 are intersubstitutable in general, the resulting logic collapses. In intuitionist logic, $\alpha \rightarrow \beta$ is also equivalent to $\beta \leftrightarrow (\alpha \vee \beta)$. So if we enrich \vdash^{ii} (or a stronger logic) with (i) the rule that says that \leftrightarrow_1 and \leftrightarrow_2 are intersubstitutable as main connectives as well as (ii) the rule that says that \vee_1 and \vee_2 are intersubstitutable in general, the resulting logic collapses.

We can prove additional collapse results for enrichments of bi-classical logic. For example:

PROPOSITION 7.4. *Suppose \vdash_{12} is a consequence relation for $\text{Sent}(C_{12}, P_{12})$ that extends \vdash^{cc} . Suppose for any $\alpha, \beta \in \text{Sent}(C_{12}, P_{12})$, $\{\alpha \vee_1 \beta\} \vdash_{12} \alpha \vee_2 \beta$ and $\{\alpha \vee_2 \beta\} \vdash_{12} \alpha \vee_1 \beta$. Then \vdash_{12} collapses.*

PROOF. $\{\alpha \rightarrow_1 \beta\} \vdash^{cc} \neg_2 \alpha \vee_1 (\alpha \rightarrow_1 \beta)$. $\{\neg_2 \alpha \vee_1 \alpha, \neg_2 \alpha \vee_1 (\alpha \rightarrow_1 \beta)\} \vdash^{cc} \neg_2 \alpha \vee_1 \beta$. $\vdash^{cc} \neg_2 \alpha \vee_2 \alpha$ and $\{\neg_2 \alpha \vee_2 \alpha\} \vdash_{12} \neg_2 \alpha \vee_1 \alpha$. By Cut, $\{\alpha \rightarrow_1 \beta\} \vdash_{12} \neg_2 \alpha \vee_1 \beta$. $\{\neg_2 \alpha \vee_1 \beta\} \vdash_{12} \neg_2 \alpha \vee_2 \beta$. $\{\neg_2 \alpha \vee_2 \beta\} \vdash^{cc} \alpha \rightarrow_2 \beta$. By Cut, $\{\alpha \rightarrow_1 \beta\} \vdash_{12} \alpha \rightarrow_2 \beta$. Similarly, $\{\alpha \rightarrow_2 \beta\} \vdash_{12} \alpha \rightarrow_1 \beta$. By Proposition 7.3, \vdash_{12} collapses. \square

Thus, if we enrich \vdash^{cc} (or a stronger logic) with the rule that says that \vee_1 and \vee_2 are intersubstitutable as main connectives, the resulting logic collapses.

In classical logic, $\alpha \rightarrow \beta$ is equivalent to $\neg(\alpha \wedge \neg\beta)$. So if we enrich \vdash^{cc} (or a stronger logic) with (i) the rule that says that \neg_1 and \neg_2 are intersubstitutable in general as well as (ii) the rule that says that \wedge_1 and \wedge_2 are intersubstitutable in general, the resulting logic collapses.

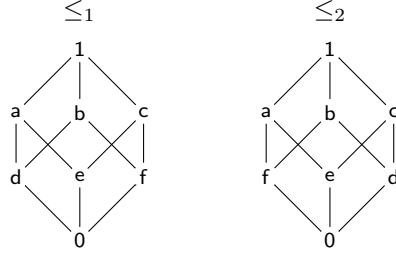
We can also prove non-collapse results for enriched logics. For example:

PROPOSITION 7.5. *Suppose \vdash_{12} is the least consequence relation that extends \vdash^{cc} such that for any $\alpha, \beta \in \text{Sent}(C_{12}, P_{12})$, (i) $\{\neg_1 \alpha\} \vdash_{12} \neg_2 \alpha$ and $\{\neg_2 \alpha\} \vdash_{12} \neg_1 \alpha$; and (ii) $\{\alpha \leftrightarrow_1 \beta\} \vdash_{12} \alpha \leftrightarrow_2 \beta$ and $\{\alpha \leftrightarrow_2 \beta\} \vdash_{12} \alpha \leftrightarrow_1 \beta$. Then in \vdash_{12} , no pair of corresponding connectives are intersubstitutable. In particular:*

- $\{p \vee_1 q\} \not\vdash_{12} p \vee_2 q$;
- $\{p \rightarrow_1 q\} \not\vdash_{12} p \rightarrow_2 q$;
- $\{\neg_2(p \leftrightarrow_1 q)\} \not\vdash_{12} \neg_2(p \leftrightarrow_2 q)$;
- $\{\neg_2(p \wedge_1 q)\} \not\vdash_{12} \neg_2(p \wedge_2 q)$;
- $\{p \vee_1 \neg_1 q\} \not\vdash_{12} p \vee_1 \neg_2 q$.

