Quasi-concepts of logic

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Abstract

A analysis of some concepts of logic is proposed, around the work of Edelcio de Souza. Two of his related issues will be emphasized, namely: opposition, and quasi-truth. After a review of opposition between logical systems [2], its extension to many-valuedness is considered following a special semantics including partial operators [13]. Following this semantic framework, the concepts of antilogic and counterlogic are translated into opposition-forming operators [15] and specified as special cases of contradictoriness and contrariety. Then quasi-truth [5] is introduced and equally translated as a product of two partial operators. Finally, the reflections proposed around opposition and quasi-truth lead to a third new logical concept: quasi-opposition, borrowing the central feature of partiality and opening the way to a potential field of new investigations into philosophical logic.

1 Oppositions

The proper contribution of Edelcio de Souza with respect to logical oppositions has been through its application to logical systems [2], beyond mere formulas or philosophical concepts in a given logical system [3]. The concept of opposition comes from Aristotle's work and consists in logical relations between bivalent formulas, in such a way that each of these formulas is to be interpreted as either true or false. For this reason, oppositions should be explained in a semantic way. However, Edelcio and the co-authors claim that every logical opposition that does not relate propositions should not be explained in semantic terms ([2], 243):

"Because the vertices of the square (...) are not propositions we reconstruct the classical oppositions accordingly. We define them in terms of relations between logics –instead of logical values".

Anyway, the authors introduce the two central notions in a classical way. Letting $\Gamma \vdash_L \varphi$ the relation of consequence from any set of premisses Γ to an arbitrary formula φ in a language L:

- $\bar{L}$ is called an *antilogic* of L if and only if it is not the case that φ is a logical consequence of Γ in $\bar{L}$: $\Gamma \nvdash_{\bar{L}} \varphi$.
- $\tilde{L}$ is called an *counterlogic* of L if and only if it is the case that $\neg\varphi$ is a logical consequence of Γ in $\tilde{L}$: $\Gamma \vdash_{\tilde{L}} \neg\varphi$.

Beyond the bivalent stance, the aim of the present paper is to re-define oppositions between logics in semantic terms and to explore the possibility of non-standard oppositions.
On the one hand, such oppositions may be formulated in the Tarskian sense of semantic consequence as a relation of truth-preservation in a model, i.e., interpretations of formulas such that these are true (symbol: t) or false (symbol: f) in a model. Thus, $\Gamma \models_L \varphi$ means that any model w of $\psi \in \Gamma$ in L is also a model of φ in L: $\Gamma \models_L \varphi$, i.e. $v(w, \psi) = t \Rightarrow v(w, \varphi) = t$. Then any model w of $\bar{L}$ can be called an *antimodel* of L, and any model w of $\tilde{L}$ can be called a *countermodel* of L, such that:

- there exists a model w of $\psi \in \Gamma$ in L that is not a model of φ in $\bar{L}$: $\Gamma \not\models_{\bar{L}} \varphi$, i.e., $v(\psi, w) = t \not\Rightarrow v(w, \varphi) = t$.
- every model w of $\psi \in \Gamma$ in L is also a model of $\neg\varphi$ in L: $\Gamma \models_{\tilde{L}} \neg\varphi$, i.e., $v(w, \psi) = t \Rightarrow v(w, \neg\varphi) = t$.

On the other hand, the bivalent interpretation of formulas in the models entails that there is no logical difference between untruth and falsity. In other words, every antimodel of φ is a model at which φ is not true, that is, φ is false: $v(w, \varphi) \neq t = v(w, \varphi) = f$; and every countermodel of φ is a model at which $\neg\varphi$ is true, that is, φ is false: $v(w, \neg\varphi) = t$ means the same as $v(w, \varphi) = f$. The difference between both antilogic and counterlogic may be easily explained in terms of how many models there are for these: an antilogic has φ false at *some (but not all)* model of it at which ψ is true, whereas a counterlogic has φ false at *every* model of it at which ψ is true.

And yet, what of such higher-order oppositions in a many-valued system where bivalence does not obtain anymore? Answering to this question will be the central task of the present section, especially because bivalence is assumed in [2]. Our aim is to extend the notion of logical oppositions into non-bivalent or many-valued systems, accordingly. For this purpose, let us consider a general domain of valuation V_n including n logical values. Bivalence includes the class of logical systems where the $m = 2$ logical values are truth and falsehood. We assume that many-valuedness relates to any logical system whose domain of interpretation V_m includes more than 2 values, such that $m > 2$.

More generally, one way to characterize many-valuedness is by taking logical values to be ordered n-tuples of elements whilst keeping in mind that the basic values of logic are t and f. A characterization of such finitely n-valued systems consists in a 2^n-valued domain of values including n ordered elements and $2^m = n$ resulting logical values. Borrowing from various works from to Jaskówski [9] to Kapsner [10] through Shramko & Wansing [17], the following wants to focus on a specific case of structured logical values analogous to Belnap's 4-valued system First Degree Entailment [1]. Thus, let V_4 a 4-valued domain of structured logical values $X = \langle x_1, x_2 \rangle$. It includes $n = 2$ elements t and f such that, for any φ, $v_4(\varphi) = \langle x_1, x_2 \rangle$ and $x_i(\varphi) \mapsto \{1, 0\}$.

Given that logical values are structured objects in V_4, their characteristic valuation function proceeds as an ordered 2-uple $\mathbf{A}(\varphi) = \langle \mathbf{a}_1(\varphi), \mathbf{a}_2(\varphi) \rangle$ wherein $\mathbf{a}_1(\varphi) = x_1$ informs about whether φ is true, and $\mathbf{a}_2(\varphi) = x_2$ about whether φ is false. Correspondingly, we will rephrase the four logical values of V_4 by translating first their basic elements t and f in terms of structured values and, then, the combination of the latter.[1]

$v(\varphi) = t$ means that $\mathbf{a}_1(\varphi) = 1$, i.e., φ is true.
$v(\varphi) \neq t$ (or $v(\varphi) = \bar{t}$) means that $\mathbf{a}_1(\varphi) = 0$, i.e., φ is not true.
$v(\varphi) = f$ means that $\mathbf{a}_2(\varphi) = 1$, i.e., φ is false.
$v(\varphi) \neq f$ (or $v(\varphi) = \bar{f}$) means that $\mathbf{a}_2(\varphi) = 0$, i.e., φ is not false.

