Supervaluationism, Modal Logic, and Weakly Classical Logic*

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Abstract

A consequence relation is strongly classical if it has all the theorems and entailments of classical logic as well as the usual meta-rules (such as Conditional Proof). A consequence relation is weakly classical if it has all the theorems and entailments of classical logic but lacks the usual meta-rules. The most familiar example of a weakly classical consequence relation comes from a simple supervaluational approach to modelling vague language. This approach is formally equivalent to an account of logical consequence according to which $\boldsymbol{\alpha_1, \ldots, \alpha_n}$ entails $\boldsymbol{\beta}$ just in case $\Box\boldsymbol{\alpha_1}, \ldots, \Box\boldsymbol{\alpha_n}$ entails $\Box\boldsymbol{\beta}$ in the modal logic S5. This raises a natural question: If we start with a different underlying modal logic, can we generate a strongly classical logic? This paper explores this question. In particular, it discusses four related technical issues: (1) Which base modal logics generate strongly classical logics and which generate weakly classical logics? (2) Which base logics generate themselves? (3) How can we directly characterize the logic generated from a given base logic? (4) Given a logic that can be generated, which base logics generate it? The answers to these questions have philosophical interest. They can help us to determine whether there is a plausible supervaluational approach to modelling vague language that yields the usual meta-rules. They can also help us to determine the feasibility of other philosophical projects that rely on an analogous formalism, such as the project of defining logical consequence in terms of the preservation of an epistemic status.

Keywords: supervaluationism, modal logic, strongly classical logic, weakly classical logic, epistemic definitions of validity

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1 Introduction

A consequence relation is called "broadly classical" if it has all of the classical theorems and entailments. A consequence relation is called "strongly classical" if, in addition, it obeys the usual meta-rules (Conditional Proof, Reasoning by Cases, Reductio ad Absurdum, and Contraposition).[1] Finally, a consequence relation is called "weakly classical" if it is broadly classical but not strongly classical.[2]

The most familiar example of a weakly classical consequence relation comes from supervaluational approaches to modelling vague language.[3] For simplicity, we can restrict attention to propositional logic with the addition of a single one-place operator "det". On a simple supervaluational approach, a model for this language consists of a set of assignments of truth values to atomic sentences. Given an assignment in a model, we can assign truth values to non-atomic sentences using the standard classical clauses for the logical connectives. A sentence of the form "det α" is true on an assignment in a model just in case α is true on every assignment in the model. A sentence is supertrue in a model just in case it is true on every assignment in the model. We define the consequence relation, $\vdash$, so that for a set of sentences Γ and a sentence α, $\Gamma \vdash \alpha$ just in case in every model, if every member of Γ is supertrue, so is α.[4] The idea behind this account of consequence is that consequence is to be understood in terms of truth-preservation, and truth is identified with supertruth. The det operator is the way of capturing truth in the object language.

Given these definitions, it is easy to show that the resulting consequence relation is broadly classical: Since the classical theorems and entailments preserve truth on each assignment, they preserve supertruth. It is also easy to show that it is weakly classical: $p \vdash \det p$ but $\nvdash p \rightarrow \det p$, so Conditional Proof fails. Similarly, $p \vdash \det p \vee \det \neg p$ and $\neg p \vdash \det p \vee \det \neg p$ but $p \vee \neg p \nvdash \det p \vee \det \neg p$, so Reasoning by Cases fails. Moreover, $p \wedge \neg \det p \vdash \bot$ but $\nvdash \neg(p \wedge \neg \det p)$, so Reductio ad Absurdum fails. Finally, $p \vdash \det p$ but $\neg \det p \nvdash \neg p$, so Contraposition fails.[5,6]

This simple supervaluational approach is equivalent to an approach based on modal logic.[7] We can think of each assignment in a supervaluational model as a world.[8] The accessibility relation between worlds is the universal relation. The det operator plays the role of a modal operator $\Box$. $\Box\alpha$ is true at a world

[1] I'll provide a more elegant characterization below.

[2] I borrow this terminology from [10].

[3] See, for example, [11], though Fine is no longer a supervaluationist.

[4] This is what [41, p. 148], calls "global validity".

[5] These examples appear in [41, pp. 151–2], as part of an argument that "supervaluations invalidate our natural mode of deductive thinking." [23, pp. 51–3] argues against supervaluationism on the ground that Conditional Proof and Reductio ad Absurdum fail. [11, p. 290], notes that Conditional Proof fails for supervaluationism, but doesn't view that as a problem.

[6] Let Replacement of Equivalents be the following: If $\alpha \vdash \beta$ and $\beta \vdash \alpha$ then $\delta \vdash \delta'$, where δ' is the result of replacing an occurrence of α in δ with β. This fails, too, since $p \vdash \det p$ and $\det p \vdash p$ but $p \rightarrow p \nvdash p \rightarrow \det p$.

[7] See [41, pp. 149–50].

[8] It may be more philosophically natural for the "worlds" in such a model not to be assignments, but to be packages of semantic rules or practices that induce such assignments (or, perhaps, package-world pairs).

just in case α is true at all accessible worlds, that is, at all worlds. A sentence α is a consequence of a set of sentences Γ just in case in all modal models, if $\Box\gamma$ is true at a world for every $\gamma \in \Gamma$, then $\Box\alpha$ is true at the world, too.

Moving from a model-theoretic to a more proof-theoretic approach, let $\vdash_{\mathrm{S5}}$ be the consequence relation for the modal logic S5. Then the weakly classical modal consequence relation characterized above can be defined as follows: $\Gamma \vdash \alpha$ just in case $\Box\Gamma \vdash_{\mathrm{S5}} \Box\alpha$, where $\Box\Gamma$ is the set $\{\Box\gamma \mid \gamma \in \Gamma\}$. This consequence relation can be axiomatized by starting with an axiomatization for S5 and adding the rule of inference $p/\Box p$.[9]

This raises a question: What happens if we start with a modal logic other than S5?[10] That is, suppose we have an arbitrary strongly classical normal modal logic $\vdash_{\mathrm{base}}$ and we generate a consequence relation as follows: $\Gamma \vdash \alpha$ just in case $\Box\Gamma \vdash_{\mathrm{base}} \Box\alpha$.[11] (In the model theory, the idea is that logical consequence is defined, not in terms of what's true at all worlds, but in terms of what's true at all accessible worlds, where the accessibility relation need not be universal.) When do we generate a weakly classical consequence relation and when do we generate a strongly classical consequence relation? When do we generate the very same consequence relation that we started with? Is there a simple way to directly characterize $\vdash$? Given a consequence relation that can be generated, which strongly classical normal modal logics generate it? The purpose of this paper is to answer these questions.

This paper focuses on these technical questions. But it is worth noting that these questions have philosophical significance. For instance, sticking with the topic of vagueness, it is natural to think that it is a serious problem facing the supervaluational approach sketched above that it yields a weakly classical consequence relation. Logical consequence is plausibly closely tied to reasoning, and in our reasoning, we freely make use of Conditional Proof, Reasoning by Cases, Reductio ad Absurdum, and the like. So having to give up these meta-rules would seem to be a serious cost of the view.[12] If we could modify the standard supervaluational apparatus by adding an accessibility relation in such a way to enable us to retain the meta-rules, that would neatly sidestep the problem.[13]

There are, of course, other potential responses to the difficulty. For instance, a different response would be to make use of an alternative account of logical consequence, such as defining consequence so that $\Gamma \vdash \alpha$ just in case for all

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[9] See [8] for this result, which is proved in a different way below.

[10] See [41, pp. 158–9] for discussion of this suggestion.

[11] This is essentially equivalent to what [7, p. 302] calls "regional validity".

[12] Indeed, if one is an inferentialist about the logical constants, it is tempting to say that Conditional Proof, Reasoning by Cases, and Reductio ad Absurdum constitute or help to determine the meaning of the conditional, disjunction, and negation, respectively.

[13] There is also a second motivation for modifying the standard supervaluational approach. One might worry that the simple supervaluational approach presented above does not properly handle higher-order vagueness, since it rules out the possibility that it can be indeterminate whether something is determinately the case. Perhaps if we start with a base logic other than S5, there is a supervaluational approach that will handle higher-order vagueness properly.

worlds in all models, if every member of Γ is true at the world, so is α.[14] This way of defining consequence would yield a strongly classical consequence relation. A worry facing this approach, however, is that it gives up on the idea of defining logical consequence in terms of the preservation of truth, at least when truth is identified with supertruth.[15]

A different response would be to try to motivate the acceptability of a weakly classical consequence relation, or even to try to claim that being forced to endorse a weakly classical logic is a feature rather than a bug.[16] On such a response, one would presumably replace Conditional Proof (and the other meta-rules) with other, weaker meta-rules. For instance, on one version of this approach, one might replace Conditional Proof with the following meta-rule: If $\Gamma \cup \{\alpha\} \vdash \beta$ then $\Gamma \vdash \det \alpha \to \beta$. (One would do something similar to the other familiar meta-rules).[17] One worry facing this proposal is that the resulting list of meta-rules is logically fairly weak. A second worry is that it requires us to sprinkle det operators throughout our reasoning, which seems awkward in practice. On a different – and perhaps more elegant – version of this approach, one might restrict Conditional Proof (and the other meta-rules) so that only classical logical inferences can be used within the relevant subproof.[18,19] A worry facing this approach is that it potentially makes reasoning much clunkier – one has to keep track of what rules one is and is not permitted to use within

[14] This is what [41, pp. 147–8] calls "local validity". See [37] and [2] for defenses of local validity, and [33] for a defense of the analogue of local validity for an account of vagueness that is formally similar to supervaluationism. See [17] for an argument for a kind of pluralism according to which both global and local validity are both legitimate notions of consequence. [37] discusses seven different (inequivalent) potential definitions of logical consequence for a supervaluational model theory – four versions of global validity, local validity, and two versions of collective validity. Varzi's list doesn't include any versions of regional validity, so there are plenty of options to choose from.

[15] See [41, p. 148] and [43] for articulations of this worry. But see [24] for a view on which logical consequence involves the preservation of truth, understood as disquotational truth.

[16] See [11, p. 290] and [18, pp. 178–9] for the claim that violations of the meta-rules are just to be expected, in part because det is a non-classical notion.

[17] This is the approach suggested by [18, pp. 179–80].

[18] See [25] for a proposal in this ballpark. More precisely, this paper suggests that the reasoning inside Conditional Proof should be restricted to "modes of inference that are known to preserve truth in each acceptable model" (p. 135). See [9] and [13] for a related suggestion: Each of these two papers suggests that the argument in [44] that supervaluationism is inconsistent with higher-order vagueness is mistaken because it applies the rule of inference $p/\det p$ within a subproof. Their diagnosis of why this is a problem is that this rule preserves truth, but permits inferring from an indeterminate premise to a false conclusion, and a conditional that can have an indeterminate antecedent and a false consequent is not a validity. One might think that this diagnosis suggests not restricting Conditional Proof but rather moving to an account of consequence according to which some premises entail a conclusion just in case if the premises are true the conclusion is guaranteed to be true and if the conclusion is false the conjunction of the premises is guaranteed to be false. However, [43, p. 526 n. 6] points out that this alternative definition of consequence still does not permit accepting an unrestricted version of Conditional Proof (and has other problems, besides). Perhaps a better alternative would be to move to a definition of consequence where truth takes wide scope: For instance, some premises entail a conclusion just in case the conditional that has the conjunction of premises as the antecedent and the conclusion as the consequent is guaranteed to be true.

[19] One might think that the problems for Conditional Proof only involve sentences containing the det operator, so we only need to rule out the use of inferences crucially involving det. However, [41, p. 152, p. 295-6 n. 25] points out that supervaluationists typically try to use their apparatus in providing an account of other parts of language, so the difficulty will be more general. [12, sec. 2] argues that if the relevant notion of consequence is not logical consequence but a more general type of consequence suitable for everyday reasoning, there will be failures of Conditional Proof even if the language does not contain any special vagueness-related operators. [43] provides additional potential examples of this phenomenon.

a subproof. It also seems in tension with a natural thought about consequence: If β really is a consequence of α, why can't one reason from α to β even within a subproof? And if one can do that, why can't one go ahead and use Conditional Proof (or Reasoning by Cases or Reductio ad Absurdum)?

There is doubtless much more that could be said about all of this. But the moral of the discussion is that it is a very natural question whether we can adjust the standard supervaluational semantics to yield a strongly classical logic. And the obvious approach – adding a (not necessarily universal) accessibility relation, taking truth = supertruth to be truth in all accessible worlds, and retaining the definition of consequence in terms of truth-preservation – is a natural one to consider.[20] Indeed, Williamson writes that this is a "natural line of thought from a standard supervaluationist perspective. If supervaluationists abandon it, they incur the suspicion that they are not serious about their identification of truth with supertruth."[21]

The technical questions considered here also arise in other areas. Consider epistemic logic. One might read "$\Box$" as an epistemic status such as "one is in a position to know that" or "one is rationally committed to its being the case that". Some philosophers have proposed characterizing validity, or a related notion, in terms of the preservation of an epistemic status.[22] It is natural to investigate the features of the resulting consequence relation, including whether it is strongly or weakly classical and how it can be axiomatized. (And similarly for other possible interpretations of "$\Box$".)

This paper will proceed as follows. In the next section, I present the relevant background on consequence relations, weakly and strongly classical logic, and modal logic. In section 3, I provide results on when specific rules are admissible in a modal logic that will be relied upon later in the paper. In section 4, I consider the question of when a consequence relation generated as above from a finitary strongly classical normal modal logic is itself strongly classical. I provide necessary and sufficient conditions for generating a strongly classical consequence relation. This result tells us that many familiar modal logics, such as K, KD, K5, and KG, generate strongly classical consequence relations. I also show that many familiar modal logics, such as KT, K4, KB, and GL generate weakly classical consequence relations. Indeed, I show that the only consistent finitary strongly classical normal extension of KT that generates a strongly classical consequence relation is Triv. I also identify the weakest finitary strongly classical normal extensions of K4 and KB (among many other modal logics) that generate a strongly classical consequence relation. In section

[20] What is the accessibility relation supposed to represent? On one picture, each world in a model does two things: (i) it specifies a privileged assignment and (ii) it specifies a set of admissible assignments with the privileged assignment located in the middle (in some sense) of the admissible assignments. The accessibility relation links each world to those worlds that have one of the admissible assignments as their privileged assignment. Given this kind of picture, the accessibility relation will presumably be reflexive. Thanks to Stephan Krämer for suggesting this approach.

[21] See [43, p. 525].

[22] This is common among inferentialists in the Dummett-Prawitz tradition. For instance, [35, p. 950] characterizes validity in terms of the preservation of knowledge. [27, p. 73] characterizes "legitimate inference" as the kind of inference that preserves the property *having conclusive evidence for*. [4, p. 168] characterizes "committive inference" as the kind of inference that preserves the property *being committed to*.

5, I consider the question of when a finitary strongly classical normal modal logic generates itself. I provide necessary and sufficient conditions for a finitary strongly classical normal modal logic to be self-generating. I identify the weakest finitary strongly classical normal extensions of KT, K4, and KB (among many other modal logics) that are self-generating. I also examine the question of when a finitary strongly classical normal modal logic generates a logic with the very same theorems, but perhaps has additional derivable rules. In section 6, I turn to the question of how to directly characterize modal logics generated from finitary strongly classical normal modal logics. I show that if we have a finitary strongly classical normal modal logic that extends KB or KT, we can provide a straightforward characterization of the modal logic it generates. I also show how to directly characterize the modal logics generated from many other modal logics. In section 7, I consider the other direction: Given a broadly classical normal modal logic, which strongly classical normal modal logics generate it? Section 8 concerns the question of generating modal logics from finitary weakly classical normal modal logics. In particular, for many familiar strongly classical normal modal logics, I determine what happens when we start with that modal logic, generate a modal logic from it, generate a modal logic from it, and iterate until we hit a fixed point. In section 9, I briefly discuss two ways to generalize the discussion. First, I discuss the generation of modal logics in a multiple-conclusion setting. Second, I discuss alternative ways of generating modal logics from modal logics. Finally, in section 10, I conclude by briefly discussing the philosophical significance of the results.

2 Background

2.1 Syntax

In this paper, we'll work with a specific language for propositional modal logic. The language has an infinite stock of atomic sentences, the familiar logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, a one-place operator $\Box$, and the parentheses, (and). We use p, q, r, and s to stand for distinct atomic sentences. We use α, β, γ, δ, ϕ, and ψ, sometimes with subscripts, to stand for arbitrary sentences. We use Γ and Δ to stand for arbitrary sets of sentences.

The formation rules for non-atomic sentences are the usual ones: If α and β are sentences, so are $\neg\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$, and $\Box\alpha$. Nothing else is a sentence. We omit parentheses in sentences when there is no danger of confusion. In particular, $\wedge$ has a higher precedence than $\vee$ which has a higher precedence than $\rightarrow$. So, for instance, $\alpha \vee \beta \wedge \gamma \rightarrow \delta$ abbreviates $((\alpha \vee (\beta \wedge \gamma)) \rightarrow \delta)$. We use $\top$ as an abbreviation for $p \rightarrow p$ and $\bot$ as an abbreviation for $\neg\top$. We use $\Box\Gamma$ as an abbreviation for $\{\Box\gamma \mid \gamma \in \Gamma\}$.

A *substitution* σ is a function from the atomic sentences to sentences. We write $\sigma\alpha$ to stand for the result of uniformly substituting every atomic sentence in α with the result of applying σ to that atomic sentence. We write $\sigma\Gamma$ to stand for the set $\{\sigma\gamma \mid \gamma \in \Gamma\}$.

2.2 Consequence Relations

A *consequence relation*, $\vdash$, is a relation holding between sets of sentences and individual sentences such that the following conditions obtain for every α, β, Γ, Δ, and σ:

Identity. $\{\alpha\} \vdash \alpha$
Weakening. If $\Gamma \vdash \alpha$ then $\Gamma \cup \Delta \vdash \alpha$
Cut. If $\Gamma \vdash \alpha$ and $\Delta \cup \{\alpha\} \vdash \beta$ then $\Gamma \cup \Delta \vdash \beta$
Uniform Substitution. If $\Gamma \vdash \alpha$ then $\sigma\Gamma \vdash \sigma\alpha$

A consequence relation is thus required to be single conclusion. It is also required to be structural in the sense of [22]. (In section 9, we'll briefly loosen these restrictions.) Notice that a consequence relation may be characterized proof-theoretically, model-theoretically, or in some other way. We use "logic" as a synonym for "consequence relation".

For ease of reading, we often drop the set brackets on the left-hand side of $\vdash$. We also write $\Gamma; \alpha$ as an abbreviation for $\Gamma \cup \{\alpha\}$.

A consequence relation $\vdash$ is *finitary* (i.e., compact) just in case if $\Gamma \vdash \alpha$ then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \alpha$.

Suppose $\vdash$ and $\vdash'$ are consequence relations. We say that $\vdash'$ *extends* $\vdash$ just in case for every Γ and α, if $\Gamma \vdash \alpha$ then $\Gamma \vdash' \alpha$. We say that a consequence relation $\vdash^*$ is *between* $\vdash$ and $\vdash'$ just in case $\vdash'$ extends $\vdash^*$ and $\vdash^*$ extends $\vdash$.

Let $\vdash$ be a consequence relation. We say that α is a *consequence* of Γ in $\vdash$ just in case $\Gamma \vdash \alpha$. Equivalently, we say that Γ *entails* α in $\vdash$. We say that α is a *theorem* of $\vdash$ just in case $\emptyset \vdash \alpha$. As usual, we write $\vdash \alpha$ for $\emptyset \vdash \alpha$.

A *rule* is an ordered pair containing a set of sentences and a sentence. For simplicity, we'll typically write the rule $\langle \Gamma, \alpha \rangle$ as Γ/α. We use θ to stand for a rule and Θ to stand for a set of rules. Given a consequence relation $\vdash$, the rule Γ/α is *derivable* in $\vdash$ just in case $\Gamma \vdash \alpha$. (By Uniform Substitution, it follows that $\sigma\Gamma/\sigma\alpha$ is also derivable in $\vdash$ for every substitution σ.) The rule Γ/α is *admissible* in $\vdash$ just in case for every substitution σ, if every member of $\sigma\Gamma$ is a theorem of $\vdash$, so is $\sigma\alpha$. In other words, an admissible rule does not yield any new theorems when added to a consequence relation.[23] A rule that is derivable in $\vdash$ is always admissible in $\vdash$, but not vice versa. For example, in the normal modal logic K, the rule $p/\Box p$ is admissible but not derivable. A consequence relation is *structurally complete* just in case every admissible rule is derivable.

A *meta-rule* is an ordered pair containing a set of rules and a rule. A consequence relation $\vdash$ *obeys* the meta-rule $\langle\{\langle\Gamma_1, \alpha_1\rangle, \ldots, \langle\Gamma_n, \alpha_n\rangle\}, \langle\Delta, \beta\rangle\rangle$ just in case for every substitution σ, if $\sigma\Gamma_1 \vdash \sigma\alpha_1$ and $\ldots$ and $\sigma\Gamma_n \vdash \sigma\alpha_n$, then $\sigma\Delta \vdash \sigma\beta$. (And similarly for the infinite case.) Equivalently, we say that the meta-rule *obtains* in $\vdash$. For reasons of familiarity, we'll typically write the meta-rule as: if $\Gamma_1 \vdash \alpha_1$ and $\ldots$ and $\Gamma_n \vdash \alpha_n$ then $\Delta \vdash \beta$.

For a given consequence relation, there will be many different proof-theoretic formal systems that can be used to characterize it. (Such formal systems include Hilbert-style axiomatizations, systems of natural deduction,

[23] The notion of an admissible rule is due to [21].

sequent calculi, and so forth.) We'll occasionally talk about the formal systems axiomatizing a given consequence relation. In this context, it will be important to distinguish between rules of inference and rules of proof. A *rule of inference* specifies that in the formal system, certain premises directly imply a certain conclusion, and similarly for applications of arbitrary substitutions. We again write a rule of inference as Γ/α. This rule of inference says that for every substitution σ, the formal system permits directly inferring from $\sigma\Gamma$ to $\sigma\alpha$. A *rule of proof* specifies that the formal system permits directly inferring from certain sentences taken to be theorems (that is, derivable from no assumptions) to a certain sentence also taken to be a theorem, and similarly for applications of arbitrary substitutions. We write a rule of proof as: If $\vdash \alpha_1$ and ... and $\vdash \alpha_n$ then $\vdash \beta$. This rule of proof says that for every substitution σ, if $\sigma\alpha_1, \ldots, \sigma\alpha_n$ are theorems, the formal system permits directly inferring from them to $\sigma\beta$ which is then a theorem, too.[24]

2.3 Weakly and Strongly Classical Consequence Relations

Given a consequence relation $\vdash$ in a language that contains at least the usual logical connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$) we say that $\vdash$ is *broadly classical* just in case it has all of the classical theorems and entailments. For a consequence relation to be broadly classical, it suffices that (i) all of the classical validities are theorems and (ii) the rule Modus Ponens ($p, p \rightarrow q/p$) is derivable. (By Uniform Substitution, it follows that $\alpha, \alpha \rightarrow \beta/\beta$ is derivable for any α and β.) A broadly classical consequence relation $\vdash$ is *strongly classical* just in case, in addition, it obeys the following meta-rule:

Conditional Proof. If $\Gamma; \alpha \vdash \beta$ then $\Gamma \vdash \alpha \rightarrow \beta$.