Therefore, \vdash_{12} does not collapse.

PROOF. We construct a coherent juxtaposed model to be a countermodel. Let $B = \{1, a, b, c, d, e, f, 0\}$. Consider the following pair of Boolean algebras on B :



Let \mathcal{B}_1 be the Boolean structure corresponding to the first of these Boolean algebras. Let \mathcal{B}_2 be the Boolean structure corresponding to the second Boolean algebra. Let $V(p^1) = a$, $V(p^2) = c$, $V(p^3) = f$, and $V(p) = 1$ for every other $p \in P_{12}$. Let the function $[\]$ be inductively defined as follows:

- $[p] = V(p)$ if $p \in P_{12}$;
- $[\neg_i \alpha] = -_i[\alpha]$;
- $[\alpha \wedge_i \beta] = [\alpha] \sqcap_i [\beta]$;
- $[\alpha \vee_i \beta] = [\alpha] \sqcup_i [\beta]$;
- $[\alpha \rightarrow_i \beta] = -_i[\alpha] \sqcup_i [\beta]$;
- $[\alpha \leftrightarrow_i \beta] = (-_i[\alpha] \sqcup_i [\beta]) \sqcap_i (-_i[\beta] \sqcup_i [\alpha])$.

Here, $-_i$, \sqcap_i , and \sqcup_i are the complement, infimum, and supremum relations on the i th Boolean algebra.

For $i \in \{1, 2\}$, let $V_i(\alpha) = [\alpha]$ if α is an i -atom. Let $\mathcal{M}_{12} = \langle \mathcal{B}_1, V_1, \mathcal{B}_2, V_2 \rangle$. It is easy to show that \mathcal{M}_{12} is a coherent bi-Boolean model. Thus, all of the classical validities for each stock of connectives are designated in this model. In \mathcal{M}_{12} , $\|\alpha\|_1 = \|\alpha\|_2 = [\alpha]$. $\mathcal{M}_{12} \models \neg_1 \alpha$ just in case $[\neg_1 \alpha] = 1$ just in case $[\alpha] = 0$ just in case $[\neg_2 \alpha] = 1$ just in case $\mathcal{M}_{12} \models \neg_2 \alpha$. $\mathcal{M}_{12} \models \alpha \leftrightarrow_1 \beta$ just in case $[\alpha \leftrightarrow_1 \beta] = 1$ just in case $[\alpha] = [\beta]$ just in case $[\alpha \leftrightarrow_2 \beta] = 1$ just in case $\mathcal{M}_{12} \models \alpha \leftrightarrow_2 \beta$.

In this model, $\|p^1 \vee_1 p^3\|_i = 1$ and $\|p^1 \vee_2 p^3\|_i = a$. $\|p^3 \rightarrow_1 p^2\|_i = 1$ and $\|p^3 \rightarrow_2 p^2\|_i = c$. $\|\neg_2(p^1 \leftrightarrow_1 p^3)\|_i = -_2 0 = 1$ and $\|\neg_2(p^1 \leftrightarrow_1 p^3)\|_i = -_2 f = c$. $\|\neg_2(p^1 \wedge_1 p^3)\|_i = -_2 0 = 1$ and $\|\neg_2(p^1 \wedge_1 p^3)\|_i = -_2 f = c$. $\|p^1 \vee_1 \neg_1 p^1\|_i = 1$ and $\|p^1 \vee_1 \neg_2 p^1\|_i = a$. \square

Thus, if we enrich \vdash^{cc} (or a weaker logic) with the rules that say that \neg_1 and \neg_2 are intersubstitutable as main connectives and \leftrightarrow_1 and \leftrightarrow_2 are intersubstitutable as main connectives, the resulting logic does not collapse.