The logical values of V_4 can be considered as ordered structured pairs such that $B = \langle t,f \rangle$, $T = \langle t,\bar{f} \rangle$, $F = \langle \bar{t},f \rangle$, and $N = \langle \bar{t},\bar{f} \rangle$[2].

$v(\varphi) = B$ means that $\mathbf{a}_1(\varphi) = \mathbf{a}_2(\varphi) = 1$, i.e., $\mathbf{A}(\varphi) = 11$.
$v(\varphi) = T$ means that $\mathbf{a}_1(\varphi) = 1$ and $\mathbf{a}_2(\varphi) = 0$, i.e., $\mathbf{A}(\varphi) = 10$.
$v(\varphi) = F$ means that $\mathbf{a}_1(\varphi) = 0$ and $\mathbf{a}_2(\varphi) = 1$, i.e., $\mathbf{A}(\varphi) = 01$.

[1] For a discussion about the meaning of such structured values and a doxastic interpretation of these, see e.g. [14].

[2] For sake of simplicity, the ordered pairs $\langle x, y \rangle$ will be rephrased as xy throughout the rest of the paper.

$v(\varphi) = N$ means that $\mathbf{a}_1(\varphi) = \mathbf{a}_2(\varphi) = 0$, i.e., $\mathbf{A}(\varphi) = 00$.

The semantic relation of consequence between a set of formulas Γ and a formula φ can also be rephrased in terms of structured logical values, such that $\Gamma \vdash_L \varphi$ means that, for every formula $\psi \in \Gamma, \mathbf{a}_1(\psi, L) = 1 \Rightarrow \mathbf{a}_1(\varphi, L) = 1$. The same does for the central notions of antilogic and counterlogic.

Antilogic: $\Gamma \vdash_{\bar{L}} \varphi$ if and only if it is not the case that $\Gamma \vdash_L \varphi$.
$\mathbf{a}_1(\Gamma, \bar{L}) = \mathbf{a}_1(\varphi, \bar{L}) = 1$ if and only if $\mathbf{a}_1(\psi, L) = 1$ and $\mathbf{a}_1(\varphi, L) = 0$.

Counterlogic: $\Gamma \vdash_{\tilde{L}} \varphi$ if and only if $\Gamma_L \vdash \neg\varphi$.
$\mathbf{a}_1(\Gamma, \tilde{L}) = \mathbf{a}_1(\varphi, \tilde{L}) = 1$ if and only if $\mathbf{a}_1(\psi, L) = \mathbf{a}_2(\varphi, L) = 1$.

Semantic consequence in a logical system can also be rephrased as a mapping function $\mathcal{F}_{\mathcal{V}}$ on values such that, for a primary logical system L where truth is preserved from premisses Γ to consequence φ

$$\mathcal{F}_{\mathcal{V}}(L) = t \mapsto t.$$

The corresponding antilogics and counterlogics can be redefined as follows, accordingly:

$$\mathcal{F}_{\mathcal{V}}(\bar{L}) = t \mapsto \bar{t};$$
$$\mathcal{F}_{\mathcal{V}}(\tilde{L}) = t \mapsto f.$$

Returning to the aforementioned paper [2], the authors gave a definition of the usual concepts of opposition whilst expressing these as set-theoretical relations of intersection $\cap$ between logical systems. Once again, we translate each of these into our semantic terms as follows: for every φ, the intersection $\vdash_{L_1} \cap \vdash_{L_2}$ is (not) empty if, and only if, φ's being true L_1 (does not) entail φ's not being true in L_2; and the intersection $\nvdash_{L_1} \cap \nvdash_{L_2}$, is (not) empty if, and only, φ's not being true L_1 (does not) entail φ's being true in L_2.[3]

[3]The second clause characterizing oppositions could be reformulated as a relation of union $\cup$ between any logical systems L_1, L_2, by virtue of the set-theoretical relation between intersection and union. Thus, $\nvdash_{L_1} \cap \nvdash_{L_2} = \varnothing$ means the same as $\vdash_{L_1} \cup \vdash_{L_2} \neq \varnothing$.

L_1 and L_2 are *contradictories* if and only if $\vdash_{L_1} \cap \vdash_{L_2} = \varnothing$ and $\nvdash_{L_1} \cap \nvdash_{L_2} = \varnothing$.
$\mathbf{a}_1(\varphi, L_1) = 1 \Rightarrow \mathbf{a}_1(\varphi, L_2) = 0$ and $\mathbf{a}_1(\varphi, L_1) = 0 \Rightarrow \mathbf{a}_1(\varphi, L_2) = 1$.

L_1 and L_2 are *contraries* if and only if $\vdash_{L_1} \cap \vdash_{L_2} = \varnothing$ and $\nvdash_{L_1} \cap \nvdash_{L_2} \neq \varnothing$.
$\mathbf{a}_1(\varphi, L_1) = 1 \Rightarrow \mathbf{a}_1(\varphi, L_2) = 0$ and $\mathbf{a}_1(\varphi, L_1) = 0 \nRightarrow \mathbf{a}_1(\varphi, L_2) = 1$.

L_1 and L_2 are *subcontraries* if and only if $\vdash_{L_1} \cap \vdash_{L_2} \neq \varnothing$ and $\nvdash_{L_1} \cap \nvdash_{L_2} = \varnothing$.
$\mathbf{a}_1(\varphi, L_1) = 1 \nRightarrow \mathbf{a}_1(\varphi, L_2) = 1$ and $\mathbf{a}_1(\varphi, L_1) = 0 \Rightarrow \mathbf{a}_1(\varphi, L_2) = 1$.

The fourth and ultimate case of *subalternation* differs from the preceding ones by being defined without the set-theoretical relation of intersection, in informal terms of 'sublogic'.

L_1 is *subaltern* to L_2 if and only if L_2 is a sublogic of L_1.

