# Prospectus to a Homotopic Metatheory of Language

Eric Schmid

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## 1 Introduction

Fernando Zalamea's book, "Synthetic Philosophy of Contemporary Mathematics," stands as a seminal work in the field of mathematical philosophy. This work is a testament to Zalamea's profound understanding of both mathematics and philosophy, bridging these two domains with remarkable skill. The book is divided into three principal parts, each targeting a different facet of the subject, and is enriched by Zalamea's lucid writing and deep insights.

## 2 Part 1: The Context of Modern Mathematics

In the opening section, Zalamea sets the stage by outlining the unique aspects of modern and contemporary mathematics. This part serves as an orientation for the reader, providing a backdrop against which the rest of the book unfolds. Zalamea argues for a new approach to understanding mathematics, one that goes beyond traditional methods. He suggests a 'synthetic' view, focusing not just on the logical or set-theoretical aspects but encompassing the full breadth of arithmetical, algebraic, geometrical, and topological constructions.

This section challenges the reader to think beyond the conventional boundaries of mathematical philosophy. Zalamea's arguments are buttressed by references to historical and contemporary sources, weaving together a narrative that is both educational and engaging. He critiques the limitations of a purely analytical approach to mathematics, advocating for a broader, more dynamic perspective.

## 3 Part 2: Case Studies in Mathematical Creativity

The second part of the book delves into detailed case studies of renowned mathematicians and their contributions. A standout chapter is dedicated to Alexander Grothendieck, a towering figure in twentieth-century mathematics.

Zalamea explores Grothendieck's profound impact on the field, particularly his work on sheaf theory and algebraic geometry. The author skillfully elucidates how Grothendieck's innovations transcended traditional boundaries, reshaping the landscape of mathematics.

Further chapters in this section explore the concept of 'Eidal Mathematics', featuring mathematicians like Serre and Langlands. Zalamea's analysis of their work is not only informative but also reveals the interconnectedness of different mathematical areas. Through these case studies, Zalamea illustrates the essence of mathematical creativity, showing how it extends beyond mere computation or logical deduction to include an artistic and deeply conceptual dimension.

# 4 Part 3: Philosophical Synthesis and Sheafification

The final section, 'Synthetic Sketches', is perhaps the most philosophically dense part of the book. Here, Zalamea ventures into the realms of transitory ontology and comparative epistemology. The concept of 'sheafification', central to this part, symbolizes the blending of diverse mathematical structures with philosophical insights. This section is not just a culmination of the preceding discussions but an expansion into new philosophical territories.

Zalamea discusses sheaf theory not merely as a mathematical tool but as a metaphor for understanding the synthesis of ideas. He uses this concept to demonstrate how different philosophical viewpoints can be 'glued' together to form a cohesive whole. This idea reflects the book's overarching theme: the synthesis of diverse mathematical and philosophical perspectives.

Zalamea also reflects on the nature of mathematical evolution, emphasizing the constant progression and transformation within the field. He challenges the static view of mathematics, advocating for an appreciation of its dynamic and ever-changing nature. This part of the book is particularly thought-provoking, inviting readers to reconsider their preconceived notions about the nature of mathematical knowledge.

# 5 Central Concepts

Throughout the book, Zalamea maintains a focus on several key themes. One is the dynamic between unity and diversity in mathematics. He portrays mathematics as a discipline in constant flux, characterized by the emergence of new ideas and the blending of various structures and concepts. Another central theme is the idea of mathematical creativity as a synthesis of invention and discovery. Zalamea argues that mathematics is not just about finding pre-existing truths but also about inventing new frameworks and languages.

The book also touches on the philosophical implications of mathematical practices. Zalamea explores how mathematical developments influence, and are influenced by, philosophical thought. He discusses the historical context of mathematical ideas, showing how they are products of their time yet transcend these boundaries to achieve a timeless relevance.

## 6 Sheaves

In "Synthetic Philosophy of Contemporary Mathematics," Fernando Zalamea discusses sheafs and sheafification as key concepts in the modern mathematical landscape. The book outlines how contemporary mathematics has witnessed a systematic geometrization across all its environments, including sheaves, homologies, cobordisms, and geometrical logic (31). This geometrization, which involves sheaves, reflects a trend where mathematical structures are increasingly viewed through a geometric lens, a shift that deeply impacts how mathematicians understand and apply these concepts.

Sheafification, as part of this geometrization process, involves adapting mathematical structures to ensure they comply with the principles of sheaf theory. This process is crucial in areas like algebraic geometry and topology, where local-to-global principles play a pivotal role. Zalamea's text points out that contemporary mathematics has moved beyond the confines of set-theoretical, algebraic, and topological restrictions, embracing more flexible and encompassing frameworks like groupoids, categories, schemas, topoi, and motifs (32). This move towards more expansive structures, including sheaves, highlights a shift from rigid, foundational approaches to more fluid and adaptable ones.

Furthermore, Zalamea notes the fluxion and deformation of traditional mathematical structures, encompassing concepts like nonlinearity, noncommutativity, nonelementarity, and quantization (33). This aspect of contemporary mathematics, where traditional boundaries are constantly evolving and being redefined, is intrinsic to the concept of sheafification. Sheafification allows for the fluid integration of local data into a cohesive global structure, accommodating the dynamic nature of modern mathematical problems and solutions.

In summary, "Synthetic Philosophy of Contemporary Mathematics" positions sheafs and sheafification as central to the modern mathematical process. These concepts are not just tools for solving problems but are fundamental to the way contemporary mathematics conceptualizes and interacts with its various structures and environments. This shift towards a more dynamic, geometrized, and interconnected approach is reflective of the broader trends in contemporary mathematical thought as outlined in Zalamea's work.

The Triumvirate of Topos Theory: Grothendieck, Mac Lane, and Lawvere

## 7 Uses and Abuses of the History of Topos Theory

In Colin McLarty's essay titled "The Uses and Abuses of the History of Topos Theory," he critically examines the historical development and understanding of topos theory and category theory. McLarty's work offers a deep dive into the intricate relationship between these mathematical concepts and the misconceptions that have arisen over time.

Topos theory, as McLarty describes, has often been misunderstood as a mere generalization of set theory. He states, "The view that toposes originated as generalized set theory is a figment of set theoretically educated common sense" (McLarty, p.2). This misconception, according to McLarty, hinders the proper understanding of category theory and the categorical foundations of mathematics. He emphasizes that problems in geometry, topology, and related algebra were the actual driving forces behind the development of categories and toposes.