It is *weakly classical* just in case it is broadly classical but not strongly classical.[25,26]

In a language that contains only the usual logical vocabulary, if a consequence relation is broadly classical it will also be strongly classical. That is what the familiar proof of the Deduction Theorem tells us. But if there are additional operators in the language, such as a modal operator, there will be broadly classical logics that are not strongly classical.

In addition to Conditional Proof, there are other familiar meta-rules:

Reasoning by Cases. If both $\Gamma; \alpha \vdash \delta$ and $\Gamma; \beta \vdash \delta$ then $\Gamma; \alpha \vee \beta \vdash \delta$.
Reductio ad Absurdum. If $\Gamma; \alpha \vdash \bot$ then $\Gamma \vdash \neg\alpha$.
Contraposition. If $\Gamma; \alpha \vdash \beta$ then $\Gamma; \neg\beta \vdash \neg\alpha$.

[24] For discussions of the distinction between rules of inference and rules of proof, see [34, p. 130], [36, p. 135], and [1, p. 134]. I don't know who first explicitly drew this distinction, but it was already familiar in the 1940s. See, for example, [26, sec. 8].

[25] This is a more restrictive notion of weak classicality than appears in [28], which permits weakly classical logics to be non-structural.

[26] These definitions can be generalized to languages that contain a set of connectives that are expressively adequate in classical logic – we can treat any missing connectives as defined rather than primitive.

Section 1 characterized a strongly classical consequence relation as a broadly classical consequence relation that obeys all of these meta-rules. This is equivalent to the characterization in terms of Conditional Proof alone. In a broadly classical logic, these meta-rules turn out to be equivalent.

Proposition 2.1 (Folklore)**.** *If $\vdash$ is a broadly classical logic, then if $\vdash$ obeys any one of Conditional Proof, Reasoning by Cases, Reductio ad Absurdum, or Contraposition, then it obeys all of them.*[27,28]

Proof We show that Reasoning by Cases, Reductio ad Absurdum, and Contraposition are each equivalent to Conditional Proof.

Reasoning by Cases: Suppose $\Gamma; \alpha \vdash \delta$ and $\Gamma; \beta \vdash \delta$. By Conditional Proof, $\Gamma \vdash \alpha \to \delta$ and $\Gamma \vdash \beta \to \delta$. In a broadly classical logic, $\alpha \to \delta, \beta \to \delta \vdash \alpha \vee \beta \to \delta$. By Cut, $\Gamma \vdash \alpha \vee \beta \to \delta$. By Modus Ponens and Cut, $\Gamma; \alpha \vee \beta \vdash \delta$.

For the other direction, suppose $\Gamma; \alpha \vdash \beta$. In a broadly classical logic, $\beta \vdash \alpha \to \beta$. So by Cut, $\Gamma; \alpha \vdash \alpha \to \beta$. In a broadly classical logic, $\neg\alpha \vdash \alpha \to \beta$. So by Weakening, $\Gamma; \neg\alpha \vdash \alpha \to \beta$. By Reasoning by Cases, $\Gamma; \alpha \vee \neg\alpha \vdash \alpha \to \beta$. In a broadly classical logic, $\vdash \alpha \vee \neg\alpha$. So by Cut, $\Gamma \vdash \alpha \to \beta$.

Reductio ad Absurdum: Suppose $\Gamma; \alpha \vdash \bot$. By Conditional Proof, $\Gamma \vdash \alpha \to \bot$. In a broadly classical logic, $\alpha \to \bot \vdash \neg\alpha$. By Cut, $\Gamma \vdash \neg\alpha$.

For the other direction, suppose $\Gamma; \alpha \vdash \beta$. By Weakening, $\Gamma; \alpha, \neg\beta \vdash \beta$. By Identity and Weakening, $\Gamma; \alpha, \neg\beta \vdash \neg\beta$. In a broadly classical logic, $\beta, \neg\beta \vdash \bot$. By Cut, $\Gamma; \alpha, \neg\beta \vdash \bot$. In a broadly classical logic, $\alpha \wedge \neg\beta \vdash \alpha$ and $\alpha \wedge \neg\beta \vdash \neg\beta$. So by Cut, $\Gamma; \alpha \wedge \neg\beta \vdash \bot$. By Reductio ad Absurdum, $\Gamma \vdash \neg(\alpha \wedge \neg\beta)$. In a broadly classical logic, $\neg(\alpha \wedge \neg\beta) \vdash \alpha \to \beta$. By Cut, $\Gamma \vdash \alpha \to \beta$.

Contraposition: Suppose $\Gamma; \alpha \vdash \beta$. By Conditional Proof, $\Gamma \vdash \alpha \to \beta$. In broadly classical logic, $\alpha \to \beta \vdash \neg\beta \to \neg\alpha$. By Cut, $\Gamma \vdash \neg\beta \to \neg\alpha$. By Modus Ponens and Cut, $\Gamma; \neg\beta \vdash \neg\alpha$.

For the other direction, suppose $\Gamma; \alpha \vdash \beta$. By the same reasoning as for Reductio ad Absurdum, $\Gamma; \alpha \wedge \neg\beta \vdash \bot$. By Contraposition, $\Gamma; \neg\bot \vdash \neg(\alpha \wedge \neg\beta)$. In a broadly classical logic, $\vdash \neg\bot$ and $\neg(\alpha \wedge \neg\beta) \vdash \alpha \to \beta$. By Cut, $\Gamma \vdash \alpha \to \beta$. ☐

Indeed, we can show that if $\vdash$ is a broadly classical logic obeying Conditional Proof, it obeys every classically valid meta-rule – that is, every meta-rule that is valid in classical propositional logic with no additional vocabulary. The

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[27] See [32, pp. 601–2] for this result and [43, p. 526 n. 5] for one direction of it (showing that Conditional Proof is derivable from each of the other listed meta-rules).

[28] Let General Replacement of Equivalents be the generalization of Replacement of Equivalents (from footnote 6) that permits side formulas. That is: If $\Gamma; \alpha \vdash \beta$ and $\Gamma; \beta \vdash \alpha$ then $\Gamma; \delta \vdash \delta'$, where δ' is the result of replacing an occurrence of α in δ with β. We can also show that if $\vdash$ obeys General Replacement of Equivalents, then it obeys Conditional Proof: Suppose $\Gamma; \alpha \vdash \beta$. In a broadly classical logic, $\Gamma; \alpha \vdash \alpha$ and $\alpha, \beta \vdash \alpha \wedge \beta$. So by Cut, $\Gamma; \alpha \vdash \alpha \wedge \beta$. In a broadly classical logic, $\Gamma; \alpha \wedge \beta \vdash \alpha$. So by General Replacement of Equivalents, $\Gamma; \alpha \to \alpha \vdash \alpha \to \alpha \wedge \beta$. In a broadly classical logic, $\vdash \alpha \to \alpha$ and $\alpha \to \alpha \wedge \beta \vdash \alpha \to \beta$. So by Cut, $\Gamma \vdash \alpha \to \beta$. We don't necessarily get the other direction, or even Replacement of Equivalents. For a simple example, consider a consequence relation $\vdash$ in the language with the usual classical logical connectives as well as the operator O. Suppose $\vdash$ is characterized model-theoretically, where the usual connectives are treated in the usual way and where $O\alpha$ is true in a model just in case α contains an occurrence of q. $\vdash$ will be strongly classical. We'll have $q \to q \vdash p \to p$ and $p \to p \vdash q \to q$, but $O(q \to q) \nvdash O(p \to p)$.

proof of this fact relies on the well-known fact that classical propositional logic is structurally complete.[29]

Proposition 2.2. *Suppose there is a classically valid meta-rule of the following form: if $\Gamma_1 \vdash \alpha_1$ and ... and $\Gamma_n \vdash \alpha_n$ then $\Delta \vdash \beta$ (where each Γ_i is finite and Δ is finite). Then if $\vdash$ is a broadly classical consequence relation obeying Conditional Proof, it obeys this meta-rule.*[30]

Proof Let $\vdash_{\text{cl}}$ be the classical consequence relation in a propositional language with the usual logical connectives but no additional vocabulary. Let γ_n be the conjunction of the members of Γ_n and let δ be the conjunction of the members of Δ. By Conditional Proof and Modus Ponens, if $\vdash_{\text{cl}} \gamma_1 \to \alpha_1, \ldots,$ and $\vdash_{\text{cl}} \gamma_n \to \alpha_n$ then $\vdash_{\text{cl}} \delta \to \beta$. By the structural completeness of classical logic, $\gamma_1 \to \alpha_1, \ldots, \gamma_n \to \alpha_n \vdash_{\text{cl}} \delta \to \beta$. Since this is an entailment, it holds in any broadly classical logic.

Now suppose $\Gamma_1 \vdash \alpha_1$ and ... and $\Gamma_n \vdash \alpha_n$. So $\vdash \gamma_1 \to \alpha_1, \ldots,$ and $\vdash \gamma_n \to \alpha_n$. By the result in the previous paragraph, $\gamma_1 \to \alpha_1, \ldots, \gamma_n \to \alpha_n \vdash \delta \to \beta$. By Cut, $\vdash \delta \to \beta$. So $\Delta \vdash \beta$. □

These results tell us that whether a broadly classical consequence relation is strongly classical depends on whether it obeys Conditional Proof. In the case of a finitary broadly classical consequence relation, we can simplify things still further.

Proposition 2.3. *A finitary broadly classical consequence relation obeys Conditional Proof just in case it obeys Conditional Proof with no side formulas: If $\alpha \vdash \beta$ then $\vdash \alpha \to \beta$.*

Proof Suppose $\Gamma; \alpha \vdash \beta$. Since $\vdash$ is finitary, for some finite $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$, $\gamma_1, \ldots, \gamma_n, \alpha \vdash \beta$. So $\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \alpha \vdash \beta$. By Conditional Proof with no side formulas, $\vdash \gamma_1 \wedge \ldots \wedge \gamma_n \wedge \alpha \to \beta$. In a broadly classical logic, $\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \alpha \to \beta \vdash \gamma_1 \wedge \ldots \wedge \gamma_n \to (\alpha \to \beta)$. By Cut, $\vdash \gamma_1 \wedge \ldots \wedge \gamma_n \to (\alpha \to \beta)$. By Modus Ponens and Cut, $\gamma_1 \wedge \ldots \wedge \gamma_n \vdash \alpha \to \beta$. So $\gamma_1, \ldots, \gamma_n \vdash \alpha \to \beta$. So by Weakening, $\Gamma \vdash \alpha \to \beta$. The other direction is trivial. □

[29] Intuitionist propositional logic and classical predicate logic are not structurally complete, so the corresponding proofs do not work in those cases.

[30] This result is compatible with the fact that obeying Conditional Proof does not guarantee obeying General Replacement of Equivalents, or even Replacement of Equivalents. This is because instances of Replacement of Equivalents may crucially involve vocabulary beyond the usual logical connectives – that is, they may not be substitution instances of a classically valid meta-rule. Consider the example in footnote 28. In that example, $\vdash$ obeys Conditional Proof but not the following meta-rule: if $p \vdash q$ and $q \vdash p$ then $Op \vdash Oq$. This meta-rule does not count as "classically valid" in the sense defined above, and is not a substitution instance of a classically valid meta-rule. By contrast, the result will apply to Reasoning by Cases, since the following meta-rule is classical valid: if $s, p \vdash r$ and $s, q \vdash r$ then $s, p \vee q \vdash r$. Obeying Conditional Proof will guarantee obeying a version of General Replacement of Equivalents restricted so that α, β, δ, and Γ do not contain any vocabulary beyond the usual vocabulary of classical propositional logic. Thanks to an anonymous referee for raising this issue.

Thus, to show that a finitary broadly classical consequence relation is strongly classical, all we need to show is that it obeys Conditional Proof with no side formulas.

Indeed, in a finitary broadly classical logic, Conditional Proof with no side formulas is equivalent to Reasoning by Cases with no side formulas, Reductio ad Absurdum with no side formulas, and Contraposition with no side formulas.[31] The proof is the same as the proof for the case of side formulas – all one has to do is to replace Γ with the empty set in the proof.

2.4 Modal Logic

A *modal logic* is a broadly classical consequence relation in a language that contains a one-place operator $\Box$ in addition to the usual logical connectives. We use $\Diamond$ as an abbreviation for $\neg\Box\neg$. We use $\Box^n$ as an abbreviation for a sequence of n $\Box$s and $\Diamond^n$ as an abbreviation for a sequence of n $\Diamond$s

We say that a modal logic is *normal* just in case it has the following property:

Normal. If $\Gamma \vdash \alpha$ then $\Box\Gamma \vdash \Box\alpha$.

For the case of a finitary strongly classical modal logic, this is equivalent to the following more familiar characterization: A finitary strongly classical modal logic is a normal modal logic just in case it has the following principle as a theorem:

K $\Box(p \to q) \to (\Box p \to \Box q)$

and obeys the following meta-rule:

Necessitation. If $\vdash \alpha$ then $\vdash \Box\alpha$.[32]

For the case of non-finitary strongly classical modal logics, Normal entails the familiar constraint but not vice-versa. (In a strongly classical logic, the familiar constraint is equivalent to Normal restricted to finite Γ.) The main reason we use Normal rather than the familiar constraint is that Normal is a more natural constraint for weakly classical modal logics. A second, more technical reason will emerge in the next section.

The weakest strongly classical normal modal logic is called "K". K can be axiomatized taking the axioms to be all of the classical tautologies and all substitution instances of the modal principle K, taking Modus Ponens to be the sole rule of inference and Necessitation to be the sole rule of proof.

Many familiar normal modal logics can be characterized as the weakest normal modal logic extending K that has a modal principle or principles as theorems. Familiar modal principles include the following:

T $\Box p \to p$
4 $\Box p \to \Box\Box p$

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[31] Conditional Proof with no side formulas also follows from Replacement of Equivalents.

[32] Necessitation is equivalent to the claim that the following rule is admissible: $p/\Box p$. In most modal logics of interest, this rule is not derivable.

5 $\Diamond p \to \Box\Diamond p$
D $\Box p \to \Diamond p$
B $p \to \Box\Diamond p$
G $\Diamond\Box p \to \Box\Diamond p$
W $\Box(\Box p \to p) \to \Box p$

It will also be useful to name the following family of modal principles:

T^n $\Box^n p \to p$ (for $n \geq 1$)

For instance, T^2 is $\Box\Box p \to p$.

When a modal logic is the weakest normal modal logic that extends K and has one or more of these principles as theorems, the modal logic is typically referred to using "K" followed by a list of the relevant modal principles. For instance, KT is the normal modal logic that results from taking K and extending it by T. KDB is the normal modal logic that results from taking K and extending it by D and B. For historical reasons, "S4" is an alternate name for KT4 and "S5" is an alternate name for KT5. (S5 is also the same logic as KT45 and KDB4, among others). "S4.2" is an alternate name for KT4G. "GL" is an alternate name for KW. (GL is also the same logic as K4W.)

All of the principles listed above are conditionals. We can use the subscript "c" to refer to their converse. So, for instance, 4_c is $\Box\Box p \to \Box p$. We can use a "!" to refer to the strengthening of the conditional to a biconditional. So, for instance, 4! is $\Box p \leftrightarrow \Box\Box p$. Interestingly, it turns out that KB! is the same modal logic as KT^2!. Finally, we can use "$\Box$" to refer to the result of prefixing a modal principle with a $\Box$. So, for instance, $\Box 4$ is $\Box(\Box p \to \Box\Box p)$.

There are two additional normal modal logics that deserve note. Triv is the extension of K by T! (i.e., $\Box p \leftrightarrow p$). Ver is the extension of K by $\Box\bot$. These are notable because every consistent strongly classical normal modal logic is extended by either Triv or Ver (or both).

Every normal modal logic named so far has been strongly classical. Using a variant of the standard proof of the Deduction Theorem, it is straightforward to show that any modal logic that is the result of adding some modal principles to K is also strongly classical. One way to generate a weakly classical normal modal logic is to add a rule of inference to K (or to a stronger normal modal logic) without adding the corresponding conditional. (We must also make sure the conditional does not end up a theorem.) For instance, if we add the rule of inference $p/\Box p$ to K, the result will be a weakly classical normal modal logic.

In what follows, we'll make use of the standard (Kripkean) relational model theory for strongly classical normal modal logics. To summarize: A *model* M is a triple $\langle W, R, V\rangle$, where W is a non-empty set (the set of worlds). R is a binary relation on W (the accessibility relation). V assigns to each atomic sentence a subset of W (the valuation function).

For worlds $w, v \in W$, we inductively define $wR^n v$ as follows: $wR^0 v$ just in case $w = v$. $wR^{n+1} v$ just in case there is a $u \in W$ such that $wR^n u$ and uRv. We write $R(w)$ for the set $\{v \mid wRv\}$ and $R^n(w)$ for the set $\{v \mid wR^n v\}$.

A *frame* F is a pair $\langle W, R\rangle$. We say that the model $\langle W, R, V\rangle$ is *based on* the frame $\langle W, R\rangle$. We inductively define *truth at the world w in the model M* as follows: An atomic sentence p is true at w just in case $w \in V(p)$. The clauses for the logical constants are just what one would expect. For instance, $\alpha \wedge \beta$ is true at w just in case both α is true at w and β is true at w. $\Box\alpha$ is true at w just in case for every world $v \in W$, if wRv then α is true at v. That is, α is true at all worlds accessible from w. We say that a sentence α is *valid on a class of models* just in case α is true at all worlds in all models in that class. We say that a set of sentence Γ *entails* the sentence α *on a class of models* just in case at all worlds in all models in that class if all the members of Γ are true, so is α.

Given a model $M = \langle W, R, V\rangle$ and a set of worlds $S \subseteq W$ we write M_S for the *submodel of M generated by* the set S. $M_S = \langle W_S, R_S, V_S\rangle$, where $W_S = \{v \mid wR^n v$ for any $w \in S$ and $n \in \mathbb{N}\}$, $R_S = R \cap W_S \times W_S$, and $V_S(p) = V(p) \cap W_S$. If $w \in W$, we write M_w for $M_{\{w\}}$, and similarly for W_w, R_w, and V_w. An important feature of generated submodels is that for every $w \in W_S$, what's true at w in M_S is exactly the same as what's true at w in M.

Given a strongly classical normal modal logic $\vdash$, we say that it is *sound with respect to a class of models* just in case if $\vdash \alpha$ then α is valid on that class. (This is equivalent to the condition that if $\Gamma \vdash \alpha$ then Γ entails α on that class.) We say that $\vdash$ is *weakly complete with respect to a class of models* just in case if α is valid on the class then $\vdash \alpha$. We say that $\vdash$ is *strongly complete with respect to a class of models* just in case if Γ entails α on the class then $\Gamma \vdash \alpha$. We say that $\vdash$ is *weakly determined with respect to a class of models* just in case $\vdash$ is sound and weakly complete with respect to the class of models. We say that $\vdash$ is *strongly determined with respect to a class of models* just in case $\vdash$ is sound and strongly complete with respect to the class of models. Similarly, we say that $\vdash$ is *sound (weakly complete, strongly complete, weakly determined, or strongly determined) with respect to a class of frames* just in case $\vdash$ is sound (weakly complete, strongly complete, weakly determined, or strongly determined, respectively) with respect to the class of all models based on a frame in the class.

Given a strongly classical normal modal logic $\vdash$, the *canonical model* for $\vdash$ is defined as follows: The set of worlds, W, is the set of all maximally consistent (in $\vdash$) sets of sentences. wRv just in case whenever $\Box\alpha \in w$, $\alpha \in v$. $V(p)$ is the set of worlds that contain the atomic sentence p. It is straightforward to show that if $\vdash$ is a finitary strongly classical normal modal logic, $\vdash$ is strongly determined with respect to the singleton of the canonical model for $\vdash$.

We say that a strongly classical normal modal logic $\vdash$ *corresponds to a class of frames* just in case $\vdash$ is strongly determined with respect to the class of frames and $\vdash$ is not strongly determined with respect to any proper superclass of the class of frames. Not every strongly classical normal modal logic corresponds to a class of frames. For instance, GL does not correspond to any class of frames. But all of the other strongly classical normal modal logics that will be discussed here correspond to a class of frames. For instance, KT corresponds to the class of reflexive frames, K4 corresponds to the class of transitive frames, KD corresponds to the class of serial frames, KB corresponds to the class of symmetric frames, S4 corresponds to the class of transitive and reflexive frames, S5 corresponds to the class of frames in which R is an equivalence relation, and Triv corresponds to the class of frames in which each world is accessible from itself and from no other world. Instead of correspondence with a class of frames, we sometimes talk about the *frame condition* for $\vdash$. For instance, KT has the frame condition that R is reflexive. One way to show that a finitary strongly classical normal modal logic corresponds to a class of frames is to show that the modal logic is sound with respect to the class of frames, it is not sound with respect to any proper superclass of the class of frames, and the canonical model for the logic is based on a frame in that class. The Sahlqvist correspondence theorem ([30]) is a general result that enables us to determine the classes of frames corresponding to a wide range of strongly classical normal modal logics.

Although GL does not correspond to a class of frames, it is weakly determined with respect to a class of frames. In particular, it is weakly determined with respect to the class of frames with finitely many worlds and a transitive and irreflexive accessibility relation. Not every strongly classical normal modal logic is weakly determined with respect to a class of frames.

3 The Admissibility of Specific Rules

The central questions of this paper are equivalent to questions about admissible rules. For instance, as we will see, a finitary strongly classical normal modal logic $\vdash_{\text{base}}$ generates a strongly classical logic just in case $\Box p \to \Box q / \Box(p \to q)$ is admissible in $\vdash_{\text{base}}$. Similarly, $\vdash_{\text{base}}$ generates itself just in case $\Box p \to \Box q / p \to q$ is admissible in $\vdash_{\text{base}}$. So it will be useful to be able to tell when some specific rules are admissible in a finitary strongly classical normal modal logic.

There are two literatures concerning the admissibility of rules in modal logics. The earlier literature focuses on specific, philosophically interesting rules and tries to answer the question of which finitary strongly classical normal modal logics these rules are admissible in.[33] The later (and more technically sophisticated) literature focuses on a more general question. For a wide range of finitary strongly classical normal modal logics extending K4, it shows that whether a rule is admissible is decidable. It also generates explicit bases for the set of admissible rules for many such logics.[34]

The results on admissible rules in this section make use of the techniques from the earlier literature on admissible rules in modal logics. This is for two reasons. First, what is of interest here is not the question of which rules are admissible in a given modal logic, but rather given a specific rule, which modal logics it is admissible in. Second, we will not be restricting our attention to modal logics extending K4.

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[33] For example, see [19, pp. 94–5], [20, pp. 44–9, 79–81], [39], [40], [6], and [42].

[34] This literature stems from the work of Rybakov (culminating in [29]). For an important contribution to this literature, see [16]. For a recent survey, see [15].

In particular, we will prove three simple results about the admissibility of rules. Each result will have the same basic form: A rule in a specified family of rules is admissible in a finitary strongly classical normal modal logic just in case the canonical model for the modal logic has a certain first-order property just in case the modal logic is weakly determined with respect to a class of models that has a related property.

