A graph-theoretic account of logics

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Abstract

A graph-theoretic account of logics is explored based on the general notion of m-graph (that is, a graph where each edge can have a finite sequence of nodes as source). Signatures, interpretation structures and deduction systems are seen as m-graphs. After defining a category freely generated by a m-graph, formulas and expressions in general can be seen as morphisms. Moreover, derivations involving rule instantiation are also morphisms. Soundness and completeness theorems are proved. As a consequence of the generality of the approach our results apply to very different logics encompassing, among others, substructural logics as well as logics with nondeterministic semantics, and subsume all logics endowed with an algebraic semantics.

1 Introduction

Diagrammatic representation has been used in several areas of knowledge ranging from basic and human sciences to engineering as can be witnessed by several conferences in very different areas dedicated to the topic every year (for instance see [25, 30, 28]). One of the reasons is because diagrams are intuitive and provide a clear view of the phenomena they explain. Moreover, they can be used to make inferences about the reality they describe (see for instance [15] for a very broad introduction to diagrammatic techniques, [29] for a specific example in justice consisting of the use of a mathematical diagrammatic layout of arguments to make inferences instead of adopting only a traditional jurisprudential model, and [11] for applications in argumentation theory). Another example is category theory [20] that provides a diagrammatic notation for abstract algebra, where, for instance, an equation is substituted by a commutative diagram.

The quest for rigorous diagrammatic reasoning has old roots and at the same time is very contemporary. For instance, L. Euler employed diagrams in order to illustrate relations between classes. J. Venn greatly improved the Euler's approach [31], and later on, an important contribution to the further development of Euler-Venn diagrams was made by C. S. Peirce [23]. Recently, some effort has been dedicated to the definition of a formal system sound and complete for reasoning with diagrams [27, 17, 18]. For a nice discussion on diagrammatic logics see [6].

The right setting for defining logic systems has deserved a lot of attention from the scientific community. The most common approach, see [14, 7], is to look at specific logic systems and try to abstract its general features following the pioneering work in [3]. A promising direction is to incorporate diagrammatic features in logical reasoning [4, 5, 1] (as in Tarski's World, Hyperproof or Openproof).

Herein, we propose diagrams as a unifying technique to present and reason with logics in an abstract way. More precisely, we use multi-graphs (or, for short, m-graphs) to define the language, the semantics and the deduction in a logic system. In signatures, the nodes of the m-graph are seen as sorts and the m-edges as language constructors. In interpretation structures, nodes are truth-values and m-edges are relations between truth-values (this approach to semantics can be seen as generalizing algebraic approaches to semantics of logics, see the overview in the classical monograph [24] and also in [14]). In deductive systems, the nodes are language expressions and the m-edges are inference rules.

However, we need a bit more of structure to define language, denotation and derivation. For this purpose, we consider the category with non empty finite products freely generated from a given m-graph. At this stage, we look at formulas and at derivation steps as morphisms in that appropriate categories (here we are close to Lambek and Scott approach to categorical logic [19]). Furthermore, in this setting, we are able to cope appropriately with schematic reasoning.

A novel feature of our approach is that interpretation structures and deductive systems are related to signatures through an abstraction process. That is, every m-graph corresponding to an interpretation structure is associated to the m-graph representing the underlying signature via an m-graph morphism. The same applies to deductive systems. This feature allows the definition of non-deterministic and partial semantics in a natural way.

As a consequence of the generality of the approach we can define in this setting very different logics including substructural logics as well as logics with nondeterministic semantics and covering all logics endowed with an algebraic semantics [22, 21, 26, 10]. Our notion of derivation allows the rigorous control of the hypotheses used. Thus, it seems worthwhile to explore in the future this fine feature for logics where hypotheses are considered as resources.

The structure of the paper is as follows. Section 2 is dedicated to defining signatures and interpretation structures as m-graphs. The central notions of m-graph and m-graph morphism are introduced in this section. Section 3 deals with formulas. They are morphisms in a category with non empty finite products freely generated from the signature m-graph. Section 4 concentrates on satisfaction and semantic entailment. In Section 5 we introduce deductive systems as an m-graph where the nodes are language expressions and m-edges include inference rules. Following this trend, in Section 6 we introduce derivation also with a diagrammatic intuition coping with the subtle notion of instantiation of schematic rules and formulas. In Section 7, we state general results for soundness and completeness of logic systems. Finally, in Section 8, we give some insight of how to accommodate provisos and quantification in our setting.

We assume a very moderate knowledge of category theory (the interested reader can consult [20]).

2 Signatures and models as m-graphs

A signature is to be seen as a multi-graph whose nodes are the sorts (indicating the relevant kinds of notions) and whose m-edges are the language constructors. For instance, a propositional signature can be seen as a multi-graph with a node, named π, representing the notion of formula and including an m-edge $\neg$ from π to π for the negation constructor and an m-edge $\supset$ for the implication constructor from $\pi\pi$ to π.

Figure 1: Multi-graph for a propositional signature.

Propositional symbols are zero-ary constructors and should also be represented in the multi-graph. For this purpose we consider a special node, named $\diamond$, and an m-edge for each propositional symbol from $\diamond$ to π.

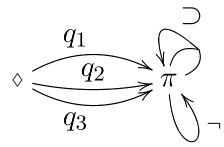

Figure 2: Multi-graph for a propositional signature with propositional symbols.

By a *multi-graph*, in short, an *m-graph*, we mean a tuple

$$G = (V, E, \mathsf{src}, \mathsf{trg})$$

where:

- V is a set (of *vertexes* or *nodes*);
- E is a set (of *m-edges*);
- $\mathsf{src} : E \to V^+$;
- $\mathsf{trg} : E \to V$;

where V^+ denotes the set of all finite non-empty sequences of V. We may write $e : s \to v$ or $e \in G(s, v)$ when $e \in E$, $\mathsf{src}(e) = s$ and $\mathsf{trg}(e) = v$, and may write $G(-, -)$ for the collection of m-edges in E.

A *language signature* or, simply, a *signature* is a tuple $\Sigma = (G, \pi, \diamond)$ where $G = (V, E, \mathsf{src}, \mathsf{trg})$ is a m-graph, π and $\diamond$ are in V, and such that no m-edge has $\diamond$ as target. The nodes in V play the role of language *sorts*, node π being the *propositions sort* (the sort of schema formulas), and node $\diamond$ being the *concrete sort*. The m-edges play the role of *constructors* for building expressions of the available sorts. The concrete sort allows the construction of concrete expressions.

Example 2.1 Let Π be a set of propositional symbols. The *propositional signature* Σ_Π is a m-graph with sorts π and $\diamond$ and the following m-edges:

- $p : \diamond \to \pi$ for each p in Π;
- $\neg : \pi \to \pi$;
- $\supset : \pi\pi \to \pi$.

The m-edges $\neg$ and $\supset$ represent the connectives negation and implication, respectively. ∇

Example 2.2 The *modal signature* $\Sigma_\Pi^\Box$ is a m-graph obtained from Σ_Π by adding the m-edge $\Box : \pi \to \pi$ for representing the modal operator $\Box$ of necessity. ∇

Example 2.3 The *propositional signature with conjunction and disjunction* $\Sigma_\Pi^{\wedge,\vee}$ is a m-graph obtained from Σ_Π by adding the m-edges $\wedge, \vee : \pi\pi \to \pi$ for representing conjunction $\wedge$ and disjunction $\vee$. ∇

Example 2.4 The *propositional signature* $\Sigma_\Pi^{\wedge,\vee,\circ}$ is a m-graph obtained from $\Sigma_\Pi^{\wedge,\vee}$ by adding the m-edge $\circ : \pi \to \pi$. ∇

Example 2.5 Let $F = \{F_n\}_{n \in N_0}$ be a family where F_n is a set (with the *function symbols of arity* n). The *equational signature* Σ_F^{EQ} is a m-graph with the sorts π, $\diamond$ and θ, and the following m-edges:

- $f : \diamond \to \theta$ for each f in F_0;
- $f : \overbrace{\theta \dots \theta}^{n} \to \theta$ for each f in F_n;
- $\approx : \theta\theta \to \pi$.

The m-edge $\approx$ represents the equality symbol. ∇

An interpretation structure, also called a model, over a signature includes an m-graph where the nodes are values and the m-edges are operations on the

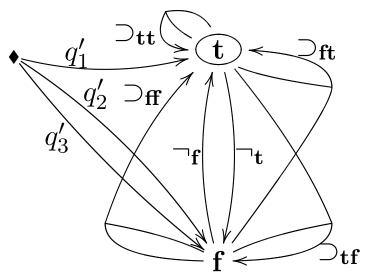

Figure 3: The operations m-graph for an interpretation structure over the propositional signature described in Figure 2.

values. For instance, in the case of propositional logic, that m-graph could be the one specified in Figure 3.

However, this is not enough because we need to know how the values are related to sorts and how operations are related to constructors, that is, we need to relate m-graphs, as depicted in Figure 4 and illustrated in Table 1 for the case of the propositional logic.