The latter is assumed to be known by the readers, in that it means a relation of consequence from the first system to the second one. That is:

$\mathbf{a}_1(\varphi, L_1) = 1 \Rightarrow \mathbf{a}_1(\varphi, L_2) = 1$ and $\mathbf{a}_1(\varphi, L_2) = 0 \Rightarrow \mathbf{a}_1(\varphi, L_1) = 0$

An alternative definition of subalternation has been proposed in [15], where oppositions are turned from relations into iterative functions. Thus, ψ is said to be 'subalternate' to φ if, and only if, ψ is the *contradictory of the contrary* of φ; and conversely, φ is 'superalternate' to ψ if, and only if, φ is the *contrary of the contradictory* of φ.
It would be interesting to see how such a functional interpretation of opposition may be implemented into the context of logical system [2]. Assuming that antilogicality and counterlogicality are special cases of contradictoriness and contrariety, respectively, then there is a discrepancy between the logical equations established in [15] and what the author said in their own symbols [2]. Thus,

(1) The antilogic of the antilogic of a given logical system L_1 is L_1 itself in [2]

$$\bar{\bar{L}} = L$$

which is confirmed in [15] by stating the contradictory of the contradictory of a given term φ is φ itself

$$cd(cd(\varphi)) = \varphi.$$

At the same time, (2) the counterlogic of the counterlogic of a given logical system L_1 does equate with L_1 itself in [2]

$$\tilde{\tilde{L}} = L$$

whereas the contrary of the contrary of a given term φ may differ from φ in [15]

$$ct(ct(\varphi)) \neq \varphi.$$

And (3) the counterlogic of the antilogic of a given logical system L_1 does equate with the antilogic of its counterlogic in [2]

$$\tilde{\bar{L}} = \bar{\tilde{L}}$$

whereas we have already seen that the contradictory of a contrary differs from the contrary of a contradictory in [14]. Indeed, the former iteration amounts to a case of subalternation

$$cd(ct(\varphi)) = sb(\varphi)$$

whereas the latter yields the converse case of superalternation

$$ct(cd(\varphi)) = sp(\varphi).$$

How to account for such a discrepancy, and what does it entail about the logical accuracy of [2] and [15]. In order to disentangle the situation, we have not only to prove that antilogicality relates to contradictoriness and counterlogicality to contradictoriness. But also, the calculus of opposition-forming operators set up in [15] leads to an important difference with respect to [2]. Indeed, such operators are not 'functions' in the strict mathematical sense of a bijection: one input value may have more than one contrary, subcontrary and subaltern (or superaltern), so that the above singular expression '*the* contrary of' is misleading. Actually, it is possible to compute the output value of such opposite-forming operators only by means by a special semantics, namely: a 'bitstring semantics' in which terms do not receive a customary 'truth-value' but, instead, a Boolean bitstring characterizing their truth-conditions in a finite set of logical spaces. It turns out that this Boolean but *not* truth-functional semantics departs from the approach of [2]. On the one hand, it matches with [2] in that every logical system has one and only one *antilogic* as a counterpart of contradictoriness ([2], 245):

"It is clear for each L there is exactly one $\bar{L}$" which can be explained set-theoretically once again:

$$\vdash_L \cap \vdash_{\bar{L}} = \varnothing;$$
$$\vdash_L \cup \vdash_{\bar{L}} = \wp(F) \times F.$$

On the other hand, it is shown in [15] that what the authors call 'counterlogic' is just a particular, truth-functional case of contrariness:

"It is clear that for each L, and for each negation operation, there is exactly one $\tilde{L}$."

The authors rightly assume that one and the same operator of negation occurs in L_1 and L_2, so that there can be only one system L_2 where φ is false whenever φ is true in L_1. A way to account for this unique case of contrariness occurs in algebraic terms of abstract operators [8,12,15]. In the second reference [12], for example, Piaget's INRC Group depicts the operation of reciprocity as mapping from an order set of conjunctive normal forms of literals *abdc* upon its reverse *cdba*. This helps why there cannot be but one of 'contrariness' once constructed in this bijective way. In [15], the same operation is applied to make sense of 'contrary' beliefs operators as ordered set of truth-conditions whilst noticing that there is one more than such one way to characterize contrariety.

And yet, one may imagine however more than one way of satisfying the clauses of antilogicality and counterlogicality once bivalence is not assumed. This requires to go beyond the Boolean approach, assumed both in [2] and [15]. For there may be more than one way of being true and false in V_4, for example, so that there may be more than one antilogic and counterlogic to an initial logical system L_1. Now going beyond bivalence is to go beyond the realm of 'classical' oppositions, which seems to lead to a *terra incognita* in the literature of logic. For what had been said thus far about 'non-classical oppositions'? Be this as it may, 'classical' oppositions may be characterized by two clauses such as *completeness* and *consistency*. Classicality is claimed and sustained in [2,247] as follows:[4]

"It is not straightforward to present oppositional structures for any logic. We will proceed by introducing some restrictions. First, we restrict ourselves to logics which accept elimination of double negation in an obvious sense. Additionally, let L be a logic with negation. We say that L is *well-behaved* if and only if for every pair (Γ, φ), it is not the case that ($\Gamma \vdash_L \varphi$ and $\Gamma \vdash_L \neg\varphi$)".

Double negation relates to completeness: $\vdash_L \varphi$ if and only if $\vdash_L \neg\neg\varphi$, whist well-behavior has to do with consistency. Both properties and their opposite may be formulated as follows:

Consistency

$$\Gamma \vdash \varphi \Rightarrow \Gamma \nvdash \neg\varphi$$

Inconsistency

$$\Gamma \vdash \varphi \nRightarrow \Gamma \nvdash \neg\varphi$$

Completeness

$$\Gamma \nvdash \varphi \Rightarrow \Gamma \vdash \neg\varphi$$

Incompleteness

$$\Gamma \nvdash \varphi \nRightarrow \Gamma \vdash \neg\varphi$$

These metalogical properties characterize what is considered as the proper features of logical oppositions ([2], 427):

"We call a square *complete* if it is a square with all four oppositions: contradiction, contrariety, sub-contrariety and subalternation. A square is *standard* if it fits any family of concepts satisfying traditional oppositions. A square is *perfect* if it is complete and standard. Moreover, any square which is not complete or/and standard is called *degenerate square.*"

Why sticking to such features, however? Let us consider in the following what non-standard squares should amount to, assuming that they might relate many-valued systems which are not well-behaved and do not accept elimination of double negation. In V_4, for example, logical systems may be incomplete or inconsistent whenever $\mathbf{A}(\varphi) = 00$ or $\mathbf{A}(\varphi) = 11$,

[4] Note that classicality need not be a synonym of bivalence, given that there may be classical theorems that do not correspond to a bivalent domain (and conversely). See e.g. [15] about this point.

respectively. Let us see what does follow from this non-bivalent situation: does it result in new kinds of oppositions? In order to answer this question, let us consider by now another issue which has been addressed by de Edelcio de Souza.