Category theory, on the other hand, emerged from practical problems in topology. McLarty notes that "Category theory arose from a complicated array of practical problems in topology" (McLarty, p.3). He further elaborates that category theory did not emerge from a contemplation of the timeless nature of mathematics but from the core of mathematical practice, offering foundational insights.

One of the central themes of McLarty's essay is the idea that arrows in category theory reveal structure. He mentions, "The main point of categorical thinking is to let arrows reveal structure" (McLarty, p.17). This approach, however, only defines objects up to isomorphism. McLarty points out that set theory, as practiced today, is unique in not generally defining its objects up to isomorphism. This perspective of set theory makes it a particularly challenging example for category theory.

McLarty also touches upon the work of prominent figures in the field. He mentions how topos theory evolved from Grothendieck's work in geometry, Tierney's interest in topology, and Lawvere's focus on the foundations of physics. He highlights that "Topos theory arose from Grothendieck's work in geometry, Tierney's interest in topology and Lawvere's interest in the foundations of physics" (McLarty, p.3).

Furthermore, McLarty delves into the common misconceptions surrounding topos theory. He discusses how many students, influenced by set theoretic thinking, often misunderstand the essence of topos theory. They tend to see objects as 'structured sets' and arrows as 'structure-preserving functions' (McLarty, p.16). This misunderstanding, McLarty argues, stems from a false history that obscures the real novelty of category theory.

In conclusion, Colin McLarty's essay offers a comprehensive exploration of the history and misconceptions surrounding topos theory and category theory. He emphasizes the importance of understanding these theories in their proper historical and mathematical contexts. By shedding light on the origins and developments of these concepts, McLarty provides valuable insights into the intricate world of mathematical foundations.

# 8 Topos Theory

Topos theory, a sophisticated branch of category theory, has been sculpted and refined by the unparalleled contributions of three mathematical luminaries: Grothendieck, Mac Lane, and Lawvere. Their collective endeavors have not only redefined our comprehension of mathematical structures but have also illuminated the intricate interplay between algebra, geometry, and logic. This review seeks to shed light on their monumental contributions, emphasizing the profound impact of their work on the evolution of topos theory.

## 9 Grothendieck: The Geometric Visionary

Alexander Grothendieck's name is often uttered in reverential tones in the corridors of algebraic geometry. His introduction of toposes as generalized spaces was nothing short of revolutionary. McLarty notes that Grothendieck's toposes were conceptualized as a "generalization of topological spaces" that could seamlessly meld algebraic and topological attributes. This was not a mere academic exercise; Grothendieck's toposes were a revelation, encapsulating nuanced properties that transcended apparent boundaries. As McLarty elucidates, Grothendieck discovered that many properties intrinsic to the category of sets could be "lifted" to the realm of toposes. This was not a premeditated design but a serendipitous discovery stemming from Grothendieck's profound engagement with the subject.

## 10 Mac Lane: Laying the Cornerstones

Saunders Mac Lane, a stalwart in the mathematical community, was instrumental in laying the foundational bricks of category theory. Collaborating with Eilenberg, he pioneered the introduction of categories, functors, and natural transformations. McLarty's account highlights Mac Lane's fervor for exploring the foundational aspects of mathematics, which inevitably led him to the intricate maze of topos theory. His lectures, as delivered at the University of Chicago and documented by McLarty, were pivotal in disseminating the core tenets of topos theory, making it accessible to a broader spectrum of scholars. Mac Lane's meticulous approach was a beacon of clarity, shedding light on the intricate nexus between logic and geometry and underscoring the pivotal role of toposes in bridging these domains.

## 11 Lawvere: A New Perspective on Foundations

F. W. Lawvere's foray into topos theory was underpinned by a keen interest in the foundational aspects of physics. McLarty recounts Lawvere's ambition to establish functorial foundations for mathematics, emphasizing not just on isolated categories but on the intricate dance of functors. This led him to probe the depths of the "category of categories."

Lawvere's contributions to topos theory are multifaceted. He ventured deep into its logical foundations, drawing parallels with other mathematical branches. One of his seminal insights, as highlighted by McLarty, was the realization that "universal and existential quantifiers were adjoints to substitution." This epiphany paved the way for a nuanced understanding of the comprehension axiom as an adjunction.

Furthermore, Lawvere's collaborative efforts with Tierney on partial map classifiers enriched the tapestry of topos theory. McLarty points out Lawvere's innovative perspective of perceiving objects in a topos as "continuously variable sets," juxtaposing them against the backdrop of classical set theory's constant sets. This vantage point offered a fresh lens to view the theory of variable sets and their intricate structures.

## 12 In Conclusion

The triumvirate of Grothendieck, Mac Lane, and Lawvere has indelibly shaped the contours of topos theory. Their synergistic efforts have crafted a robust framework that has revolutionized our understanding of mathematical structures. While Grothendieck's geometric insights laid the groundwork, Mac Lane's foundational prowess and Lawvere's innovative perspectives have enriched and expanded the horizons of topos theory. Their collective legacy serves as a testament to the transformative power of collaboration and the boundless vistas that open up when brilliant minds converge on a singular idea.

## 13 Apperception, Reason, and Abduction

- Agent 1 apperceives
- Agent 1 reasons as a judgment
- Agent 2 understands and abducts
- Etc

(Generalize this now to cases n and m concurrently: in the background is the inferential network which institutes agential roles yet is defined through interaction between "multi agents", ie a generalization of the base case of agent 1 and agent 2 to agent n and agent m, all interrelated to each other)

The concise explanation of my forthcoming book involves a three part process. An agent apperceives the physical world from existing mathematics. They ratiocinate on physics forming a novel judgment. Then another agent understands and abducts from the novel (exact) judgment (purportedly expressing an explication of a past unrigorous episteme from yesteryear into exactitude but this explication is only partially ratiocination because of the wellspring of yet to be intelligible mathematical discoveries which will relativize philosophical thinking once again—call it Geist, the mathematical Absolute or the unknown whichever you prefer. This process defines a relatived a priori. Note the loophole in the abductive mode of thinking, which allows the process to unfold historically through the becoming of philosophy of science through dialectics or if you prefer generational research programmes. If you accept the hypostasis of Ideas over matter, ultimately we are approaching Absolute mind, the mathematical Absolute or world historical spirit in the background processes.