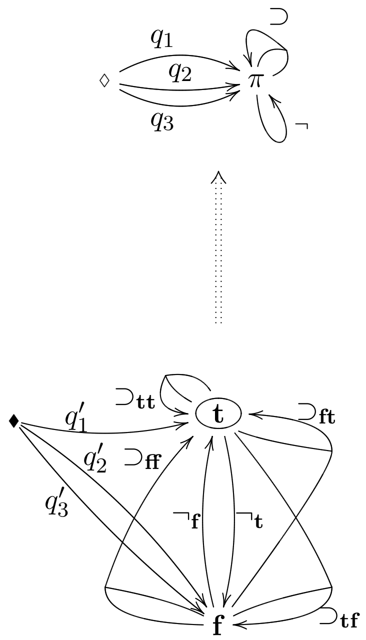

Figure 4: Abstraction map from the operations m-graph presented in Figure 3 and the signature m-graph presented in Figure 2.

By a *m-graph morphism* $h : G_1 \to G_2$ we mean a pair of maps

$$\begin{cases} h^{\mathsf{v}} : V_1 \to V_2 \\ h^{\mathsf{e}} : E_1 \to E_2 \end{cases}$$

such that:

$$
\begin{array}{rcl}
\mathbf{f}, \mathbf{t} & \rightsquigarrow & \pi \\
\blacklozenge & \rightsquigarrow & \diamond \\
q'_1 & \rightsquigarrow & q_1 \\
q'_2 & \rightsquigarrow & q_2 \\
q'_3 & \rightsquigarrow & q_3 \\
\neg_{\mathbf{f}}, \neg_{\mathbf{t}} & \rightsquigarrow & \neg \\
\supset_{\mathbf{ff}}, \supset_{\mathbf{ft}}, \supset_{\mathbf{tf}}, \supset_{\mathbf{tt}} & \rightsquigarrow & \supset
\end{array}
$$

Table 1: Correspondence between the constructors and the operations for propositional logic.

- $\mathsf{src}_2 \circ h^{\mathsf{e}} = h^{\mathsf{v}} \circ \mathsf{src}_1$;
- $\mathsf{trg}_2 \circ h^{\mathsf{e}} = h^{\mathsf{v}} \circ \mathsf{trg}_1$.

In the sequel we denote by **mGraph** the category of m-graphs and their morphisms where identities and compositions are defined as expected. Moreover, given a set S and $s \in S^+$, we denote by $|s|$ the length of s and, for each $i = 1, \ldots, |s|$, we denote by $(s)_i$ the i-th element of s. Furthermore, given a map $f : S \to R$, we let f^+ be the map $\lambda\, s\,.\, f((s)_1) \ldots f((s)_n) : S^+ \to R^+$. For the sake of simplicity, we tend to write f for f^+ when no confusion arises.

We now define the concept of interpretation structure, which departs from a novel perspective in which semantics is abstracted into the syntax and not the other way around. In many cases an interpretation structure is an algebra (that is, it includes operations and sets for each sort) and the denotation consists of assigning to each logical constructor an operator over the appropriate sort. In other words, in many cases, denotation is a concretization process. In our case, we adopt a dual approach. We instead use a graph-theoretic approach (more general than an algebra) for representing truth-values and, possibly nondeterministic operations, and then we assign them to sorts and constructors. In a sense we abstract from the truth-values and operations the linguistic expressions assigned to them.

An *interpretation structure I over a signature* $(G, \pi, \diamond)$ is a tuple

$$(G', \alpha, D, \blacklozenge)$$

such that G' is an m-graph (the *operations graph*), $\alpha : G' \to G$ is an m-graph morphism (the *abstraction morphism*), $D \subseteq (\alpha^{\mathsf{v}})^{-1}(\pi)$ is a non-empty set and $\blacklozenge \in (\alpha^{\mathsf{v}})^{-1}(\diamond)$.

The set V' of nodes of the operations graph is called the *universe*. Observe that V' is partitioned by α: we denote by V'_v the *domain* $(\alpha^{\mathsf{v}})^{-1}(v)$ of values for each v in V. The elements of V'_π are the *truth values* and the elements of $V'_\diamond$ are the *concrete values*. The elements of the set D are the *distinguished truth values*. The requirement on D excludes trivial cases.

Given s in V^+ we denote by V'^+_s the subset of V'^+ consisting of the set $((\alpha^{\mathsf{v}})^+)^{-1}(s)$, that is, $\{s' : (\alpha^{\mathsf{v}})^+(s') = s\}$. The set E' of m-edges of the operations graph is also partitioned by α: we denote by E'_e the set $(\alpha^{\mathsf{e}})^{-1}(e)$ for each e in E. In the sequel, we may call the pair (G', α) a *basis* over a m-graph G.

An *interpretation structure* is a pair (Σ, I) where Σ is a signature and I is an interpretation structure over Σ. An *interpretation system* $\mathcal{I}$ is a pair $(\Sigma, \mathfrak{I})$ where Σ is a signature and $\mathfrak{I}$ is a class of interpretation structures over Σ.

Example 2.6 *Interpretation structure for propositional logic.*
Consider the signature Σ_Π as introduced in Example 2.1 where $\Pi = \{q_1, q_2, q_3\}$. Let $v : \{q_1, q_2, q_3\} \to \{\mathbf{f}, \mathbf{t}\}$ be a classical valuation such that $v(q_1) = \mathbf{t}$ and $v(q_2) = v(q_3) = \mathbf{f}$. The interpretation structure $(G', \alpha, D, \blacklozenge)$ over Σ_Π corresponding to v is as follows:

- G' is such that[1]:

  $V' = \{\mathbf{f}, \mathbf{t}\} \cup \{\blacklozenge\}$;
  $E' = \{q'_1, q'_2, q'_3, \neg_{\mathbf{f}}, \neg_{\mathbf{f}}, \supset_{\mathbf{ff}}, \supset_{\mathbf{ft}}, \supset_{\mathbf{tf}}, \supset_{\mathbf{tt}}\}$;
  src' and trg' are such that:

  $q'_1 : \blacklozenge \to \mathbf{t}$;
  $q'_2 : \blacklozenge \to \mathbf{f}$;
  $q'_3 : \blacklozenge \to \mathbf{f}$;
  $\neg_{\mathbf{f}} : \mathbf{f} \to \mathbf{t}$;
  $\neg_{\mathbf{t}} : \mathbf{t} \to \mathbf{f}$;
  $\supset_{\mathbf{ff}} : \mathbf{f}\,\mathbf{f} \to \mathbf{t}$;
  $\supset_{\mathbf{ft}} : \mathbf{f}\,\mathbf{t} \to \mathbf{t}$;
  $\supset_{\mathbf{tf}} : \mathbf{t}\,\mathbf{f} \to \mathbf{f}$;
  $\supset_{\mathbf{tt}} : \mathbf{t}\,\mathbf{t} \to \mathbf{t}$.

- $\alpha : G' \to G$ is such that:

  $\alpha^{\mathsf{v}}(\mathbf{f}) = \pi$;
  $\alpha^{\mathsf{v}}(\mathbf{t}) = \pi$;
  $\alpha^{\mathsf{v}}(\blacklozenge) = \diamond$;
  $\alpha^{\mathsf{e}}(q'_i) = q_i$ for $i = 1, 2, 3$;
  $\alpha^{\mathsf{e}}(\neg_{v'}) = \neg$ for each v' in V'_π;
  $\alpha^{\mathsf{e}}(\supset_{v'_1 v'_2}) = \supset$ for each v'_1 and v'_2 in V'_π.

- $D = \{\mathbf{t}\}$.

Observe that $V'_\pi = \{\mathbf{f}, \mathbf{t}\}$ and, for instance,

$$E'_\neg = \{\neg_{\mathbf{f}} : \mathbf{f} \to \mathbf{t}, \neg_{\mathbf{t}} : \mathbf{t} \to \mathbf{f}\}$$

where the m-edges $\neg_{\mathbf{f}}$ and $\neg_{\mathbf{t}}$ represent the pairs $(\mathbf{f}, \mathbf{t})$ and $(\mathbf{t}, \mathbf{f})$, respectively. ∇

[1]Using module 2 arithmetical operations within V'.

Example 2.7 *Interpretation structure for modal logic* T*.*
Consider the signature $\Sigma_{\Pi}^{\Box}$ as introduced in Example 2.2 where $\Pi = \{q_1, q_2, q_3\}$. Let $(A, \wedge, \vee, -, \perp, \top, \Box)$ be a modal algebra for modal logic T, and v a valuation over the algebra, that is, a map from $\{q_1, q_2, q_3\} \to A$ (see [9]). The interpretation structure $(G', \alpha, D, \blacklozenge)$ over $\Sigma_{\Pi}^{\Box}$ corresponding to the algebra and the valuation is as follows:

- G' is such that:

  $V' = A \cup \{\blacklozenge\}$;

  $E' = \{q'_1, q'_2, q'_3\} \cup \{\neg_a : a \in A\} \cup \{\supset_{a_1 a_2} : a_1 \in A \text{ and } a_2 \in A\} \cup \{\Box_a : a \in A\}$;

  src' and trg' are such that:

  $q'_i : \blacklozenge \to v(q_i)$ for $i = 1, 2, 3$;
  $\neg_a : a \to -a$ for each a in A;
  $\supset_{a_1 a_2} : a_1\, a_2 \to ((-a_1) \vee a_2)$ for each a_1 and a_2 in A;
  $\Box_a : a \to \Box a$ for each a in A.