Notice that if we enrich \vdash^{ii} (or a stronger logic) with the rules that says that \leftrightarrow_1 and \leftrightarrow_2 are intersubstitutable as main connectives, the resulting logic is left-extensional. (This is because $\{q \leftrightarrow_i r\}$ is a regular equivalence set over C_i for \vdash^{ii} . So if we enrich \vdash^{ii} with the rules that says that \leftrightarrow_1 and \leftrightarrow_2 are intersubstitutable as main connectives, the resulting consequence relation is regularly equivalential (simpliciter). By Proposition 6.36, any regularly equivalential consequence relation is left-extensional.) Thus, it follows from Proposition 7.5 that if we enrich

\vdash^{cc} (or a weaker logic) with left-extensionality, the resulting consequence relation does not collapse.

7.4. Weak Collapse. It is worth briefly considering a weaker kind of collapse. Let f be the bijection from $\text{Sent}(C_1, P_{12})$ to $\text{Sent}(C_2, P_{12})$ that maps each sentence $\alpha \in \text{Sent}(C_1, P_{12})$ to the sentence that results from uniformly substituting each connective in α with the corresponding connective from C_2 . We say that a consequence relation, \vdash_{12} , for $\text{Sent}(C_{12}, P_{12})$ *weakly collapses* just in case for every $\Gamma \subseteq \text{Sent}(C_1, P_{12})$ and $\alpha \in \text{Sent}(C_1, P_{12})$, $\Gamma \vdash_{12} \alpha$ just in case $f(\Gamma) \vdash_{12} f(\alpha)$.

It is immediate that \vdash^{cc} and \vdash^{ii} weakly collapse. If \vdash_{12} collapses then \vdash_{12} weakly collapses. So by Proposition 7.3, if we enrich \vdash^{ic} with the rules that say that \rightarrow_1 and \rightarrow_2 are intersubstitutable as main connectives, the resulting logic weakly collapses.

Since \vdash^{ic} is a strong conservative extension over both intuitionist and classical logic, \vdash^{ic} does not weakly collapse. Moreover, if we enrich \vdash^{ic} with the rules that say that \neg_1 and \neg_2 are intersubstitutable as main connectives and \leftrightarrow_1 and \leftrightarrow_2 are intersubstitutable as main connectives, the resulting logic still does not weakly collapse:

PROPOSITION 7.6. *Suppose \vdash_{12} is the least consequence relation that extends \vdash^{ic} such that for any $\alpha, \beta \in \text{Sent}(C_{12}, P_{12})$, (i) $\{\neg_1\alpha\} \vdash_{12} \neg_2\alpha$ and $\{\neg_2\alpha\} \vdash_{12} \neg_1\alpha$; and (ii) $\{\alpha \leftrightarrow_1 \beta\} \vdash_{12} \alpha \leftrightarrow_2 \beta$ and $\{\alpha \leftrightarrow_2 \beta\} \vdash_{12} \alpha \leftrightarrow_1 \beta$. Then \vdash_{12} does not weakly collapse.*

PROOF. Let B be a countably infinite set. Let 0 and 1 be elements of B . Let \mathcal{B}_1 be a Heyting structure that is not also a Boolean structure with carrier set B and with least element 0 and greatest element 1. Let \mathcal{B}_2 be a Boolean structure with carrier set B and with least element 0 and greatest element 1.

Let \sqcap_i , \sqcup_i , and \Rightarrow_i be the infimum, supremum, and implication relations on the i th algebra. Let \mathbf{a} be an element of B such that $\mathbf{a} \sqcup_1 (\mathbf{a} \Rightarrow_1 \mathbf{0}) \neq 1$. Such an element is guaranteed to exist since \mathcal{B}_1 is not a Boolean structure. Let V be a function from P_{12} to B such that $V(p^1) = \mathbf{a}$. Let the function $[\]$ be defined inductively as follows:

- $[p] = V(p)$ if $p \in P_{12}$;
- $[\neg_i\alpha] = [\alpha] \Rightarrow_i \mathbf{0}$;
- $[\alpha \wedge_i \beta] = [\alpha] \sqcap_i [\beta]$;
- $[\alpha \vee_i \beta] = [\alpha] \sqcup_i [\beta]$;
- $[\alpha \rightarrow_i \beta] = [\alpha] \Rightarrow_i [\beta]$;
- $[\alpha \leftrightarrow_i \beta] = ([\alpha] \Rightarrow_i [\beta]) \sqcap_i ([\beta] \Rightarrow_i [\alpha])$.