The existence of the interaction between agent 1 and agent 2 is key. The bridge between syntax and semantics is interaction. Negarestani states through logician Jean-Yves Girard's ludics interactions between agents can be computationally encoded. But my claim is that the inferential network which institutes these roles is expressed through singular and formal terms a la Brandom vis-a-vis linear dependent Homotopy Type Theory (dlHoTT) thanks to David Corfield's suggestion. DlHoTT has the expressive power to encode predicates in the object language but moreover substitutable terms in higher dimensions of the infinity-topos. The tripartite structure of (1) meta science followed by (2) revolution followed by post-hoc (3) philosophy is one historical philosophy of science as per Michael Friedman aside from Thomas Kuhn. But instead of a neo-Kantian argument, I claim this teleology is Hegelian, with apperception and judgment typifying formal Kantianism a la pure type theory and the bind between conceptual inference from apperception of mathematical intuition. Note though, this interaction is contingent upon a prior language to encode the apperception-i.e. a prior mathematical theory which is refashioned into physics or in the local case, minimal syntax.

My point is that this global view can lead to the following local view and VICE VERSA: language is co-extensive with sociality, general intelligence is realizable if and only if language is understood inferentially in terms of semantic social norms defined interactionally between agents through a minimal syntax generating the whole system of rule-based roles. The computational trinity states predicates in proof theory are equivalent to types in computation. The semantic ascent from a language to a metalanguage of say expressing "redness" (substitutable, in order to define red linguistically, you need even more concepts to express it black boxed functionally, such as ROYGBIV, yet singular in term because of its base language understanding of the datum of red qua red) can be instituted through quantum natural language processing which can be described categorically with string diagrams defining sentence structure in a network of words. This categorical approach moreover can be made equivalent possibly to linear dependent Homotopy Type Theory as Corfield suggests. If mind is understood computational-functionally, self-consciousness is co-extensive with public language through social interactions of multiagents, where inherently sociality realizes general intelligence since natural semantics are generated by syntax coupled with interaction.

Negarestani writes: "Syntax, under the right conditions, is indeed sufficient for semantics, and meaning can be conferred upon syntactic expression if such conditions are satisfied. These conditions are what the inferentialist theory of meaning, as a species of social-pragmatics or the use-theory of meaning, attempts to capture. It argues that meaning is ultimately, at its most basic level, the justified use of mere expressions in social discursive linguistic practices; and that what counts as thejustification of an expression is what counts as its meaning. While syntax by itself does not yield semantics, it does so when coupled with interaction. In this sense, pragmatics—at least in the sense defined by Brandom s inferentialist pragmatism—can be understood as a bridge between syntax and semantics. Broadly speaking, semantics (content) is concerned with what is said, while pragmatism (use) is concerned with what one is doing in saying it (i.e., discursive practices-or-abilities that count as deploying a vocabulary, conferring or applying meaning). More precisely, semantics asks what it is that one believes (or knows, claims) when one believes that p (a content), whereas pragmatism asks what it is that one must know how to do in order to count as producing a performance that expresses that content." (Negarestani)

## 14 Dr. David Corfield in a private message to me

I think there's great potential in redoing Brandom with dependent type theory. As mentioned in that recent tweet, before we come to ask for reasons from someone, we must hold them to be meaningful, and part of the act of doing this is to parse what someone says. I think this can be understood in type theoretic terms. We have the grammar correct if replacement of terms by others in the same type preserves meaningfulness (not meaning).

It would be interesting to compare the Quantum Natural Language Processing approach to one via dependent type theory. From what I understand, one difference amounts to that between categorising entities according to different types, and so non-comparable, and embedding them all in some large vector space, where great distance between elements corresponds to there being quite distinct. So, for the latter we might have cat, dog and cup in a space, with cat and dog near each other, while both far from cup. For the former, we'd have cat, dog: Animal and cup: Utensil.

Perhaps then to the charge that in some sentences we can replace 'cat' by 'cup', so better to have them in the same space, we might say, that they belong to a supertype, Object, or something like that. This explains the limited range of replacements possible: 'cup' and 'dog' can replace 'cat' in 'I see a cat', but only 'dog' in 'The cat is walking away', etc.

(Maybe linear HoTT for the best of both worlds,
https://twitter.com/DavidCorfield8/status/1592862711532384259 )

But vast tracts of Brandom's Making it Explicit could be given the type theoretic treatment in terms of Intro and Elimination rules.

There's also the structural inferentialism that HoTT provides.

## 15 Computational Trinity

I finally understand that (dependent) function-types, i.e. $\prod$-types, are equivalent to intuitionistic universal quantifiers of predicates and also equivalent to the space of sections. Dependent pair-types, $\sum$-types are equivalent to existential quantification in IL and total spaces (above the base space). Families of types

| Types | Logic | Sets | Homotopy |
|---|---|---|---|
| $A$ | proposition | set | space |
| $a : A$ | proof | element | point |
| $B(x)$ | predicate | family of sets | fibration |
| $b(x) : B(x)$ | conditional proof | family of elements | section |
| $\mathbf{0}, \mathbf{1}$ | $\bot, \top$ | $\emptyset, \{\emptyset\}$ | $\emptyset, *$ |
| $A + B$ | $A \vee B$ | disjoint union | coproduct |
| $A \times B$ | $A \wedge B$ | set of pairs | product space |
| $A \to B$ | $A \Rightarrow B$ | set of functions | function space |
| $\sum_{(x:A)} B(x)$ | $\exists_{x:A} B(x)$ | disjoint sum | total space |
| $\prod_{(x:A)} B(x)$ | $\forall_{x:A} B(x)$ | product | space of sections |
| $\mathsf{Id}_A$ | equality $=$ | $\{ (x, x) \mid x \in A \}$ | path space $A^I$ |

**Table 1: Comparing points of view on type-theoretic operations**

Figure 1: Taken from the HoTT book

are equivalent to a predicate and to a fibration. A type becomes a proposition which is also a space. Terms x : b(x) of families of types B(x) becomes a conditional proof and also a section. A term of a type becomes a proof which is a point. Functions become intuitionistic implication which become a function space.