2 Quasi-truth

Indeed, one of de Souza's main contributions to the reflection in philosophical logic relates to the concept of *quasi-truth* [4], inherited from da Costa's seminal work. Roughly speaking, quasi-truth is to be viewed as a set of *partial* structures such that the predicates are seen as triples of pairwise disjoint sets $\{R^+, R^-, R^u\}$: the set of tuples which satisfies, does not satisfy and may satisfy or not a predicate in a given model. Our attention will be focused on the third subset R^u, since it stands for the 'partial' features of structures and leads to the notion of quasi-truth. R^u may be taken to be the set of undeterminate logical values, {11,00}, such that logical value of φ is neither determinately true nor determinately false. Although quasi-truth is usually interpreted into a 3-valued domain $V_{3^+} = \{11, 10, 01\}$ or $V_{3^-} = \{10, 01, 00\}$ –depending upon whether the additional third value is designated or not, it makes sense to consider as two proper cases of quasi-truth the situations in which there is evidence both for and against a given formula or neither for nor against, respectively. The concrete upshot is the same as the one when there is evidence neither for nor against the formula, in the sense that it leads to the same *practical* stance of indecision. Likewise, the coming 4-valued framework accommodates with the 3-valued definition of quasi-truth by treating gappy and glutty values (00 and 11) as two *pragmatic* variants of the same partial structure: underdetermined and overdetermined logical values amount to the same result of remaining undecided about φ, insofar as the logical value of formulas relate to what agents should do in the light of such informational data.

We propose to reconstruct both logical values and relations of opposition between logical systems into a common framework $\mathbf{AR}_{4[O_i]}$ [13]. It includes a number of logical systems distinguished by two sets of unary operators of *affirmation* $[O_i]$ and *negation* $[N_i]$. The language of $\mathbf{AR}_{4[O_i]}$ can be described by means of the usual Backus-Naur form:

$$\varphi ::= \quad [O_i]p \mid [O_i](\varphi \bullet \psi) \mid [O_i]\varphi \bullet [O_i]\psi \mid \neg_1[O_i]\varphi \mid [O_i]\neg_2\varphi$$

The lower case variable i of $[O_i]$ means that there is a plurality of affirmative and negative operators in $\mathbf{AR}_{4[O_i]}$. Roughly speaking, both categories of operators constitute a variety of ways to restrict the logical values of formulas in V_4. Affirmative operators are not redundant by excluding logical values whilst always affirming their input value, whereas negative operators always exclude the input value. Their general definitions are the following, for any pairs of values $\{x_i, x_j\}$ in V_n:

Affirmative operators
$[A_i]\varphi : x_i \mapsto \overline{x_j}$

Negative operators
$[N_i]\varphi : x_i \mapsto \overline{x_i}$

An essential feature of $[A_i]$ and $[N_i]$ is that these are *partial*: they turn some, but not necessarily all input values into output values of the entire domain V_4.[5]

Given any domain of valuation V_n, there is a set of $i = 2^n - 1$ affirmative operators. In the present case of V_4, there are $2^4 - 1 = 15$ affirmative and negative operators which obey double negation in a metalogical sense of the word: $\bar{\bar{x}} = x$.

$[A_1]\varphi : t \mapsto \bar{f}$	$[N_1]\varphi : t \mapsto \bar{t}$
$[A_2]\varphi : f \mapsto \bar{t}$	$[N_2]\varphi : f \mapsto \bar{f}$
$[A_3]\varphi : \bar{t} \mapsto f$	$[N_3]\varphi : \bar{t} \mapsto t$
$[A_4]\varphi : \bar{f} \mapsto t$	$[N_4]\varphi : \bar{f} \mapsto f$
$[A_5]\varphi : t \mapsto \bar{f} \otimes f \mapsto \bar{t}$	$[N_5]\varphi : t \mapsto \bar{t} \otimes f \mapsto \bar{f}$
$[A_6]\varphi : t \mapsto \bar{f} \otimes \bar{t} \mapsto f$	$[N_6]\varphi : t \mapsto \bar{t} \otimes \bar{t} \mapsto t$
$[A_7]\varphi : t \mapsto \bar{f} \otimes \bar{f} \mapsto t$	$[N_7]\varphi : t \mapsto \bar{t} \otimes \bar{f} \mapsto f$
$[A_8]\varphi : f \mapsto \bar{f} \otimes \bar{t} \mapsto t$	$[N_8]\varphi : f \mapsto \bar{f} \otimes \bar{t} \mapsto t$
$[A_9]\varphi : f \mapsto \bar{f} \otimes \bar{f} \mapsto f$	$[N_9]\varphi : f \mapsto \bar{f} \otimes \bar{f} \mapsto f$
$[A_{10}]\varphi : \bar{t} \mapsto f \otimes \bar{f} \mapsto t$	$[N_{10}]\varphi : \bar{t} \mapsto t \otimes \bar{f} \mapsto f$
$[A_{11}]\varphi : t \mapsto \bar{f} \otimes f \mapsto \bar{t} \otimes \bar{t} \mapsto f$	$[N_{11}]\varphi : t \mapsto \bar{t} \otimes f \mapsto \bar{f} \otimes \bar{t} \mapsto t$
$[A_{12}]\varphi : t \mapsto \bar{f} \otimes f \mapsto \bar{t} \otimes \bar{f} \mapsto t$	$[N_{12}]\varphi : t \mapsto \bar{t} \otimes f \mapsto \bar{f} \otimes \bar{f} \mapsto f$
$[A_{13}]\varphi : t \mapsto \bar{f} \otimes \bar{t} \mapsto f \otimes \bar{f} \mapsto t$	$[N_{13}]\varphi : t \mapsto \bar{t} \otimes \bar{t} \mapsto t \otimes \bar{f} \mapsto f$
$[A_{14}]\varphi : f \mapsto \bar{t} \otimes \bar{t} \mapsto f \otimes \bar{f} \mapsto t$	$[N_{14}]\varphi : f \mapsto \bar{f} \otimes \bar{t} \mapsto t \otimes \bar{f} \mapsto f$
$[A_{15}]\varphi : t \mapsto \bar{f} \otimes f \mapsto \bar{t} \otimes \bar{t} \mapsto f \otimes \bar{f} \mapsto t$	$[N_{15}]\varphi : t \mapsto \bar{t} \otimes f \mapsto \bar{f} \otimes \bar{t} \mapsto t \otimes \bar{f} \mapsto f$

[5] Another way to characterize these operators is to take these as a combination of redundant and non-redundant mappings: they turn some (but not all) of their input values into some other output values.