For $i \in \{1, 2\}$, let $V_i(\alpha) = [\alpha]$ if α is an i -atom. Let $\mathcal{M}_{12} = \langle \mathcal{B}_1, V_1, \mathcal{B}_2, V_2 \rangle$. It is easy to show that \mathcal{M}_{12} is a coherent Heyting-Boolean model. Thus, all of the intuitionist validities are designated for the first stock of connectives and all of the classical validities are designated for the second. In \mathcal{M}_{12} , $\|\alpha\|_1 = \|\alpha\|_2 = [\alpha]$. $\mathcal{M}_{12} \models \neg_1\alpha$ just in case $[\neg_1\alpha] = 1$ just in case $[\alpha] = \mathbf{0}$ just in case $[\neg_2\alpha] = 1$ just in case $\mathcal{M}_{12} \models \neg_2\alpha$. $\mathcal{M}_{12} \models \alpha \leftrightarrow_1 \beta$ just in case $[\alpha \leftrightarrow_1 \beta] = 1$ just in case $[\alpha] = [\beta]$ just in case $[\alpha \leftrightarrow_2 \beta] = 1$ just in case $\mathcal{M}_{12} \models \alpha \leftrightarrow_2 \beta$.

In this model, $\|p^1 \vee_1 \neg_1 p^1\|_i \neq 1$ and $\|p^1 \vee_2 \neg_2 p^1\|_i = 1$. Therefore \vdash_{12} does not weakly collapse. \square

It follows from this result that if we enrich \vdash^{ic} with left-extensionality, the resulting consequence relation does not weakly collapse.

Taken together, our collapse and non-collapse results are somewhat surprising. They show that the issue of when a logic collapses (or weakly collapses) is very delicate. A full catalogue of such results must await another occasion.

7.5. Meta-Rules. There is a final topic worth discussing – namely, the status of the familiar classical and intuitionist meta-rules. Here is a list of standard natural deduction meta-rules:

Conditional Introduction. If $\Gamma \cup \{\alpha\} \vdash \beta$ then $\Gamma \vdash \alpha \rightarrow \beta$;

Biconditional Introduction. If $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\beta\} \vdash \alpha$ then $\Gamma \vdash \alpha \leftrightarrow \beta$;

Reasoning by Cases. If $\Gamma \cup \{\alpha\} \vdash \delta$ and $\Gamma \cup \{\beta\} \vdash \delta$ then $\Gamma \cup \{\alpha \vee \beta\} \vdash \delta$;

Intuitionist Reductio. If $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\alpha\} \vdash \neg\beta$ then $\Gamma \vdash \neg\alpha$;

Classical Reductio. If $\Gamma \cup \{\neg\alpha\} \vdash \beta$ and $\Gamma \cup \{\neg\alpha\} \vdash \neg\beta$ then $\Gamma \vdash \alpha$.

Given our results, it is easy to show that if we enrich \vdash^{ii} (or a stronger logic) with Conditional Introduction for both of \rightarrow_1 and \rightarrow_2 , the resulting logic collapses:

PROPOSITION 7.7. *Let \vdash_{12} be a consequence relation for $\text{Sent}(C_{12}, P_{12})$ that extends \vdash^{ii} and obeys Conditional Introduction for both conditionals. Then \vdash_{12} collapses.⁵⁰*

PROOF. $\{\alpha, \alpha \rightarrow_1 \beta\} \vdash^{ii} \beta$. By Conditional Introduction for \rightarrow_2 , $\{\alpha \rightarrow_1 \beta\} \vdash_{12} \alpha \rightarrow_2 \beta$. Similarly, $\{\alpha \rightarrow_2 \beta\} \vdash_{12} \alpha \rightarrow_1 \beta$. By Proposition 7.3, \vdash_{12} collapses. \square

This result has an important consequence. We know that \vdash^{cc} , \vdash^{ii} , and \vdash^{ic} do not collapse. So none of these consequence relations obey Conditional Introduction for both conditionals. (Indeed, they do not obey Conditional Introduction for either conditional.) Although juxtaposition preserves entailments, it does not preserve the validity of meta-rules.⁵¹ What this shows is that the meta-rules are in an important sense stronger than the corresponding entailments that they license.