- $\alpha : G' \to G$ is such that:

  $\alpha^{\mathsf{v}}(a) = \pi$;
  $\alpha^{\mathsf{v}}(\blacklozenge) = \diamond$;
  $\alpha^{\mathsf{e}}(q'_i) = q_i$ for $i = 1, 2, 3$;
  $\alpha^{\mathsf{e}}(\neg_a) = \neg$;
  $\alpha^{\mathsf{e}}(\supset_{a_1 a_2}) = \supset$;
  $\alpha^{\mathsf{e}}(\Box_a) = \Box$.

- $D = \{\top\}$. ∇

Substructural logics can also be represented in our graph-theoretic context as we illustrate in the next example.

Example 2.8 *Interpretation structure for relevance logic* **R***.*
Consider the signature $\Sigma_{\Pi}^{\wedge,\vee}$ as introduced in Example 2.3 where $\Pi = \{q_1, q_2\}$. Let $m = (W, R, 0, *, v)$ be an **R**-frame for relevance logic **R** (see [12]) with a valuation v. The interpretation structure $(G', \alpha, D, \blacklozenge)$ over $\Sigma_{\Pi}^{\wedge,\vee}$ corresponding to m is defined as follows:

- G' is such that:

  $V' = \wp W \cup \{\blacklozenge\}$;

  $E' = \{q'_1, q'_2\} \cup \{\neg_b : b \in \wp W\} \cup \{\supset_{b_1 b_2} : b_1, b_2 \in \wp W\} \cup \{\wedge_{b_1 b_2} : b_1, b_2 \in \wp W\} \cup \{\vee_{b_1 b_2} : b_1, b_2 \in \wp W\}$;

  src' and trg' are such that:

  $q'_1 : \blacklozenge \to \emptyset$;
  $q'_2 : \blacklozenge \to W$;

$\neg_b : b \to \{w \in W : w^* \notin b\}$;
$\supset_{b_1 b_2} : b_1\, b_2 \to \{w \in W : Rww_1w_2$ and $w_1 \in b_1$ implies $w_2 \in b_2\}$;
$\wedge_{b_1 b_2} : b_1\, b_2 \to b_1 \cap b_2$;
$\vee_{b_1 b_2} : b_1\, b_2 \to b_1 \cup b_2$.

- $\alpha : G' \to G$ is such that:

  $\alpha^{\mathsf{v}}(b) = \pi$;
  $\alpha^{\mathsf{v}}(\blacklozenge) = \diamond$;
  $\alpha^{\mathsf{e}}(q_i') = q_i$ for $i = 1, 2$;
  $\alpha^{\mathsf{e}}(\neg_b) = \neg$;
  $\alpha^{\mathsf{e}}(\supset_{b_1 b_2}) = \supset$;
  $\alpha^{\mathsf{e}}(\wedge_{b_1 b_2}) = \wedge$;
  $\alpha^{\mathsf{e}}(\vee_{b_1 b_2}) = \vee$.

- D is the set of all subsets of W containing 0. ∇

Although in the examples above the graph-theoretic interpretation structures are algebraic in nature, that is not necessarily so. Indeed, graph-theoretic interpretation structures can even be non deterministic or partial. This is the case with the interpretation structure that we now consider.

Example 2.9 *Non-deterministic interpretation structure.*
Consider the signature $\Sigma_{\Pi}^{\wedge,\vee,\circ}$ as introduced in Example 2.4 for $\Pi = \{q_1, q_2\}$, and the interpretation structure $I_{\mathrm{nd}} = (G', \alpha, D, \blacklozenge)$ over $\Sigma_{\Pi}^{\wedge,\vee,\circ}$ (inspired by [2]) where:

- G' is such that[2]:

  $V' = \{\mathbf{t}, \mathbf{I}, \mathbf{f}\} \cup \{\blacklozenge\}$;
  E' is composed of the following m-edges (note that src' and trg' are also being defined):
  $q_1' : \blacklozenge \to \mathbf{f}$;
  $q_2' : \blacklozenge \to \mathbf{I}$;
  $\neg_{v_1' v_2'} : v_1' \to v_2'$ where v_1' is in $\{\mathbf{I}, \mathbf{f}\}$ and v_2' is in D;
  $\neg_{\mathbf{tf}} : \mathbf{t} \to \mathbf{f}$;
  $\circ_{v_1' v_2'} : v_1' \to v_2'$ where v_1' is in $\{\mathbf{t}, \mathbf{f}\}$ and v_2' is in V_π';
  $\circ_{\mathbf{If}} : \mathbf{I} \to \mathbf{f}$;
  $\supset_{v_1' v_2' v'} : v_1'\, v_2' \to v'$ where v_1' is $\mathbf{f}$ or v_2' is in D, and v' is in D;
  $\supset_{v' \mathbf{ff}} : v'\, \mathbf{f} \to \mathbf{f}$ for v' is in D;
  $\wedge_{v_1' v_2' v'} : v_1'\, v_2' \to v'$ where v_1', v_2' and v' are in D;

[2] Intuitively speaking, $\mathbf{t}$ represents a consistently true formula, that is, a true formula whose negation is false; $\mathbf{I}$ represents a inconsistently true formula, that is, a true formula whose negation is true; $\mathbf{f}$ represents a false formula. Observe that in I_{nd} is not possible to have both a formula and its negation as false.

$\wedge_{v_1' v_2' \mathbf{f}} : v_1'\, v_2' \to \mathbf{f}$ where v_1' is $\mathbf{f}$ or v_2' is $\mathbf{f}$;
$\vee_{v_1' v_2' v'} : v_1'\, v_2' \to v'$ where v_1' or v_2' are in D, and v' is in D;
$\vee_{\mathbf{fff}} : \mathbf{f}\,\mathbf{f} \to \mathbf{f}$.

- $\alpha : G' \to G$ is such that:

  $\alpha^{\mathsf{v}}(v') = \pi$ with v' in $\{\mathbf{t}, \mathbf{I}, \mathbf{f}\}$;
  $\alpha^{\mathsf{v}}(\blacklozenge) = \lozenge$;
  $\alpha^{\mathsf{e}}(q_1') = q_1$;
  $\alpha^{\mathsf{e}}(q_2') = q_2$;
  $\alpha^{\mathsf{e}}(\neg_{v_1' v_2'}) = \neg$ for every $\neg_{v_1' v_2'}$ in E';
  $\alpha^{\mathsf{e}}(\circ_{v_1' v_2'}) = \circ$ for every $\circ_{v_1' v_2'}$ in E';
  $\alpha^{\mathsf{e}}(\supset_{v_1' v_2' b}) = \supset$ for every $\supset_{v_1' v_2' b}$ in E';
  $\alpha^{\mathsf{e}}(\wedge_{v_1' v_2' b}) = \wedge$ for every $\wedge_{v_1' v_2' b}$ in E';
  $\alpha^{\mathsf{e}}(\vee_{v_1' v_2' b}) = \vee$ for every $\vee_{v_1' v_2' b}$ in E'.

- $D = \{\mathbf{t}, \mathbf{I}\}$.

A graphical perspective of part of the interpretation structure I_{nd}, comprising negation and propositional symbols, can be seen in Figure 5.

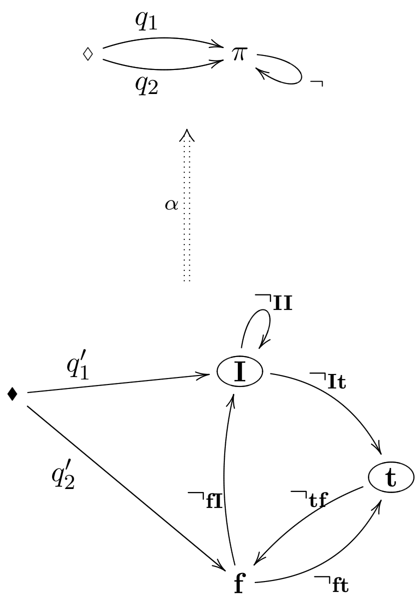

Figure 5: Part of interpretation structure I_{nd} described in Example 2.9.

Observe that the denotation of the paraconsistent negation $\neg$ and of the consistency connective $\circ$ is not deterministic. ∇

Semantics of logics using several sorts can also be expressed very intuitively in our setting as we illustrate in the following example.

Example 2.10 *Interpretation structure for equational logic.*
Consider the signature Σ_F^{EQ} as introduced in Example 2.5. Let $(A, \{F_{nA} : n \geq 0\})$ be an algebra for (one-sorted) equational logic EQ where $f_A : A^n \to A$ for each $f_A \in F_{nA}$ (see [8, 16]). The interpretation structure $(G', \alpha, D, \blacklozenge)$ over Σ_F^{EQ} corresponding to the algebra is as follows:

- G' is such that:

  $V' = A \cup \{\blacklozenge\} \cup \{0, 1\}$;
  $E' = \{f_{a_1 \ldots a_n} : a_1, \ldots, a_n \in A\} \cup \{\approx_{a_1 a_2} : a_1, a_2 \in A\}$;
  src' and trg' are such that:
  $f_{a_1 \ldots a_n} : a_1 \ldots a_n \to f_A(a_1 \ldots a_n)$;
  $\approx_{a_1 a_2} : a_1 a_2 \to b$ where b is 1 iff a_1 is equal to a_2.