Proposition 3.1. *Suppose $\vdash$ is a finitary strongly classical normal modal logic and n and m are natural numbers. Then the following are equivalent:*

- *The rule $\Box^n p/\Box^m p$ is admissible in $\vdash$.*
- *The canonical model for $\vdash$ has the following property: $\forall w(\exists u\, uR^m w \rightarrow \exists u^*\, u^* R^n w)$.*[35]
- *$\vdash$ is weakly determined with respect to a class C of models that has the following related property: For every model $M = \langle W, R, V\rangle$ in C and $w \in W$, if $\exists u \in W\, uR^m w$ then there is an $M' = \langle W', R', V'\rangle$ in C (perhaps identical to M) such that $M'_w = M_w$ and $\exists u^* \in W'\, u^* R'^n w$.*

Proof Suppose the rule $\Box^n p/\Box^m p$ is admissible in $\vdash$. Suppose for some world w in the canonical model for $\vdash$, there is a world u such that $uR^m w$. We show that the set $\{\Diamond^n \alpha \mid \alpha \in w\}$ is consistent in $\vdash$. Suppose not. Then $\vdash \neg(\Diamond^n\alpha_1 \wedge \ldots \wedge \Diamond^n\alpha_k)$ for some $\alpha_1, \ldots, \alpha_k \in w$. So $\vdash \Box^n\neg\alpha_1 \vee \ldots \vee \Box^n\neg\alpha_k$. So $\vdash \Box^n\neg(\alpha_1 \wedge \ldots \wedge \alpha_k)$. By the admissible rule, $\vdash \Box^m\neg(\alpha_1 \wedge \ldots \wedge \alpha_k)$. So $\Box^m\neg(\alpha_1 \wedge \ldots \wedge \alpha_k) \in u$. But since $uR^m w$, $\neg\Box^m\neg(\alpha_1 \wedge \ldots \wedge \alpha_k) \in u$. So u is inconsistent. Contradiction! So the set is consistent. So it is a subset of a maximal consistent set u^*. The set u^* is a world in the canonical model. By the definition of the canonical model, $u^* R^n w$.

Suppose the canonical model for $\vdash$ has the specified property. Since $\vdash$ is weakly determined with respect to the singleton of its canonical model, $\vdash$ is weakly determined with respect to a class of models that has the related property.

Suppose $\vdash$ is weakly determined with respect to a class C of models that has the related property. Suppose $\nvdash \Box^m\alpha$ for some sentence α. So there is a model $M = \langle W, R, V\rangle$ in C and a world $u \in W$ such that $\Box^m\alpha$ is false at u in M. So there is a w such that $uR^m w$ and α is false at w. By the property of C, there is a model $M' = \langle W', R', V'\rangle$ in C such that $M'_w = M_w$ and $\exists u^* \in W'\, u^* R'^n w$. Since $M'_w = M_w$, α is false at w in M'. So $\Box^n\alpha$ is false at u^* in M'. Since $\vdash$ is weakly determined with respect to C, $\nvdash \Box^n\alpha$. So the rule $\Box^n p/\Box^m p$ is admissible in $\vdash$. $\Box$

Notice that, by Necessitation, if $n \leq m$, the rule $\Box^n p/\Box^m p$ is admissible in any broadly classical normal modal logic.

Proposition 3.2. *Suppose $\vdash$ is a finitary strongly classical normal modal logic and l, m, n, and o are natural numbers. Then the following are equivalent:*

- *The rule $\Box^o(\Box^n p \rightarrow \Box^n q)/\Box^l(\Box^m p \rightarrow \Box^m q)$ is admissible in $\vdash$.*
- *The canonical model for $\vdash$ has the following property: $\forall w, v, x(wR^l v \wedge vR^m x \rightarrow \exists u, u^*(uR^o u^* \wedge u^* R^n x \wedge R^n(u^*) \subseteq R^m(v)))$.*

[35]The quantifiers here range over the set of worlds in the model.

- $\vdash$ *is weakly determined with respect to a class C of models that has the following related property: For every model $M = \langle W, R, V\rangle$ in C and $w, v, x \in W$, if $wR^l v$ and $vR^m x$ then there is an $M' = \langle W', R', V'\rangle$ in C (perhaps identical to M) such that $M'_v = M_v$ and $\exists u, u^* \in W'$ such that $uR'^o u^*$, $u^* R'^n x$, and $R'^n(u^*) \subseteq R'^m(v)$.*

Proof Suppose the rule $\Box^o(\Box^n p \to \Box^n q)/\Box^l(\Box^m p \to \Box^m q)$ is admissible in $\vdash$. Suppose for three worlds w, v, x in the canonical model for $\vdash$, $wR^l v$ and $vR^m x$. We show that the set $\{\Diamond^o(\Box^n\alpha \wedge \Diamond^n\beta) \mid \Box^m\alpha \in v \text{ and } \beta \in x\}$ is consistent in $\vdash$. Suppose not. Then $\vdash \neg(\Diamond^o(\Box^n\alpha_1 \wedge \Diamond^n\beta_1) \wedge \ldots \wedge \Diamond^o(\Box^n\alpha_k \wedge \Diamond^n\beta_k))$ for $\Box^m\alpha_1, \ldots, \Box^m\alpha_k \in v$ and $\beta_1, \ldots, \beta_k \in x$. So $\vdash \Box^o(\Box^n\alpha_1 \to \Box^n\neg\beta_1) \vee \ldots \vee \Box^o(\Box^n\alpha_k \to \Box^n\neg\beta_k)$. So $\vdash \Box^o(\Box^n(\alpha_1 \wedge \ldots \wedge \alpha_k) \to \Box^n\neg(\beta_1 \wedge \ldots \wedge \beta_k))$. By the admissible rule, $\vdash \Box^l(\Box^m(\alpha_1 \wedge \ldots \wedge \alpha_k) \to \Box^m\neg(\beta_1 \wedge \ldots \wedge \beta_k))$. So this sentence is a member of w. Since $wR^l v$, $\Box^m(\alpha_1 \wedge \ldots \wedge \alpha_k) \to \Box^m\neg(\beta_1 \wedge \ldots \wedge \beta_k) \in v$. The antecedent of this conditional is a member of v, so the consequent is, too. Since $vR^m x$, $\neg(\beta_1 \wedge \ldots \wedge \beta_k) \in x$. But $\beta_1 \wedge \ldots \wedge \beta_k \in x$. So x is inconsistent. Contradiction! So the set is consistent. So it is a subset of a maximal consistent set u.

Now consider the set $\{\Box^n\alpha \mid \Box^m\alpha \in v\} \cup \{\Diamond^n\beta \mid \beta \in x\} \cup \{\gamma \mid \Box^o\gamma \in u\}$. (The first set in this union is what will ensure that $R^n(u^*) \subseteq R^m(v)$, the second is what will ensure $u^* R^n x$, and the third is what will ensure that $uR^o u^*$.) We show that this set is consistent. Suppose not. Then $\vdash \neg(\Box^n\alpha_1 \wedge \ldots \wedge \Box^n\alpha_i \wedge \Diamond^n\beta_1 \wedge \ldots \wedge \Diamond^n\beta_j \wedge \gamma_1 \wedge \ldots \wedge \gamma_k)$ for some $\Box^m\alpha_1, \ldots, \Box^m\alpha_i \in v$, $\beta_1, \ldots, \beta_j \in x$, and $\Box^o\gamma_1, \ldots, \Box^o\gamma_k \in u$. So $\vdash \Box^n(\alpha_1 \wedge \ldots \wedge \alpha_i) \wedge \Diamond^n(\beta_1 \wedge \ldots \wedge \beta_j) \to \neg(\gamma_1 \wedge \ldots \wedge \gamma_k)$. By the construction of u, $\Diamond^o(\Box^n(\alpha_1 \wedge \ldots \wedge \alpha_i) \wedge \Diamond^n(\beta_1 \wedge \ldots \beta_j)) \in u$. So there is a world z such that, $uR^o z$ and $\Box^n(\alpha_1 \wedge \ldots \wedge \alpha_i) \wedge \Diamond^n(\beta_1 \wedge \ldots \beta_j) \in z$. $\Box^n(\alpha_1 \wedge \ldots \wedge \alpha_i) \wedge \Diamond^n(\beta_1 \wedge \ldots \wedge \beta_j) \to \neg(\gamma_1 \wedge \ldots \wedge \gamma_k) \in z$. So $\neg(\gamma_1 \wedge \ldots \wedge \gamma_k) \in z$, But $\Box^o(\gamma_1 \wedge \ldots \wedge \gamma_k) \in u$. So $\gamma_1 \wedge \ldots \wedge \gamma_k \in x$. Contradiction! So the set is consistent. So it is a subset of a maximal consistent set u^*. It is straightforward to show that $uR^o u^*$, $u^* R^n x$, and $R^n(u^*) \subseteq R^m(v)$.

Suppose the canonical model for $\vdash$ has the specified property. Since $\vdash$ is weakly determined with respect to the singleton of its canonical model, $\vdash$ is weakly determined with respect to a class of models that has the related property.

Suppose $\vdash$ is weakly determined with respect to a class C of models that has the related property. Suppose $\nvdash \Box^l(\Box^m\alpha \to \Box^m\beta)$. Then there is a model $M = \langle W, R, V\rangle$ in C and $w, v, x \in W$ such that $wR^l v$, $vR^m x$, α is true at every world in $R^m(v)$, and β is false at x. By the property of C, there is a model $M' = \langle W', R', V'\rangle$ in C such that $M'_v = M_v$ and $\exists u, u^* \in W'$ such that $uR'^o u^*$, $u^* R'^n x$, and $R'^n(u^*) \subseteq R'^m(v)$. Since $M'_v = M_v$ and α is true at every world in $R^m(v)$, α is true at every world in $R'^{m(v)}$. Since $R'^n(u^*) \subseteq R'^m(v)$, α is true at every world in $R'^n(u^*)$, and so $\Box^n\alpha$ is true at u^* in M'. Since $u^* R'^n x$, $\Box^n\beta$ is false at u^* in M'. So $\Box^n\alpha \to \Box^n\beta$ is false at u^* in M'. Since $uR'^o u^*$, $\Box^o(\Box^n\alpha \to \Box^n\beta)$ is false at u in M'. So $\nvdash_{\text{base}} \Box^o(\Box^n\alpha \to \Box^n\beta)$. So the rule $\Box^o(\Box^n p \to \Box^n q)/\Box^l(\Box^m p \to \Box^m q)$ is admissible in $\vdash$.[36] $\Box$

Proposition 3.3. *Suppose $\vdash$ is a finitary strongly classical normal modal logic and l, m, and n are natural numbers. Then the following are equivalent:*

- *The rule $\Box^n p \to \Box^n q/p \wedge \Diamond^l\Box^m p \to q$ is admissible in $\vdash$.*

[36] Part of this proof follows the strategy of [39, p. 306] for the case of the rule $\Box p \to \Box q/p \to q$.

- *The canonical model for* $\vdash$ *has the following property:* $\forall w, v(wR^l v \rightarrow \exists u(uR^n w \wedge R^n(u) \subseteq R^m(v) \cup \{w\}))$.
- $\vdash$ *is weakly determined with respect to a class* C *of models that has the following related property: For every model* $M = \langle W, R, V\rangle$ *in* C *and* $w, v \in W$, *if* $wR^l v$ *then there is an* $M' = \langle W', R', V'\rangle$ *in* C *(perhaps identical to* M*) such that* $M'_w = M_w$ *and* $\exists u \in W'$ *such that* $uR'^n w$ *and* $R'^n(u) \subseteq R'^m(v) \cup \{w\}$.

Proof Suppose the rule $\Box^n p \rightarrow \Box^n q / p \wedge \Diamond^l \Box^m p \rightarrow q$ is admissible in $\vdash$. Suppose for two worlds w, v in the canonical model for $\vdash$, $wR^l v$. We show that the set $\{\Box^n \alpha \mid \Box^m \alpha \in v \text{ and } \alpha \in w\} \cup \{\neg\Box^n\neg\beta \mid \beta \in w\}$ is consistent in $\vdash$. (The first set in this union is what will ensure that $R^n(u) \subseteq R^m(v) \cup \{w\}$ and the second is what will ensure $uR^n w$.) Suppose not. Then $\vdash \neg(\Box^n\alpha_1 \wedge \ldots \wedge \Box^n\alpha_j \wedge \neg\Box^n\neg\beta_1 \wedge \ldots \wedge \neg\Box^n\neg\beta_k)$ for some $\alpha_1, \ldots, \alpha_j$ such that for each α_i, $\Box^m\alpha_i \in v$ and $\alpha_i \in w$, and $\beta_1, \ldots, \beta_k \in w$. So $\vdash \Box^n(\alpha_1 \wedge \ldots \wedge \alpha_j) \rightarrow \Box^n\neg(\beta_1 \wedge \ldots \wedge \beta_k)$. By the admissible rule, $\vdash \alpha_1 \wedge \ldots \wedge \alpha_j \wedge \Diamond^l\Box^m(\alpha_1 \wedge \ldots \wedge \alpha_j) \rightarrow \neg(\beta_1 \wedge \ldots \wedge \beta_k)$. So $\alpha_1 \wedge \ldots \wedge \alpha_j \wedge \Diamond^l\Box^m(\alpha_1 \wedge \ldots \wedge \alpha_j) \rightarrow \neg(\beta_1 \wedge \ldots \wedge \beta_k) \in w$. $\alpha_1 \wedge \ldots \wedge \alpha_j \in w$. $\Box^m(\alpha_1 \wedge \ldots \wedge \alpha_j) \in v$. Since $wR^l v$, $\Diamond^l\Box^m(\alpha_1 \wedge \ldots \wedge \alpha_j) \in w$. So $\alpha_1 \wedge \ldots \wedge \alpha_j \wedge \Diamond^l\Box^m(\alpha_1 \wedge \ldots \wedge \alpha_j) \in w$. So $\neg(\beta_1 \wedge \ldots \wedge \beta_k) \in w$. But $\beta_1 \wedge \ldots \wedge \beta_k \in w$. Contradiction! So the set is consistent. So it is a subset of a maximal consistent set u. It is straightforward to show that $uR^n w$ and $R^n(u) \subseteq R^m(v) \cup \{w\}$.

Suppose the canonical model for $\vdash$ has the specified property. Since $\vdash$ is weakly determined with respect to the singleton of its canonical model, $\vdash$ is weakly determined with respect to a class of models that has the related property.

Suppose $\vdash$ is weakly determined with respect to a class C of models that has the related property. Suppose $\nvdash \alpha \wedge \Diamond^l\Box^m\alpha \rightarrow \beta$. Then there is a model $M = \langle W, R, V\rangle$ in C and worlds $w, v \in W$ such that $wR^l v$, α is true at every world in $R^m(v) \cup \{w\}$, and β is false at w. By the property of C, there is a model $M' = \langle W', R', V'\rangle$ in C such that $M'_w = M_w$ and $\exists u \in W'$ such that $uR'^n w$ and $R'^n(u) \subseteq R'^m(v) \cup \{w\}$. Since $M'_w = M_w$ and α is true at every world in $R^m(v) \cup \{w\}$, α is true at every world in $R'^m(v) \cup \{w\}$ in M'. Since $R'^n(u) \subseteq R'^m(v) \cup \{w\}$, $\Box^n\alpha$ is true at u in M'. Since $uR'^n w$, $\Box^n\beta$ is false at u in M'. So $\Box^n\alpha \rightarrow \Box^n\beta$ is false at u in M'. So $\nvdash \Box^n\alpha \rightarrow \Box^n\beta$. So the rule $\Box^n p \rightarrow \Box^n q / p \wedge \Diamond^l\Box^m p \rightarrow q$ is admissible in $\vdash$. $\Box$

These results can be generalized further, but they are (more than) sufficient for the results to follow.[37]

4 Strongly Classical Generated Consequence Relations

Suppose $\vdash_{\text{base}}$ is a broadly classical normal modal logic and $\vdash$ is generated from it as follows: $\Gamma \vdash \alpha$ just in case $\Box\Gamma \vdash_{\text{base}} \Box\alpha$. It is straightforward to show that

[37] For instance, it is straightforward albeit tedious to unify and generalize the three results to cover all rules α/β, where α and β are each equivalent to sentences built only using atomic sentences and their negations, $\top$, $\bot$, $\vee$, $\Box$, and $\Diamond$, with the proviso that a $\Box$ never occurs within the scope of a $\Diamond$. It would be interesting to see just how far this result can be generalized.

$\vdash$ is a broadly classical normal modal logic. In fact, we can show something stronger:

Theorem 4.1. *If $\vdash$ is generated from a broadly classical normal modal logic $\vdash_{\text{base}}$, then $\vdash$ is a broadly classical normal modal logic that extends $\vdash_{\text{base}}$. If $\vdash_{\text{base}}$ is finitary, so is $\vdash$.*

Proof Identity, Weakening, Cut, and Substitution Invariance for $\vdash$ all follow from the corresponding properties for $\vdash_{\text{base}}$. So $\vdash$ is a consequence relation.

Suppose $\Gamma \vdash_{\text{base}} \alpha$. Since $\vdash_{\text{base}}$ is normal, $\Box\Gamma \vdash_{\text{base}} \Box\alpha$. So $\Gamma \vdash \alpha$. So $\vdash$ extends $\vdash_{\text{base}}$.

Since $\vdash_{\text{base}}$ is broadly classical and $\vdash$ extends $\vdash_{\text{base}}$, $\vdash$ is broadly classical.

Suppose $\Gamma \vdash \alpha$. So $\Box\Gamma \vdash_{\text{base}} \Box\alpha$. Since $\vdash_{\text{base}}$ is normal, $\Box\Box\Gamma \vdash_{\text{base}} \Box\Box\alpha$. So $\Box\Gamma \vdash \Box\alpha$. So $\vdash$ is normal, too.

If $\vdash_{\text{base}}$ is finitary, it easily follows that $\vdash$ is finitary, too. $\Box$

This result explains why we define normality for broadly classical logics as we do.

For the rest of this section (and for the following two), we'll assume that $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic. We make this assumption so we can make use of the standard (Kripkean) relational model theory for $\vdash_{\text{base}}$. We'll also assume that $\vdash$ is generated from $\vdash_{\text{base}}$ as above. So $\vdash$ is a finitary broadly classical normal modal logic extending $\vdash_{\text{base}}$. The question we'll be considering in the remainder of this section is: When is $\vdash$ also strongly classical?

Our first result provides necessary and sufficient conditions for $\vdash_{\text{base}}$ to generate a strongly classical logic. Some definitions will be useful. We say that a model is *single-minded* just in case $\forall v \exists u(R(u) = \{v\})$. We say that a model is *quasi-single-minded* just in case $\forall w, v(wRv \to \exists u(R(u) = \{v\})$. In other words, a model is single-minded just in case for every world v, there is a world u such that v and only v is accessible from u. A model is quasi-single-minded just in case for every world v that is accessible from any world, there is a world u such that v and only v is accessible from u.

We say that a class of models is *single-minded* just in case it has the following property: If M is a model in the class and v is a world in M then there is a model M' in the class such that $M'_v = M_v$ and there is a world u in M' such that $R(u) = \{v\}$. We say that a class of models is *quasi-single-minded* just in case it has the following property: If M is a model in the class and v is a world in M accessible from some world in M then there is a model M' in the class such that $M'_v = M_v$ and there is a world u in M' such that $R(u) = \{v\}$. (In both cases, M may be identical to M'.) Notice that any class of single-minded models is a single-minded class of models (but not necessarily vice-versa), and any class of quasi-single-minded models is a quasi-single-minded class of models (but not necessarily vice-versa).

Theorem 4.2. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic and $\vdash$ is generated from $\vdash_{\text{base}}$. Then the following are equivalent:*

- *$\vdash$ is strongly classical.*
- *The rule $\Box p \to \Box q / \Box(p \to q)$ is admissible in $\vdash_{\text{base}}$.*
- *The canonical model for $\vdash_{\text{base}}$ is quasi-single-minded.*
- *$\vdash_{\text{base}}$ is weakly determined with respect to a quasi-single-minded class of models.*

Proof Since $\vdash$ is finitary, $\vdash$ is a strongly classical logic just in case if $\Box\alpha \vdash_{\text{base}} \Box\beta$ then $\vdash_{\text{base}} \Box(\alpha \to \beta)$. Since $\vdash_{\text{base}}$ is strongly classical, this obtains just in case if $\vdash_{\text{base}}$ $\Box\alpha \to \Box\beta$ then $\vdash_{\text{base}} \Box(\alpha \to \beta)$. This is equivalent to the rule $\Box p \to \Box q/\Box(p \to q)$ being admissible in $\vdash_{\text{base}}$.

The remainder of the result follows from Proposition 3.2, for the case where $m = o = 0$ and $l = n = 1$. $\Box$

We can leverage the familiar frame correspondence results to show that various modal logics are weakly determined with respect to a quasi-single-minded class of models, and thus generate strongly classical logics.

Corollary 4.3. *The modal logics* K*,* KD*,* KT^n_c *for any $n > 0$, and* $\mathrm{KT}^n!$ *for any $n > 0$ each generate a strongly classical logic.*

Proof K corresponds to the class of all frames. The class of models based on this class of frames is quasi-single-minded.

KD corresponds to the class of serial frames. The class of models based on this class of frames is quasi-single-minded.

KT^n_c corresponds to the class of frames in which for all worlds w, v, if $wR^n v$ then $w = v$. We can show that the class of models based on this class of frames is quasi-single-minded. Let M be a model in the class and w, v be worlds in M such that wRv. We show that there is a model M' in the class just like M except that there is a world u from which v and only v is accessible. Since wRv, either (i) v is only accessible from w, or (ii) v is accessible from an additional world, x, too. In the former case, let $u = w$. In the latter case, there is no world z such that $vR^{n-1}z$, since if there were, both $z = w$ and $z = x$. So if we define M' as the result of adding a world u to M such that u is not accessible from any world, and v and no other world is accessible from u, M' will be in the class of models.

$\mathrm{KT}^n!$ corresponds to the class of frames in which for all worlds w, v, $wR^n v$ just in case $w = v$. These frames include single reflexive worlds, loops with size a factor of n, and disjoint unions of such frames. Every model based on this class of frames is quasi-single-minded. $\Box$

It follows from this result that K, KD, KT^n_c for any $n > 0$, and $\mathrm{KT}^n!$ for any $n > 0$ all generate strongly classical logics.

We could continue in this way, going through modal logics one at a time and showing that many of them generate strongly classical logics. But there is a more general result we can prove. Call a sentence a $\Diamond\Box$-*sentence* if it is of the form $\Diamond\phi_1 \wedge \ldots \wedge \Diamond\phi_m \to \Box\psi_1 \vee \ldots \vee \Box\psi_n$ for $m, n \geq 0$, where ϕ_i and ψ_i are arbitrary sentences.[38] (By convention, a conjunction with no conjuncts is $\top$. A disjunction with no disjuncts is $\bot$.) Notice that each $\Diamond\Box$-sentence is logically equivalent to a sentence of the form $\Box\delta_1 \vee \ldots \vee \Box\delta_k$ (and vice-versa).

Proposition 4.4. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic that generates a strongly classical logic. Suppose $\vdash^+_{\text{base}}$ is the weakest strongly classical normal modal logic extending $\vdash_{\text{base}}$ that has a given set Δ of $\Diamond\Box$-sentences as theorems. Then $\vdash^+_{\text{base}}$ also generates a strongly classical logic.*

Proof Since $\vdash_{\text{base}}$ generates a strongly classical logic, by Theorem 4.2, its canonical model, M, is quasi-single-minded. Let M^+ be the canonical model for $\vdash^+_{\text{base}}$. We show that M^+ is quasi-single-minded, too.

Suppose w, v are worlds in M^+ such that wRv. Since $\vdash^+_{\text{base}}$ extends $\vdash_{\text{base}}$, M^+ is a submodel of M. So w, v are worlds in M. Since M is quasi-single-minded, there is a world u in M such that v and only v is accessible from u. To show that M^+ is quasi-single-minded, we show that u is a member of M^+. To do that, we show that $\{\beta \mid \vdash^+_{\text{base}} \beta\} \subseteq u$. To do that, we show that $\Delta^* \subseteq u$, where Δ^* is the smallest set that contains all substitution instances of members of Δ and is closed under the operation of prefixing a sentence with a $\Box$.