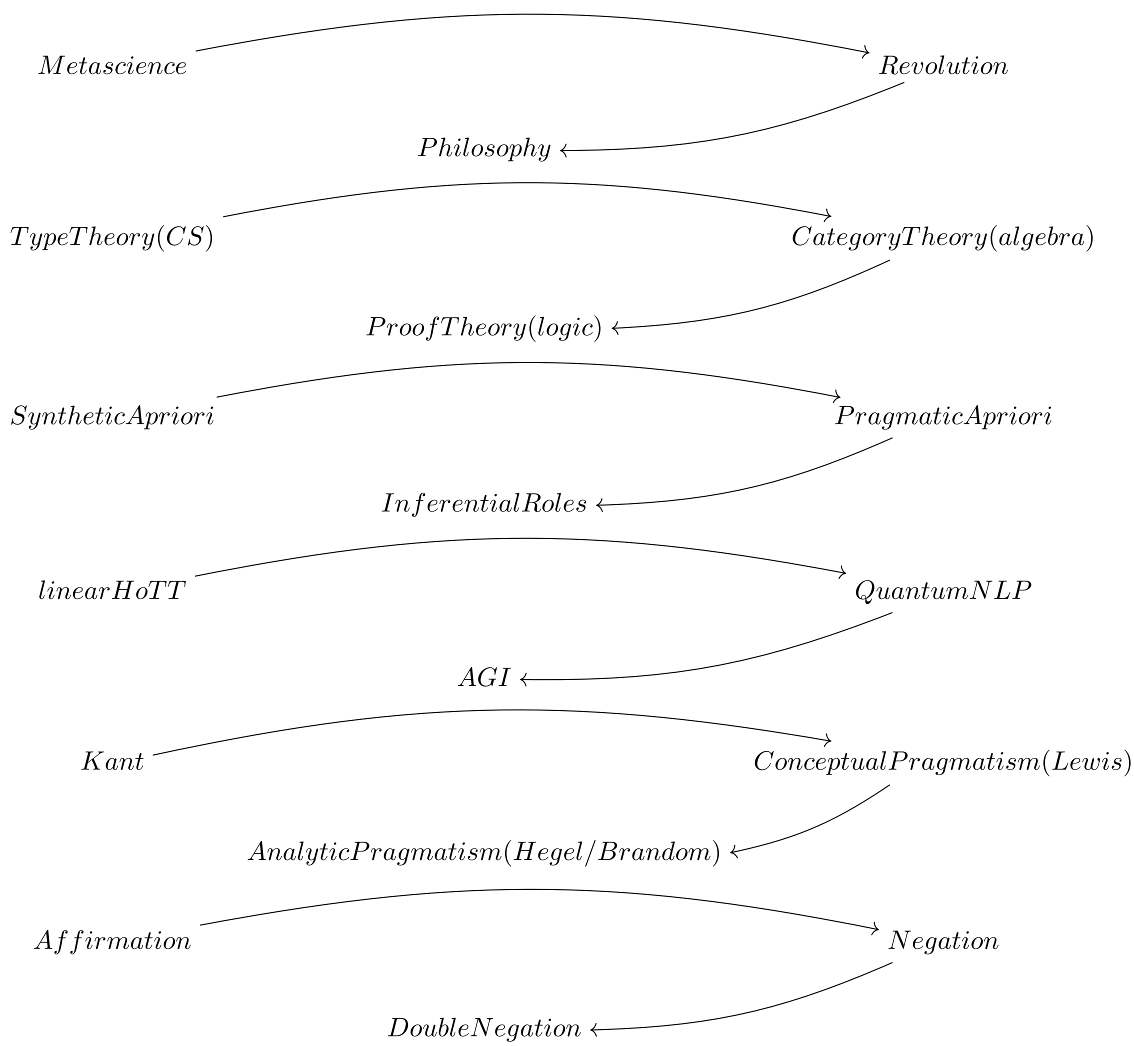
## 16 Condensed Thesis

Following Reza Negarestani, the aim of this book is to argue for a Kantian-Hegelian idealism, but where historical epochs are divided into the three temporal periods of Michael Friedman's "dynamics of reason": metascience, revolution and philosophy. The first temporal period, metascience, is explicitly Kantian in terms of the (mathematical) presuppositions behind formulating physical theories (Kant had said geometry was a priori). This time period (the epoch of metascience) is structured by a transcendental logic which necessitates a "synthetic a priori" of existent mathematical knowledge in order to "philosophically refashion" it into scientific theory. While universally valid during this period, this a priori is relativized as new scientific revolutions occur. The second period is scientific revolution. Then, the final period is deliberation on the aftermath, which Friedman calls the period of "philosophy." I claim the overarching temporal teleology/progress through the entirety of the three periods is explicitly Brandomian-Hegelian, while the first period is Kantian. **The central thesis of this book is that this metatheory for scientific progress can be expressed through the language of Homotopy Type Theory/Univalent**

**Foundations (HoTT/UF).**

Moreover, due to the wide scope of (in particular *linear*) homotopy type theory (using quantam natural language processing), **this metatheory can be applied not just to scientific progress, but ordinary language or any public language** defined by sociality/social agents as the precondition for the realizability of (general) intelligence via an inferential network from which judgement can be made. How this metatheory of science generalizes to public language is through the recent advances of quantum natural language processing, but the traditional metalogical encodings (of Tarski) is relatively comparable through universes such as the **type of types** or **type of types of types** found in the inherent inferentialism of UF/HoTT via $\infty$-groupoids. The "computational trinity" of proofs=programs=algebra in HoTT also means reason is defined functionally as "what it does" by what it computes (proofs are programs). Following Negarestani's recent book, through the self's self-realizability, achieving self-consciousness and consciousness beyond selfhood, "geistig manifestation" is achieved in the form of general intelligence, but only through an inferential network of social agents intrinsically encoded through computation.