The standard meta-rules are closely related to one another:

PROPOSITION 7.8. *Suppose \vdash is a consequence relation that extends the intuitionist consequence relation (in a perhaps larger language). Suppose \vdash obeys Conditional Introduction. Then:*

1. \vdash obeys Biconditional Introduction, Reasoning by Cases, and Intuitionist Reductio;
2. If \vdash extends the classical consequence relation, then \vdash obeys Classical Reductio.

PROOF. Biconditional Introduction: Suppose $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\beta\} \vdash \alpha$. By Conditional Introduction, $\Gamma \vdash \alpha \rightarrow \beta$ and $\Gamma \vdash \beta \rightarrow \alpha$. $\{\alpha \rightarrow \beta, \beta \rightarrow \alpha\} \vdash \alpha \leftrightarrow \beta$. By Cut, $\Gamma \vdash \alpha \leftrightarrow \beta$.

⁵⁰See Harris (1982) for a different proof of this result.

⁵¹See Coniglio (2007) for a variant of fibring designed to preserve meta-rules.

Reasoning by Cases: Suppose $\Gamma \cup \{\alpha\} \vdash \delta$ and $\Gamma \cup \{\beta\} \vdash \delta$. By Conditional Introduction, $\Gamma \vdash \alpha \rightarrow \delta$ and $\Gamma \vdash \beta \rightarrow \delta$. $\{\alpha \rightarrow \delta, \beta \rightarrow \delta, \alpha \vee \beta\} \vdash \delta$. By Cut, $\Gamma \cup \{\alpha \vee \beta\} \vdash \delta$.

Intuitionist Reductio: Suppose $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\alpha\} \vdash \neg\beta$. By Conditional Introduction, $\Gamma \vdash \alpha \rightarrow \beta$ and $\Gamma \vdash \alpha \rightarrow \neg\beta$. $\{\alpha \rightarrow \beta, \alpha \rightarrow \neg\beta\} \vdash \neg\alpha$. By Cut, $\Gamma \vdash \neg\alpha$.

Classical Reductio: Suppose $\Gamma \cup \{\neg\alpha\} \vdash \beta$ and $\Gamma \cup \{\neg\alpha\} \vdash \neg\beta$. By Conditional Introduction, $\Gamma \vdash \neg\alpha \rightarrow \beta$ and $\Gamma \vdash \neg\alpha \rightarrow \neg\beta$. In classical logic, $\{\neg\alpha \rightarrow \beta, \neg\alpha \rightarrow \neg\beta\} \vdash \alpha$. By Cut, $\Gamma \vdash \neg\alpha$. \square

More importantly for our purposes here, we also have the following relations:

PROPOSITION 7.9. *Suppose \vdash is a consequence relation that extends the intuitionist consequence relation (in a perhaps larger language). Then:*

1. *If \vdash obeys Biconditional Introduction or Classical Reductio, then \vdash obeys Conditional Introduction;*
2. *If \vdash extends the classical consequence relation and \vdash obeys Reasoning by Cases or Intuitionist Reductio, then \vdash obeys Conditional Introduction.*

PROOF. Biconditional Introduction: Suppose $\Gamma \cup \{\alpha\} \vdash \beta$. $\{\alpha, \beta\} \vdash \alpha \wedge \beta$. By Cut, $\Gamma \cup \{\alpha\} \vdash \alpha \wedge \beta$. $\Gamma \cup \{\alpha \wedge \beta\} \vdash \alpha$. By Biconditional Introduction, $\Gamma \vdash \alpha \leftrightarrow (\alpha \wedge \beta)$. $\{\alpha \leftrightarrow (\alpha \wedge \beta)\} \vdash \alpha \rightarrow \beta$. By Cut, $\Gamma \vdash \alpha \rightarrow \beta$.