This language includes two main negations, the Boolean one $\neg_1$ and the Morganian one $\neg_2$, in addition with a set of binary connectives $\bullet = \{\wedge, \vee, \rightarrow\}$. Products $\otimes$ are idempotent, commutative, transitive and associative operators that merely add different mappings of the same kind to each other. For example, $[A_7]$ proceeds in such a way that every formula is unfalse whenever true *and* true whenever unfalse, whereas $[A_8]$ means that every formula is false whenever untrue *and* untrue whenever false. The single values occurring in boldface in the below matrix correspond to the outputs altered by the affirmative operators, the other ones remaining unchanged.

φ	$[A_7]\varphi$	$[A_8]\varphi$
11	1**0**	**0**1
10	**10**	10
01	01	0**1**
00	**1**0	**01**

Both $[A_7]$ and $[A_8]$ are *bivalence*-forming, or *normalization* operators: they reintroduce bivalence by restricting the output values in different ways, such that the resulting logical values are either 10 or 01. That is, every true formula is thereby not false and conversely. The aforementioned case of Boolean negation correspond to a single negative operator, that is:

$$\neg_1\varphi = [N_{15}]\varphi : t \mapsto \bar{t} \otimes \bar{t} \mapsto t \otimes f \mapsto \bar{f} \otimes \bar{f} \mapsto f.$$

At the same time, the structuration of such unary operators is such that it helps to see to what extent Morganian negation is not a 'pure' negation. Rather, it is case of 'mixed' operator conflating both affirmative and operators into mappings of the form $x_i \mapsto \overline{\overline{x_j}} = x_i \mapsto x_j$. The corresponding process is a *fusion* of the partial operators of affirmation and negation, thus resulting in 'affirmed negations' $[AN]$ or, equivalently. 'negated affirmations' $[NA]$:

$$\neg_2\varphi = [NA_{15}]\varphi = [AN_{15}]\varphi : t \mapsto f \otimes \bar{t} \mapsto \bar{f} \otimes f \mapsto t \otimes \bar{f} \mapsto \bar{t}.^6$$

[6]Fusion of partial operators differs both from their product $\otimes$ and the following operation of composition or iteration, $\circ$. It could be also shown that two other kinds of *redundancy*-making operators are equivalent with each other in $\mathbf{AR}_{4[O_i]}$, namely: $[NN]\varphi = [AA]\varphi$. The proof of such equivalences can be established as follows:

It turns out that antilogics and counterlogics are may be constructed by means of the unary operators of *Boolean* negation $\neg_1$ and *Morganian* negation $\neg_2$, following the definitions given in [13] and leading to the following truth-tables:

φ	$\neg_1\varphi$	$\neg_2\varphi$
11	00	11
10	10	01
01	01	10
00	11	00

According to this, Boolean negation $\neg_1$ turns logics L into *antilogics* $\tilde{L}$ whenever they turn true (or false) formulas into untrue (or unfalse) ones; and Morganian negation $\neg_2$ turn logics L into *counterlogics* $\tilde{L}$ whenever they turn true (or false) formulas into false (or true) ones. Antilogics correspond to situations in which a set of formulas belonging to L do not belong to another language $\bar{L}$, and this may be obtained by more than negative operator –not only $[N_{15}] = \neg_1$, but also every negative operator including the clauses of $[N_1]$ and $[N_2]$: $t \mapsto \bar{t} \otimes \bar{t} \mapsto t$. In the same vein, counterlogics correspond to situations in which the negations of a set of formulas belonging to L do belong to another language $\tilde{L}$, and this may be obtained by more than mixed operator –not only $[AN_{15}] = \neg_2$, but also every negative operator including the clauses of $[AN_1]$ and $[AN_2]$: $t \mapsto f \otimes f \mapsto t$.

Furthermore, it can be shown by now how the equations established in [2] may be validated or not according to the kind of partial operator selected in $\mathbf{AR}_{4[O_i]}$. The expressions 'antilogic of antilogic' and 'counterlogic of antilogic' correspond to cases of iteration or *composition* $\circ$, which are to be clearly distinguished from those of product $\otimes$ and mixed operators. Whilst the difference between product and composition can be easily shown by induction upon truth-tables,[7] it also helps to see

Proof.
$[NA]\varphi : x_i \mapsto [A]\overline{x_i} = x_i \mapsto \overline{\overline{x_j}} = x_i \mapsto x_j$.
$[AN]\varphi : x_i \mapsto [N]\overline{x_j} = x_i \mapsto \overline{\overline{x_j}} = x_i \mapsto x_j$.
Therefore $[AN]\varphi = [NA]\varphi$.
$[AA]\varphi : x_i \mapsto [A]\overline{x_j} = x_i \mapsto \overline{\overline{x_i}} = x_i \mapsto x_i$.
$[NN]\varphi : x_i \mapsto [N]\overline{x_i} = x_i \mapsto \overline{\overline{x_i}} = x_i \mapsto x_i$.
Therefore $[AA]\varphi = [NN]\varphi$. $\square$

[7] Let $[A_3]$ and $[A_4]$ be two such partial operations. Then the following truth-tables

that the following equations hold only when the corresponding operators proceed by iteration of specific operators –Boolean negation as an antilogic-forming operator and Morganian negation as a counterlogic-forming operator, once again.

$\bar{\bar{L}} = L$, that is, $[N_{15}][N_{15}]\varphi = \varphi$

$\tilde{\tilde{L}} = L$, that is, $[AN_{15}][AN_{15}]\varphi = \varphi$

$\bar{\tilde{L}} = \tilde{\bar{L}}$, that is, $[AN_{15}][N_{15}]\varphi = [N_{15}][AN_{15}]\varphi$.

Again, it must be recalled that all of these equations fail whenever antilogicality and counterlogicality are rephrased into $\mathbf{AR}_{4[O_i]}$ by partial operators which satisfy lesser semantic constraints whilst behaving as proper contradictory- and contrary-forming operators. This means that antilogic does not go on par with contradictoriness and counterlogic does not go on a par with contrariness –they are so only in a bivalent frame, where the unique negative operator is both Boolean and Morganian.