# 17 Diagrams only possible after logician William Lawvere's formalization of Hegel via categories




# 18 Reclaiming the Kantian synthetic a priori:

I am interested in focusing on Carnap's critiques of Wittgenstein, specifically the deracination of 'meaning' and substitution of 'meaning' with syntax. Analytic philosophy was created at the height of neo-Kantianism in the aftermath of German Idealism. Obscure terms and concepts were much overused at the time (such as Hegel's philosophy). But I am interested in a return to an 'analytic pragmatism' as advocated by Brandom and his readings of Hegel. There was a flood of what Reza Negarestani calls 'metaphysical bloatware' (inflationary metaphysics) at the turn of the 20th century. Russell's pupil Wittgenstein famously said that where we cannot know something, we must remain in silence (Tractatus). Is analyticity only meaningful when it is devoid of all content and meaning (i.e. only positivistic logico-philosophico propositions where all deductions/math are implied from itself deductively like a succession of dominos)? All math is tautology? "5.133 All deductions are made a priori" (Tractatus) And is analyticity only meaningful when philosophy is truly 'tolerant' (in Carnap's sense) for a multitude of interpretative frameworks (metalanguages) for instituting the object language of scientific theory (the language where one has propositions or the arithmetic, e.g. Peano arithmetic) through the supporting "ocean of metalanguages" to use Steve Awodey's term (e.g. Goedel's encoding of the Peano Arithmetic through prime numbers) backing this object language. My plan is to argue for the inferentialist account of meaning (Brandom), that meaning is understood in terms of use and that semantics is inherently an ethical question tied to commitments to discursive norms.

I am interested in surveying the history of early analytic philosophy and then connecting semantic inferentialism to intuitionistic type theory (connecting philosophy to mathematics). I am interested in drawing a connection between the philosophical (semantic anti-realism) and the logical (with the codification of intuitionism in Martin Lof type theory + recent work in HoTT by Awodey and many others). Frege established the analytic demarcation of a priori reasoning which would inform Russell's logicism and Wittgenstein's construction of fact or tautology. Carnap would make the logical framework robust in the notion of analyticity–a radical opposition to the synthetic a priori. But what about the morning star and the evening star? How does one identify the same sense of different denotations? Frege's work on incomplete arithmetic expressions provides a hint of the functional paradigm after the work of Church's lambda calculus, defined by functionals. Tarski posed a great challenge to Carnap with the undefinability of truth theorem wherein every metalanguage necessitates a metalanguage to encode prior object language; the model theoretic paradigm begins. Yet I am interested in an alternative reading of the split between Tarski and Carnap. Through the work of intuitionistic logic, propositions are taken as if the witness matters. This would later culminate in type theories such as Martin-Lof and substructural logics such as Girard's linear logic. The semantic anti-realist position (where the opposition between transcendent truth and constructive truth can be situated between Platonist and intuitionist philosophy of math) harkens back to Carnap's original principle of verification via the witness of intuitionistic logic, yet situated in an inferential network. Through the constructive possibilities of Homotopy Type Theory, a new paradigm for the foundation of mathematics, the identity of types with topology extends the synthetic lineage of the a priori through a mathematical codification of intuitionism (which remain mere mental constructions).

I am interested in the split between Carnap and Tarski over metalogic regarding semantics. Carnap crystallized the bare minimum of structure for an

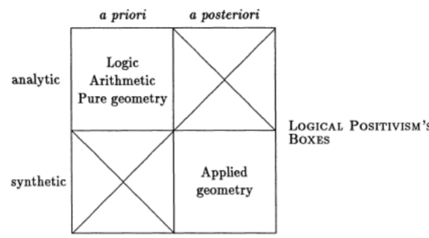



Figure 2: Taken from "The Taming of the True"

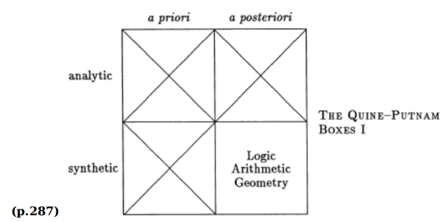



Figure 3: Taken from "The Taming of the True"

alien civilization to understand our language based on propositions and variables defining such propositions. Carnap argued that we should be tolerant of a multitude of metalanguages for instituting the object language in a logical syntax of language. What are the possibilities of defining a universally-quantified language or is such an endeavor doomed to fail because of Goedel's theorem regarding incompleteness? What about the recent advancements in the foundation of mathematics, i.e. Homotopy Type Theory? There has been a continual casting away of the synthetic a priori by Russell, the positivists and later Quine and Putnam. The central question of the book will be whether to admit ontologically, metaphysical realism or anti-realism and then epistemologically, semantic realism or anti-realism?

Per Martin-Löf in his tracts on the philosophy of math accepts the synthetic

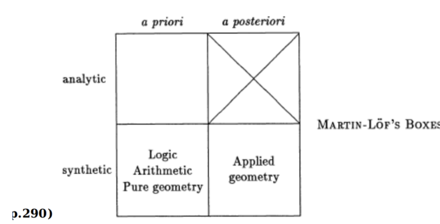



Figure 4: Taken from "The Taming of the True"

a priori, which Kant postulated as existent with geometry and arithmetic. I will be arguing for the synthetic a priori in terms of synthetic geometry such as Homotopy Type Theory, which is at the nexus between theoretical computer science and algebraic topology. The computational types can be transported directly into geometric terms through Steve Awodey's interpretation of the Univalence Axiom.

One of the main readings of Frege's logicism, which is the school of philosophy of math which believes in the fundamentality of atomic propositions, will be Michael Dummett's work regarding the bivalence of truth statements and the justificatory power of demonstrating a proof to a witness via the assertion. This is what Robert Harper calls "logic as if people matter." The key disagreement between Martin-Löf and Dummett is whether proof of a statement is an ontological or epistemological claim with Martin-Löf arguing for the latter and Dummett for the former. Dummett was the key proponent of the semantic anti-realist school and argued that all language can be reduced to its logical basis in the philosophy of math: either one is a Platonist or one is a intuitionist. Dummett was a proponent of verificationism, which Carnap and the logical positivists believed was central to establishing the veracity of empirical phenomenon. David Corfield argues that that inferentialist school of Brandom would be the most faithful way to read the philosophy of Homotopy Type Theory:

I believe that there are grounds to hope that large portions of Brandom's program can be illuminated by type theory. Since Brandom's inferentialism derives in part from the constructive, proof-theoretic outlook of Gentzen, Prawitz, and Dummett, this might not be thought to be such a bold claim. However, his emphasis is generally on material inference, only some aspects of which he takes to be treatable as formal inference. For instance, the argument he gives in *Making it Explicit* (Brandom 1994, Chap. 6) for why we have substitutable singular terms and directed inferences between predicates, is presented with a minimal formal treatment, and yet this phenomenon makes very good sense in the context of HoTT when understood as arising from the different properties of terms and types. Indeed, the kind of category whose internal structure our type theory describes, an $(\infty, 1)$-topos, presents both aspects—the '1' corresponding to unidirected inference between types (morphisms between objects) and the '$\infty$' referring to the reversible substitutability of terms (2-morphisms and higher between morphisms). On the other hand, his frequent use of formalisms, even very briefly to category theory (for example, Brandom 2010, p. 14), tells us that he recognizes the organizing power of logical languages. Elsewhere (Brandom 2015, p. 36), he notes that while both Wittgenstein and Sellars emphasized that much of our language is deployed otherwise than for empirical description, whereas Wittgenstein addressed this excess as an assortment, Sellars viewed much of it more systematically as 'broadly metalinguistic locutions'. These additional functions of language pertain to the very framework that allows for our descriptive practices, and are what Brandom looks to make explicit in much of his work. My broader suggestion in this book is that we look to locate these locutions in features of our type theory, especially those modalities we will add in Chap. 4. (Corfield 2020)

Through inferentialism, truths are not related necessarily to its meaning, but rather meaningfulness in the sense of the meaning in context or meaning-as-use. The book will be situated in the history and philosophy of language and mathematics. I hope to further the research on Homotopy Type Theory being done by Corfield, Andrei Roden and Awodey and extend this mathematical subfield into the larger philosophical lineage of philosophy of language. If time permits, I would like to compare the pragmatist reading of philosophy of language with the rationalist nominalist reading of Jean Cavailles. My book would examine the universality of mathematical necessity through the mathematical philosophy of Cavailles. Through the mathematical Absolute we arrive at both the contingent accidents and mathematical necessity that allows for the development of new theorems, but moreover mathematical experience as a determined concept. Can we found universality (as in Cavailles) on the generality of infinity-groupoids in Homotopy Type Theory, i.e. globally, or is it necessary that metalinguistic locutions be situated in terms of the inferentialism of Homotopy Type Theory, i.e. locally?

## 19 Following Reza Negarestani

Intelligence needs to be made intelligible, the mind needs to be approached functionally in terms of "what it does" (behaviors and functions), which needs to exist within a "history of histories" (spirit).

## 20 Brandom's Claim

Brandom makes the claim that not only Geisteswissenschaften exists within history but moreover Naturwissenschaften. Science inherently has a historical dimension, which is not to say that it is socially constructed but rather the semantic truth of scientific theory involves context-dependent norms that make determinate scientific truth.

## 21 Michael Friedman in *Dynamics of Reason*

Michael Friedman argues that there is a contextual milieu that provides the relative a-priori for each and every scientific revolution. David Corfield says, "If we are to follow Friedman's schema, then the period we are currently in is his 'metascientific' one, where thinkers refashion mathematics and reformulate physical principles in a philosophically-minded way. Think of Helmholtz, Mach, Clifford, Klein, Poincaré,..., Einstein." This a-priori is both universally valid and relativized.

## 22 Brandom's Argument

Brandom argues that most statements in a language need to be materially good and are not substitutable, i.e., not formally inferred. Following David Corfield's book on the philosophy of HoTT, it is agreeable that Homotopy Type Theory can correct this through "terms and types" of ($\infty$,1)-topos where 2-morphisms, 3-morphisms, 4-morphisms, etc., are reversibly substitutable, and therefore formally inferred. Corfield has pointed me (in a private email) in the direction of Quantum Natural Language Processing, in particular linear homotopy type theory, and he says it provides a means of defining a topological metric between similar words, and formal substitution is defined by such a metric.

## 23 Semantic Ascent vs. Semantic Descent

As opposed to J.N. Findlay, who argues for semantic ascent in Hegel from the object-language to the meta-language (a move championed by Tarski) in order to define a judgment, Brandom argues for semantic descent to the bottom level or the materially good inferences.

## 24 Theorem

Apart from the original morphisms and objects directed in the lowest level, there exists a transit up and down and vice versa between object language and metalanguage through ($\infty$,1)-topoi. In Agda (the proof-checker/proof-assistant language for Univalent Foundations) **a type of types** or **type of types of types** corresponds to varying universe levels. For example, see Agda documentation: "Agda' type system includes an infinite hierarchy of universes $Set_i$ : $Set_{i+1}$. This hierarchy enables quantification over arbitrary types without running into the inconsistency that follows from $Set : Set$."

## 25 The Universally-Valid A Priori

The universally valid a-priori prior to a scientific revolution, which informs said revolution, is context-dependent and exists within a history of histories.

## 26 The Curry-Howard Correspondence

The Curry-Howard Correspondence states that $\prod$ types are equivalent to $\forall$ intuitionistic quantifiers, $\sum$ types are equivalent to $\exists$ in logic, and maps are equivalent to intuitionistic implication.

- "Types correspond to logical formulas (aka propositions)."
- "Programs correspond to logical proofs."

- "Evaluation corresponds to simplification of proofs."

Mind becomes only what it does functionally and therefore what is computable. Spirit becomes context-dependent codifications of public language in an inferential network of $\infty$-topoi.

# 27 Univalence Axiom and Awodey's interpretation

#### 27.0.1 $(\mathbf{A} = \mathbf{B}\ ) \cong (\ \mathbf{A} \cong \mathbf{B})$

#### 27.0.2 Awodey (as quoted in private email): "The Univalence Axiom was indeed the work of Voevodsky, but the interpretation of identity types as topological path spaces, which forms the basis of HoTT, was in fact due to me."

# 28 Pure Geometry, Arithmetic, Logic

Following Per Martin-Löf, it is agreeable that pure geometry, arithmetic and logic are modes of synthetic a priori knowledge epistemologically.