Choose an arbitrary member of Δ^*. This sentence will be logically equivalent to a sentence of the form $\Diamond\phi_1 \wedge \ldots \wedge \Diamond\phi_m \to \Box\psi_1 \vee \ldots \vee \Box\psi_n$. We work in the model M. Suppose $\Diamond\phi_1 \wedge \ldots \wedge \Diamond\phi_m$ is true at u. Since v is the only world accessible from u, $\phi_1, \ldots, \phi_m$ are true at v. Since wRv, $\Diamond\phi_1 \wedge \ldots \wedge \Diamond\phi_m$ is true at w. Since w is also a world in M^+, $\Diamond\phi_1 \wedge \ldots \wedge \Diamond\phi_m \to \Box\psi_1 \vee \ldots \vee \Box\psi_n \in w$. So $\Diamond\phi_1 \wedge \ldots \wedge \Diamond\phi_m \to \Box\psi_1 \vee \ldots \vee \Box\psi_n$ is true at w. So $\Box\psi_1 \vee \ldots \vee \Box\psi_n$ is true at w. So at least one of $\psi_1, \ldots, \psi_n$ is true at v. So at least one of $\Box\psi_1, \ldots, \Box\psi_n$ is true at u. So $\Diamond\phi_1 \wedge \ldots \wedge \Diamond\phi_m \to \Box\psi_1 \vee \ldots \vee \Box\psi_n$ is true at u. So $\Diamond\phi_1 \wedge \ldots \wedge \Diamond\phi_m \to \Box\psi_1 \vee \ldots \vee \Box\psi_n \in u$. Since we chose an arbitrary member of Δ^*, $\Delta^* \subseteq u$. So M^+ is quasi-single-minded. By Theorem 4.2, $\vdash^+_{\text{base}}$ generates a strongly classical logic. $\Box$

Corollary 4.5. *Any modal logic that is the result of extending* K, KD, KT^n_c, *or* $\text{KT}^n!$ *(for any $n > 0$) with one or more $\Diamond\Box$-sentences generates a strongly classical normal modal logic.*

Many familiar modal principles are $\Diamond\Box$-sentences. Such principles include 5, G, D_c, and any other principle of the form $\Diamond\alpha \to \Box\beta$. Any principle of the form $\Box\alpha$, such as $\Box$T and $\Box$4, is logically equivalent to a $\Diamond\Box$-sentence. So any modal logic that is the result of extending K, KD, KT^n_c, or $\text{KT}^n!$ (for any $n > 0$) with one or more of these principles generates a strongly classical logic. For instance, Triv and Ver each generate a strongly classical logic.

We can also find strongly classical normal modal logics such that every extension of them generates a strongly classical logic. A simple example is that if $\vdash_{\text{base}}$ is an extension of KK_c, then $\vdash$ is strongly classical. This is because in such a modal logic, $\Box p \to \Box q / \Box(p \to q)$ is derivable and so admissible. It

[38] The notion of a $\Diamond\Box$-sentence is unrelated to the notion of a $\Box\Diamond$-formula in [29, p. 247].

is easy to show that KD_c is the same logic as KK_c. So any extension of KD_c generates a strongly classical logic.

For a more interesting example, for any fixed $n > 0$, consider the two principles $\Diamond p \to \Diamond^n \Box p$ and $\Box^n \Diamond \top$. We can show that if $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic that has these two principles as theorems, then it generates a strongly classical logic. The argument is as follows: By the Sahlqvist correspondence theorem, these two principles are canonical. The first principle corresponds to the frame condition $\forall w, v(wRv \to \exists u(wR^n u \wedge R(u) \subseteq \{v\}))$. The second principle corresponds to the frame condition of quasi^n-seriality $\forall w, v(wR^n v \to \exists uvRu)$. Suppose in the canonical model for $\vdash_{\text{base}}$ there are two worlds w, v such that wRv. Then by the first condition, there is a world u such that $wR^n u$, and if any world is accessible from u, it is v. By the second condition, there is a world accessible from u. So v and only v is accessible from u.[39]

As we've already seen, though, not every generated consequence relation $\vdash$ is strongly classical. The example presented above was the consequence relation generated from the modal logic S5, but there are many other examples. In what follows, for several familiar modal logics, we'll identify the weakest strongly classical normal modal logic that extends the logic and generates a strongly classical logic. We'll start with KT.

For most applications of a generated modal logic, it is natural to take the base modal logic to extend KT. For instance, if "$\Box$" is interpreted as "it is determinately true that" or "one is in a position to know that", it is difficult to see how one could avoid accepting T. But we can show that there is only one consistent strongly classical normal modal logic extending KT that generates a strongly classical logic.

Proposition 4.6. *The only consistent strongly classical normal modal logic extending* KT *that generates a strongly classical logic is* Triv.[40,41]

Proof Suppose $\vdash_{\text{base}}$ is a consistent strongly classical normal modal logic that extends KT. Suppose $\vdash$ is strongly classical. So $\Box p \to \Box q / \Box(p \to q)$ is admissible in $\vdash_{\text{base}}$. $\Box(\alpha \wedge \neg\Box\alpha) \to \Box\bot$ is a theorem of KT. So $\vdash_{\text{base}} \Box(\alpha \wedge \neg\Box\alpha) \to \Box\bot$. By the admissible rule, $\vdash_{\text{base}} \Box(\alpha \wedge \neg\Box\alpha \to \bot)$. By T and Modus Ponens, $\vdash_{\text{base}} \alpha \wedge \neg\Box\alpha \to \bot$. So $\vdash_{\text{base}} \alpha \to \Box\alpha$. This is T_c, so $\vdash_{\text{base}}$ is an extension of Triv. There is no consistent strongly classical normal modal proper extension of Triv, so $\vdash_{\text{base}}$ must be identical to Triv.[42] $\Box$

[39] One might try to generalize this result by finding a modal principle that defines the class of quasi-single-minded models. Unfortunately, there is no such principle because the property of being a quasi-single-minded model is not preserved by taking generated submodels.

[40] [41, p. 297 n. 32] shows that KT generates a weakly classical logic. Also see [43, p. 526] for a version of this result.

[41] We can extend this result further. Suppose $\vdash_{\text{base}}$ is a consistent broadly classical normal modal logic that extends K such that $\Box\alpha/\alpha$ is a derivable rule. Then if $\vdash_{\text{base}}$ generates a strongly classical logic, $\vdash_{\text{base}}$ is Triv. *Proof* $\Box(\alpha \wedge \neg\Box\alpha) \vdash_{\text{base}} \Box\alpha$. $\Box(\alpha \wedge \neg\Box\alpha) \vdash_{\text{base}} \Box\neg\Box\alpha$. $\Box\neg\Box\alpha \vdash_{\text{base}} \neg\Box\alpha$. $\{\Box\alpha, \neg\Box\alpha\} \vdash_{\text{base}} \Box\bot$. So $\Box(\alpha \wedge \neg\Box\alpha) \vdash_{\text{base}} \Box\bot$. Since $\vdash_{\text{base}}$ generates a strongly classical logic, $\vdash_{\text{base}} \Box(\alpha \wedge \neg\Box\alpha \to \bot)$. So $\vdash_{\text{base}} \alpha \wedge \neg\Box\alpha \to \bot$. So $\vdash_{\text{base}} \alpha \to \Box\alpha$. By analogous reasoning, $\Box(\neg\alpha \wedge \Box\alpha) \vdash_{\text{base}} \Box\bot$, so $\vdash_{\text{base}} \Box(\neg\alpha \wedge \Box\alpha \to \bot)$, so $\vdash_{\text{base}} \neg\alpha \wedge \Box\alpha \to \bot$, and so $\vdash_{\text{base}} \Box\alpha \to \alpha$.

[42] The proof follows the same strategy as a proof of a related result in [39, pp. 301–2].

The modal logic Triv is not suitable for most applications, since in this logic α and $\Box\alpha$ are logically equivalent.

It follows from this result that many familiar modal logics, such as KT, KTB, S4, S4.2, and S5, generate weakly classical consequence relations.

There are similar results for other modal logics. For example:

Proposition 4.7. *The weakest strongly classical normal modal logic extending* KB *that generates a strongly classical logic is* KB$\Box$B$_c$.

Proof Suppose $\vdash_{\text{base}}$ is a strongly classical normal modal logic that extends KB. Suppose $\vdash$ is strongly classical. So $\Box p \to \Box q / \Box(p \to q)$ is admissible in $\vdash_{\text{base}}$. Since $\Box\alpha \to \Box\Diamond\Box\alpha$ is a theorem of KB, $\vdash_{\text{base}} \Box\alpha \to \Box\Diamond\Box\alpha$. By the admissible rule, $\vdash_{\text{base}} \Box(\alpha \to \Diamond\Box\alpha)$. Substituting $\neg\alpha$ for α, it follows that $\vdash_{\text{base}} \Box(\Box\Diamond\alpha \to \alpha)$. This is the principle $\Box$B$_c$. So $\vdash_{\text{base}}$ is an extension of KB$\Box$B$_c$.

We can show that KB$\Box$B$_c$ generates a strongly classical logic model-theoretically. It is easy to show that this logic is strongly determined by the class of all models where the accessibility relation is both (i) symmetric and (ii) is such that $\forall w, v(wRv \to \exists u(vRu \wedge R(u) \subseteq \{v\}))$. Any model satisfying these properties is quasi-single-minded. So by Theorem 4.2, KB$\Box$B$_c$ generates a strongly classical logic. $\Box$

Since $\Box$B$_c$ is not a theorem of S5, it follows that no strongly classical normal modal logic between KB and S5 generates a strongly classical logic.

Corollary 4.8. *For each of the following pairs, the weakest strongly classical normal modal logic extending the first that generates a strongly classical logic is the second.*

- KDB*,* KB!
- KB5*,* Triv

Proof KDB: Since this modal logic extends KB, any strongly classical normal modal logic extending it that generates a strongly classical logic must extend KDB$\Box$B$_c$. It is straightforward to show that KDB$\Box$B$_c$ is the same modal logic as KB!. It is also straightforward to show that it is the same modal logic as KT^2!, which by Corollary 4.3 generates a strongly classical logic.

KB5: Since this modal logic extends KB, any strongly classical normal modal logic extending it that generates a strongly classical logic must extend KB5$\Box$B$_c$. It is straightforward to show that KB5$\Box$B$_c$ is the same modal logic as Triv. Triv generates a strongly classical logic. $\Box$

For extensions of K4, the following result is useful:

Proposition 4.9. *Suppose* $\vdash_{\text{base}}$ *is a finitary strongly classical normal modal logic that generates a strongly classical logic. Let* $\vdash^+_{\text{base}}$ *be the weakest strongly classical normal modal logic extending* $\vdash_{\text{base}}$ *that has* $\Box(\alpha \to \beta)$ *as a theorem (for given sentences* α *and* β*). Then the weakest strongly classical normal* *modal logic extending $\vdash_{\text{base}}$ that has $\Box\alpha \to \Box\beta$ as a theorem and that generates a strongly classical logic is $\vdash^{+}_{\text{base}}$.*

Proof Suppose $\vdash'_{\text{base}}$ is a strongly classical normal modal logic extending $\vdash_{\text{base}}$ that has $\Box\alpha \to \Box\beta$ as a theorem and that generates a strongly classical logic. So $\Box p \to \Box q/\Box(p \to q)$ is admissible in $\vdash'_{\text{base}}$. Since $\vdash'_{\text{base}} \Box\alpha \to \Box\beta$, $\vdash'_{\text{base}} \Box(\alpha \to \beta)$. So any strongly classical normal modal logic extending $\vdash_{\text{base}}$ that has $\Box\alpha \to \Box\beta$ as a theorem and that generates a strongly classical logic must extend $\vdash^{+}_{\text{base}}$. By K and Modus Ponens, $\vdash^{+}_{\text{base}}$ has $\Box\alpha \to \Box\beta$ as a theorem. Moreover, since $\Box(\alpha \to \beta)$ begins with a $\Box$, by Proposition 4.4, $\vdash^{+}_{\text{base}}$ generates a strongly classical logic. $\Box$

We can use this result to show the following:

Corollary 4.10. *For each of the following pairs, the weakest strongly classical normal modal logic extending the first that generates a strongly classical logic is the second.*

- K4*,* K$\Box$T$_c$
- KD4*,* KD$\Box$T$_c$
- KG4*,* KG$\Box$T$_c$
- KDG4*,* KDG$\Box$T$_c$
- K45*,* K5$\Box$T$_c$
- KD45*,* KD5$\Box$T$_c$
- GL*,* K$\Box^2\bot$

Proof Each of these cases immediately follows from Proposition 4.9, except for the case of GL. By Proposition 4.9, the weakest strongly classical normal modal logic extending GL that generates a strongly classical logic is K with the addition of the theorem $\Box((\Box p \to p) \to p)$. A substitution instance of this is $\Box((\Box\bot \to \bot) \to \bot)$, which is logically equivalent to $\Box^2\bot$. K$\Box^2\bot$ extends GL since it corresponds to the class of frames in which $\forall w R^2(w) = \emptyset$, and GL is valid on all such frames. Since $\Box^2\bot$ begins with a $\Box$, by Proposition 4.4, K$\Box^2\bot$ generates a strongly classical logic. $\Box$

This result tells us that the weakest strongly classical normal modal logic extending K4 that generates a strongly classical logic is K$\Box$T$_c$. Since $\Box$T$_c$ is not a theorem of S5, it follows that no strongly classical normal modal logic between K4 and S5 generates a strongly classical logic.

Corollary 4.11. *For each of the following pairs, the weakest strongly classical normal modal logic extending the first that generates a strongly classical logic is the second.*

- K4$_c$*,* K$\Box$T
- K5$_c$*,* KD$\Box$T$_c$

Proof K4$_c$: This follows from Proposition 4.9.

K5$_c$: $\Box(\Diamond\alpha \wedge \neg\alpha) \rightarrow \Box\bot$ is a theorem of K5$_c$. So any strongly classical normal modal logic extending K5$_c$ that generates a strongly classical logic has $\Box(\Diamond\alpha \wedge \neg\alpha \rightarrow \bot)$ as a theorem. This sentence is logically equivalent to $\Box(\Diamond\alpha \rightarrow \alpha)$. Substituting $\neg\alpha$ for α, it follows that $\Box$T$_c$ is a theorem. K5$_c$ also has D as a theorem. So any strongly classical normal modal logic extending K5$_c$ that generates a strongly classical logic extends KD$\Box$T$_c$. KD$\Box$T$_c$ extends K5$_c$. By Corollary 4.5, KD$\Box$T$_c$ is strongly classical. $\Box$

5 Self-Generating Consequence Relations

When a finitary strongly classical normal modal logic $\vdash_{\text{base}}$ generates a strongly classical logic $\vdash$, in some cases $\vdash_{\text{base}}$ generates itself and in some cases it does not. When is $\vdash_{\text{base}}$ self-generating? We can provide necessary and sufficient conditions, analogous to Theorem 4.2. Instead of quasi-single-mindedness, this result instead concerns single-mindedness.

Theorem 5.1. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic and $\vdash$ is generated from $\vdash_{\text{base}}$. Then the following are equivalent:*

- *$\vdash$ is identical to $\vdash_{\text{base}}$.*
- *The rule $\Box p \rightarrow \Box q / p \rightarrow q$ is admissible in $\vdash_{\text{base}}$.*
- *The canonical model for $\vdash_{\text{base}}$ is single-minded.*
- *$\vdash_{\text{base}}$ is weakly determined with respect to a single-minded class of models.*

Proof Suppose $\vdash$ is identical to $\vdash_{\text{base}}$. Suppose $\vdash_{\text{base}} \Box\alpha \rightarrow \Box\beta$. By Modus Ponens and Cut, $\Box\alpha \vdash_{\text{base}} \Box\beta$. By the definition of $\vdash$, $\alpha \vdash \beta$. Since $\vdash$ is identical to $\vdash_{\text{base}}$, $\alpha \vdash_{\text{base}} \beta$. Since $\vdash_{\text{base}}$ is strongly classical, $\vdash_{\text{base}} \alpha \rightarrow \beta$. So the rule $\Box p \rightarrow \Box q / p \rightarrow q$ is admissible in $\vdash_{\text{base}}$.

For the other direction, suppose the rule $\Box p \rightarrow \Box q / p \rightarrow q$ is admissible in $\vdash_{\text{base}}$. By Theorem 4.1, if $\Gamma \vdash_{\text{base}} \alpha$ then $\Gamma \vdash \alpha$. To show if $\Gamma \vdash \alpha$ then $\Gamma \vdash_{\text{base}} \alpha$, suppose $\Gamma \vdash \alpha$. So $\Box\Gamma \vdash_{\text{base}} \Box\alpha$. Since $\vdash_{\text{base}}$ is finitary, $\Box\gamma \vdash_{\text{base}} \Box\alpha$ for some conjunction γ of members of Γ. Since $\vdash_{\text{base}}$ is strongly classical, $\vdash_{\text{base}} \Box\gamma \rightarrow \Box\alpha$. By the admissible rule, $\vdash_{\text{base}} \gamma \rightarrow \alpha$. By Modus Ponens and Cut, $\gamma \vdash_{\text{base}} \alpha$. So $\Gamma \vdash_{\text{base}} \alpha$. So $\vdash$ is identical to $\vdash_{\text{base}}$.

The remainder of the result follows from Proposition 3.2, for the case where $l = m = o = 0$ and $n = 1$. $\Box$

Notice that if the rule $\Box p \rightarrow \Box q / p \rightarrow q$ is admissible in a modal logic, so is the rule $\Box p / p$.

We can use this result to show that various modal logics generate themselves.

Corollary 5.2. *The modal logics* K, KD, KD$_c$, KDG, *and* KT^n! *for any $n > 0$ are self-generating.*

Proof K corresponds to the class of all frames. The class of models based on this class of frames is single-minded.

KD corresponds to the class of serial frames. The class of models based on this class of frames is single-minded.

KD_c corresponds to the class of frames in which at most one world is accessible from each world. The class of models based on this class of frames is single-minded.

KDG corresponds to the class of serial and convergent frames. The class of models based on this class of frames is single-minded.

$\mathrm{KT}^n!$ corresponds to the class of frames in which wR^nv just in case $w = v$. These frames include single reflexive worlds, loops with size a factor of n, and disjoint unions of such frames. Every model based on this class of frames is single-minded. □

We can also find strongly classical normal modal logics such that every extension of them is self-generating. For any fixed $n \geq 0$, consider the two principles $p \to \Diamond^n\Box p$ and $\Box^n\Diamond\top$. We can show that if $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic that has these two principles as theorems, then it is self-generating. The argument is as follows: By the Sahlqvist correspondence theorem, these two principles are canonical. The first principle corresponds to the frame condition $\forall v\exists u(vR^nu \land R(u) \subseteq \{v\})$. The second principle corresponds to the frame condition of quasi^n-seriality $\forall w, v(wR^nv \to \exists uvRu)$. Suppose v is a world in the canonical model for $\vdash_{\text{base}}$. By the first condition, there is a world u such that wR^nu, and if any world is accessible from u, it is v. By the second condition, there is a world accessible from u. So v and only v is accessible from u.

We can also show that many strongly classical normal modal logics are not self-generating.[43] As before, we do this by finding the weakest strongly classical normal modal logic that extends a given logic that is self-generating.

Proposition 5.3.

- *The only consistent strongly classical normal modal logic extending any of* KT*,* KT_c*,* K4*,* $\mathrm{K4}_c$*,* K5*, or* $\mathrm{K5}_c$ *that is self-generating is Triv.*[44]
- *The weakest strongly classical normal modal logic extending* KB *that is self-generating is* KB!*.*[45]
- *The weakest strongly classical normal modal logic extending* KG *that is self-generating is* KDG*.*
- *The only strongly classical normal modal logic extending* GL *that is self-generating is the inconsistent logic* $\perp$*.*
- *The weakest strongly classical normal modal logic extending* KT^n_c *that is self-generating is* $\mathrm{KT}^n!$*.*
- *The only strongly classical normal modal logic extending* Ver *that is self-generating is the inconsistent logic* $\perp$*.*

[43] Exercise 5.33 in [5, p. 182] asks one to show that the rule $\Box p \to \Box q/p \to q$ is not admissible in several familiar strongly classical normal modal logics, from which it follows by Theorem 5.1 that they are not self-generating.

[44] The case of KT is proved in [39, p. 305].

[45] The case of KB follows from Proposition 3.1 in [42, p. 186].

Proof First note that Triv is self-generating. Triv extends KT, KT_c, K4, $K4_c$, K5, and $K5_c$. There is no consistent strongly classical normal modal logic that extends Triv.

KT: By Proposition 4.6, the weakest strongly classical normal modal logic extending KT that is self-generating is Triv.

KT_c: $\Box$T is a theorem of KT_c. Any self-generating logic that has $\Box$T as a theorem has T as a theorem. By the result for KT, the weakest strongly classical normal modal logic extending KT that is self-generating is Triv.

K4: By Theorem 5.1, any self-generating logic that has 4 as a theorem has T_c as a theorem. By the result for KT_c, the weakest strongly classical normal modal logic extending KT_c that is self-generating is Triv.

$K4_c$: By Theorem 5.1, any self-generating logic that has 4_c as a theorem has T as a theorem. By the result for KT, the weakest strongly classical normal modal logic extending KT that is self-generating is Triv.

K5: $\Box$T is a theorem of K5. Any self-generating logic that has $\Box$T as a theorem has T as a theorem. By the result for KT, the weakest strongly classical normal modal logic extending KT that is self-generating is Triv.

$K5_c$: By Corollary 4.11, the weakest strongly classical normal modal logic extending 5_c has $\Box T_c$ as a theorem. Any self-generating logic that has $\Box T_c$ as a theorem has T_c as a theorem. By the result for KT_c, the weakest strongly classical normal modal logic extending KT_c that is self-generating is Triv.

KB: $\Box\alpha \to \Box\Diamond\Box\alpha$ is a theorem of KB. By Theorem 5.1, any self-generating logic that has this sentence as a theorem has $\alpha \to \Diamond\Box\alpha$ as a theorem. This sentence is logically equivalent to B_c. Since KB! is the same modal logic as KT^2!, KB! is self-generating.

KG: $\Box$D is a theorem of KG. Any self-generating logic that has $\Box$D as a theorem has D as a theorem. KDG is self-generating.

GL: 4 is a theorem of GL. By the result for K4, the weakest strongly classical normal modal logic extending K4 is Triv. $\Box(\bot \to \bot) \to \Box\bot$ is a theorem of GL. In Triv, this is equivalent to $(\bot \to \bot) \to \bot$, which is equivalent to $\bot$.

KT^n_c: $\Box^n T^n$ is a theorem of KT^n_c. Any self-generating logic that has $\Box^n T^n$ as a theorem has T^n as a theorem. KT^n! is self-generating.

Ver: The case of Ver is obvious. $\Box$

Even if a finitary strongly classical normal modal logic is not self-generating, it may generate a logic with the very same theorems as itself. (In such a case, the generated logic will have additional derivable rules. These rules will be admissible but not derivable in the base logic.) We can provide simple necessary and sufficient conditions for $\vdash_{\text{base}}$ to generate a logic with the very same theorems:

Theorem 5.4. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic and $\vdash$ is generated from $\vdash_{\text{base}}$. Then the following are equivalent:*

- *$\vdash \alpha$ just in case $\vdash_{\text{base}} \alpha$.*
- *The rule $\Box p/p$ is admissible in $\vdash_{\text{base}}$.*
- *The canonical model for $\vdash_{\text{base}}$ has the following property: $\forall w \exists u\, uRw$.*
- *$\vdash_{\text{base}}$ is weakly determined with respect to a class of models C that has the following property: For every model $M = \langle W, R, V\rangle$ in C and $w \in W$, there*

is an $M' = \langle W', R', V' \rangle$ *in* C *(perhaps identical to* M*) such that* $M'_w = M_w$ *and* $\exists u \in W'$ *such that* $uR'w$*.*

Proof Suppose $\vdash \alpha$ just in case $\vdash_{\text{base}} \alpha$. Suppose $\vdash_{\text{base}} \Box\alpha$. So $\vdash \alpha$. So $\vdash_{\text{base}} \alpha$. So the rule $\Box p/p$ is admissible in $\vdash_{\text{base}}$.