Classical Reductio: We first show that if \vdash obeys Classical Reductio, then \vdash extends classical logic. $\{\neg\neg\alpha, \neg\alpha\} \vdash \neg\alpha$. $\{\neg\neg\alpha, \neg\alpha\} \vdash \neg\neg\alpha$. By Classical Reductio, $\{\neg\neg\alpha\} \vdash \alpha$. Thus, \vdash extends classical logic. We next show that \vdash obeys Conditional Introduction. Suppose $\Gamma \cup \{\alpha\} \vdash \beta$. In classical logic, $\Gamma \cup \{\neg(\alpha \rightarrow \beta)\} \vdash \alpha$. By Cut, $\Gamma \cup \{\neg(\alpha \rightarrow \beta)\} \vdash \beta$. In classical logic, $\Gamma \cup \{\neg(\alpha \rightarrow \beta)\} \vdash \neg\beta$. By Classical Reductio, $\Gamma \vdash \alpha \rightarrow \beta$.

Reasoning by Cases: Suppose $\Gamma \cup \{\alpha\} \vdash \beta$. $\{\beta\} \vdash \alpha \rightarrow \beta$. By Cut, $\Gamma \cup \{\alpha\} \vdash \alpha \rightarrow \beta$. In classical logic, $\Gamma \cup \{\neg\alpha\} \vdash \alpha \rightarrow \beta$. By Reasoning by Cases, $\Gamma \cup \{\alpha \vee \neg\alpha\} \vdash \alpha \rightarrow \beta$. In classical logic, $\vdash \alpha \vee \neg\alpha$. By Cut, $\Gamma \vdash \alpha \rightarrow \beta$.

Intuitionist Reductio: Suppose $\Gamma \cup \{\alpha\} \vdash \beta$. In classical logic, $\Gamma \cup \{\neg(\alpha \rightarrow \beta)\} \vdash \alpha$. By Cut, $\Gamma \cup \{\neg(\alpha \rightarrow \beta)\} \vdash \beta$. In classical logic, $\Gamma \cup \{\neg(\alpha \rightarrow \beta)\} \vdash \neg\beta$. By Intuitionist Reductio, $\Gamma \vdash \neg\neg(\alpha \rightarrow \beta)$. In classical logic, $\{\neg\neg(\alpha \rightarrow \beta)\} \vdash \alpha \rightarrow \beta$. By Cut, $\Gamma \vdash \alpha \rightarrow \beta$. \square

It follows that if we enrich \vdash^{ii} (or a stronger logic) with two copies of Biconditional Introduction or with two copies of Classical Reductio, the resulting logic collapses. If we enrich \vdash^{cc} (or a stronger logic) with two copies of Reasoning by Cases or two copies of Intuitionist Reductio, the resulting logic collapses.⁵²

There is another meta-rule worth discussing. Recall the definition of Entailment Congruence:

Entailment Congruence. If $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\beta\} \vdash \alpha$ then for any $\delta \in \text{Sent}(C, P)$ and p occurring in δ , $\Gamma \cup \{\delta[\alpha/p]\} \vdash \delta[\beta/p]$.

This meta-rule is not an introduction or elimination rule. It does not govern the behavior of any particular connective. Roughly speaking, it governs how

⁵²See McGee (2000) for a proof of the claim that if \vdash_{12} is a consequence relation for $\text{Sent}(C_{12}, P_{12})$ that obeys the usual natural deduction rules for intuitionist logic for each stock of connectives, then \vdash_{12} collapses.

logical equivalents behave within embeddings. It is perhaps better thought of as something like a structural rule.

We can prove the following result:

PROPOSITION 7.10. *Suppose \vdash is a consequence relation that extends the intuitionist consequence relation (in a perhaps larger language). Then \vdash obeys Entailment Congruence just in case \vdash obeys Conditional Introduction.*

PROOF. Suppose $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\beta\} \vdash \alpha$. By Conditional Introduction, $\Gamma \vdash \alpha \rightarrow \beta$ and $\Gamma \vdash \beta \rightarrow \alpha$. $\{\alpha \rightarrow \beta, \beta \rightarrow \alpha, \delta[\alpha/p]\} \vdash \delta[\beta/p]$. By Cut, $\Gamma \cup \{\delta[\alpha/p]\} \vdash \delta[\beta/p]$.