Coming back to the central section of the present issue, quasi-truth, it has been shown in [13] that the affirmative operators $[A_7]\varphi$ and $[A_8]$ are plausible candidates for being four-valued counterparts of the modalities of necessity and possibility in S5. Letting τ be a translation function from S5 to $\mathbf{AR}_{4[O_i]}$ and including a redundant-forming operator $[AA_{15}] = [NN_{15}]$ such that

$[AA_{15}]\varphi = [NN_{15}]\varphi : t \mapsto t \otimes \bar{t} \mapsto \bar{t} \otimes f \mapsto f \otimes \bar{f} \mapsto \bar{f}$.

It follows from this that

$\tau(\varphi, S5) = [AA_{15}]\varphi = [NN_{15}]\varphi$;

show both that their product differs from their composition and that, unlike product, composition is not a symmetrical operation.

φ	$[A_3]\varphi$	$[A_4]\varphi$	$[A_3]\varphi \otimes [A_4]\varphi$	$[A_3]\varphi \circ [A_4]\varphi$	$[A_4]\varphi \circ [A_3]\varphi$
11	11	11	11	11	11
10	10	10	10	10	10
01	01	01	01	01	01
00	01	10	11	10	01

$\tau(\Box\varphi, S5) = [A_8]\varphi$;
$\tau(\Diamond\varphi, S5) = [A_7]\varphi$.

We are going to use the two many-valued translations of necessity and possibility in the following, in order to propose a many-valued counterpart of *quasi-truth* in $\mathbf{AR}_{4[O_i]}$. On the other hand, it has been claimed in [5] that there is a connection between the concepts of quasi-truth and *contingency*, ∇. According to the author ([6],176),

> non-mathematical justifications are not able to lead to necessary but, rather, only to contingent truths. If there does not exist any demonstration about the truth of a proposition, then there is no certainty. Therefore, the proposition is not entitled to be acknowledged as true necessarily.

In other words, quasi-true formulas are those for which there is no conclusive evidence and that remain possibly false without being so determinately ([6], 180):

> Logics of justification – on its two approaches – can be used in order to define and think about the concept of quasi-truth. This was proposed by Newton da Costa in (1986) because, as a matter of fact, whenever we stand outside mathematics and logic we cannot talk exactly in terms of necessary truth, but only in terms of contingent truth, that is, quasi-truth.

Our main idea is to render da Costa & Bueno & Souza's insightful idea of quasi-truth as *partial structures* in semantic terms of quasi-truth as a *partial operator*, whereas some affirmative operators $[A_i]$ proceed as normalization-forming operators by restoring normal structures through partial ones. Assuming that quasi-truth proceeds as a contingency operator, and given our preceding translations of S5-modal necessity and possibility into $\mathbf{AR}_{4[O_i]}$, let us characterize quasi-truth QT as a conjunction of possibility and unnecessity.

Quasi-truth (as contingency)
$\nabla\varphi \Leftrightarrow \Diamond\varphi \wedge \neg\Box\varphi$
$\tau(QT(\varphi)) = [A_7]\varphi \wedge \neg_1[A_8]\varphi$.[8]

[8]Only Boolean negation $\neg_1$ has a wide scope in $\mathbf{AR}_{4[O_i]}$, but note that the above translation of negated possibility would result in the same truth-table had the corresponding operator of negation been the Morganian one $\neg_2$ –due to the bivalent behavior of QT. Moreover, the logical constants of $\mathbf{AR}_{4[O_i]}$ have not been defined

φ	$[A_7]\varphi$	$[A_8]\varphi$	$\neg_1[A_8]\varphi$	$QT\varphi$
11	10	01	10	10
10	10	10	01	01
01	01	01	10	01
00	10	01	10	10

The above matrix accounts for quasi-truth as being false with every formula whose logical value is determinately true or determinately false –i.e., $\mathbf{A}(QT\varphi)) = 01$ whenever $\mathbf{A}(\varphi) \in \{10, 01\}$.
Such an operator may also be seen as a proper translation by satisfying the main negative features of quasi-truth, namely:

(i) $\not\models QT\varphi \rightarrow \varphi$
(ii) $QT\varphi, QT\neg\varphi \not\models \psi$
(iii) $QT\varphi \not\models \neg QT\neg\varphi$.[9]

Our final consideration will consist in combining the previous two issues of the paper, opposition and quasi-truth, in order to pave the way to a third new topic: *quasi-oppositions*. This will answer to the question about whether there could be further non-standard relations of opposition in a non-bivalent frame like V_4.

3 Quasi-oppositions

Following [15], we assume that consequence and opposition can be treated either as relations $R(x, y)$ or as operators $f(x) = y$ (without any specification about the nature of the objects x and y). Consequence $Cn(\Gamma, \varphi)$

thus far, given that these are useless for the present purpose. However, contingency requires some words on conjunction since the latter makes part of its definition. So let $max(x, y)$ and $min(x, y)$ be the functions selecting the greater and lesser value among x and y, respectively, given that $1 > 0$. Then:

$$\mathbf{A}(\varphi \wedge \psi) = \langle min(\mathbf{a}_1(\varphi), \mathbf{a}_1(\psi)), max(\mathbf{a}_2(\varphi), \mathbf{a}_2(\psi))\rangle.$$

See [12,13,15] for more information about these 4-valued logical constants.

[9] The translations of the formulas (i)-(iv) into $\mathbf{AR}_{4[O_i]}$ and their corresponding counter-models are the following, given the rules established in [14] and our previous definition of QT:
$\tau(i)$ $\not\models QT\varphi \rightarrow [AA_{15}]\varphi$ (counter-model: $\mathbf{A}(\varphi) = 00$.)
$\tau(ii)$ $QT\varphi, QT\neg_2\varphi \not\models \psi$ (counter-model: $\mathbf{A}(\varphi) = 11$.)
$\tau(iii)$ $QT\varphi \not\models \neg_1 QT\neg_2\varphi$ (counter=model: $\mathbf{A}(\varphi) = 11$.)

has been studied since Tarski though several features like monotonicity, closure or structurality; and it has also be viewed as a possible operator mapping from given sets to close sets. Opposition $Op(\varphi, \psi)$ is traditionally considered as a relation between truth-values, and it has also been turned into an operator $op(\varphi) = \psi$ in the above reference. Given that logical oppositions are set of truth- and falsity-conditions between 'opposed' terms, truth-values constitute an essential feature in order to make sense of them. In the present context of a 4-valued domain, our main concern will be something like this: what sort of opposition is there between one formula which is neither-true-nor-false and another one which is both-true-and-false, for example?