For the other direction, suppose the rule $\Box p/p$ is admissible in $\vdash_{\text{base}}$. $\vdash \alpha$ just in case $\vdash_{\text{base}} \Box\alpha$. By Necessitation and the admissible rule, this obtains just in case $\vdash_{\text{base}} \alpha$.

The remainder of the result follows from Proposition 3.1, for the case where $m = 0$ and $n = 1$. $\Box$

Corollary 5.5. *Any finitary strongly classical normal modal logic that extends* KT *or* KDB *generates a modal logic with the same theorems as itself. Each of* K4*,* KD4*,* KDG4*,* GL*,* $\mathrm{K4}_c$*, and* $\mathrm{K5}_c$ *generates a modal logic with the same theorems as itself.*[46]

Proof In any normal modal logic extending KT, $\Box p/p$ is derivable and hence admissible. The canonical model for any normal modal logic extending KDB is serial and symmetric, so $\forall w \exists u \, uRw$.

K4 corresponds to the class of all transitive frames. KD4 corresponds to the class of serial transitive frames. KDG4 corresponds to the class of frames that are serial, convergent, and transitive. GL is weakly determined with respect to the class of all finite frames that have transitive and irreflexive accessibility relations. Consider the class of models based on any of these classes of frames. Given any model M in the class and world w in the model, consider the model that is just like M except for the addition of a world u such that $R(u) = R(w) \cup \{w\}$. This model will also be in the class. So the class of models has the property listed in Theorem 5.4.

$\mathrm{K4}_c$ corresponds to the class of dense frames. $\mathrm{K5}_c$ corresponds to the class of frames such that for $\forall w \exists v (wRv \wedge R(v) \subseteq R(w))$. Consider the class of models based on either of these classes of frames. Given any model M in the class and world w in the model, consider the model that is just like M except for the addition of a world u such that uRu and uRw. This model will also be in the class. So the class of models has the property listed in Theorem 5.4. $\Box$

We can show that many strongly classical normal modal logics do not generate logics with the very same theorems. Again, we do this by finding the weakest strongly classical normal modal logic extending a given logic that generates a logic with the very same theorems as the given logic.

Corollary 5.6. *For each of the following pairs, the weakest strongly classical normal modal logic extending the first that generates a logic with the very same theorems as itself is the second.*

- K5*,* S5

[46] See [5, pp. 99–100, 168] for results concerning the admissibility of the rule $\Box p/p$ in some familiar modal logics. See [20, pp. 44–6, 79–90] for results concerning the admissibility of the stronger rule of disjunction in some familiar modal logics.

- KG, KDG
- KG4, KDG4
- KB, KDB
- KT^n_c, $\mathrm{KT}^n!$
- Ver, $\perp$

Proof K5: $\Box$T is a theorem of K5. So any strongly classical normal modal logic extending K5 that generates a logic with the very same theorems has T as a theorem. By Corollary 5.5, S5 generates a logic with the very same theorems.

KG: $\Box$D is a theorem of KG. So any strongly classical normal modal logic extending KG that generates a logic with the very same theorems has D as a theorem. By Corollary 5.2, KDG is self-generating and so generates a logic with the very same theorems.

KG4: By the case of KG, any strongly classical normal modal logic extending KG4 that generates a logic with the very same theorems has D as a theorem. By Corollary 5.5, KDG4 generates a logic with the very same theorems.

KB: $\Box$D is a theorem of KB. By Corollary 5.5, KDB generates a logic with the very same theorems.

KT^n_c: $\Box^n\mathrm{T}^n$ is a theorem of KT^n_c. So any strongly classical normal modal logic extending KT^n_c that generates a logic with the very same theorems has T^n as a theorem. By Corollary 5.2, $\mathrm{KT}^n!$ is self-generating and so generates a logic with the very same theorems.

Ver: The case of Ver is obvious. $\Box$

In the next section, we'll consider the issue of how to directly characterize the logic generated from a given finitary strongly classical normal modal logic. But before we turn to that issue, there is a natural technical question to consider. The results of this paper so far tell us for various modal logics (i) what the weakest *finitary strongly classical* normal modal logic is extending the given logic that generates a strongly classical logic, (ii) what the weakest *finitary* normal modal logic is extending the given logic that is strongly classical and self-generating, and (iii) what the weakest *finitary strongly classical* normal modal logic is extending the given logic that generates a logic with the very same theorems as itself. One might wonder if we can find still weaker logics with the relevant properties if we permit the generating logic to be *non-finitary* or *weakly* classical. It turns out that the answer is no:

Proposition 5.7. *Suppose* $\vdash_{\text{base}}$ *is a broadly classical normal modal logic. Then there is a finitary strongly classical normal modal logic* $\vdash^-_{\text{base}}$ *such that:*

(i) $\vdash_{\text{base}}$ *extends* $\vdash^-_{\text{base}}$.

(ii) $\vdash^-_{\text{base}}$ *extends every finitary strongly classical normal modal logic that* $\vdash_{\text{base}}$ *extends.*

(iii) *If* $\vdash_{\text{base}}$ *generates a strongly classical logic, so does* $\vdash^-_{\text{base}}$.

(iv) *If* $\vdash_{\text{base}}$ *generates a strongly classical logic and self-generates,* $\vdash^-_{\text{base}}$ *also self-generates.*

(v) *The logic generated from* $\vdash^-_{\text{base}}$ *has the very same theorems as the logic generated from* $\vdash_{\text{base}}$.

(vi) *If* $\vdash_{\text{base}}$ *is finitary and generates a strongly classical logic,* $\vdash^-_{\text{base}}$ *generates the very same logic.*

Proof (i) Let $\vdash^-_{\text{base}}$ be defined as follows: $\Gamma \vdash^-_{\text{base}} \alpha$ just in case $\vdash_{\text{base}} \gamma \to \alpha$ for some conjunction γ of members of Γ. It is easy to see that $\vdash^-_{\text{base}}$ is a finitary strongly classical normal modal logic and that $\vdash_{\text{base}}$ extends $\vdash^-_{\text{base}}$.

(ii) Suppose $\vdash_{\text{base}}$ extends a finitary strongly classical normal modal logic $\vdash_{\text{given}}$. Suppose $\Gamma \vdash_{\text{given}} \alpha$. Since $\vdash_{\text{given}}$ is finitary and strongly classical, $\vdash_{\text{given}} \gamma \to \alpha$, where Γ is a conjunction of members of Γ. Since $\vdash_{\text{base}}$ extends $\vdash_{\text{given}}$, $\vdash_{\text{base}} \gamma \to \alpha$. By the definition of $\vdash^-_{\text{base}}$, $\Gamma \vdash^-_{\text{base}} \alpha$. So $\vdash^-_{\text{base}}$ extends $\vdash_{\text{given}}$.

(iii) Suppose $\vdash_{\text{base}}$ generates a strongly classical logic. Suppose $\Box\alpha \vdash^-_{\text{base}} \Box\beta$. Since $\vdash_{\text{base}}$ extends $\vdash^-_{\text{base}}$, $\Box\alpha \vdash_{\text{base}} \Box\beta$. Since $\vdash_{\text{base}}$ generates a strongly classical logic, $\vdash_{\text{base}} \Box(\alpha \to \beta)$. By the definition of $\vdash^-_{\text{base}}$, $\vdash^-_{\text{base}} \Box(\alpha \to \beta)$. So, since $\vdash^-_{\text{base}}$ is finitary, it generates a strongly classical logic.

(iv) Suppose $\vdash_{\text{base}}$ generates a strongly classical logic and self-generates. Suppose $\Box\Gamma \vdash^-_{\text{base}} \Box\alpha$. Since $\vdash_{\text{base}}$ is finitary $\Box\gamma \vdash^-_{\text{base}} \Box\alpha$, where γ is a conjunction of members of Γ. By the definition of $\vdash^-_{\text{base}}$, $\vdash_{\text{base}} \Box\gamma \to \Box\alpha$. Since $\vdash_{\text{base}}$ generates a strongly classical logic, $\vdash_{\text{base}} \Box(\gamma \to \alpha)$. Since $\vdash_{\text{base}}$ self-generates, $\vdash_{\text{base}} \gamma \to \alpha$. By the definition of $\vdash^-_{\text{base}}$, $\gamma \vdash^-_{\text{base}} \alpha$. So $\Gamma \vdash^-_{\text{base}} \alpha$. So $\vdash^-_{\text{base}}$ self-generates.

(v) Since $\vdash_{\text{base}}$ extends $\vdash^-_{\text{base}}$, if $\vdash^-_{\text{base}} \Box\alpha$ then $\vdash_{\text{base}} \Box\alpha$. By the definition of $\vdash^-_{\text{base}}$, if $\vdash_{\text{base}} \Box\alpha$ then $\vdash^-_{\text{base}} \Box\alpha$. So the logic generated from $\vdash^-_{\text{base}}$ has the very same theorems as the logic generated from $\vdash_{\text{base}}$.

(vi) Suppose $\vdash_{\text{base}}$ is finitary and generates a strongly classical logic. Suppose $\Box\Gamma \vdash_{\text{base}} \Box\alpha$. Since $\vdash_{\text{base}}$ is normal and finitary, $\Box\gamma \vdash_{\text{base}} \Box\alpha$, where γ is a conjunction of members of Γ. Since $\vdash_{\text{base}}$ generates a strongly classical logic, $\vdash_{\text{base}} \Box(\gamma \to \alpha)$. By the definition of $\vdash^-_{\text{base}}$, $\vdash^-_{\text{base}} \Box(\gamma \to \alpha)$. Since $\vdash^-_{\text{base}}$ is normal, $\Box\gamma \vdash^-_{\text{base}} \Box\alpha$, and so $\Box\Gamma \vdash^-_{\text{base}} \Box\alpha$.

For the other direction, suppose $\Box\Gamma \vdash^-_{\text{base}} \Box\alpha$. Since $\vdash_{\text{base}}$ extends $\vdash^-_{\text{base}}$, $\Box\Gamma \vdash_{\text{base}} \Box\alpha$. So $\vdash^-_{\text{base}}$ generates the very same logic as $\vdash_{\text{base}}$. $\Box$

6 Characterizing Generated Consequence Relations

Given a finitary strongly classical normal modal logic $\vdash_{\text{base}}$ and a consequence relation $\vdash$ generated from it, is there a way to directly characterize $\vdash$? We've already seen cases in which $\vdash$ is identical to $\vdash_{\text{base}}$. But what happens when it is not? The task of this section is to investigate this question.

For a wide variety of base logics, we can provide a characterization of the logics they generate by adding a meta-rule to $\vdash_{\text{base}}$:

Theorem 6.1. *Suppose* $\vdash_{\text{base}}$ *is a finitary strongly classical normal modal logic. Let* $\vdash^+$ *be the weakest broadly classical normal modal logic that extends* $\vdash_{\text{base}}$ *such that the following meta-rule obtains for every* α *and* β*: if* $\vdash^+ \Box\alpha \to \Box\beta$ *then* $\alpha \vdash^+ \beta$*. Then the following are equivalent:*

- $\vdash$ *is identical to* $\vdash^+$.
- $\Box(\Box p \to \Box q)/\Box p \to \Box q$ *is admissible in* $\vdash_{\text{base}}$.

- *The canonical model for $\vdash_{\text{base}}$ has the following property: $\forall w, v(wRv \to \exists u, u^*(uRu^* \wedge u^*Rv \wedge R(u^*) \subseteq R(w)))$.*
- *$\vdash_{\text{base}}$ is weakly determined with respect to a class C of models that has the following related property: For every model $M = \langle W, R, V\rangle$ in C and $w, v \in W$, if wRv then there is an $M' = \langle W', R', V'\rangle$ in C (perhaps identical to M) such that $M'_w = M_w$ and $\exists u, u^* \in W'$ such that $uR'u^*$, $u^*R'v$, and $R'(u^*) \subseteq R'(w)$.*

Proof Suppose the rule $\Box(\Box p \to \Box q)/\Box p \to \Box q$ is admissible in $\vdash_{\text{base}}$. We first show that if $\Gamma \vdash \alpha$ then $\Gamma \vdash^+ \alpha$. Suppose $\Gamma \vdash \alpha$. So $\Box\Gamma \vdash_{\text{base}} \Box\alpha$. Since $\vdash_{\text{base}}$ is finitary, $\Box\gamma \vdash_{\text{base}} \Box\alpha$, where γ is the conjunction of some finite subset of Γ. Since $\vdash_{\text{base}}$ is strongly classical, $\vdash_{\text{base}} \Box\gamma \to \Box\alpha$. So $\vdash^+ \Box\gamma \to \Box\alpha$. By the definition of $\vdash^+$, $\gamma \vdash^+ \alpha$. So $\Gamma \vdash^+ \alpha$.

We next show that if $\Gamma \vdash^+ \alpha$ then $\Gamma \vdash \alpha$. Since $\vdash$ extends $\vdash_{\text{base}}$, all we need to show is that the meta-rule obtains for $\vdash$. Suppose $\vdash \Box\alpha \to \Box\beta$. So $\vdash_{\text{base}} \Box(\Box\alpha \to \Box\beta)$. By the admissible rule, $\vdash_{\text{base}} \Box\alpha \to \Box\beta$. By Modus Ponens and Cut, $\Box\alpha \vdash_{\text{base}} \Box\beta$. So $\alpha \vdash \beta$.

For the other direction, suppose $\vdash$ is identical to $\vdash^+$. Suppose $\vdash_{\text{base}} \Box(\Box\alpha \to \Box\beta)$. So $\vdash \Box\alpha \to \Box\beta$. So $\vdash^+ \Box\alpha \to \Box\beta$. By the meta-rule, $\alpha \vdash^+ \beta$. Since $\vdash$ is identical to $\vdash^+$, $\alpha \vdash \beta$. So $\Box\alpha \vdash_{\text{base}} \Box\beta$. So $\vdash_{\text{base}} \Box\alpha \to \Box\beta$.

The remainder of the result follows from Proposition 3.2, for the case where $l = 0$ and $m = n = o = 1$. $\Box$

Incidentally, notice that if $\vdash_{\text{base}}$ is a strongly classical normal modal logic, the converse of the meta-rule (namely, if $\alpha \vdash \beta$ then $\vdash \Box\alpha \to \Box\beta$) obtains for the generated logic $\vdash$. So if $\vdash_{\text{base}}$ satisfies the conditions listed in Theorem 6.1, $\alpha \vdash \beta$ just in case $\vdash \Box\alpha \to \Box\beta$. We also get the other direction: If $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic such that $\alpha \vdash \beta$ just in case $\vdash \Box\alpha \to \Box\beta$, the conditions listed in Theorem 6.1 are satisfied.

We can use Theorem 6.1 to characterize the logics generated from many familiar modal logics:

Corollary 6.2. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic. Suppose $\vdash_{\text{base}}$ is an extension of* KT, *an extension of* KB, *or is one of* K4, KD4, KDG4, K45, KD45, GL, K4$_c$, *or* K5$_c$. *Then $\vdash$ is the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the following meta-rule obtains for every α and β: if $\vdash \Box\alpha \to \Box\beta$ then $\alpha \vdash \beta$.*

Proof Suppose $\vdash_{\text{base}}$ is an extension of KT or is one of K4, KD4, KDG4, GL, K4$_c$, or K5$_c$. By Corollary 5.5, $\vdash_{\text{base}}$ generates a logic with the same theorems as itself. So by Theorem 5.4, the rule $\Box p/p$ is admissible in $\vdash_{\text{base}}$. So $\Box(\Box p \to \Box q)/\Box p \to \Box q$ is admissible in $\vdash_{\text{base}}$. So by Theorem 6.1, the result holds for these cases.

Suppose $\vdash_{\text{base}}$ is an extension of KB. The canonical model for $\vdash_{\text{base}}$ is symmetric. Suppose wRv in the canonical model. Then by symmetry, vRw. Let $u = v$ and $u^* = w$. Then $uRu^* \wedge u^*Rv \wedge R(u^*) \subseteq R(w)$. So by Theorem 6.1, the result holds for these cases.

Suppose $\vdash_{\text{base}}$ is K45 or KD45. These modal logics correspond to the class of transitive and Euclidean frames and the class of serial, transitive, and Euclidean frames, respectively. Let C be the class of models based on the appropriate class of frames. Suppose $M \in C$ and wRv in M. Let M' be a model that is just like M except for the addition of a new world u such that $R'(u) = R(v) \cup \{v\}$. Then $M' \in C$. So C has the related property (letting $u^* = v$). So by Theorem 6.1, the result holds for these cases, too.[*] □

It is not always the case that $\vdash$ can be characterized as the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the meta-rule obtains. For instance, if $\vdash_{\text{base}}$ is an extension of K5 that is not an extension of K4, then $\Box(\Box p \to \Box\Box p)$ is a theorem of $\vdash_{\text{base}}$ but $\Box p \to \Box\Box p$ is not, so $\Box(\Box p \to \Box q)/\Box p \to \Box q$ is not admissible.

When $\vdash$ can be characterized in this way, such a characterization can be useful for providing a sequent calculus for axiomatizing $\vdash$. We can axiomatize $\vdash$ by adding the following sequent rule to a sequent calculus for $\vdash_{\text{base}}$:

$$\frac{\Rightarrow \Box\alpha \to \Box\beta}{\alpha \Rightarrow \beta}$$

For other purposes, though, it would be good to be able to characterize $\vdash$ as the result of adding a derivable rule (or rules) to $\vdash_{\text{base}}$. In many cases, we can do exactly that.

There is a simple way to characterize $\vdash$ when it is generated from an extension of KT. In fact, it suffices that $\Box(\Box p \to \Box q) \to (\Box p \to \Box q)$ is a theorem of $\vdash_{\text{base}}$.[47] (Notice that this is the conditional corresponding to the admissible rule appearing in Theorem 6.1.)

Theorem 6.3. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic. Let $\vdash^+$ be the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the rule $\Box p \to \Box q, p/q$ is derivable. Then $\vdash$ is identical to $\vdash^+$ just in case $\vdash_{\text{base}} \Box(\Box p \to \Box q) \to (\Box p \to \Box q)$.*

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[*] This result can be extended still further. Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logics. It is easy to see that the result applies if $\vdash_{\text{base}}$ has $\Box(\Box p \to \Box q) \to (\Box p \to \Box q)$ as a theorem. This includes all extensions of KT and all extensions of K45. More generally, it applies if $\vdash_{\text{base}}$ has $\Box^n(\Box p \to \Box q) \to (\Box p \to \Box q)$ as a theorem (for any $n \geq 1$): Suppose $\vdash_{\text{base}} \Box(\Box p \to \Box q)$, so by Necessitation $\vdash_{\text{base}} \Box^n(\Box p \to \Box q)$, and so $\vdash_{\text{base}} \Box p \to \Box q$. This includes all extensions of KT^n. The result also applies if $\vdash_{\text{base}}$ extends KT_c: Suppose w and v are worlds in the canonical model for $\vdash_{\text{base}}$ such that wRv. By the construction of the canonical model, $w = v$. Let $u = u^* = w$. So the canonical model for $\vdash_{\text{base}}$ has the property listed in Theorem 6.1. (This includes the case of Ver.) As we've seen, the result also applies if $\vdash_{\text{base}}$ is an extension of KB. In fact, the result applies to every strongly classical normal modal logic listed in the first column of Table 1 below except K5 and KD5. Here are proofs of the remaining cases: (i) K, KD, KG, and KDG are strongly determined with respect to the classes of all models, all serial models, all convergent models, and all serial and convergent models, respectively. We can show that these classes are closed under the property listed in Theorem 6.1. Suppose M is a model in one of these classes and w and v are worlds in M such that wRv. Let M' be a model just like M with the addition of a world u such that $R(u) = \{w\}$. Let $u^* = w$. It is easy to show that $uR'u^*$, $u^*R'v$, and $R'(u^*) \subseteq R'(w)$, and that M' is is a member of the relevant class of models. (ii) KG4 is strongly determined with respect to the class of all transitive and convergent models. Suppose M is a model in this class and w and v are worlds in M such that wRv. Let M' be a model just like M with the addition of a world u such that $R(u) = W_w$. Let $u^* = w$. Again, we can show that $uR'u^*$, $u^*R'v$, and $R'(u^*) \subseteq R'(w)$, and that M' is a member of the class of models.

[47] By the Sahlqvist correspondence theorem, this sentence is valid on all and only those frames such that $\forall w, v(wRv \to \exists u(wRu \wedge uRv \wedge R(u) \subseteq R(w))$.

Proof Suppose $\vdash_{\text{base}} \Box(\Box p \to \Box q) \to (\Box p \to \Box q)$. We first show that if $\Gamma \vdash \alpha$ then $\Gamma \vdash^+ \alpha$. Suppose $\Gamma \vdash \alpha$. So $\vdash_{\text{base}} \Box\gamma \to \Box\alpha$, where γ is the conjunction of some finite subset of Γ. So $\vdash^+ \Box\gamma \to \Box\alpha$. By the definition of $\vdash^+$, $\Box\gamma \to \Box\alpha, \gamma \vdash^+ \alpha$. By Cut, $\gamma \vdash^+ \alpha$. So $\Gamma \vdash^+ \alpha$.

We next show that if $\Gamma \vdash^+ \alpha$ then $\Gamma \vdash \alpha$. Since $\vdash$ extends $\vdash_{\text{base}}$, all we need to show is that $\Box\alpha \to \Box\beta, \alpha \vdash \beta$. $\vdash_{\text{base}} \Box(\Box\alpha \to \Box\beta) \to (\Box\alpha \to \Box\beta)$. By Modus Ponens and Cut, $\Box(\Box\alpha \to \Box\beta), \Box\alpha \vdash_{\text{base}} \Box\beta$. So $\Box\alpha \to \Box\beta, \alpha \vdash \beta$.

For the other direction, suppose $\vdash$ is identical to $\vdash^+$. $\Box\alpha \to \Box\beta, \alpha \vdash^+ \beta$. So $\Box\alpha \to \Box\beta, \alpha \vdash \beta$. So $\Box(\Box\alpha \to \Box\beta), \Box\alpha \vdash_{\text{base}} \Box\beta$. So $\vdash_{\text{base}} \Box(\Box\alpha \to \Box\beta) \to (\Box\alpha \to \Box\beta)$. $\Box$

In particular, if $\vdash_{\text{base}}$ is an axiomatizable finitary strongly classical normal modal logic that has $\Box(\Box p \to \Box q) \to (\Box p \to \Box q)$ as a theorem, we can axiomatize $\vdash$ by adding the rule of inference $\Box p \to \Box q, p/q$ to the axiomatization.* By Corollary 5.5, if $\vdash_{\text{base}}$ extends KT, $\vdash_{\text{base}}$ has the same theorems as $\vdash$, so in that case, adding the rule of inference does not generate any new theorems.