## 29 *Hegel in Mathematics*, Alexander Prähauser (May 2022):

**29.0.1** **"It is ironic that the concepts Russell used in his attack, infinitesimals and "continuity", have been particularly useful, once re-evaluated through the recent formalization of Hegel's thought in the context of modal homotopy type theory. This was work started by William Lawvere in the 1985s [16]"**

**29.0.2** **"In early 1985, while I was studying the foundations of homotopy theory, it occurred to me that the explicit use of a certain simple categorical structure might serve as a link between mathematics and philosophy."**

**29.0.3** **"Lawvere went on to provide a formally strict logical calculus that tries to cap- ture Hegelian dialectics and started the formalization of Hegel's objective logic [15]. However, the mathematical power of Lawvere's formalization, though already considerable was restricted by the mathematics it was founded on.1 More recently, a new foundation of mathematics was developed under the initiative of Vladimir Voevodsky in homotopy type theory, which provides an alternative to set theory and a setting for logic based on a radical interpretation of equality, which amplifies its power and has been found to show remarkable similarity to Hegel's thought. However, the importance of these devopments exceeds math- ematics and reaches into philosopy. Lawvere stated in 1992 [13]:"**

**29.0.4** **"It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathe- matical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophi-cal questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy."**

Therefore, the formal order of self-consciousness (the logical $I=I$) first and foremost points to the underlying structure of what appears as a simple identity relation. This underlying structure is nothing other than the transformation afforded by conceiving $I$ and $I^*$ as identity maps ($I=I^*$). Adopting the Hegel-inspired mathematical formalism of William Lawvere, this map or composition of maps can be represented as follows.[31]

If we treat $I$ and $I^*$ as objects with their respective identity maps, then $I=I^*$ is really:

$$I \xrightarrow{f} I^*$$

meaning that $I$ (formal self-consciousness) is the identity map of the domain $I^*$ (concrete self-consciousness or the assertion that there is reality in excess of the self or mind) and $I^*$ is the identity map of the codomain of $I$ (the freedom of self-consciousness as conceived from a reality that is of nowhere and nowhen, a concrete freedom in which self-consciousness only exists in every respect for another self-consciousness). It then universally and necessarily follows that:

$$I \xrightarrow{f} I^* \Rightarrow I \xrightarrow{I} I \wedge I \xrightarrow{f} I^* \wedge I^* \xrightarrow{I^*} I^* \wedge If = f = fI^*$$

In concretely rendering reality intelligible, in expanding the domain of the intelligible and hence that of reality, in acting on the intelligible or intervening in reality, the formal condition of intelligence ($I$) is realized as intelligence ($I^*$). Formal self-consciousness only becomes self-consciousness in satisfying another self-consciousness,[32] in extending over into the intelligibility of a reality which in its unrestrictedness establishes the truth of I, the mind, or intelligence. Yet the achievement of this truth ($I^*$) is also impossible

31 F.W. Lawvere, *Functorial Semantics of Algebraic Theories* (New York: Columbia University Press, 1963).

32 'Self-consciousness attains its satisfaction only in another self-consciousness'—and without this satisfaction it is only a consciousness that finds its 'satisfaction in mere dirt and water'. Hegel, *Phenomenology of Spirit*, §8, §175.

33

Figure 5: Taken from Reza Negarestani, *Intelligence and Spirit*

## 4 Calculus of Hegelian dialectics

Using category theory, we can explain Schreiber's formulation of Lawvere's formalization of Hegelian dialectics. For this, what Hegel calls a "Moment" is formalized as a (co)modality on a category [6]$C$: an operator $\Box$ that maps an object $X$ of $C$ to an object $\Box X$ called the *aspect* of $X$ under $\Box$, maps a morphism $Y \xrightarrow{f} X$ to a morphism $\Box Y \xrightarrow{\Box(f)} \Box X$ between the aspects of $X$ and $Y$, and is idempotent: $\Box\Box = \Box$, together with, for each object $X$, a morphism between $X$ and the image of $X$ under the operator: $\Box X \to X$ or $X \to \bigcirc(X)$. The direction of this morphism depends on whether the moment is a modality or a comodality. We will call moments of the form $\Box(X) \to X$ *previous* moments and denote them by $\Box$, while we call moments of the form $X \to \bigcirc(X)$

[4] However, a mathematical object can live in different categories at once, if it allows, or necessitates, several different kinds of structures. For instance, the "real numbers" are a field, and as such live in the category of fields, but they are also a total order, so they live in the category total orders. If an object lives in several categories at once, the internal properties of the object often give rise to relations between the categories it is living in.

[5] Of course, this is a *mathematical* universe that houses *mathematical* structures, we will only get to our physical universe at the very end.

[6] In later sections we will mainly look at universes of types, but the potential applicability of Lawvere's calculus is much more general.

Figure 6: Taken from Alexander Prähauser

*successive* moments.[78]

Due to idempotency, any type that lies in the image of a moment is *purely of that moment* (and vice versa): $Y = \Box X \Leftrightarrow \Box Y = Y$. Thus, the types that are invariant under a moment form their own sub-category: $C_\Box \hookrightarrow C$. The image $\Box X$ of a type $X$ under that moment should be thought of as "the best possible approximation of $X$ in $C_\Box$" or the $\Box$-aspect of $X$, and the transformation $\Box X \to X$ (resp. $X \to \bigcirc X$), depending on the exact nature of the moment, either a deduction of $X$ from $\Box X$ ($\bigcirc X$ from $X$), an embedding of $\Box X$ into $X$ ($X$ into $\bigcirc X$), or a construction/deformation that makes $\Box X$ into $X$ ($X$ into $\bigcirc X$).

From a category-theoretic perspective, this state of affairs can be understood as saying that a moment $\Box$ can be decomposed into a *projection* of $C$ onto another category $C_\Box$ :

$$C \twoheadrightarrow C_\Box$$

and an *embedding* of $C_\Box$ into $C$

$$C_\Box \hookrightarrow C.$$

Now, the concept of a *unity of opposites* is translated into a pair of moments $\triangle_1 \dashv \triangle_2$ that fulfill the *adjointness condition*: for any two objects $X, Y$ and each morphism $\triangle_1 Y \to X$, there exists a morphism $Y \to \triangle_2 X$, and these morphisms are subject to a further naturality condition. Adjunctions are deep structures that can be expressed in a variety of ways, this being the most concrete, and the reader cannot be expected to immediately grasp the meaning of the adjointness condition. However, some of its consequences can be used for better understanding. For instance, it follows that if one of the $\triangle_i$ is a preceding moment $\Box$, then the other is a successive moment $\bigcirc$ and vice versa, so that a unity of opposites is either a *unity of a preceding to an opposite successive moment* (or *ps-unity* for short) of the form $\Box \dashv \bigcirc$ or a *unity of a successive to an opposite preceding moment* (or *sp*-unity) $\bigcirc \dashv \Box$. So a unity of opposites is made up of moments that are actually of opposite kinds (preceding to successive), and two opposite kinds of unities of opposites exist. Furthermore, one part of a unity of opposites uniquely determines the other in that, if, for a moment $\Box$, $\Box \dashv \bigcirc_1$ and $\Box \dashv \bigcirc_2$, then $\bigcirc_1$ and $\bigcirc_2$ are equal, and the other way around (however, a moment can have both a left and a right opposite, so the opposition is actually directed). Each type $X$ sits in between its preceeding and successive aspects (let us call this the *aspect sequence of the unity*):