Now suppose $\Gamma \cup \{\alpha\} \vdash \beta$. $\{\alpha, \beta\} \vdash \alpha \wedge \beta$. By Cut, $\Gamma \cup \{\alpha\} \vdash \alpha \wedge \beta$. $\Gamma \cup \{\alpha \wedge \beta\} \vdash \alpha$. By Entailment Congruence, $\Gamma \cup \{\alpha \rightarrow \alpha\} \vdash \alpha \rightarrow (\alpha \wedge \beta)$. $\vdash \alpha \rightarrow \alpha$. $\{\alpha \rightarrow (\alpha \wedge \beta)\} \vdash \alpha \rightarrow \beta$. By Cut, $\Gamma \vdash \alpha \rightarrow \beta$. \square

It follows that \vdash^{cc} , \vdash^{ii} , and \vdash^{ic} do not obey Entailment Congruence. It also follows that juxtaposition does not preserve Entailment Congruence.⁵³ Finally, it follows that if \vdash^{ii} (or a stronger logic) is enriched with Entailment Congruence, the resulting logic collapses.

§8. Discussion. Let us return to the issues that this paper started with: What should we make of the appeal to collapse arguments in motivating philosophical theses? Should we conclude that the logical constants in our language have determinate extensions? Should we conclude that disputes in ontology are genuine disputes? Should we conclude that either intuitionism is correct and classical logic is incoherent or classical logic is correct and the intuitionists fail to recognize genuinely valid entailments?

One of the morals of this paper is that the answers to these questions are not very clear. What has emerged from the discussion above is that the success of collapse arguments, perhaps surprisingly, depends on the status of the meta-rules – including the standard natural deduction rules as well as Entailment Congruence. If we endorse classical logic (say), should we accept only the classically valid theorems and entailments or should we also accept the classical meta-rules? If we are working in a language with only the familiar logical constants, there is no way to accept the classically valid theorems and entailments without also accepting the meta-rules. But the difference looms large when we expand our language with additional vocabulary, for instance by adding a second stock of logical constants. The crux of the matter, then, seems to be this: When we expand our language, should we endorse the meta-rules as applying to the expanded language? If the answer is no, the collapse arguments are not worth very much. If the answer is yes, the collapse arguments are difficult to resist.⁵⁴

This issue is bound up with deep issues concerning the nature of logic. On one picture of the nature of logic, logic is intimately tied to reasoning. Logic is a theory of good reasoning: It captures how we ought to reason deductively. This conception of logic fits very naturally with the claim that the standard natural deduction rules are of central importance. Such rules tell us how we

⁵³Algebraic fibring does not preserve Entailment Congruence, either. For an example, see Zanardo et al. (2001), page 434.

⁵⁴We might give different answers to different proposed appeals to collapse arguments.

ought to reason.⁵⁵ We should hold on to such rules no matter how we expand our language.⁵⁶

On an alternative picture of the nature of logic, logic is not normative but descriptive. Logic is the theory of the logical consequence relation. It is the theory of what follows from what.⁵⁷ This conception of logic fits very naturally with the claim that what is central to logic is not the natural deduction rules but the logical truths and entailment claims. We need not hold on to the meta-rules as we expand our language. The role of the meta-rules is simply to encode entailments in a user-friendly way.⁵⁸

What we should conclude about the use of collapse arguments in arguing for philosophically striking theses thus depends on how we should conceive of the nature of logic. This a difficult and central issue in the philosophy of logic.

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⁵⁵Indeed, the natural deduction rules were originally motivated (in part) as a way of connecting logic with psychologically-realistic patterns of reasoning. See Gentzen (1935).

⁵⁶Endorsing this general picture of logic does not, however, require that we hold on to the meta-rules in their full generality as we expand our language. For instance, we might accept that natural deduction rules capture how we ought to reason, but claim that the relevant rules have the side condition that inferences involving new vocabulary not appear in sub-derivations.

⁵⁷See Harman (1986), chapter 2, for arguments for this view of logic.

⁵⁸Endorsing this general picture of logic does not, however, force us to take this position. For instance, we might think that the natural deduction meta-rules encode important patterns among entailments, patterns that continue to hold even as we expand our language. Or we might think that Entailment Congruence states a central fact about logical consequence.

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