- $\alpha : G' \to G$ is such that:

  $\alpha^{\mathsf{v}}(a) = \theta$;
  $\alpha^{\mathsf{v}}(\blacklozenge) = \diamond$;
  $\alpha^{\mathsf{v}}(0) = \pi$;
  $\alpha^{\mathsf{v}}(1) = \pi$;
  $\alpha^{\mathsf{e}}(f_{a_1 \ldots a_n}) = f$;
  $\alpha^{\mathsf{e}}(\approx_{a_1 a_2})$ is $\approx$.

- $D = \{1\}$. ∇

3 Formulas as paths

At first sight a formula can be seen as a path over the signature m-graph. For instance, in the context of the signature for propositional logic presented in Example 2.1, the formula $(\neg\, q_1) \supset q_2$ corresponds to the path described in Figure 6. It is convenient however to work on the richer setting of the category

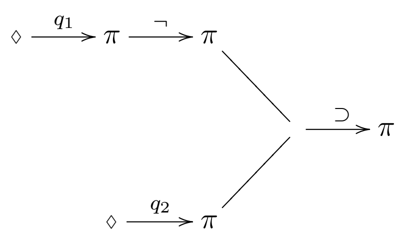

Figure 6: Formula $(\neg\, q_1) \supset q_2$ as a path in the signature m-graph.

generated by the signature m-graph. In this setting sequences of sorts are first class citizens as well as pairing of morphisms. Moreover projections will be available (they are very useful for dealing with schema formulas). In this

$$\diamond \xrightarrow{\supset \circ \langle \neg \circ q_1, q_2 \rangle} \pi$$

Figure 7: Formula $(\neg\, q_1) \supset q_2$ as a morphism in the category generated by the signature m-graph.

context, the formula $(\neg\, q_1) \supset q_2$ corresponds to the morphism presented in Figure 7.

Before proceeding with the study of language expressions in the graph-theoretic account of logics proposed herein, we have to present first some technical preliminaries (we illustrate some of the constructions with the running example of the formula $(\neg\, q_1) \supset q_2$).

By a *non-empty path* $e_n \dots e_1$ over a m-graph G we mean a finite and non-empty sequence of elements of E such that $\mathsf{src}(e_{k+1}) = \mathsf{trg}(e_k)$ for $k = 1, \dots, n-1$. The source of a non-empty sequence $e_n \dots e_1$ is $\mathsf{src}(e_1)$ and the target of that sequence is $\mathsf{trg}(e_n)$. To each element s of V^+ we associate an *empty path*, denoted by ϵ_s. The source and target of an empty sequence ϵ_s is s. A path w can be written as $w : s \to t$ whenever the source of w is s and the target of w is t. We denote by $paths(G)$ the set of all paths over the m-graph G.

The main objective now is to freely generate a category with non empty finite products out of a given m-graph. The idea is that the objects of the generated category are non-empty finite sequences of vertexes of the m-graph and that each path $w : s \to t$ induces a morphism $\widehat{w} : s \to t$. Moreover, the object $v_1 \dots v_n$ is the object $v_1 \times \dots \times v_n$ in the obtained category. The construction is done in several steps. (i) From a m-graph G we obtain a (classical) graph $G^\dagger$ where the vertexes are in V^+ and the edges besides containing the m-edges in G contain also additional edges for projections and tuples; (ii) from $G^\dagger$ we freely generate a category $G^\ddagger$ whose objects are the same as the vertexes of $G^\dagger$ and including morphisms for edges, paths, projections and tuples; (iii) from $G^\ddagger$ we get the envisaged category G^+ by making a quotient over the class of morphisms ensuring that projections and tuples have the required universal properties.

Before presenting the construction we introduce some notation. Let **fpCat** be the category of categories with non empty finite products. As usual in a category with products, we denote by $\mathsf{p}_i^{b_1 \times \dots \times b_n}$ the i-th canonical projection of the product $b_1 \times \dots \times b_n$ for $n \geq 1$. Given morphisms $f_1 : b \to b_1, \dots, f_n : b \to b_n$, we refer to

$$\langle f_1, \dots, f_n \rangle : b \to (b_1 \times \dots \times b_n)$$

as the unique morphism such that $\mathsf{p}_i^{b_1 \times \dots \times b_n} \circ \langle f_1, \dots, f_n \rangle = f_i$ for every i. If $f_1 : b_1 \to b_1', \dots, f_n : b_n \to b_n'$ are morphisms then

$$f_1 \times \dots \times f_n : b_1 \times \dots \times b_n \to b_1' \times \dots \times b_n'$$

will stand for the morphism $\langle f_1 \circ \mathsf{p}_1^{b_1 \times \dots \times b_n}, \dots, f_n \circ \mathsf{p}_n^{b_1 \times \dots \times b_n} \rangle$. As usual, $\langle f_1, \dots, f_n \rangle$ and $f_1 \times \dots \times f_n$ will be identified with f_1 when n is 1.

The aim now is to define the category with non empty finite products G^+ from a m-graph G, following the steps sketched above.

i. From a m-graph G to a graph $G^\dagger$. We start by defining a family

$$\{G^\dagger_k = (V^+, E^\dagger_k, \mathsf{src}^\dagger_k, \mathsf{trg}^\dagger_k)\}_{k\geq 1}$$

of m-graphs such that

- $E^\dagger_1 = E \cup \{\mathsf{p}_i^{v_1\ldots v_n} : v_1, \ldots, v_n \in V, n \geq 2, i = 1, \ldots, n\}$;
- $\mathsf{src}^\dagger_1(e) = \mathsf{src}(e)$ and $\mathsf{trg}^\dagger_1(e) = \mathsf{trg}(e)$ whenever e is in E, $\mathsf{src}^\dagger_1(\mathsf{p}_i^{v_1\ldots v_n}) = v_1 \ldots v_n$ and $\mathsf{trg}^\dagger_1(\mathsf{p}_i^{v_1\ldots v_n}) = v_i$;
- $E^\dagger_k$ is the union of $E^\dagger_{k-1}$ with $\cup_{j=2,\ldots,k}\{\langle w_1, \ldots, w_j\rangle : w_1, \ldots, w_j$ are paths over $G^\dagger_{k-1}$ with target in V and with the same source$\}$;
- $\mathsf{src}^\dagger_k(e) = \mathsf{src}^\dagger_{k-1}(e)$ and $\mathsf{trg}^\dagger_k(e) = \mathsf{trg}^\dagger_{k-1}(e)$ if $e \in E^\dagger_{k-1}$, otherwise e is $\langle w_1, \ldots, w_j\rangle$, $\mathsf{src}^\dagger_k(e) = \mathsf{src}^\dagger_{k-1}(w_1)$ and $\mathsf{trg}^\dagger_k(e) = \mathsf{trg}^\dagger_{k-1}(w_1) \ldots \mathsf{trg}^\dagger_{k-1}(w_j)$.

So $G^\dagger$ is $(V^+, E^\dagger, \mathsf{src}^\dagger, \mathsf{trg}^\dagger)$ where $E^\dagger$ is $\cup_{k\in N} E^\dagger_k$, $\mathsf{src}^\dagger(e) = \mathsf{src}^\dagger_j(e)$ and $\mathsf{trg}^\dagger(e) = \mathsf{trg}^\dagger_j(e)$ for e in $E^\dagger_j$.

$$\diamond \xrightarrow{\langle \neg q_1, q_2\rangle} \pi\pi \xrightarrow{\supset} \pi$$

Figure 8: Formula $(\neg\, q_1) \supset q_2$ represented as a path over the graph $G^\dagger$ generated by the signature m-graph G described in Example 2.1.

ii. From a graph $G^\dagger$ to a category $G^\ddagger$. Given a graph $G^\dagger$, $G^\ddagger$ is the category freely generated by graph $G^\dagger$. That is, the category obtained as follows:

- the objects are the vertexes of $G^\dagger$;
- each path $w : s \to t$ in over $G^\dagger$ determines a unique morphism $w^\ddagger : s \to t$ in $G^\ddagger$ in such a way that if w is in E we set $w^\ddagger = w$;
- the identity morphism $\mathsf{id}_s : s \to s$ is ${\epsilon_s}^\ddagger$;
- $(w_2)^\ddagger \circ (w_1)^\ddagger = (w_2 w_1)^\ddagger$ whenever $w_2 : s \to t$ and $w_1 : r \to s$.

$$\begin{tikzcd} \diamond \arrow[r, "\langle \neg q_1, q_2\rangle"] \arrow[rr, bend left, "\supset \circ \langle \neg q_1, q_2\rangle"] & \pi\pi \arrow[r, "\supset"] & \pi \end{tikzcd}$$

Figure 9: Formula $(\neg\, q_1) \supset q_2$ represented as a morphism in the category $G^\ddagger$ generated from the signature m-graph G described in Example 2.1.

iii. From a category $G^\ddagger$ to a category G^+ with non empty finite products. Given a category $G^\ddagger$, the category G^+ is defined as follows:

- the set of objects of G^+ is the same as the set of objects of $G^\ddagger$, i.e., is V^+;
- the collection $G^+(-,-)$ of morphisms in G^+ is the quotient $G^\ddagger(-,-)/\Delta^\ddagger$ where $\Delta^\ddagger \subseteq G^\ddagger(-,-)^2$ is the least equivalence relation such that:
  - $((\mathsf{p}_i^{v_1\ldots v_n}\langle w_1,\ldots,w_n\rangle)^\ddagger, {w_i}^\ddagger)$ is in $\Delta^\ddagger$ for $i=1,\ldots,n$, where $w_j : s \to v_j$ are paths over $G^\dagger$ and v_j is in V for $j=1,\ldots,n$;
  - $(w^\ddagger, \langle u_1,\ldots,u_n\rangle^\ddagger)$ is in $\Delta^\ddagger$ if $((\mathsf{p}_i^{v_1\ldots v_n} w)^\ddagger, {u_i}^\ddagger)$ is in $\Delta^\ddagger$ where $w : s \to v_1\ldots v_n$ and $u_i : s \to v_i$ are paths over $G^\dagger$ and $v_i \in V$, $i=1,\ldots,n$;
  - $((w_2w_1)^\ddagger, (u_2u_1)^\ddagger)$ is in $\Delta^\ddagger$ if $({w_2}^\ddagger, {u_2}^\ddagger)$ and $({w_1}^\ddagger, {u_1}^\ddagger)$ are in $\Delta^\ddagger$ where $w_2, u_2 : s_1 \to t$ and $w_1, u_1 : s \to s_1$ are paths over $G^\dagger$;
- in G^+ the identity in s is the morphism $[{\epsilon_s}^\ddagger]_{\Delta^\ddagger}$;
- in G^+ the operation $\circ$ is such that $[{w_2}^\ddagger]_{\Delta^\ddagger} \circ [{w_1}^\ddagger]_{\Delta^\ddagger} = [(w_2w_1)^\ddagger]_{\Delta^\ddagger}$.

We denote by

$$\widehat{w}$$

the equivalence class $[w^\ddagger]_{\Delta^\ddagger}$. The first clause of the equivalence relation establishes that the i-th projection has the expected behavior when applied to a tuple, that is, is equivalent to the i-th component. The second clause imposes the universal property of the product. Finally, the third clause asserts that composition preserves equivalence.

$$\begin{tikzcd} \diamond \arrow[r, "\langle \widehat{\neg} \circ \widehat{q_1}, \widehat{q_2} \rangle"] \arrow[rr, bend left, "\widehat{\supset} \circ \langle \widehat{\neg} \circ \widehat{q_1}, \widehat{q_2} \rangle"] & \pi\pi \arrow[r, "\widehat{\supset}"] & \pi \end{tikzcd}$$

Figure 10: Formula $(\neg\, q_1) \supset q_2$ as a morphism in the category G^+ generated from the signature m-graph G described in Example 2.1.

The previous construction deserves some comments. Firstly, note that for any path $e_n \ldots e_1$ over $G^\dagger$ where e_i is in $E^\dagger$ for $i=1,\ldots,n$, if e_k is a projection and $k > 1$ then e_{k-1} is a tuple. Secondly, it is immediate to see that the domains and codomains of the morphisms in G^+ are well defined. In fact it is very easy to prove the following lemma by induction on $\Delta^\ddagger$:

Lemma 3.1 Given a m-graph G, if $({w_1}^\ddagger, {w_2}^\ddagger)$ is in $\Delta^\ddagger$ and $w_1 : s \to t$ then w_2 is also a path from s to t.

Thirdly, note that G^+ is, by construction, a category with non empty finite products.

Proposition 3.2 The category G^+ has non empty finite products.

Proof: For simplicity, consider the objects v_1, v_2 which are sequences of length one. Their product is

$$(v_1 v_2, \widehat{\mathsf{p}}_1^{v_1 v_2}, \widehat{\mathsf{p}}_1^{v_1 v_2}).$$

Given morphisms $\widehat{w}_1 : s \to v_1$ and $\widehat{w}_2 : s \to v_2$. We will show that $\widehat{\langle w_1, w_2 \rangle}$ is the unique morphism in G^+ such that $\widehat{\mathsf{p}}_i^{v_1 v_2} \circ \widehat{\langle w_1, w_2 \rangle} = \widehat{w}_i$ for $i = 1, 2$.

(a) $\widehat{\mathsf{p}}_1^{v_1 v_2} \circ \widehat{\langle w_1, w_2 \rangle} = \widehat{w}_1$. Note that

$$\widehat{\mathsf{p}}_1^{v_1 v_2} \circ \widehat{\langle w_1, w_2 \rangle} = [\mathsf{p}_1^{v_1 v_2 \ddagger}]_{\Delta^\ddagger} \circ [\langle w_1, w_2 \rangle^\ddagger]_{\Delta^\ddagger}$$

which is $[\mathsf{p}_1^{v_1 v_2} \langle w_1, w_2 \rangle^\ddagger]_{\Delta^\ddagger} = [w_1{}^\ddagger]_{\Delta^\ddagger} = \widehat{w}_1$.

(b) Unicity. Assume that $\widehat{u} : s \to v_1 v_2$ such that $\widehat{\mathsf{p}}_i^{v_1 v_2} \circ \widehat{u} = \widehat{w}_i$ for $i = 1, 2$. Hence, $[\mathsf{p}_i^{v_1 v_2} u^\ddagger]_{\Delta^\ddagger} = \widehat{\mathsf{p}}_i^{v_1 v_2} \circ \widehat{u} = \widehat{w}_i = [w_i{}^\ddagger]_{\Delta^\ddagger}$. Therefore, $((\mathsf{p}_i^{v_1 v_2} u)^\ddagger, w_i{}^\ddagger)$ is in $\Delta^\ddagger$ for $i = 1, 2$ and so $(u^\ddagger, \langle w_1, w_2 \rangle^\ddagger)$ is in $\Delta^\ddagger$. Hence $[u^\ddagger]_{\Delta^\ddagger} = [\langle w_1, w_2 \rangle^\ddagger]_{\Delta^\ddagger}$. That is, $\widehat{u} = \widehat{\langle w_1, w_2 \rangle}$. QED

It is worthwhile to note that $\langle \widehat{w}_1, \ldots, \widehat{w}_n \rangle$ is $\widehat{\langle w_1, \ldots, w_n \rangle}$ when $w_i : s \to v_i$ and $v_i \in V$ according to Proposition 3.2. Given the path $w_i : s \to s_i$ over $G^\dagger$ where s_i has length m_i, for $i = 1, \ldots, n$, the tuple $\langle \widehat{w}_1, \ldots, \widehat{w}_n \rangle$ is $\widehat{\langle \mathsf{p}_1^{s_1} w_1, \ldots, \mathsf{p}_{m_1}^{s_1} w_1, \ldots, \mathsf{p}_1^{s_n} w_n, \ldots, \mathsf{p}_{m_n}^{s_n} w_n \rangle}$. Moreover, for $i = 1, \ldots, n$, let s_i in V^+ be $v_{i1} \ldots v_{im_i}$, where $v_{i1}, \ldots, v_{im_i}$ are in V. Then the product of $s_1, \ldots, s_n$ denoted by

$$(s_1 \times \ldots \times s_n, \mathsf{p}_1^{s_1 \times \ldots \times s_n}, \ldots, \mathsf{p}_n^{s_1 \times \ldots \times s_n})$$

can be taken to be the object $v_{11} \ldots v_{1m_1} \ldots v_{n1} \ldots v_{nm_n}$ with the morphisms $\widehat{\langle \mathsf{p}_{m_1 + \ldots + m_{i-1} + 1}^{v_{11} \ldots v_{1m_1} \ldots v_{n1} \ldots v_{nm_n}}, \ldots, \mathsf{p}_{m_1 + \ldots + m_{i-1} + m_i}^{v_{11} \ldots v_{1m_1} \ldots v_{n1} \ldots v_{nm_n}} \rangle}$ for $i = 1, \ldots, n$.

Given a signature $\Sigma = (G, \pi, \diamond)$, the objects of G^+ are the finite and non-empty sequences of sorts in the signature Σ and the morphisms of G^+ play the role of *expressions* (schema formulas, schema terms, whatever) over Σ, and constitute the *language* generated by the signature, also denoted by $L(\Sigma)$. More precisely, each morphism $\widehat{w} : s \to t$ in G^+ represents an *expression* of type $s \to t$. Note that a morphism in G^+ corresponds to a path over the signature m-graph G.