We can prove similar results for every extension of KB and for some extensions of K4. To do so, it's useful to prove some more general results:

Proposition 6.4. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic and l and m are natural numbers. Let $\vdash^+$ be the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the rule $p/\Diamond^l\Box^m p$ is derivable. Then:*

(i) *If $\Box p \to \Box q/p \wedge \Diamond^l\Box^m p \to q$ is admissible in $\vdash_{\text{base}}$ then $\vdash \subseteq \vdash^+$.*
(ii) *$\vdash^+ \subseteq \vdash$ just in case $\vdash_{\text{base}} \Box p \to \Box\Diamond^l\Box^m p$.*

Proof (i) Suppose $\Box p \to \Box q/p \wedge \Diamond^l\Box^m p \to q$ is admissible in $\vdash_{\text{base}}$. Suppose $\Gamma \vdash \alpha$. So $\vdash_{\text{base}} \Box\gamma \to \Box\alpha$, where γ is the conjunction of some finite subset of Γ. By the admissible rule, $\vdash_{\text{base}} \gamma \wedge \Diamond^l\Box^m\gamma \to \alpha$. So $\gamma, \Diamond^l\Box^m\gamma \vdash_{\text{base}} \alpha$. So $\gamma, \Diamond^l\Box^m\gamma \vdash^+ \alpha$. By the definition of $\vdash^+$, $\gamma \vdash^+ \Diamond^l\Box^m\gamma$. By Cut, $\gamma \vdash^+ \alpha$. So $\Gamma \vdash^+ \alpha$.

(ii) Suppose $\vdash_{\text{base}} \Box p \to \Box\Diamond^l\Box^m p$. Since $\vdash$ extends $\vdash_{\text{base}}$, this suffices to show that $\vdash^+ \subseteq \vdash$. For the other direction, suppose $\vdash^+ \subseteq \vdash$. By the definition of $\vdash^+$, $p \vdash^+ \Diamond^l\Box^m p$. So $p \vdash \Diamond^l\Box^m p$. So $\Box p \vdash_{\text{base}} \Box\Diamond^l\Box^m p$. So $\vdash_{\text{base}} \Box p \to \Box\Diamond^l\Box^m p$. $\Box$

Proposition 6.5. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic. Then the following are equivalent:*

- *$\Box p \to \Box q/p \wedge \Diamond^l\Box^m p \to q$ is admissible in $\vdash_{\text{base}}$.*
- *The canonical model for $\vdash$ has the following property: $\forall w, v(wR^l v \to \exists u(uRw \wedge R(u) \subseteq R^m(v) \cup \{w\}))$.*
- *$\vdash$ is weakly determined with respect to a class C of models that has the following related property: For every model $M = \langle W, R, V\rangle$ in C and $w, v \in W$, if $wR^l v$ then there is an $M' = \langle W', R', V'\rangle$ in C (perhaps identical to M) such that $M'_w = M_w$ and $\exists u \in W'$ such that $uR'w$ and $R'(u) \subseteq R'^m(v) \cup \{w\}$.*

*Notice that if $\vdash_{\text{base}}$ extends K4!, it will have $\Box(\Box p \to \Box q) \to (\Box p \to \Box q)$ as a theorem. For more on such logics, see the *-ed footnote on page 34.

Proof This result follows from Proposition 3.3, for the case where $n = 1$. $\square$

Fact 6.6. *$\vdash_{\text{base}} \Box p \to \Box\Diamond^n\Box^m p$ just in case the canonical model for $\vdash_{\text{base}}$ has the following property: $\forall w, v(wRv \to \exists u(vR^l u \wedge R^m(u) \subseteq R(w)))$.*

Proof This result follows from the Sahlqvist correspondence theorem. $\square$

Using these results, we can characterize every extension of KB and some extensions of K4 (including every extension of S4).

Theorem 6.7. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic that extends* KB. *Let $\vdash^+$ be the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the rule $p/\Diamond\Box p$ is derivable. Then $\vdash$ is identical to $\vdash^+$.*

Proof By Proposition 6.4, we only need to show (i) $\Box p \to \Box q/p \wedge \Diamond\Box p \to q$ is admissible in $\vdash_{\text{base}}$ and (ii) $\vdash_{\text{base}} \Box p \to \Box\Diamond\Box p$.

For the former, by Proposition 6.5, it suffices to show that in the canonical model for $\vdash_{\text{base}}$, $\forall w, v(wRv \to \exists u(uRw \wedge R(u) \subseteq R(v) \cup \{w\}))$. Since $\vdash_{\text{base}}$ extends KB, the canonical model for $\vdash_{\text{base}}$ is symmetric. So if wRv, we can take $u = v$. By symmetry, uRw. Moreover, $R(u) = R(v) \subseteq R(v) \cup \{w\}$.

For the latter, by Fact 6.6, it suffices to show that in the canonical model for $\vdash_{\text{base}}$, $\forall w, v(wRv \to \exists u(vRu \wedge R(u) \subseteq R(w)))$. Suppose wRv. We can take $u = w$. By symmetry, vRu. Moreover, $R(u) = R(w) \subseteq R(w)$. $\square$

In particular, if $\vdash_{\text{base}}$ is an axiomatizable finitary strongly classical normal modal logic extending KB, we can axiomatize $\vdash$ by adding the rule of inference $p/\Diamond\Box p$ to the axiomatization.

KB has $\Box$D as a theorem. So any modal logic extending KB generates a modal logic with D as a theorem. It is straightforward to show that in any broadly classical normal modal logic extending KD, the rule $p/\Diamond\Box p$ is admissible.[48] So if $\vdash_{\text{base}}$ is an axiomatizable finitary strongly classical normal modal logic that extends KB, we can axiomatize $\vdash$ by first adding D to the axiomatization (if it is not already a theorem) and then adding the rule of inference $p/\Diamond\Box p$. This is a useful axiomatization of the resulting logic because, since the rule is admissible in extensions of KD, adding the rule of inference to the axiomatization including D does not yield any new theorems.

For some extensions of K4, including K4 itself, KD4, KDG4, GL, and any finitary strongly classical normal modal logic that extends S4, there is also a simple characterization:

Theorem 6.8. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic that extends* K4 *and is such that the following rule is admissible: $\Box p \to$*

[48] Suppose α is a theorem. By two uses of Necessitation, $\Box\Box\alpha$ is a theorem. $\Box\Box\alpha \to \Diamond\Box\alpha$ is a theorem of KD. So $\Diamond\Box\alpha$ is a theorem, as well.

$\Box q/p \wedge \Box p \to q$. Let $\vdash^+$ be the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the rule $p/\Box p$ is derivable. Then $\vdash$ is identical to $\vdash^+$.

Proof By Proposition 6.4, we only need to show that $\vdash_{\text{base}} \Box p \to \Box\Box p$. This is true in any extension of K4. □

Corollary 6.9. *Suppose $\vdash_{\text{base}}$ is* K4, KD4, KDG4, GL*, or a finitary strongly classical normal modal logic that extends* S4*. Let $\vdash^+$ be the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the rule $p/\Box p$ is derivable. Then $\vdash$ is identical to $\vdash^+$.*

Proof By Theorem 6.8, it suffices to show that the rule $\Box p \to \Box q/p \wedge \Box p \to q$ is admissible in $\vdash_{\text{base}}$.

K4, KD4, KDG4, and GL: K4 corresponds to the transitive frames, KD4 corresponds to the serial and transitive frames, and KDG4 corresponds to the serial, convergent, and transitive frames. GL is weakly determined with respect to the class of finite frames that have transitive and irreflexive accessibility relations. Suppose $\alpha \wedge \Box\alpha \to \beta$ is false at a world v in a model M based on a frame in the relevant class of frames for $\vdash_{\text{base}}$. Let $M' = \langle W', R', V' \rangle$ be defined so that $W' = W \cup \{w\}$ (for some $w \notin W$), $R'(w) = W_v$, $R'(x) = R(x)$ for $x \in W$, and $V'(p) = V(p)$. It is easy to show that M' is based on a frame for $\vdash_{\text{base}}$. $\Box\alpha \to \Box\beta$ is false at w.

Extensions of S4: By T, the rule $\Box p \to \Box q/p \wedge \Box p \to q$ is derivable (and hence admissible) in any extension of KT, including every extension of S4. □

In particular, if $\vdash_{\text{base}}$ is K4, KD4, KDG4, GL, or an axiomatizable finitary strongly classical normal modal logic extending S4, then we can axiomatize $\vdash$ by adding the rule of inference $p/\Box p$ to the axiomatization of $\vdash_{\text{base}}$. Notice that since the rule $p/\Box p$ is admissible in any normal modal logic, adding $p/\Box p$ as a derivable rule to a normal modal logic doesn't yield any new theorems. So the modal logics generated from these base logics do not have any additional theorems.

If $\vdash_{\text{base}}$ is KG4, it is easy to see that $\vdash$ can be axiomatized by taking KDG4 and adding the rule of inference $p/\Box p$: □D is a theorem of KG, so $\vdash$ extends KDG4 with the addition of $p/\Box p$ as a derivable rule. But since KDG4 generates KDG4 with the addition of $p/\Box p$ as a derivable rule, $\vdash$ is extended by KDG4 with the addition of $p/\Box p$ as a derivable rule.*

*It is worth discussing the case where $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic extending K4!. It is easy to see that $\vdash$ can be characterized as the weakest extension of $\vdash_{\text{base}}$ such that the rules $p/\Box p$ and $\Box p/p$ are both derivable. (Both of these rules are admissible in K4!, so if $\vdash_{\text{base}}$ is K4! itself, $\vdash$ has the same theorems as $\vdash_{\text{base}}$.) We can say more about such logics. Over K4, 4*c* is equivalent to $\Box(\Box p \to \Box q) \to (\Box p \to \Box q)$. So by Theorem 6.1, $\vdash$ can be characterized as the weakest extension of $\vdash_{\text{base}}$ such that the following meta-rule obtains: if $\vdash \Box\alpha \to \Box\beta$ then $\alpha \vdash \beta$. By Theorem 6.3, $\vdash$ can be characterized as the weakest extension of $\vdash_{\text{base}}$ such that the rule $\Box p \to \Box q, p/q$ is derivable. (Notice that it is easy to directly show that in any extension of K4!, the rule $\Box p \to \Box q, p/q$ is derivable just in case the rules $p/\Box p$ and $\Box p/p$ are both derivable.) Moreover, if $\vdash_{\text{base}}$ is an extension of K4□T, then $\vdash$ has T as a theorem. So $\vdash$ can be characterized as the weakest extension of $\vdash_{\text{base}}$ that has T as a theorem and in which the rule $p/\Box p$ is derivable. Since this rule is admissible in all normal modal logics, $\vdash$ will have the same theorems as $\vdash_{\text{base}}$ with the addition of T. (Since K45 extends K4□T, this result applies to the extensions of K45.)

We can also provide direct characterizations of several other modal logics:

Corollary 6.10. *Suppose* $\vdash_{\text{base}}$ *is a finitary strongly classical normal modal logic that extends* K5 *and is extended by* KD5*. Then* $\vdash$ *is* S5*.*

Proof In K5, $\Box$T is a theorem. So $\vdash$ has T as a theorem. Since $\vdash$ extends K5, it extends S5.

To show S5 extends $\vdash$, suppose $\Gamma \nvdash_{\text{S5}} \alpha$. S5 corresponds to the class of frames in which R is an equivalence relation. So there is a world v in such a model at which the members of Γ are true but α is not. Add a world w to this model such that v and only v is accessible from w. It is straightforward to show that the resulting model is serial and Euclidean, so is a model of KD5. At w, the members of $\Box\Gamma$ are true but $\Box\alpha$ is false. So $\Box\Gamma \nvdash_{\text{KD5}} \Box\alpha$. So $\Box\Gamma \nvdash_{\text{base}} \Box\alpha$. So $\Gamma \nvdash \alpha$. $\Box$

Corollary 6.11. *Suppose* $\vdash_{\text{base}}$ *is a finitary strongly classical normal modal logic that extends* K45 *and is extended by* S5*. Then* $\vdash$ *is* S5 *with the addition of the derivable rule* $p/\Box p$*.*

Proof Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic that extends K45 and is extended by S5. Any finitary strongly classical normal modal logic that extends K4 generates a logic in which the rule $p/\Box p$ is derivable. K5 generates S5. So $\vdash$ must be at least as strong as S5 with the addition of the derivable rule. It cannot be stronger than this, since S5 itself generates S5 with the addition of the derivable rule. $\Box$

For instance, since KB5 has 4 as a theorem, the logic it generates is S5 with the addition of the derivable rule $p/\Box p$. As before, since $p/\Box p$ is admissible in any normal modal logic, S5 with the addition of the derivable rule $p/\Box p$ does not have any theorems beyond those of S5.

Corollary 6.12. *Suppose* $\vdash_{\text{base}}$ *is a finitary strongly classical normal modal logic that extends* KG *and is extended by* KDG*. Then* $\vdash$ *is* KDG*.*

Proof In KG, $\Box(\Box p \to \Diamond p)$ is a theorem. So $\vdash_{\text{base}} \Box(\Box p \to \Diamond p)$. So $\vdash$ has D as a theorem. Since $\vdash$ extends KG, it extends KDG.

To show KDG extends $\vdash$, suppose $\Gamma \nvdash_{\text{KDG}} \alpha$. KDG corresponds to the class of frames in which R is serial and convergent. So there is a world v in a serial and convergent model at which the members of Γ are true but α is not. Add a world w to this model such that v and only v is accessible from w. It is straightforward to show that the resulting model is also serial and convergent, and so is a model of KDG. At w, the members of $\Box\Gamma$ are true but $\Box\alpha$ is false. So $\Box\Gamma \nvdash_{\text{KDG}} \Box\alpha$ and thus $\Box\Gamma \nvdash_{\text{base}} \Box\alpha$. So $\Gamma \nvdash \alpha$. $\Box$

Finally, if $\vdash_{\text{base}}$ is an extension of S4, $\vdash$ can be characterized as the weakest extension of $\vdash_{\text{base}}$ in which the rule $p/\Box p$ is derivable. In this case, $\vdash$ will have the same theorems as $\vdash_{\text{base}}$.

Corollary 6.13. *Suppose* $\vdash_{\text{base}}$ *is* KT_c*. Then* $\vdash$ *is* Triv.[49]

Proof $\Box\mathrm{T}$ is a theorem of KT_c. So T is a theorem of $\vdash$. $\vdash$ extends KT_c. So $\vdash$ extends KT! = Triv.

To show Triv extends $\vdash$, suppose $\Gamma \nvdash_{\text{Triv}} \alpha$. Triv corresponds to the class of frames in which each world is accessible from exactly itself. So there is a world v in a model based on such a frame at which the members of Γ are true but α is not. This model is also based on a frame for KT_c. At v, the members of $\Box\Gamma$ are true but $\Box\alpha$ is not. So $\Box\Gamma \nvdash_{\text{base}} \Box\alpha$. So $\Gamma \nvdash \alpha$. □

Finally, we can easily characterize the modal logic generated from Ver: Ver generates the inconsistent modal logic.

Thus, we can directly characterize the modal logics generated from many familiar modal logics. Table 1 below lists direct characterizations of the modal logics generated from many familiar modal logics. It also summarizes many of the results about individual modal logics from the previous two sections.

7 Generating a Consequence Relation

In the previous section, we studied the question of how to characterize the broadly classical normal modal logic $\vdash$ generated from a given strongly classical normal modal logic $\vdash_{\text{base}}$. In this section, we consider the other direction: Given a broadly classical normal modal logic $\vdash$ which strongly classical normal modal logics generate it?

We can show that if $\vdash$ is finitary and is generated from a strongly classical normal modal logic, then there is a weakest strongly classical normal modal logic that generates it. We can also show that every finitary strongly classical normal modal logic is generated from a finitary strongly classical normal modal logic.

To show this, a bit of notation will be useful. Given a rule with finitely many premises $\theta = \langle\{\alpha_1, \ldots, \alpha_n\}, \beta\rangle$, let $\Box_{\to}\theta$ abbreviate the sentence $\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n \to \Box\beta$. Given a set of rules Θ with finitely many premises, let $\Box_{\to}\Theta$ abbreviate the set of sentences $\{\Box_{\to}\theta \mid \theta \in \Theta\}$.

Proposition 7.1.

(i) *Suppose* $\vdash$ *is the weakest finitary broadly classical normal modal logic that has the members of* Δ *as theorems and the members of* Θ *as derivable rules, where the rules in* Θ *have finitely many premises. Suppose* $\vdash$ *is generated from some strongly classical normal modal logic. Then the weakest strongly classical normal modal logic that generates* $\vdash$ *is the weakest broadly classical normal modal logic extending K that has the members of* $\Box\Delta \cup \Box_{\to}\Theta$ *as theorems. This logic is finitary.*

[49] It would be nice to have a direct characterization of the modal logic generated from KT^n_c for arbitrary $n > 0$. It is straightforward to show that this modal logic extends $\mathrm{KT}^n_c\Box^{n-1}\mathrm{T}^n$ and is extended by KT^n!. For $n = 1$, these coincide. It would be nice to pin it down further when $n > 1$.

Table 1 Results for Specific Modal Logics

$\vdash_{\text{base}}$	Ext:Strong	Ext:Self	Ext:Thms	$\vdash$
K	-	-	-	K
KD	-	-	-	KD
K5	-	Triv	S5	S5
KD5	-	Triv	S5	S5
KG	-	KDG	KDG	KDG
KDG	-	-	-	KDG
KT	Triv	Triv	-	KT + $\Box p \to \Box q, p/q$
KB	KB$\Box$B$_c$	KB!	KDB	KDB + $p/\Diamond\Box p$
KDB	KB!	KB!	-	KDB + $p/\Diamond\Box p$
KTB	Triv	Triv	-	KTB + $p/\Diamond\Box p$
K4	K$\Box$T$_c$	Triv	-	K4 + $p/\Box p$
KD4	KD$\Box$T$_c$	Triv	-	KD4 + $p/\Box p$
KG4	KG$\Box$T$_c$	Triv	KDG4	KDG4 + $p/\Box p$
KDG4	KDG$\Box$T$_c$	Triv	-	KDG4 + $p/\Box p$
S4	Triv	Triv	-	S4 + $p/\Box p$
S4.2	Triv	Triv	-	S4.2 + $p/\Box p$
KB5	Triv	Triv	S5	S5 + $p/\Box p$
K45	K5$\Box$T$_c$	Triv	S5	S5 + $p/\Box p$
KD45	KD5$\Box$T$_c$	Triv	S5	S5 + $p/\Box p$
S5	Triv	Triv	-	S5 + $p/\Box p$
GL	K$\Box^2\bot$	$\bot$	-	GL + $p/\Box p$
KD$_c$	-	-	-	KD$_c$
KT$_c$	-	Triv	Triv	Triv
KT^n!	-	-	-	KT^n! (for $n > 0$, $n = 1$ is Triv)
Ver	-	$\bot$	$\bot$	$\bot$

The first column lists the base logic. The second column specifies the weakest broadly classical normal modal logic extending the base logic that generates a strongly classical logic. The third column specifies the weakest strongly classical normal modal logic extending the base logic that is self-generating. The fourth column specifies the weakest broadly classical normal modal logic extending the base logic that generates a logic with the very same theorems. A hyphen represents that the entry is the same as the base logic. The final column provides a direct characterization of the generated logic in terms of a strongly classical normal modal logic with the addition (if needed) of a derivable rule. The characterization has been chosen so that the rule is admissible in the strongly classical logic, so adds no theorems.

(ii) *If $\vdash$ is a finitary strongly classical normal modal logic, then $\vdash$ is generated from some strongly classical normal modal logic.*

Proof (i) If a strongly classical normal modal logic has the members of $\Box\Delta \cup \Box_{\to}\Theta$ as theorems, it will generate a broadly classical normal modal logic at least as strong as $\vdash$. Any strongly classical normal modal logic generating $\vdash$ must have the members of $\Box\Delta \cup \Box_{\to}\Theta$ as theorems. So if $\vdash$ is generated from a strongly classical normal modal logic, the weakest such logic is the weakest strongly classical normal modal logic that has the members of $\Box\Delta \cup \Box_{\to}\Theta$ as theorems. It is easy to see that this logic is the weakest broadly classical normal modal logic extending K that has the members of $\Box\Delta \cup \Box_{\to}\Theta$ as theorems. It is also easy to see that this logic is finitary.

(ii) Suppose $\vdash$ is a finitary strongly classical normal modal logic. So it is the weakest finitary strongly classical normal modal logic that has the members of some set Δ as theorems. Consider the weakest strongly classical normal modal logic $\vdash_{\text{base}}$

that has the members of $\Box\Delta$ as theorems. It is easy to see that $\vdash_{\text{base}}$ generates a logic that extends $\vdash$. To show that $\vdash_{\text{base}}$ generates a logic that is extended by $\vdash$, suppose $\vdash \alpha$. We show $\vdash_{\text{base}} \Box\alpha$. (We can restrict attention to theorems since $\vdash$ is finitary and strongly classical.) Since $\vdash \alpha$, $\Delta^* \vdash_{\text{K}} \alpha$, where Δ^* is the smallest set that contains all substitution instances of members of Δ and is closed under the operation of prefixing a sentence with a $\Box$. Since K is normal, $\Box\Delta^* \vdash_{\text{K}} \Box\alpha$. The members of $\Box\Delta^*$ are theorems of $\vdash_{\text{base}}$. So by Cut, $\vdash_{\text{base}} \Box\alpha$. $\Box$

To illustrate this result, consider, for instance, S5. It follows from this result that the weakest strongly classical normal modal logic that generates S5 is K$\Box$T$\Box$5. Now consider S5 with the addition of $p/\Box p$ as a derivable rule. We know from the previous section that there is a strongly classical normal modal logic that generates this logic. It follows from the result that the weakest strongly classical normal modal logic generating this logic is K$\Box$T$\Box$5 with the addition of the sentence $\Box p \to \Box\Box p$, which is just the 4 principle. So the weakest strongly classical normal modal logic that generates S5 with the addition of $p/\Box p$ as a derivable rule is K4$\Box$T$\Box$5.

This result raises the question of whether every finitary weakly classical normal modal logic can be generated from some finitary strongly classical normal modal logic. The answer is no. Every logic that is generated from a strongly classical normal modal logic has K as a theorem, since it extends any logic that generates it. But not every finitary weakly classical normal modal logic has K as a theorem. Moreover, every logic that is generated from a strongly classical normal modal logic obeys the following meta-rule: If $\alpha \vdash \beta$ then $\vdash \Box\alpha \to \Box\beta$. (Suppose $\alpha \vdash \beta$. So $\Box\alpha \vdash_{\text{base}} \Box\beta$. So $\vdash_{\text{base}} \Box\alpha \to \Box\beta$. So $\vdash \Box\alpha \to \Box\beta$.) But not every finitary weakly classical normal modal logic obeys this meta-rule. For instance, K with the addition of $p/\Box p$ as a derivable rule does not obey this meta-rule, since in this logic, $\alpha \vdash \Box\alpha$ but it is not the case that 4 is a theorem. (4 is not a theorem because $p/\Box p$ is admissible in K, so adding the derivable rule does not yield any new theorems.) An interesting question is whether every finitary broadly classical normal modal logic that has K as a theorem and that obeys this meta-rule can be generated from some finitary strongly classical normal modal logic. I do not know the answer to this question.

We also have the following easy result:

Proposition 7.2. *If a broadly classical normal modal logic $\vdash$ is generated from two strongly classical normal modal logics, $\vdash_1$ and $\vdash_2$ such that $\vdash_2$ extends $\vdash_1$, then every strongly classical normal modal logic between $\vdash_1$ and $\vdash_2$ generates $\vdash$.*

Given a broadly classical normal modal logic $\vdash$ that can be generated from a strongly classical normal modal logic, is there a strongest strongly classical normal modal logic that generates $\vdash$? We can provide a partial answer to this question:

Proposition 7.3. *Suppose $\vdash$ is a finitary broadly classical normal modal logic.*

(i) *If* $\vdash$ *is generated from a finitary strongly classical normal modal logic extending KT, then there is a strongest finitary strongly classical normal modal logic generating* $\vdash$.