Figure 7: Taken from Alexander Prähauser

$$\Box X \to X \to \bigcirc X.$$

Finally, the two sub-universes $C_\Box$, $C_\bigcirc$ determined by the modalities are both equal *without the context of the surrounding universe* $C$, so their opposition actually lies in their relationship to the larger category, and to be opposites they have to be the same. More precisely, we can, recalling the decomposition of moments into a projection and an embedding we saw before, say that the opposites of a unity share each one of their morphisms, and describe the situation in the following way: a *ps*-unity $\Box \dashv \bigcirc$ consists of *two* embeddings of the same sub-universe $C_\Box = C_\bigcirc$ into the universe $C$ and *one* projection from $C$ onto $\mathcal{V}$:

$$C_\Box \;\substack{\hookrightarrow \\ \twoheadleftarrow \\ \hookrightarrow}\; C$$

while an *sp*-unity consists of *one* embedding of $\mathcal{V}$ into $C$ and *two* projections of $C$ onto $\mathcal{V}$:

$$C_\Box \;\substack{\twoheadleftarrow \\ \hookrightarrow \\ \twoheadleftarrow}\; C$$

From this follows in particular that in an *sp*-unity $\Box \bigcirc X = \Box\Box Y = \Box Y = \bigcirc X$ and vice versa, so $\Box\bigcirc = \bigcirc$ and $\bigcirc\Box = \Box$. Thus, a type is purely of one moment if and only if it is purely of the other.

Adjunctions are category-theoretic concepts and thus can be applied in any suitable 2-category, so that an adjunction between a unity of opposites might itself be adjoint to a another unity of opposites [3]. This *opposite to a unity* of opposites $\triangle_1 \dashv \triangle_2$ is another unity of opposites, and their relation will be denoted

$$\begin{array}{ccc} \triangle_3 & \dashv & \triangle_4 \\ \perp & & \perp \\ \triangle_1 & \dashv & \triangle_2 \end{array}.$$

From the uniqueness of adjoints follows that in such a configuration, the left moment of the first unity has to be equal to the right moment of the second unity: $\triangle_1 = \triangle_4$. So each unity of opposites of unities takes the form of a string of modalities $\lozenge \dashv \Box \dashv \bigcirc$ or $\bigstar \dashv \bigcirc \dashv \Box$. We could ask for even higher opposites, but examples are rare and largely in categories with exotic or no logics.[9]

The other significant relation between unities of opposites is that of Aufhebung. Following (Schreiber's formalization of) Hegel, we say a unity of opposites $\triangle_3 \dashv \triangle_4$ is a *higher sphere*[10] of a unity of opposites $\triangle_2 \dashv \triangle_1$, denoted

Figure 8: Taken from Alexander Prähauser

$$\begin{array}{ccc} \triangle_3 & \dashv & \triangle_4 \\ \vee & & \vee \\ \triangle_1 & \dashv & \triangle_2 \end{array}$$

if $\triangle_1$ is contained in $\triangle_3$: $\triangle_3\triangle_1 = \triangle_1$ and $\triangle_2$ is contained in $\triangle_4$: $\triangle_4\triangle_2 = \triangle_2$. A higher sphere $\Box_2 \to \bigcirc_2$ of a unity $\Box_1 \to \bigcirc_1$ is a *right Aufhebung* of $\Box_1 \to \bigcirc_1$ if furthermore $\Box_1$ is contained in $\bigcirc_2$: $\bigcirc_2\Box_1 = \Box_1$ and a *left Aufhebung* if $\bigcirc_1$ is contained in $\Box_2$: $\Box_2\bigcirc_1 = \bigcirc_1$. Both kinds can be referred to as *Aufhebungen* We can similarly define Aufhebungen for *sp*-unities $\bigcirc \dashv \Box$, however, since in an *sp*-unity both moments project into the same sub-universe, each higher sphere of $\bigcirc \dashv \Box$ is already both a left and a right Aufhebung. We will denote a right Aufhebung as

$$\begin{array}{cccc} \Box_2 & \dashv & \bigcirc_2 & \\ \vee & & \vee & \\ \Box_1 & & \dashv & \bigcirc_1 \end{array}$$

and a left Aufhebung as

$$\begin{array}{cccc} & \Box_2 & \dashv & \bigcirc_2 \\ & \vee & & \vee \\ \Box_1 & \dashv & & \bigcirc_1 \end{array}$$

So an Aufhebung of $\Box_1 \dashv \bigcirc_1$ is a unity of opposites $\Box_2 \dashv \bigcirc_2$ such that $\Box_1$ is a special aspect of $\Box_2$ and $\bigcirc_1$ is a special aspect of $\bigcirc_2$, but where *also* both $\bigcirc_1$ and $\bigcirc_2$ are special aspects of one of the aspects of $\Box_2 \dashv \bigcirc_2$, let's say $\bigcirc_2$, so that they are unified in $\bigcirc_2$ and both on the same side of a greater opposition. From this follows that the opposition $\Box_2 \dashv \bigcirc_2$ is trivial on, in this case, $\Box_1$: $\Box_2\Box_1 = \Box_1 = \bigcirc_2\Box_1$. Please note that this construction captures all three meanings of the German word *Aufheben*: to lift, to preserve and to negate.

Generally, Aufhebungen are not unique, the same unity can have several of them. The *minimal (left/right) Aufhebung* of a unity $\Box \dashv \bigcirc$, the smallest sphere that fulfills the Aufhebungs-condition would be unique, but does not always exist.[11]

Figure 9: Taken from Alexander Prähauser