For instance, using the constructors of signature Σ_Π, the morphism

$$\widehat{\supset} \circ \langle \widehat{\neg} \circ \widehat{q}_1, \widehat{q}_2 \rangle$$

corresponds to the path $\supset \langle \neg\, q_1, q_2 \rangle$ over $G^\dagger$ where q_1 and q_2 are *propositional symbols*, that is, are m-edges in E of type $\diamond \to \pi$, since:

- $q_1, q_2, \supset, \neg \in E_1^\dagger$;
- $\langle \neg\, q_1, q_2 \rangle \in E_2^\dagger$;

hence $\supset \langle \neg\, q_1, q_2 \rangle$ is a path over $G_2^\dagger$ and so over $G^\dagger$. Moreover, it is straightforward to see that $\widehat{\supset} \circ \langle \widehat{\neg} \circ \widehat{q}_1, \widehat{q}_2 \rangle$ is $\supset \widehat{\langle \neg\, q_1, q_2 \rangle}$. Indeed,

$$\begin{aligned} \widehat{\supset} \circ \langle \widehat{\neg} \circ \widehat{q}_1, \widehat{q}_2 \rangle &= \widehat{\supset} \circ \langle \widehat{\neg\, q_1}, \widehat{q}_2 \rangle \\ &= \widehat{\supset} \circ \widehat{\langle \neg\, q_1, q_2 \rangle} \\ &= \supset \circ \widehat{\langle \neg\, q_1, q_2 \rangle}. \end{aligned}$$

In the sequel, when there is no ambiguity, we may denote a morphism $\widehat{e}$ of G^+ where e is a m-edge of E simply by e. Expressions with the object $\diamond$ as source are said to be *concrete expressions*. Thus, $G^+(\diamond, \pi)$ is the set of all *concrete formulas*, or simply the set of all *formulas* in the language of Σ. This set corresponds to the traditional (set-theoretic) notion of language of propositions over Σ.

For instance, the morphism:

$$\supset \circ \langle \neg \circ p_1, \supset \circ \langle p_2, p_1 \rangle \rangle : \diamond \to \pi$$

is an expression of type $\diamond \to \pi$ and so is a formula, represented more simply as $((\neg p_1) \supset (p_2 \supset p_1))$. In the sequel we may simplify the representation of morphisms in a similar way. Clearly, it is possible to write expressions with a non-concrete object as source. Such expressions are said to be *schema expressions* because only part of their structure is known (or determined), and when its target is π we may call them *schema formulas*. So by a schema formula we mean a morphism in G^+ whose target is π and with no constraints over the source. Schema variables are projections from $\pi \ldots \pi$ to π, or from $\pi \ldots \pi\diamond$ to π, where the π-sequence at the source is non-empty, and are denoted by $\xi, \xi', \xi'', \ldots, \xi_1, \xi_1', \xi_1'', \ldots, \xi_2, \xi_2', \xi_2'', \ldots$.

Example 3.3 Consider the signature Σ_Π defined in Example 2.1. The schema formula

$$(\xi_1 \supset (\xi_1 \supset \xi_1)) \supset \xi_2$$

is the morphism

$$\supset \circ \langle \supset \circ \langle \xi_1, \supset \circ \langle \xi_1, \xi_1 \rangle \rangle, \xi_2 \rangle : \pi\pi \to \pi$$

where ξ_i is $\widehat{\mathsf{p}}_i^{\pi\pi}$, for $i = 1, 2$; and the schema formula

$$(\xi_3 \supset (\xi_1 \supset \xi_2)) \supset \xi_4$$

is the morphism

$$\supset \circ \langle \supset \circ \langle \xi_3, \supset \circ \langle \xi_1, \xi_2 \rangle \rangle, \xi_4 \rangle : \pi\pi\pi\pi \to \pi$$

where ξ_i is $\widehat{\mathsf{p}}_i^{\pi\pi\pi\pi}$, for $i = 1, \ldots, 4$. Given the propositional symbol $p : \diamond \to \pi$, the morphism

$$\supset \circ \langle \supset \circ \langle p \circ \widehat{\mathsf{p}}_3^{\pi\pi\diamond}, \supset \circ \langle \xi_1, p \circ \widehat{\mathsf{p}}_3^{\pi\pi\diamond} \rangle \rangle, \xi_2 \rangle : \pi\pi\diamond \to \pi$$

where ξ_i is $\widehat{\mathsf{p}}_i^{\pi\pi\diamond}$, for $i = 1, 2$, corresponds to the schema formula

$$(p \supset (\xi_1 \supset p)) \supset \xi_2.$$

Finally, given the propositional symbols $p, q : \diamond \to \pi$, the morphism

$$\supset \circ \langle \supset \circ \langle p \circ \widehat{\mathsf{p}}_2^{\pi\diamond}, \supset \circ \langle \xi_1, q \circ \widehat{\mathsf{p}}_2^{\pi\diamond} \rangle \rangle, \xi_1 \rangle : \pi\diamond \to \pi$$

represents the schema formula

$$(p \supset (\xi_1 \supset q)) \supset \xi_1$$

where ξ_1 is $\widehat{\mathsf{p}}_1^{\pi\diamond}$. ∇

Non-concrete expressions are very useful for setting up deductive rules that can be instantiated using substitutions. Deductive rules with non-concrete expressions are called *schema rules*. Expression instantiation and rule instantiation is achieved using morphism composition. Given the expressions $\widehat{w} : s_2 \to s_3$ and $\widehat{u} : s_1 \to s_2$, the *expression instantiation* of the former by the latter is the expression $\widehat{w} \circ \widehat{u}$.

Example 3.4 Let φ be the schema formula

$$\supset \circ \langle \supset \circ \langle \neg \circ \xi, \xi' \rangle, \supset \circ \langle \xi, \xi'' \rangle \rangle$$

where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$. We can interchange ξ with ξ' by instantiating φ with $\langle \xi', \xi, \xi'' \rangle : \pi\pi\pi \to \pi\pi\pi$, obtaining the following schema formula

$$\begin{aligned} \varphi \circ \langle \xi', \xi, \xi'' \rangle &= \supset \circ \langle \supset \circ \langle \neg \circ \xi, \xi' \rangle \circ \langle \xi', \xi, \xi'' \rangle, \supset \circ \langle \xi, \xi'' \rangle \circ \langle \xi', \xi, \xi'' \rangle \rangle \\ &= \supset \circ \langle \supset \circ \langle \neg \circ \xi', \xi \rangle, \supset \circ \langle \xi', \xi'' \rangle \rangle \end{aligned}$$

from $\pi\pi\pi$ to π. On the other hand, if we want to make concrete the second slot of φ we could consider the propositional symbol $p : \diamond \to \pi$, and then instantiate φ with $\langle \xi_1, p \circ \widehat{\mathsf{p}}_3^{\pi\pi\diamond}, \xi_2 \rangle : \pi\pi\diamond \to \pi\pi\pi$ where $\xi_j = \widehat{\mathsf{p}}_j^{\pi\pi\diamond}$ for $j = 1, 2$ obtaining the following schema formula

$$\begin{aligned} &\varphi \circ \langle \xi_1, p \circ \widehat{\mathsf{p}}_3^{\pi\pi\diamond}, \xi_2 \rangle = \\ &\qquad = \supset \circ \langle \supset \circ \langle \neg \circ \xi, \xi' \rangle \circ \langle \xi_1, p \circ \widehat{\mathsf{p}}_3^{\pi\pi\diamond}, \xi_2 \rangle, \supset \circ \langle \xi, \xi'' \rangle \circ \langle \xi_1, p \circ \widehat{\mathsf{p}}_3^{\pi\pi\diamond}, \xi_2 \rangle \rangle \\ &\qquad = \supset \circ \langle \supset \circ \langle \neg \circ \xi_1, p \circ \widehat{\mathsf{p}}_3^{\pi\pi\diamond} \rangle, \supset \circ \langle \xi_1, \xi_2 \rangle \rangle \end{aligned}$$

from $\pi\pi\diamond$ to π. ∇

4 Satisfaction as a path

The main objective of the section is to introduce the notion of denotation of an expression, and the notion of entailment of an expression from a set of expressions. Intutively, denotation of a formula in the context of an interpretation structure, is expected to be the set of the targets of all paths in the operations m-graph that are mapped by the abstraction map to the formula. Consider the denotation of the formula $(\neg q_1) \supset q_2$ in the interpretation structure described in Example 2.6. Then, it is not difficult to see that the denotation of that formula is the target of the path in Figure 11, that is, the truth value $\mathbf{t}$. Equivalently, as we will see, denotation of a formula can also be defined as the set of targets of all morphisms in the category G'^{+}, corresponding to the formula $(\neg q_1) \supset q_2$. As an example see Figure 12.

As a consequence, we need to extend the abstraction map α to a functor α^+ from the category G'^{+} generated from the operations m-graph to the category G^+ generated from the signature m-graph.

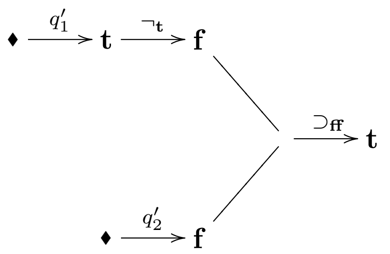

Figure 11: Path in the operations m-graph G', introduced in Example 2.6 for propositional logic, corresponding to formula $(\neg\, q_1) \supset q_2$.

$$\begin{tikzcd}
\blacklozenge \arrow[r, "\langle \widehat{\neg}_{\mathbf{f}} \circ \widehat{q}_1', \widehat{q}_2' \rangle"] \arrow[rr, bend left, "\widehat{\supset}_{\mathbf{ff}} \circ \langle \widehat{\neg}_{\mathbf{f}} \circ \widehat{q}_1', \widehat{q}_2' \rangle"] & \mathbf{ff} \arrow[r, "\widehat{\supset}_{\mathbf{ff}}"] & \mathbf{t}
\end{tikzcd}$$

Figure 12: Morphism in G'^{+} denoting formula $(\neg\, q_1) \supset q_2$.