(ii) *If* $\vdash$ *is the weakest broadly classical logic extending the strongly classical normal modal logic* $\vdash_{\mathrm{S}}$ *by the addition of some derivable rules that are admissible in* $\vdash_{\mathrm{S}}$ *and* $\vdash_{\mathrm{S}}$ *generates* $\vdash$, *then* $\vdash_{\mathrm{S}}$ *is the strongest strongly classical normal modal logic generating* $\vdash$.

Proof (i) Suppose $\vdash$ is generated from a finitary strongly classical normal modal logic $\vdash_{\text{base}}$ extending KT. Consider the weakest finitary strongly classical normal modal logic $\vdash^{+}_{\text{base}}$ that extends every finitary strongly classical normal modal logic that generates $\vdash$. We show that $\vdash^{+}_{\text{base}}$ generates $\vdash$. Since $\vdash^{+}_{\text{base}}$ extends $\vdash_{\text{base}}$, the logic generated from $\vdash^{+}_{\text{base}}$ extends $\vdash$. For the other direction, suppose $\vdash^{+}_{\text{base}} \Box\alpha$. (We can restrict our attention to theorems, since $\vdash^{+}_{\text{base}}$ is finitary and strongly classical.) So $\Delta \vdash_{\text{base}} \Box\alpha$, where Δ is the union of the sets of theorems of the finitary strongly classical normal modal logics generating $\vdash$. Since $\vdash_{\text{base}}$ extends KT, $\Box\Delta \vdash_{\text{base}} \Box\alpha$. So $\Delta \vdash \alpha$. Every member of Δ is a theorem of $\vdash$. So $\vdash \alpha$.

(ii) If there is strongly broadly classical normal modal logic generating $\vdash$ that is not extended by $\vdash_{\mathrm{S}}$, then it will have a theorem α such that $\nvdash_{\mathrm{S}} \alpha$. But since any broadly classical normal modal logic is extended by the logic it generates, $\vdash \alpha$, which is impossible since $\vdash$ has the same theorems as $\vdash_{\mathrm{S}}$. $\Box$

To illustrate the second part of this result, let $\vdash$ be S5 with the addition of $p/\Box p$ as a derivable rule. Since S5 generates $\vdash$ and since $p/\Box p$ is admissible in S5, it follows from the result that S5 is the strongest strongly classical normal modal logic generating $\vdash$. So the strongly classical normal modal logics that generate S5 with the addition of $p/\Box p$ as a derivable rule are exactly the strongly classical normal modal logics between K4□T□5 and S5.

We can apply this result to determine exactly which strongly classical normal modal logics generate some broadly classical normal modal logics of interest:

Corollary 7.4.

(i) *Suppose* $\vdash$ *is the weakest broadly classical normal modal logic extending* KT *that has* $\Box p \to \Box q, p/q$ *as a derivable rule and has the members of some set* Δ *as theorems. Then the strongly classical normal modal logics generating* $\vdash$ *are exactly the strongly classical normal modal logics between* K□T *with* $\Box(\Box p \to \Box q) \to (p \to q)$ *and the members of* $\Box\Delta$ *as theorems and* KT *with the members of* Δ *as theorems.*

(ii) *Suppose* $\vdash$ *is the weakest broadly classical normal modal logic extending* KDB *that has* $p/\Diamond\Box p$ *as a derivable rule and has the members of some set* Δ *as theorems. Then the strongly classical normal modal logics generating* $\vdash$ *are exactly the strongly classical normal modal logics between* K□D□B *with* $\Box p \to \Box\Diamond\Box p$ *and the members of* $\Box\Delta$ *as theorems and* KDB *with the members of* Δ *as theorems.*

(iii) *Suppose* $\vdash$ *is the weakest broadly classical normal modal logic extending* K4 *that has* $p/\Box p$ *as a derivable rule and has the members of* Δ *as theorems, where* Δ *is* $\emptyset$, $\{\mathrm{D}\}$, $\{\mathrm{D}, \mathrm{G}\}$, $\{\mathrm{W}\}$, *or any set of sentences containing* T. *Then the strongly classical normal modal logics generating* $\vdash$ *are exactly the strongly classical normal modal logics between* K4 *with the members of* $\Box\Delta$ *as theorems and* K4 *with the members of* Δ *as theorems.*

Proof This result follows from applying Propositions 7.1 and 7.3 to Theorem 6.3, Theorem 6.7, and Corollary 6.9, respectively, noting that the rules listed are admissible in the relevant normal modal logics. □

Notice that it immediately follows that the only strongly classical normal modal logic generating K4 with the addition of $p/\Box p$ as a derivable rule is K4.

The results in this section so far tell us exactly which strongly classical normal modal logics generate the generated broadly classical normal modal logics appearing in the last column of Table 1, with one exception: They don't tell us which strongly classical normal modal logics generate S5. (They also don't tell us which strongly classical normal modal logics generate other familiar strongly classical normal modal logics, such as KT, S4, GL, and the like.) One might wonder what the strongest strongly classical normal modal logic is that generates S5, or even whether there is a strongest strongly classical normal modal logic that generates S5. I don't know the answers to these questions. We can, however, determine some of the strongly classical normal modal logics that generate S5 using the following result:

Proposition 7.5. *Suppose* $\vdash$ *is the weakest strongly classical normal modal logic that extends the strongly classical normal modal logic* $\vdash_{\mathrm{S}}$ *and has the members of* Δ *as theorems. Suppose the strongest strongly classical normal modal logic that is extended by the logic that* $\vdash$ *generates is* $\vdash$ *itself. Let* Γ *be any set of theorems of* $\vdash$ *and let* $\vdash_{\mathrm{new}}$ *be the weakest strongly classical normal modal logic extending* $\vdash_{\mathrm{S}}$ *with the members of* $\Gamma \cup \Box\Delta$ *as theorems. Then if* $\vdash_{\mathrm{new}}$ *generates a strongly classical logic,* $\vdash_{\mathrm{new}}$ *generates* $\vdash$.

Proof Suppose $\vdash_{\mathrm{new}}$ generates a strongly classical logic. Since $\vdash_{\mathrm{new}}$ is extended by $\vdash$ and since the strongest strongly classical normal modal logic extended by the logic $\vdash$ generates is $\vdash$ itself, $\vdash_{\mathrm{new}}$ generates a logic extended by $\vdash$. Since $\vdash_{\mathrm{new}}$ extends $\vdash_{\mathrm{S}}$ and has the members of $\Box\Delta$ as theorems, $\vdash_{\mathrm{new}}$ has the members of $\{\Box\alpha \mid \vdash \alpha\}$ as theorems. So $\vdash_{\mathrm{new}}$ generates a logic extending $\vdash$. So $\vdash_{\mathrm{new}}$ generates $\vdash$. □

Define the modal principle T^- to be $(\Box p \to p) \vee (\Box q \leftrightarrow \Diamond q)$. This is the disjunction of T with D!, using different atomic sentences in the two disjuncts. S5 can be characterized as K5 with the addition of T. S5 generates S5 with the addition of $p/\Box p$ as a derivable rule, so the strongest strongly classical normal modal logic that is extended by the logic that S5 generates is S5 itself. T^- is a theorem of S5. Define $\vdash_{\mathrm{new}}$ to be $\mathrm{K5}\Box\mathrm{TT}^-$. It is straightforward to show that $\vdash_{\mathrm{new}}$ is strongly determined with respect to the class of Euclidean frames such that every non-reflexive world has exactly one world accessible from it. The class of models based on this class of frames is quasi-single-minded. So by Theorem 4.2, $\vdash_{\text{new}}$ generates a strongly classical logic. So by Proposition 7.5, K5□TT$^-$ generates S5. Since □T is a theorem of K5, K5□TT$^-$ is the same logic as KT$^-$5. So, we can conclude that every strongly classical normal modal logic between K□T□5 and KT$^-$5 generates S5.

For any modal principle P, let P^- be the disjunction of P with $\Box q \leftrightarrow \Diamond q$, replacing q with a different atomic sentence if needed to avoid using an atomic sentence appearing in P. By similar reasoning, we can show that KT can be generated from any strongly classical normal modal logic between K□T and K□TT$^-$, KB can be generated from any strongly classical normal modal logic between K□B and K□BB$^-$, KDB can be generated from any strongly classical normal modal logic between K□D□B and KD□BB$^-$, K4 can be generated from any strongly classical normal modal logic between K□4 and K□44$^-$, S4 can be generated from any strongly classical normal modal logic between K□T□4 and K□TT$^-$□44$^-$, and so forth for many familiar strongly classical normal modal logics.

We've so far examined the question of which strongly classical normal modal logics generate a given broadly classical normal modal logic. A natural question is which broadly classical normal modal logics generate a given broadly classical normal modal logic. To partially answer this question, we can show the following:

Proposition 7.6. *Suppose $\vdash$ is a broadly classical normal modal logic. Then:*

(i) *Suppose $\vdash$ is the weakest finitary broadly classical normal modal logic that has the members of Δ as theorems and the members of Θ as derivable rules, where the rules in Θ have finitely many premises. Suppose $\vdash$ is generated from some broadly classical normal modal logic. Then the weakest broadly classical normal modal logic that generates $\vdash$ is the weakest broadly classical normal modal logic that has the members of $\Box\Delta \cup \Box_{\to}\Theta$ as theorems. This logic is finitary.*

(ii) *If $\vdash$ is generated from two broadly classical normal modal logics, $\vdash_1$ and $\vdash_2$ such that $\vdash_2$ extends $\vdash_1$, then every broadly classical normal modal logic between $\vdash_1$ and $\vdash_2$ generates $\vdash$.*

(iii) *If $\vdash$ is self-generating, $\vdash$ is the strongest broadly classical normal modal logic that generates $\vdash$.*

(iv) *Suppose $\vdash$ is the weakest broadly classical normal modal logic that extends the broadly classical normal modal logic $\vdash_{\text{L}}$ and has the members of Δ as theorems. Suppose the logic $\vdash_{\text{L}}$ generates is extended by $\vdash$. Let $\vdash_{\text{new}}$ be the weakest broadly classical normal modal logic extending $\vdash_{\text{L}}$ with the members of $\Box\Delta$ as theorems. Then $\vdash_{\text{new}}$ generates $\vdash$.*

Proof The proof of (i) is just like the proof of the first part of Proposition 7.1. The proofs of (ii) and (iii) are easy.

(iv) Suppose $\Gamma \vdash \alpha$. So $\Gamma \cup \Delta^* \vdash_{\mathrm{L}} \alpha$, where Δ^* is the smallest set that contains all substitution instances of members of Δ and is closed under the operation of prefixing a sentence with a $\Box$. So, since $\vdash_{\mathrm{L}}$ is normal, $\Box\Gamma \cup \Box\Delta^* \vdash_{\mathrm{L}} \Box\alpha$. So $\Box\Gamma \vdash_{\text{new}} \Box\alpha$.

For the other direction, suppose $\Box\Gamma \vdash_{\text{new}} \Box\alpha$. So $\Box\Gamma \cup \Box\Delta^* \vdash_{\mathrm{L}} \Box\alpha$. So, since the logic $\vdash_{\mathrm{L}}$ generates is extended by $\vdash$, $\Gamma \cup \Delta^* \vdash \alpha$. So $\Gamma \vdash \alpha$. $\Box$

To illustrate this result, let $\vdash$ be S5 with the addition of $p/\Box p$ as a derivable rule. It follows from the first part of this result that the weakest broadly classical normal modal logic generating this logic is $\Box$K$\Box$T$\Box$5 with the addition of the theorem $\Box p \rightarrow \Box\Box p$, which is just the 4 principle.[50] We'll see below that $\vdash$ is self-generating. So it follows from the first three parts of this result that the broadly classical normal modal logics that generate S5 with the addition of $p/\Box p$ as a derivable rule are exactly the broadly classical normal modal logics between $\Box$K4$\Box$T$\Box$5 and S5 with the addition of $p/\Box p$ as a derivable rule. In the next section, we'll show that K4 with the addition of $p/\Box p$ as a derivable rule is also self-generating. It follows from the fourth part of this result that if $\vdash$ is the weakest broadly classical normal modal logic extending this logic that contains the members of some arbitrary set Δ as theorems, then $\vdash$ is generated from some broadly classical normal modal logic. In particular, it is generated from K4 with the addition of $p/\Box p$ as a derivable rule and the members of $\Box\Delta$ as theorems.

8 Iterative Generation Sequences

Our results so far have largely focused on the generation of modal logics from strongly classical base logics. (They also focus on the case where the base logics are finitary.) But Theorem 4.1 tells us that if $\vdash$ is generated from any broadly classical normal modal logic $\vdash_{\text{base}}$, then $\vdash$ is a broadly classical normal modal logic that extends $\vdash_{\text{base}}$. So it is natural to consider generating modal logics from weakly classical normal modal logics, as well.[51]

It is difficult to directly study the generation of modal logics from weakly classical base logics using the techniques of this paper. There are two reasons for this. First, the results showing equivalences between claims about generated logics (e.g., that they are strongly classical, identical to the base logic, etc.) and claims about the admissibility of rules in the base logic rely on Conditional Proof for the base logic. So we cannot rely on them in studying the modal logics generated from weakly classical logics. Second, the standard relational model theory for modal logic only applies to finitary strongly classical normal modal logics, so we cannot directly make use of it in studying modal logics generated from weakly classical logics.

There is, however, a way to study the generation of modal logics from weakly classical logics using the techniques of this paper in a more indirect way, at least for a range of cases. Notice that the generation process can

[50] $\Box$K$\Box$T$\Box$5 is the weakest broadly classical normal modal logic having $\Box$K, $\Box$T, and $\Box$5 as theorems. It is not strongly classical because it doesn't have K as a theorem.

[51] Thanks to an anonymous referee for raising this issue.

be iterated. Given a strongly classical normal modal logic, it will generate a (perhaps distinct) broadly classical normal modal logic, which will generate a (perhaps distinct) broadly classical normal modal logic, and so on. We can make use of the standard relational model theory for modal logic to study the modal logics generated from weakly classical normal modal logics, so long as those weakly classical normal modal logics were themselves generated from finitary strongly classical normal modal logics (either directly or indirectly). We can apply the model theory to the initial strongly classical base logic.

In this section, we will study what happens when we start with a finitary strongly classical normal modal logic and iteratively apply the generation procedure. What is the sequence of broadly classical normal modal logics generated in this way? Does the sequence ultimately hit a fixed point? What does the fixed point look like? And so forth. These questions are interesting for two reasons: First, it is intrinsically interesting to see what happens when we start with a strongly classical logic and iterate the generation procedure. And second, examining such sequences can tell us what happens when we generate a logic from a weakly classical normal modal logic, at least in many cases of interest.

In a bit more detail: Given a broadly classical normal modal logic $\vdash_{\text{base}}$, we inductively define $\vdash_\iota$ (for ordinals ι) as follows:

- $\Gamma \vdash_0 \alpha$ just in case $\Gamma \vdash_{\text{base}} \alpha$
- Successor ordinals: $\Gamma \vdash_{\iota+1} \alpha$ just in case $\Box\Gamma \vdash_\iota \Box\alpha$
- Limit ordinals: $\Gamma \vdash_\lambda \alpha$ just in case $\Gamma \vdash_\iota \alpha$ for some $\iota < \lambda$

Call these sequences "iterated generation sequences". Notice that if $\vdash_{\text{base}}$ is finitary, so is each member of its iterated generation sequence.

It is easy to see that every iterated generation sequence ultimately hits a fixed point. By Theorem 4.1, $\vdash_{\iota+1}$ extends $\vdash_\iota$. By the definition of $\vdash_\lambda$, $\vdash_\lambda$ extends $\vdash_\iota$ for each $\iota < \lambda$. So by a simple cardinality argument, each iterated generation sequence will eventually hit a fixed point. Indeed, it is easy to directly show that $\vdash_{\omega+1}$ is identical to $\vdash_\omega$, so each iterated generation sequence will hit a fixed point at some ordinal less than or equal to ω. The fixed point will be a broadly classical normal modal logic that is self-generating.

We have already seen some examples of such fixed points. In particular, K, KD, KDG, KD_c, and $\text{KT}^n!$ (for any $n > 0$) are strongly classical normal modal logics that are self-generating and so are fixed points. There are also weakly classical normal modal logics that are self-generating. For instance, it is easy to see that any finitary broadly classical normal extension of K in which both $p/\Box p$ and $\Box p/p$ are derivable rules is self-generating. This includes finitary broadly classical normal extensions of KT in which $p/\Box p$ is derivable as well as finitary broadly classical normal extensions of KT_c in which $\Box p/p$ is derivable, among other logics. So these modal logics, too, will be fixed points of iterated generation sequences.

We can make progress on studying iterated generation sequences by noticing that the central results of this paper can all be generalized to logics that are generated from finitary strongly classical normal modal logics in n steps. Here are the relevant generalizations. (The proofs are straightforward generalizations of the proofs provided in sections 4 through 6, making use of the results on admissibility in section 3.)

Theorem 8.1. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic. Then the following are equivalent:*

- *$\vdash_n$ is strongly classical.*
- *The rule $\Box^n p \to \Box^n q / \Box^n (p \to q)$ is admissible in $\vdash_{\text{base}}$.*
- *The canonical model for $\vdash_{\text{base}}$ has the following property: $\forall w, v(wR^n v \to \exists u(R^n(u) = \{v\}))$.*
- *$\vdash_{\text{base}}$ is weakly determined with respect to a class C of models that has the following property: If $M \in C$ and w, v are worlds in M such that $wR^n v$ then there is an $M' \in C$ (perhaps identical to M) such that $M'_v = M_v$ and there is a world u in M' such that $R'^n(u) = \{v\}$.*

Theorem 8.2. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic. Then the following are equivalent:*

- *$\vdash_{n+1}$ is identical to $\vdash_n$.*
- *The rule $\Box^{n+1} p \to \Box^{n+1} q / \Box^n p \to \Box^n q$ is admissible in $\vdash_{\text{base}}$*
- *The canonical model for $\vdash_{\text{base}}$ has the following property: $\forall w, v(wR^n v \to \exists u(uR^{n+1} v \wedge R^{n+1}(u) \subseteq R^n(w)))$.*
- *$\vdash_{\text{base}}$ is weakly determined with respect to a class C of models that has the following property: If $M \in C$ and w, v are worlds in M such that $wR^n v$ then there is an $M' \in C$ (perhaps identical to M) such that $M'_w = M_w$ and there is a world u in M' such that $uR'^{n+1} v$ and $R'^{n+1}(u) \subseteq R'^n(w)$.*

Theorem 8.3. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic. Then the following are equivalent:*

- *$\vdash_{n+1} \alpha$ just in case $\vdash_n \alpha$.*
- *The rule $\Box^{n+1} p / \Box^n p$ is admissible in $\vdash_{\text{base}}$.*
- *The canonical model for $\vdash_{\text{base}}$ has the following property: $\forall w(\exists u\, uR^n w \to \exists u^*\, u^* R^{n+1} w)$.*
- *$\vdash_{\text{base}}$ is weakly determined with respect to a class of models C that has the following property: If $M \in C$ and w, u are worlds in M such that $uR^n w$ then there is an $M' \in C$ (perhaps identical to M) such that $M'_w = M_w$ and there is a world u^* in M' such that $u^* R'^{n+1} w$.*

Theorem 8.4. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic. Let $\vdash^+$ be the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the following meta-rule obtains for every α and β: if $\vdash^+ \Box^n \alpha \to \Box^n \beta$ then $\alpha \vdash^+ \beta$. Then the following are equivalent:*

- *$\vdash_n$ is identical to $\vdash^+$.*

- $\Box^n(\Box^n p \to \Box^n q)/\Box^n p \to \Box^n q$ *is admissible in* $\vdash_{\text{base}}$.
- *The canonical model for* $\vdash$ *has the following property:* $\forall w, v(wR^n v \to \exists u, u^*(uR^n u^* \wedge u^* R^n v \wedge R^n(u^*) \subseteq R^n(w)))$.
- $\vdash$ *is weakly determined with respect to a class* C *of models that has the following related property: If* $M \in C$ *and* w, v *are worlds in* M *such that* $wR^n v$ *then there is an* $M' \in C$ *(perhaps identical to* M*) such that* $M'_w = M_w$ *and there are worlds* u, u *in* M' *such that* $uR'^n u^*$, $u^* R'^n v$, *and* $R'^n(u^*) \subseteq R'^n(w)$.

Proposition 8.5. *Suppose* $\vdash_{\text{base}}$ *is a finitary strongly classical normal modal logic and* l *and* m *are natural numbers. Let* $\vdash^+$ *be the weakest broadly classical normal modal logic that extends* $\vdash_{\text{base}}$ *such that the rule* $p/\Diamond^l \Box^m p$ *is derivable. Then:*

(i) *If* $\Box^n p \to \Box^n q/p \wedge \Diamond^l \Box^m p \to q$ *is admissible in* $\vdash_{\text{base}}$ *then* $\vdash_n \subseteq \vdash^+$.
(ii) $\vdash^+ \subseteq \vdash_n$ *just in case* $\vdash_{\text{base}} \Box^n p \to \Box^n \Diamond^l \Box^m p$.

We can use these results to prove results about iterated generation sequences that begin with extensions of KT, KB, and K4. For example, if $\vdash_{\text{base}}$ is a finitary strongly classical normal modal extension of KT, we can give a characterization of each $\vdash_n$. (Indeed, instead of T, we can use the weaker principle $\Box^n(\Box^n p \to \Box^n q) \to (\Box^n p \to \Box^n q)$.) We can show that for extensions of KT, we get no additional theorems as we iterate the generation procedure. We can also show that for the cases of KT and KTB, there is no fixed point in the iterated generation sequence below ω.

Theorem 8.6. *Suppose* $\vdash_{\text{base}}$ *is a finitary strongly classical normal modal logic. Let* $\vdash_n^+$ *be the weakest broadly classical normal modal logic that extends* $\vdash_{\text{base}}$ *such that the rule* $\Box^n p \to \Box^n q, p/q$ *is derivable. Then:*

(i) $\vdash_n$ *is identical to* $\vdash_n^+$ *just in case* $\vdash_{\text{base}} \Box^n(\Box^n p \to \Box^n q) \to (\Box^n p \to \Box^n q)$.
(ii) *If* $\vdash_{\text{base}}$ *extends* KT, *then* $\vdash_n$ *has the very same theorems as* $\vdash_{\text{base}}$.
(iii) *If* $\vdash_{\text{base}} \Box^n(\Box^n p \to \Box^n q) \to (\Box^n p \to \Box^n q)$ *and* $\vdash_{\text{base}}$ *is extended by* KTB *then* $\vdash_{n+1}$ *is strictly stronger than* $\vdash_n$.

Proof (i) The proof of this is a straightforward generalization of the proof of Theorem 6.3.

(ii) This follows from Theorem 8.3.