One simple way to make an end to this discussion until its very opening is by applying the rationale urged by Roman Suszko, thereby rejecting the logical relevance of many-valuedness and reducing it to only two possible values: *designated*, or *not designated*. Thus, formulas are said 'designated' whenever they include the value of truth; they are 'not designated', otherwise. There are at least two ways not to follow this path, otherwise. Firstly, philosophical arguments –including those about quasi-truth, gave some reason to develop a set of many-valued inferences beyond Suszko's strictly bivalent policy. Following this stance introduced by Malinowski [11] and extended by Frankowski [7], there may be more than one way to characterize semantic consequence (or 'entailment') beyond the Tarskian classical pattern of truth-preservation. Here is a remainder of the four ways of dealing with consequence in a many-valued framework:

(Cn_t) $\quad \Gamma \models_t \varphi$ iff $\forall v[(\forall\psi \in \Gamma : v(\psi) \in \mathcal{D}^+) \Rightarrow v(\varphi) \in \mathcal{D}^+]$

(Cn_f) $\quad \Gamma \models_f \varphi$ iff $\forall v[(\forall\psi \in \Gamma : v(\psi) \notin \mathcal{D}^-) \Rightarrow v(\varphi) \notin \mathcal{D}^-]$

(Cn_q) $\quad \Gamma \models_q \varphi$ iff $\forall v[(\forall\psi \in \Gamma : v(\psi) \notin \mathcal{D}^-) \Rightarrow v(\varphi) \in \mathcal{D}^+]$

(Cn_p) $\quad \Gamma \models_p \varphi$ iff $\forall v[(\forall\psi \notin \Gamma : v(\psi) \notin \mathcal{D}^-) \Rightarrow v(\varphi) \in \mathcal{D}^+]$

In addition to the Tarskian pattern (Cn_t), the other three extensions depict semantic consequence as either a relation of non-falsity presentation (Cn_f), or a derivation of truth from non-refuted premises (Cn_q), or a derivation or mere plausibility from truth (Cn_p).

Following the developments around 4-valued inference by Blasio & Marcos & Wansing [4], three central issues will be approached in this last section: (a) What does truth and falsity mean into such a 4-valued frame? (b) How to systematize the kind of semantic consequence en-

dorsed by Malinowski's line? (c) How to express the logical difference between the relations of consequence and opposition into one and the same framework?
With respect to (a), our 4-valued framework is such that the two main sets of logical values $\mathcal{D}^+$ and $\mathcal{D}^-$ will receive a special interpretation. For although these are generally taken to be exclusive from each other, the domain of values V_4 motivates another treatment. For let $\mathcal{D}^+ = \{11, 10\}$ be the subset of *designated* values that are cases of truth, and $\mathcal{D}^- = \{11, 01\}$ the subset of *antidesignated* values that are cases of falsehood. Then the glutty value 11 is both designated and antidesignated whereas the gappy value 00 is none, which entails that

$$\mathcal{D}^+ \cap \mathcal{D}^- \neq \varnothing$$
$$\mathcal{D}^+ \cup \mathcal{D}^- \neq \wp(F)$$

This means that $\mathcal{D}^-$ is not the mere complementary of $\mathcal{D}^+$, due to the overlapping relation of truth and falsity in V_4.
With respect to (b), one can make abstraction from the intuitive meaning of truth-values and conceive an exhaustive set of relations between designated and anti-designated sets. The reason why there are four kinds of entailment can be explained in a combinatorial way, given that it relies upon two clauses: belonging to the set of true formulas, and not belonging to the set of false formulas. This results in a set of $2^2 = 4$ possibles clauses for entailment, and we are going now to see how to extend this set to further semantic clauses. Starting from an initial set of two sets of formulas, i.e. designated and anti-designated, one can conceive of further relations between formulas and whose clauses of satisfaction do not consist in tracking truth whilst avoiding falsehood. Such is precisely the case with opposition, insofar as the latter essentially consists in tracking falsehood for a given formula whenever its 'opposed' term is true.
By thus introducing the additional two clauses of belonging to the set of false formulas and not belonging to the set of true formulas, it results in a set of $2^4 = 16$ kinds of relations. Letting $\mathcal{O}$ be a general meta-operator mapping between sets or their complementaries, two main interpretations of $\mathcal{O}$ will be naturally of interest in the following: consequence Cn, and opposition Op. Here is an exhaustive list of possible relations between subsets of values $\mathcal{D}^i = \{\mathcal{D}^+, \mathcal{D}^-\} \in V_4$:

- from D^+ onto D^+

(*i*) $v(\varphi) \in D^+ \Rightarrow v(\psi) \in D^+$
(*ii*) $v(\varphi) \notin D^+ \Rightarrow v(\psi) \notin D^+$
(*iii*) $v(\varphi) \in D^+ \Rightarrow v(\psi) \notin D^+$
(*iv*) $v(\varphi) \notin D^+ \Rightarrow v(\psi) \in D^+$

- from D^+ onto D^-

  (*v*) $v(\varphi) \in D^+ \Rightarrow v(\psi) \in D^-$
  (*vi*) $v(\varphi) \notin D^+ \Rightarrow v(\psi) \notin D^-$
  (*vii*) $v(\varphi) \in D^+ \Rightarrow v(\psi) \notin D^-$
  (*viii*) $v(\varphi) \notin D^+ \Rightarrow v(\psi) \in D^-$

- from D^- onto D^+

  (*ix*) $v(\varphi) \in D^- \Rightarrow v(\psi) \in D^+$
  (*x*) $v(\varphi) \notin D^- \Rightarrow v(\psi) \notin D^+$
  (*xi*) $v(\varphi) \in D^- \Rightarrow v(\psi) \notin D^+$
  (*xii*) $v(\varphi) \notin D^- \Rightarrow v(\psi) \in D^+$

- from D^- onto D^-

  (*xiii*) $v(\varphi) \in D^- \Rightarrow v(\psi) \in D^-$
  (*xiv*) $v(\varphi) \notin D^- \Rightarrow v(\psi) \notin D^-$
  (*xv*) $v(\varphi) \in D^- \Rightarrow v(\psi) \notin D^-$
  (*xvi*) $v(\varphi) \notin D^- \Rightarrow v(\psi) \in D^-$

With respect to (c), let us recall that the framework assumed in [2] was bivalent. This gave rise to a standard view of the square of opposition, in which whatever is not true is false and conversely. That is, in terms of structured values:

$\mathbf{a}_1(\varphi) = 1 \Rightarrow \mathbf{a}_2(\varphi) = 0$ and $\mathbf{a}_1(\varphi) = 0 \Rightarrow \mathbf{a}_2(\varphi) = 1$.