So, given an m-graph morphism $h : G' \to G$, we now present a general way to induce a functor $h^+ : G'^+ \to G^+$. First, we need to induce a graph morphism $h^\dagger$ from a m-graph morphism h. Given a m-graph morphism $h : G' \to G$ we define inductively the graph morphism $h^\dagger : \mathit{paths}(G'^\dagger) \to \mathit{paths}(G^\dagger)$ as follows:

- $h^\dagger(\epsilon_{v_1' \ldots v_n'}) = \epsilon_{h^{\mathsf{v}}(v_1') \ldots h^{\mathsf{v}}(v_n')}$ for $v_1', \ldots, v_n'$ in V';
- $h^\dagger(e'w') = h^{\mathsf{e}}(e')h^\dagger(w')$ where e' is a m-edge in E';
- $h^\dagger(\mathsf{p}_i^{v_1' \ldots v_n'} w') = \mathsf{p}_i^{h^{\mathsf{v}}(v_1') \ldots h^{\mathsf{v}}(v_n')} h^\dagger(w')$;
- $h^\dagger(\langle w_1', \ldots, w_n' \rangle w_0') = \langle h^\dagger(w_1'), \ldots, h^\dagger(w_n') \rangle h^\dagger(w_0')$.

Note that, if $w' : s' \to t'$ then $h^\dagger(w') : (h^{\mathsf{v}})^+(s') \to (h^{\mathsf{v}})^+(t')$. The main result to be stated is the following one:

Proposition 4.1 Given a m-graph morphism $h : G' \to G$, the pair

$$h^+ = ((h^{\mathsf{v}})^+, (h^{\mathsf{e}})^+),$$

where $(h^{\mathsf{e}})^+(\widehat{w}') = \widehat{h^\dagger(w')}$ and $(h^{\mathsf{v}})^+$ is the extension of h^{v} to sequences, is a functor from G'^+ to G^+.

In order to prove the result above, we need an auxiliary technical lemma stating that $(h^{\mathsf{e}})^+$ is well defined (that is, its value does not depend on the particular chosen representative of the equivalence class).

Lemma 4.2 Given a m-graph morphism $h : G' \to G$, if $({w_1'}^\ddagger, {w_2'}^\ddagger)$ is in $\Delta'^\ddagger$ then $(h^\dagger(w_1')^\ddagger, h^\dagger(w_2')^\ddagger)$ is in $\Delta^\ddagger$.

Proof: We show by induction on $\Delta'^{\ddagger}$ that $(h^\dagger(w'_1)^\ddagger, h^\dagger(w'_2)^\ddagger)$ is in $\Delta^\ddagger$:

- $({w'_1}^\ddagger, {w'_2}^\ddagger)$ is such that w'_1 is $\mathsf{p}_i^{v'_1\dots v'_n}\langle u'_1,\dots,u'_n\rangle$ and w'_2 is u'_i. The result follows since $(\mathsf{p}_i^{h^\mathsf{v}(v'_1)\dots h^\mathsf{v}(v'_n)}\langle h^\dagger(u'_1),\dots,h^\dagger(u'_n)\rangle^\ddagger, h^\dagger(u'_i)^\ddagger)$ is in $\Delta^\ddagger$, $h^\dagger(w'_1)^\ddagger$ is the first element of that pair and $h^\dagger(w'_2)^\ddagger$ is the second;

- $({w'_1}^\ddagger, {w'_2}^\ddagger)$ is such that w'_2 is $\langle u'_1,\dots,u'_n\rangle$ and $((\mathsf{p}_i^{v'_1\dots v'_n} w'_1)^\ddagger, {u'_i}^\ddagger) \in \Delta'^{\ddagger}$ for $i = 1,\dots,n$. So $((\mathsf{p}_i^{h^\mathsf{v}(v'_1)\dots h^\mathsf{v}(v'_n)} h^\dagger(w'_1))^\ddagger, h^\dagger(u'_i)^\ddagger) \in \Delta^\ddagger$ for $i = 1,\dots,n$ by induction hypothesis. Then $(h^\dagger(w'_1)^\ddagger, \langle h^\dagger(u'_1),\dots,h^\dagger(u'_n)\rangle^\ddagger)$ is in $\Delta^\ddagger$ by definition of $\Delta^\ddagger$. The result follows since $\langle h^\dagger(u'_1),\dots,h^\dagger(u'_n)\rangle$ is $h^\dagger(\langle u'_1,\dots,u'_n\rangle)$;

- $({w'_1}^\ddagger, {w'_2}^\ddagger)$ is such that w'_1 is $g'_1 f'_1$, w'_2 is $g'_2 f'_2$, and $({g'_1}^\ddagger, {g'_2}^\ddagger)$ and $({f'_1}^\ddagger, {f'_2}^\ddagger)$ are in $\Delta'^{\ddagger}$. Hence, by induction hypothesis, $(h^\dagger(g'_1)^\ddagger, h^\dagger(g'_2)^\ddagger)$ and $(h^\dagger(f'_1)^\ddagger, h^\dagger(f'_2)^\ddagger)$ are in $\Delta^\ddagger$, and so $(h^\dagger(g'_1)h^\dagger(f'_1)^\ddagger, h^\dagger(g'_2)h^\dagger(f'_2)^\ddagger)$ is also in $\Delta^\ddagger$. The result follows since $h^\dagger(w'_1)^\ddagger$ is the first element of that pair and $h^\dagger(w'_2)^\ddagger$ is the second.

- $({w'_1}^\ddagger, {w'_2}^\ddagger)$ is such that ${w'_1}^\ddagger = {w'_2}^\ddagger$. Then by the uniqueness of the representation $w'_1 = w'_2$ and so the result follows straightforwardly.

- $({w'_1}^\ddagger, {w'_2}^\ddagger)$ is such that $({w'_2}^\ddagger, {w'_1}^\ddagger)$ is in $\Delta'^{\ddagger}$. The result follows straightforwardly by induction hypothesis.

- $({w'_1}^\ddagger, {w'_2}^\ddagger)$ is such that $({w'_1}^\ddagger, {g'}^\ddagger)$ and $({g'}^\ddagger, {w'_2}^\ddagger)$ are in $\Delta'^{\ddagger}$. Then the result follows straightforwardly by induction hypothesis. QED

Proof: *(of Proposition 4.1)*
The map $(h^\mathsf{e})^+$ is well defined by Lemma 4.2, and preserves identities since $(h^\mathsf{e})^+(\mathsf{id}_{v'_1\dots v'_n}) = (h^\mathsf{e})^+(\widehat{\epsilon}_{v'_1\dots v'_n}) = \widehat{h^\dagger(\epsilon_{v'_1\dots v'_n})} = \widehat{\epsilon}_{h^\mathsf{v}(v'_1)\dots h^\mathsf{v}(v'_n)} = \mathsf{id}_{h^\mathsf{v}(v'_1)\dots h^\mathsf{v}(v'_n)}$. The map $(h^\mathsf{e})^+$ preserves compositions since $(h^\mathsf{e})^+(\widehat{w}'_2 \circ \widehat{w}'_1) = (h^\mathsf{e})^+(\widehat{w'_2 w'_1}) = \widehat{h^\dagger(w'_2 w'_1)} = \widehat{h^\dagger(w'_2) h^\dagger(w'_1)} = \widehat{h^\dagger(w'_2)} \circ \widehat{h^\dagger(w'_1)} = (h^\mathsf{e})^+(\widehat{w}'_2) \circ (h^\mathsf{e})^+(\widehat{w}'_1)$. QED

Finally, we observe that, from the results above, we obtain a functor $\cdot^+ : \mathbf{mGraph} \to \mathbf{fpCat}$ defined in the obvious way.

So, we are now able to define denotation of a path, and then, denotation of a morphism. With this purpose in mind, we have to give, for each sort, the starting values so that we can define denotation inductively. Observe that, as hinted before, the denotation of a path is a set of values. Recall that the *concatenation* $A \cdot B$, or even AB, of the sets of sequences A and B is the set of sequences $\{ab : a \in A \text{ and } b \in B\}$. Moreover, given an interpretation structure I over a signature Σ, and $v_1,\dots,v_n$ in V, a subset S of $V'^{+}_{v_1\dots v_n}$ is a *concatenation of basic sets* whenever there exist $S_1 \subseteq V'_{v_1},\dots,S_n \subseteq V'_{v_n}$ such that S is $S_1\dots S_n$. Given a concatenation of sets $S_1\dots S_n$ we denote by $(S_1\dots S_n)_i$ its i-th component, that is, the set S_i.

The *denotation of a concrete path* $w : \diamond \to t$ over $G^\dagger$ at I, represented by

$$[\![w]\!]^I$$

is a concatenation of basic sets contained in V'^{+}_t, inductively defined on the complexity of the path w as follows:

- $[\![\epsilon_\diamond]\!]^I$ is $\{\blacklozenge\}$;
- $[\![\mathsf{p}_i^{v_1\ldots v_m} w_1]\!]^I$ is $([\![w_1]\!]^I)_i$ where $v_1,\ldots,v_m$ are in V;
- $[\![\langle w_1,\ldots,w_n\rangle w_0]\!]^I$ is $[\![w_1 w_0]\!]^I \ldots [\![w_n w_0]\!]^I$;
- $[\![e w_1]\!]^I$ is the union of $\mathsf{trg}'(E'_e(v',-))$ for each v' in $[\![w_1]\!]^I$, when e is in E.