(iii) By (i), $\Box^{n+1} p \to \Box^{n+1} q, p \vdash_{n+1} q$. We show $\Box^{n+1} p \to \Box^{n+1} q, p \nvdash_n q$. That is, $\Box^n(\Box^{n+1} p \to \Box^{n+1} q), \Box^n p \nvdash_{\text{base}} \Box^n q$. We show this using a countermodel $M = \langle W, R, V \rangle$. Let $W = \{w_0, w_1, \dots, w_{n+1}\}$. For each $i \leq n$, let $R(w_i) = \{w_i, w_{i+1}\}$ and let $R(w_{n+1}) = \{w_{n+1}\}$. Let $V(p) = \{w_i \mid i \leq n\}$ and $V(q) = \emptyset$. This is a model for KTB. $\Box^{n+1} p$ is false at every $w \in W$. So $\Box^{n+1} p \to \Box^{n+1} q$ is true at every $w \in W$. $\Box^n(\Box^{n+1} p \to \Box^{n+1} q)$ is true at w_0. $\Box^n p$ is true at w_0. But $\Box^n q$ is false at w_0. □

It follows from this result that if $\vdash_{\text{base}}$ is KT or KTB, there is no fixed point in the iterated generation sequence for $\vdash_{\text{base}}$ below ω. In these cases, we can characterize $\vdash_\omega$ as the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that for every $n \in \mathbb{N}$, the rule $\Box^n p \to \Box^n q, p/q$ is derivable.

Next, we can prove an analogous claim about extensions of KB.

Theorem 8.7. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic that extends* KB*. Let $\vdash_n^+$ be the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the rule $p/\Diamond^n\Box^n p$ is derivable. Then:*

(i) *$\vdash_n$ is identical to $\vdash_n^+$.*

(ii) *For $n > 0$,* D *is a theorem of $\vdash_n$.*

(iii) *If $\vdash_{\text{base}}$ extends* KDB*, then $\vdash_n$ has the very same theorems as $\vdash_{\text{base}}$. If $\vdash_{\text{base}}$ does not extend* KDB*, then for $n > 0$, $\vdash_n$ has the very same theorems as $\vdash_1$.*

(iv) *If $\vdash_{\text{base}}$ is between* KB *and* KTB*, then $\vdash_{n+1}$ is strictly stronger than $\vdash_n$.*

Proof (i) The proof of this is a straightforward generalization of the proof of Theorem 6.7.

(ii) KB has $\Box$D as a theorem, so $\vdash_1$D. So $\vdash_n$D for any $n > 0$.

(iii) In any broadly classical normal modal logic $\vdash_{\text{base}}$ extending KD, the rule $p/\Diamond^n\Box^n p$ is admissible: Suppose α is a theorem. By Necessitation, $\Box^n\alpha$ is a theorem. By repeated use of Necessitation, D, and Modus Ponens, $\Diamond^n\Box^n\alpha$ is a theorem. Since this rule is admissible, adding it as a derivable rule does not yield any new theorems. So if $\vdash_{\text{base}}$ extends KDB, $\vdash_n$ has the very same theorems as $\vdash_{\text{base}}$. If $\vdash_{\text{base}}$ does not extend KDB, by (ii), $\vdash_1$D. And so $\vdash_n$ has the very same theorems as $\vdash_1$.

(iv) By (i), $p \vdash_{n+1} \Diamond^{n+1}\Box^{n+1}p$. We show $p \nvdash_n \Diamond^{n+1}\Box^{n+1}p$. That is, $\nvdash_{\text{base}} \Box^n p \to \Box^n\Diamond^{n+1}\Box^{n+1}p$. We show this using a countermodel $M = \langle W, R, V\rangle$. Let $W = \{w_0, w_1, \ldots, w_{n+1}\}$. Let $R(w_0) = \{w_0, w_1\}$. For $1 \leq i \leq n$, let $R(w_i) = \{w_{i-1}, w_i, w_{i+1}\}$. Let $R(w_{n+1}) = \{w_n, w_{n+1}\}$. Let $V(p) = \{w_i \mid i \leq n\}$. This is a model for KTB. $\Box^n p$ is true at w_0. But at no world is $\Box^{n+1}p$ true. So at no world is $\Diamond^{n+1}\Box^{n+1}p$ true. So $\Box^n\Diamond^{n+1}\Box^{n+1}p$ is false at w_0. $\Box$

It follows from this result that if $\vdash_{\text{base}}$ is KB, KDB, or KTB, there is no fixed point in the iterated generation sequence for $\vdash_{\text{base}}$ below ω. In these cases, we can characterize $\vdash_\omega$ as the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that for every $n \in \mathbb{N}$, the rule $p/\Diamond^n\Box^n p$ is derivable. Notice that for the case of KTB, we have two ways to characterize $\vdash_n$ (and $\vdash_\omega$). We can characterize $\vdash_n$ as KTB with the addition of the derivable rule $\Box^n p \to \Box^n q, p/q$ or as KTB with the addition of the derivable rule $p/\Diamond^n\Box^n p$.

Finally, we can examine extensions of K4. Unlike for the cases of KT, KB, KDB, and KTB, for many familiar extensions of K4, we reach a fixed point at $\vdash_1$.*

*It is also straightforward to show that for any extension of K4!, we reach a fixed point at $\vdash_1$. In particular, let $\vdash_{\text{base}}$ be a finitary strongly classical normal modal logic that extends K4!. Let $\vdash^+$ be the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the rules $p/\Box p$ and $\Box p/p$ are derivable. Then, for $n > 0$, $\vdash_n$ is identical to $\vdash^+$.

Theorem 8.8. *Suppose $\vdash_{\text{base}}$ is a finitary strongly classical normal modal logic that extends* K4. *Let $\vdash^+$ be the weakest broadly classical normal modal logic that extends $\vdash_{\text{base}}$ such that the rule $p/\Box p$ is derivable. Then:*

(i) *If the rule $\Box^n p \to \Box^n q/p \wedge \Box p \to q$ is admissible in $\vdash_{\text{base}}$ then for $n > 0$, $\vdash_n$ is identical to $\vdash^+$.*

(ii) *If $\vdash_{\text{base}}$ is* K4, KD4, KDG4, *or* GL, *or extends* S4, *then for $n > 0$, $\vdash_n$ is identical to $\vdash^+$.*

Proof (i) The proof of this is a straightforward generalization of the proof of Theorem 6.8.

(ii) By (i), it suffices to show that the rule $\Box^n p \to \Box^n q/p \wedge \Box p \to q$ is admissible in $\vdash_{\text{base}}$.

K4, KD4, KDG4, and GL: K4 corresponds to the transitive frames, KD4 corresponds to the serial and transitive frames, and KDG4 corresponds to the serial, convergent, and transitive frames. GL is weakly determined with respect to the class of finite frames that have transitive and irreflexive accessibility relations. Suppose $\alpha \wedge \Box\alpha \to \beta$ is false at a world v in a model M based on a frame in the relevant class of frames for $\vdash_{\text{base}}$. Let $M' = \langle W', R', V' \rangle$ be defined so that $W' = W \cup \{w_1, \ldots, w_n\}$ (for some $w_1, \ldots, w_n \notin W$), $R'(w_i) = W_v \cup \{w_j \mid j > i\}$, $R'(x) = R(x)$ for $x \in W$, and $V'(p) = V(p)$. It is easy to show that M' is based on a frame for $\vdash_{\text{base}}$. $\Box^n \alpha \to \Box^n \beta$ is false at w_1.

Extensions of S4: The rule $\Box^n p \to \Box^n q/p \wedge \Box p \to q$ is derivable (and hence admissible) in any extension of S4. □

Using these results, as well as the results from previous sections, we can determine the iterated generation sequences for many familiar modal logics. These results are summarized in Table 2.

These results also tell us about what is generated from some finitary weakly classical normal modal logics. For instance, if $\vdash_{\text{base}}$ is the result of taking a strongly classical normal modal logic extending KT and adding $\Box^n p \to \Box^n q, p/q$ as a derivable rule, $\vdash$ is $\vdash_{\text{base}}$ with the addition of $\Box^{n+1} p \to \Box^{n+1} q, p/q$ as a derivable rule. If $\vdash_{\text{base}}$ is the result of taking a strongly classical normal modal logic extending KB and adding $p/\Box^n \Diamond^n p$ as a derivable rule, $\vdash$ is $\vdash_{\text{base}}$ with the addition of $p/\Box^{n+1} \Diamond^{n+1} p$ as a derivable rule. If $\vdash_{\text{base}}$ is the result of taking a strongly classical normal modal logic extending K4 in which $\Box^n p \to \Box^n q/p \wedge \Box p \to q$ is admissible and adding $p/\Box p$ as a derivable rule, $\vdash$ is identical to $\vdash_{\text{base}}$. In particular, if $\vdash_{\text{base}}$ is the result of taking a strongly classical normal modal logic extending S4 and adding $p/\Box p$ as a derivable rule, $\vdash$ is identical to $\vdash_{\text{base}}$.

9 Generalizations

9.1 Multiple Conclusions

This paper has focused on single-conclusion consequence relations, which hold between a set of sentences and a sentence. It is natural to wonder how things

Table 2 Iterated Generation Sequences for Specific Modal Logics

$\vdash_0$	FP	$\vdash_1$	$\vdash_2$	...	$\vdash_n$
K	0				
KD	0				
K5	2	S5	S5 + $p/\Box p$		
KD5	2	S5	S5 + $p/\Box p$		
KG	1	KDG			
KDG	0				
KT	ω	KT + $\Box p \to \Box q, p/q$	KT + $\Box^2 p \to \Box^2 q, p/q$		KT + $\Box^n p \to \Box^n q, p/q$
KB	ω	KDB + $p/\Diamond\Box p$	KDB + $p/\Diamond^2\Box^2 p$		KDB + $p/\Diamond^n\Box^n p$
KDB	ω	KDB + $p/\Diamond\Box p$	KDB + $p/\Diamond^2\Box^2 p$		KDB + $p/\Diamond^n\Box^n p$
KTB	ω	KTB + $p/\Diamond\Box p$	KTB + $p/\Diamond^2\Box^2 p$		KTB + $p/\Diamond^n\Box^n p$
K4	1	K4 + $p/\Box p$			
KD4	1	KD4 + $p/\Box p$			
KG4	1	KDG4 + $p/\Box p$			
KDG4	1	KDG4 + $p/\Box p$			
S4	1	S4 + $p/\Box p$			
S4.2	1	S4.2 + $p/\Box p$			
KB5	1	S5 + $p/\Box p$			
K45	1	S5 + $p/\Box p$			
KD45	1	S5 + $p/\Box p$			
S5	1	S5 + $p/\Box p$			
GL	1	GL + $p/\Box p$			
KD$_c$	0				
KT$_c$	1	Triv			
KT^n!	0				
Ver	1	$\bot$			

The first column lists the base logic. The second column lists the smallest n such that $\vdash_n$ is self-generating. The third column characterizes $\vdash_1$ if it is distinct from $\vdash_0$. The fourth column characterizes $\vdash_2$ if it is distinct from $\vdash_1$. The final column characterizes $\vdash_n$ for the cases where there is no finite n such that $\vdash_n$ is self-generating. The characterizations have been chosen so that the rule is admissible in the strongly classical normal modal logic listed, so adds no theorems.

change if we move to a setting in which consequence relations are multiple conclusion, that is, hold between a set of sentences and a set of sentences.[52]

Let $\vdash_{\text{base}}$ be a multiple-conclusion consequence relation. There are two natural options for defining the consequence relation $\vdash$ generated from $\vdash_{\text{base}}$. On the first option, $\Gamma \vdash \Delta$ just in case $\Box\Gamma \vdash_{\text{base}} \Box\Delta$. That is, $\{\Box\gamma \mid \gamma \in \Gamma\} \vdash_{\text{base}} \{\Box\delta \mid \delta \in \Delta\}$. This is perhaps the most natural approach to take. But it has a serious cost. We would presumably like $\Gamma \vdash \alpha_1, \alpha_2$ to be equivalent to $\Gamma \vdash \alpha_1 \vee \alpha_2$. However, for most modal logics of interest, this will fail: $\vdash_{\text{base}} \Box(p \vee \neg p)$ so $\vdash p \vee \neg p$. But if $\vdash_{\text{base}}$ does not extend D$_c$, $\nvdash_{\text{base}} \Box p \vee \Box\neg p$. So $\nvdash_{\text{base}} \Box p, \neg\Box p$, and so $\nvdash p, \neg p$.[53]

On the second option, $\Gamma \vdash \Delta$ just in case $\Box(\gamma_1 \wedge \ldots \wedge \gamma_m) \vdash_{\text{base}} \Box(\delta_1 \vee \ldots \vee \delta_n)$ for some $\gamma_1, \ldots, \gamma_m \in \Gamma$ and $\delta_1, \ldots \delta_n \in \Delta$. This approach avoids the problem for the first approach. The cost of this approach is that it essentially returns us to a single-conclusion setting.

[52] Thanks to an anonymous referee for raising this question.
[53] This observation appears in both [14] and [37].

9.2 Alternative Generation Procedures

This paper has focused on generating logics from base modal logics by defining $\Gamma \vdash \alpha$ as $\Box\Gamma \vdash_{\text{base}} \Box\alpha$. It is natural to look at alternative definitions, too. For instance, a natural idea is to define $\alpha \vdash' \beta$ as $\Diamond\alpha \vdash_{\text{base}} \Diamond\beta$, and more generally, to define $\Gamma \vdash' \alpha$ as $\Diamond(\gamma_1 \wedge \ldots \wedge \gamma_n) \vdash_{\text{base}} \Diamond\alpha$ for some finite subset $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$. (On this account, $\vdash' \alpha$ is defined as $\Diamond\top \vdash_{\text{base}} \Diamond\alpha$.)[54]

Since $\Diamond$ is an abbreviation for $\neg\Box\neg$, $\vdash'$ is equivalent to the contraposition of $\vdash$. So if $\vdash_{\text{base}}$ is a finitary broadly classical normal modal logic, $\vdash'$ is identical to $\vdash$ just in case $\vdash$ obeys Contraposition just in case (by the results in section 2.3) $\vdash$ is strongly classical. So there is an intimate connection between $\vdash$ and $\vdash'$.

It is straightforward to show that if $\vdash_{\text{base}}$ is a strongly classical normal modal logic, $\vdash'$ obeys Identity, Weakening, Uniform Substitution, and extends $\vdash_{\text{base}}$. It also obeys Conditional Proof and Reasoning by Cases. However, it may not obey Contraposition. It may not obey Reductio ad Absurdum. And, more interestingly, while it does obey Transitivity (in the sense that if both $\Gamma \vdash' \alpha$ and $\alpha \vdash' \beta$ then $\Gamma \vdash' \beta$), it may not obey Cut. So this definition yields logics that resemble a strongly classical normal modal logic except that we may lose something different than what we lose using the definition of $\vdash$.

Other alternative generating procedures also yield logics that resemble a strongly classical normal modal logic except that they may be missing a familiar property or properties. Suppose we define $\alpha \vdash^* \beta$ as $\Diamond\alpha \vdash_{\text{base}} \Box\beta$, and more generally, $\Gamma \vdash^* \alpha$ as $\{\Diamond\gamma \mid \gamma \in \Gamma\} \vdash_{\text{base}} \Box\alpha$. In this case, if $\vdash_{\text{base}}$ is a strongly classical normal modal logic, $\vdash^*$ has all the theorems of $\vdash_{\text{base}}$ and obeys Weakening, Cut, Conditional Proof, Reductio ad Absurdum, and Contraposition. However, it may be missing many classical rules of inference, such as Modus Ponens, Conjunction Introduction and Elimination, and even Identity.

A different idea is to define $\Gamma \vdash^\star \alpha$ as $\Box\Gamma \vdash_{\text{base}} \Diamond\alpha$. If $\vdash_{\text{base}}$ is a strongly classical normal modal logic extending KD, this can be thought of as something like the modal analogue of the three-valued semantics for the logic ST.[55] In this case, $\vdash^\star$ obeys Identity, Weakening, Uniform Substitution, and extends $\vdash_{\text{base}}$. It also obeys Conditional Proof, Reductio ad Absurdum, and Contraposition. But it may not obey Reasoning by Cases. And, more interestingly, it may not obey Transitivity.

Thus, we can use modal logic to generate logics in the vicinity of classical logic, but that are weaker in interesting respects.

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[54] This is close to the subvaluational account of vagueness defended in [14], except that the usual subvaluational semantics treats the premises independently, so defines $\gamma_1, \ldots, \gamma_n \vdash' \alpha$ as $\Diamond\gamma_1, \ldots, \Diamond\gamma_n \vdash_{\text{base}} \Diamond\alpha$.

[55] See [28]. On this semantics, some premises entail a conclusion just in case when the premises all have the top value, the conclusion does not have the bottom value. If we think of the top value as truth in all worlds and the bottom value as truth in no world, we end up with this definition.

10 Conclusion

Let's return to the philosophical motivations for investigating an account of consequence on which $\Gamma \vdash \alpha$ is defined as $\Box\Gamma \vdash_{\text{base}} \Box\alpha$ for some broadly classical normal modal logic $\vdash_{\text{base}}$. One motivation was to see if there is a plausible supervaluational approach to vagueness that yields a strongly classical logic. A second motivation was to investigate epistemic characterizations of logical consequence, such as the view that an inference is valid if it preserves the epistemic status *one is in a position to know* or the epistemic status *one is rationally committed to its being the case*. Given the results in this paper, what should we make of these motivations?

One upshot of the discussion is that the project of finding a supervaluational approach to vagueness that yields a strongly classical logic by moving to a background logic other than S5 does not look very promising. If we interpret "$\Box$" as standing for "it is determinately true that", then presumably if it is determinately true that α then α. Presumably, then, the base logic will extend KT. By Proposition 4.6, the only strongly classical normal modal logic extending KT that generates a strongly classical logic is Triv. So to generate a strongly classical logic, the base logic will have to be Triv. Since Triv is self-generating, the generated modal logic will be Triv, too. This is a problem, since in Triv, α and $\Box\alpha$ are logically equivalent. In other words, there will be no difference between α and it being determinately true that α. This seems incompatible with the motivations for a supervaluational treatment of vagueness.

The issue can be put in more model-theoretic terms. On a supervaluational model theory in which models are equipped with an accessibility relation, presumably the "worlds" are truth assignments, or packages of semantic rules that induce truth assignments, or the like. The accessibility relation is some kind of nearness relation on such worlds. Presumably, this relation is reflexive – every truth assignment or package of semantic rules is maximally near itself. Since the modal logic KT corresponds to the constraint on frames that the accessibility relation be reflexive, this means that for such a supervaluational approach to generate a strongly classical logic, the base logic will have to be Triv. That corresponds to the constraint on frames that every world is accessible from itself and from no other world. If that's the accessibility relation, then what's supertrue at a world is just what is true at that world. This is to give up on the idea that supertruth at a world is stronger than truth at a world, which is one of the main ideas behind this kind of supervaluationism.

There is, however, a potential way to wiggle out of this difficulty. It seems central to our understanding of determinateness that "if it is determinately true that α then α" should be a theorem. But a theorem of which modal logic? Must it be a theorem of the base logic or is it enough that it is a theorem of the generated logic? If the latter suffices, then there is a potential way out. We can take the base logic to be an extension of K$\Box$T that is not an extension of KT. In model-theoretic terms, the idea is to constrain the accessibility relation to be quasi-reflexive but not necessarily reflexive. K$\Box$T generates KT, so any extension of K☐T will generate an extension of KT. So it will be a theorem of the generated logic that if it is determinately true that α then α. Some of these logics will generate a strongly classical logic. For instance, by Corollary 4.5 K☐T itself generates a strongly classical logic. So perhaps there is a way to find an improved supervaluational approach, after all.

The trouble with this suggestion, though, is that it is difficult to make sense of the resulting view. In the base logic, it is a theorem that it is determinately true that if it is determinately true that α then α. But it is not a theorem that if it is determinately true that α then α. How should we understand this? Relatedly, in the corresponding model theory, each world that is accessible from some world is accessible from itself. But not every world need be accessible from itself. Why is this the case? Why wouldn't a truth assignment (or package of semantic rules that induces a truth assignment) be accessible from itself? The account is philosophically very murky.[56]

Now consider the second project, that of defining logical consequence in terms of the preservation of an epistemic status. The results here suggest that some versions of this project, too, face trouble.[57] If the epistemic status is factive, such as *one is in a position to know* or *one has conclusive reason to believe*, and if the generated consequence relation is supposed to be strongly classical, then the same kind of difficulty will arise here as arises for the case of vagueness. Presumably, the relevant epistemic base logic will be an extension of KT. By Proposition 4.6, the only strongly classical normal modal logic extending KT that generates a strongly classical logic is Triv. Triv generates Triv. But Triv is clearly mistaken when "☐" stands for "one is in a position to know" or "one has conclusive reason to believe". As before, one could try to use an extension of K☐T as the base logic rather than an extension of KT. But again, as before, it is not clear how to make intuitive sense of such a view.[58]

A perhaps better suggestion would be to define logical consequence in terms of the preservation of a non-factive epistemic status, such as *one is rationally committed to* or *one has justification for believing*. The epistemic modal logics for these statuses do not extend KT, so aren't saddled with the same problem. There are interesting questions about just what the correct epistemic modal logics are for these statuses. Should they include D? What about B, 4, or 5?

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[56] But see [38] for a somewhat different formalism that may avoid this problem. On this alternative approach, each model has (in effect) a set of actual worlds. What's true in a model is what's true at all of its actual worlds. Consequence is global validity, that is, it is defined in terms of the preservation of truth in each model. The advantage of this formalism is that for the base logic (in my sense) to be K☐T corresponds to the frame condition of reflexivity (rather than quasi-reflexivity) in such models. This is a far more natural condition to impose. A problem with this approach is, as [43, p. 527 n. 6] points out, that it breaks the connection between ☐ and supertruth, which "flouts the point" of having the modal operator in the language.

[57] See [31] for related problems facing epistemic characterizations of validity.

[58] Inferentialists about logic often endorse intuitionistic rather than classical logic. On the typical inferentialist intuitionistic view, meta-rules such as Conditional Proof constitute or determine the meanings of the logical constants. So on such a view, then, it will be important that the generated logic be strongly intuitionist rather than weakly intuitionist. Many of the results here can be generalized to the case of intuitionistic logic. For instance, the intuitionistic analogue of Proposition 4.6 states that any strongly intuitionistic normal modal logic extending KT that generates a strongly intuitionistic logic will have $\neg(\alpha \wedge \neg\Box\alpha)$ as a theorem. Since this claim is clearly mistaken when "☐" stands for "one is in a position to know that" or "one has conclusive reason to believe that", moving to an intuitionistic base logic doesn't avoid the problem.

Some choices of a base epistemic modal logic yield an overly strong generated logic. For instance, if the base logic includes 4 (e.g., if one has justification for believing a claim then one has justification for believing that one has justification for believing the claim), then the generated logic will include $p/\Box p$ as a derivable rule (e.g., from a claim it follows that one has justification for believing it). On the face of it, this is implausible.[59] One way to see this is to notice that the logical consequence relation presumably tells us not only what follows from believed claims but also what holds under claims that are merely hypothetically supposed. It is implausible that under the supposition of a claim α, one always has justification for believing it.[60] But given a weak enough base epistemic modal logic, perhaps a version of this approach could be developed that avoids these difficulties. Or perhaps a proponent of an epistemic characterization of consequence should reject the requirement to generate a strongly classical (or strongly intuitionistic) logic.[*]

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[59] But see [3] for a defense of the inference from α to $\Box\alpha$ when $\Box$ is understood as the epistemic must.

[60] A second concern with this derivable rule is that given a strongly classical logic, $p \rightarrow \Box p$ turns out to be a theorem, which is implausible. Of course, if the base logic includes 4, we should presumably reject the idea that the generated logic should be strongly classical. That's because the weakest base logic including 4 that generates a strongly classical logic has $\Box\mathrm{T}_c$ as a theorem, which is a highly implausible principle.

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