Such a normal or complete square may be depicted as follows, thereby fulfilling the clauses of consistency and completeness.

$$\begin{tikzcd}
\mathbf{a}_1(\varphi)=0 \arrow[r, dash, "ct"] \arrow[d, "sb"] \arrow[dr, dash, "cd"] & \mathbf{a}_2(\varphi)=0 \arrow[d, "sb"] \arrow[dl, dash, "cd"'] \\
\mathbf{a}_2(\varphi)=1 \arrow[r, dash, "sct"] & \mathbf{a}_1(\varphi)=1
\end{tikzcd}$$

The situation is sensibly different into a 'non-standard square', that is, a non-bivalent set of relations where the aforementioned clauses are not followed:

$\mathbf{a}_1(\varphi) = 1 \not\Rightarrow \mathbf{a}_2(\varphi) = 0$ or $\mathbf{a}_1(\varphi) = 0 \not\Rightarrow \mathbf{a}_2(\varphi) = 1$

So what should such a non-standard square look like? Given that the extension of logical values and their subsequent logical relations must complicate the resulting picture, one may begin answering to the above question by making a list of the possible relations of consequence and opposition. It appears that each of the four aforementioned relations of many-valued consequence corresponds to one case of the exhaustive list of the 16 $\mathcal{O}$-relations (i)-(xvi) . Thus,

Many-valued consequence

(Cn_t)	$\varphi \in D^+ \Rightarrow \psi \in D^+$	(i)
(Cn_f)	$\varphi \notin D^- \Rightarrow \psi \notin D^-$	(xvi)
(Cn_q)	$\varphi \in D^+ \Rightarrow \psi \notin D^-$	(vii)
(Cn_p)	$\varphi \notin D^- \Rightarrow \psi \in D^+$	(xii)

Bueno & Souza [5] depicted quasi-truth in terms of partial structures whose final conclusion is open, which means that the formula into consideration may be true without being definitely so through the justification process [4]. For this reason, the above three non-Tarskian characterizations of consequence Cn_f, Cn_q, Cn_p may be taken to be various sorts of *quasi-consequence*. Likewise, the introduction of untrue and unfalse sets with $\mathcal{D}^+$ and $\mathcal{D}^-$ also seems to be in position make sense of the coming *quasi-oppositions*.

Roughly speaking, each case of 'quasi'-X is a situation in which the assessed object (proposition, concept, logical system, or whatever) is not X whilst being possibly so. Let us take the case of contrariness. According to the standard definition, any two objects are contrary to each other if, and only if, they cannot be true together in such a way that the second is false whenever the first is true. In a case of of *quasi-contrariness*, however, the second term is merely not true (or untrue) whenever the first is true. Assuming that being almost or being still in position to be (true or false) affords an intuitive meaning of the 'quasi'-phrase, here is the list of quasi-oppositions Op_f, Op_q, Op_p that correspond to the remaining cases of non-consequence relations (or operators) $\mathcal{O}$.

Many-valued opposition

Contrariness

(Ct_t)	$\varphi \in D^+ \Rightarrow \psi \in D^-$	(v)
(Ct_f)	$\varphi \notin D^- \Rightarrow \psi \notin D^+$	(x)
(Ct_q)	$\varphi \in D^+ \Rightarrow \psi \notin D^+$	(iii)
(Ct_p)	$\varphi \notin D^- \Rightarrow \psi \in D^-$	(xvi)

Contradictoriness

(Cd_t)	$\varphi \in D^+ \Rightarrow \psi \in D^-$ and $\varphi \in D^- \Rightarrow \psi \in D^+$	$(v) \otimes (ix)$
(Cd_f)	$\varphi \notin D^- \Rightarrow \psi \notin D^+$ and $\varphi \notin D^+ \Rightarrow \psi \notin D^-$	$(x) \otimes (vi)$
(Cd_q)	$\varphi \in D^+ \Rightarrow \psi \notin D^+$ and $\varphi \notin D^+ \Rightarrow \psi \in D^+$	$(iii) \otimes (iv)$
(Cd_p)	$\varphi \notin D^- \Rightarrow \psi \in D^-$ and $\varphi \in D^- \Rightarrow \psi \notin D^-$	$(xvi) \otimes (xv)$

Subcontrariness

(Sct_t)	$\varphi \in D^- \Rightarrow \psi \in D^+$	(ix)
(Sct_f)	$\varphi \notin D^+ \Rightarrow \psi \notin D^-$	(vi)
(Sct_q)	$\varphi \notin D^+ \Rightarrow \psi \in D^+$	(iv)
(Sct_p)	$\varphi \in D^- \Rightarrow \psi \notin D^-$	(xv)

Subalternation

(Sb_t)	$\varphi \in D^+ \Rightarrow \psi \in D^+$	(i)
(Sb_f)	$\varphi \notin D^- \Rightarrow \psi \notin D^-$	(xvi)
(Sb_q)	$\varphi \in D^+ \Rightarrow \psi \notin D^-$	(vii)
(Sb_p)	$\varphi \notin D^- \Rightarrow \psi \in D^+$	(xii)

It clearly appears that subalternation and consequence are one and the same logical relation (or operator), at least when these resort to the same non-standard kind Cn_x and Sb_x. This amounts to say that every such $\mathcal{O}$-mapping is a single case of opposition, reminding that subalternation can be parsed as the iteration of two simple opposite-forming operators

[14].

Two future investigations might be pursued with respect to this new concept of quasi-opposition, provided that the latter turn out to be a relevant issue. One first work would have to do with the philosophical applications to it into informal contexts use, just as q-entailment and p-entailment had been interpreted by their authors in terms of plausibility and degrees of truth [7,11]. Another work would be about a calculus of quasi-operators, thus extending the work devoted to consequence-forming operators [15].
Thanks already to Edelcio for opening the way towards these potential tools of logic.

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