For instance, for evaluating ew_1 over I, we start by evaluating w_1 and getting a set of values. For each value s' in the evaluation of w_1, we pick all the m-edges in G' with source s' and which are mapped into e. Finally, the envisaged denotation is obtained by taking the collection of targets of such m-edges.

Denotation is now extended to non-concrete paths. The denotation of the schema variables is given by an assignment, which must be also a component in the denotation process. An *assignment*

$$\rho$$

for an interpretation structure I over a signature Σ is a family $\{\rho_s\}_{s\in V^+}$ such that ρ_s is $[\![w_s]\!]^I$ for some concrete path $w_s : \diamond \to s$. Observe that ρ_s is contained in $V_s'^+$ and is a concatenation of basic sets, and $\rho_\diamond = \{\blacklozenge\}$.

The *denotation of a path* $w : s \to t$ over $G^\dagger$ at I and ρ, denoted by

$$[\![w]\!]^{I\rho}$$

is a concatenation of basic sets contained in $V_t'^+$, inductively defined on the complexity of the path w similarly to the denotation of a concrete path with the exception that $[\![\epsilon_s]\!]^{I\rho}$ is ρ_s.

Example 4.3 Consider the interpretation structure I_{nd} in Example 2.9. Then

- $[\![q_1 \wedge (\neg\, q_1)]\!]^{I_{\mathrm{nd}}\rho} = \{\mathbf{I},\mathbf{t}\}$ since

$$\begin{array}{rcl}
[\![\epsilon_\diamond]\!]^{I_{\mathrm{nd}}\rho} & = & \{\blacklozenge\} \\
[\![q_1]\!]^{I_{\mathrm{nd}}\rho} & = & \mathsf{trg}'(E'_{q_1}([\![\epsilon_\diamond]\!]^{I_{\mathrm{nd}}\rho},-)) \\
& = & \mathsf{trg}'(E'_{q_1}(\{\blacklozenge\},-)) \\
& = & \mathsf{trg}'(\{q'_1 : \blacklozenge \to \mathbf{I}\}) \\
& = & \{\mathbf{I}\}
\end{array}$$

$$\begin{array}{rcl}
[\![\neg\, q_1]\!]^{I_{\mathrm{nd}}\rho} & = & \mathsf{trg}'(E'_\neg([\![q_1]\!]^{I_{\mathrm{nd}}\rho},-)) \\
& = & \mathsf{trg}'(E'_\neg(\mathbf{I},-)) \\
& = & \mathsf{trg}'(\{\neg_{\mathbf{II}} : \mathbf{I}\to\mathbf{I}, \neg_{\mathbf{It}} : \mathbf{I}\to\mathbf{t}\} \\
& = & \{\mathbf{I},\mathbf{t}\}
\end{array}$$

$$\begin{array}{rcl}
[\![q_1 \wedge (\neg\, q_1)]\!]^{I_{\mathrm{nd}}\rho} & = & \mathsf{trg}'(E'_\wedge([\![q_1]\!]^{I_{\mathrm{nd}}\rho}[\![\neg\, q_1]\!]^{I_{\mathrm{nd}}\rho},-)) \\
& = & \mathsf{trg}'(E'_\wedge(\{\mathbf{I}\}\{\mathbf{I},\mathbf{t}\},-)) \\
& = & \mathsf{trg}'(E'_\wedge(\{\mathbf{II},\mathbf{It}\},-)) \\
& = & \mathsf{trg}'(\{\wedge_{\mathbf{III}} : \mathbf{II}\to\mathbf{I}, \wedge_{\mathbf{IIt}} : \mathbf{II}\to\mathbf{t}, \\
& & \qquad \wedge_{\mathbf{ItI}} : \mathbf{It}\to\mathbf{I}, \wedge_{\mathbf{Itt}} : \mathbf{It}\to\mathbf{t}\}) \\
& = & \{\mathbf{I},\mathbf{t}\};
\end{array}$$

- $[\![\circ q_1]\!]^{I_{\text{nd}}\rho} = \{\mathbf{f}\}$ since

$$\begin{array}{rcl}[\![\circ q_1]\!]^{I_{\text{nd}}\rho} & = & \mathsf{trg}'(E'_\circ([\![q_1]\!]^{I_{\text{nd}}\rho}, -)) \\ & = & \mathsf{trg}'(E'_\circ(\{\mathbf{I}\}, -)) \\ & = & \mathsf{trg}'(\{\circ_{\mathbf{If}} : \mathbf{I} \to \mathbf{f}\}) \\ & = & \{\mathbf{f}\}\end{array}$$

∇

As it is expected, the denotation of a path that is concrete does not depend on the assignments as the following result states.

Proposition 4.4 Given an interpretation structure (Σ, I), assignments ρ_1 and ρ_2 over I, and a concrete path w, $[\![w]\!]^{I\rho_1} = [\![w]\!]^{I\rho_2}$.

The next step is to extend denotation of paths to morphisms in order to evaluate expressions and, in particular, formulas. But first we have to state some technical lemmas.

Proposition 4.5 Given an interpretation structure (Σ, I) and an assignment ρ over I such that $\rho_s = [\![w_s]\!]^I$, then $[\![w]\!]^{I\rho} = [\![ww_s]\!]^I$ for any path $w : s \to t$ over $G^\dagger$.

Proof: The proof follows by induction on the complexity of w:

- w is ϵ_s. Then $[\![w]\!]^{I\rho} = \rho_s = [\![w_s]\!]^I = [\![ww_s]\!]^I$;

- w is $\mathsf{p}_i^{\hat{s}} w_0$. Then $[\![w]\!]^{I\rho} = [\![\mathsf{p}_i^{\hat{s}} w_0]\!]^{I\rho} = ([\![w_0]\!]^{I\rho})_i = ([\![w_0 w_s]\!]^I)_i = [\![\mathsf{p}_i^{\hat{s}} w_0 w_s]\!]^I = [\![ww_s]\!]^I$;

- w is $\langle u_1, \ldots, u_n \rangle u_0$. Then $[\![w]\!]^{I\rho} = [\![\langle u_1, \ldots, u_n \rangle u_0]\!]^{I\rho} = [\![u_1 u_0]\!]^{I\rho} \ldots [\![u_n u_0]\!]^{I\rho} = [\![u_1 u_0 w_s]\!]^I \ldots [\![u_n u_0 w_s]\!]^I = [\![\langle u_1, \ldots, u_n \rangle u_0 w_s]\!]^I = [\![ww_s]\!]^I$;

- w is ew_0. The thesis follows since $[\![w]\!]^{I\rho} = [\![ew_0]\!]^{I\rho} = \cup_{v' \in [\![w_0]\!]^{I\rho}} \mathsf{trg}'(E'_e(v', -)) = \cup_{v' \in [\![w_0 w_s]\!]^I} \mathsf{trg}'(E'_e(v', -)) = [\![ew_0 w_s]\!]^I = [\![ww_s]\!]^I$. QED

In the sequel, we use $\rho_{s/[\![w]\!]^{I\rho}}$ to refer to the assignment obtained from ρ by replacing ρ_s by the set $[\![w]\!]^{I\rho}$. Note that, by Poposition 4.5, ρ is well defined. The first result, in Proposition 4.6, is a substitution lemma adapted to our setting.

Proposition 4.6 Given an interpretation structure (Σ, I) and an assignment ρ over I, $[\![w_2 w_1]\!]^{I\rho} = [\![w_2]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}}$ for paths $w_1 : s_1 \to s_2$ and $w_2 : s_2 \to s_3$ over $G^\dagger$.

Proof: The proof follows by induction on the complexity of w_2:

- w_2 is ϵ_s. So $[\![w_2 w_1]\!]^{I\rho} = [\![w_1]\!]^{I\rho} = (\rho_{s/[\![w_1]\!]^{I\rho}})_s = [\![w_2]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}}$;

- w_2 is $\mathsf{p}_i^s w_0$. So $[\![w_2 w_1]\!]^{I\rho} = [\![\mathsf{p}_i^s w_0 w_1]\!]^{I\rho} = ([\![w_0 w_1]\!]^{I\rho})_i = ([\![w_0]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}})_i = [\![\mathsf{p}_i^s w_0]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}} = [\![w_2]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}}$;

- w_2 is $\langle u_1,\ldots,u_n\rangle u_0$. Then $[\![w_2w_1]\!]^{I\rho} = [\![\langle u_1,\ldots,u_n\rangle u_0 w_1]\!]^{I\rho} = [\![u_1u_0w_1]\!]^{I\rho}\ldots [\![u_nu_0w_1]\!]^{I\rho} = [\![u_1u_0]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}}\ldots [\![u_nu_0]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}} = [\![\langle u_1,\ldots,u_n\rangle u_0]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}} = [\![w_2]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}}$;

- w_2 is ew_0. Therefore $[\![w_2w_1]\!]^{I\rho} = [\![ew_0w_1]\!]^{I\rho} = \mathsf{trg}'(E'_e([\![w_0w_1]\!]^{I\rho},-)) = \mathsf{trg}'(E'_e([\![w_0]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}},-)) = [\![ew_0]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}} = [\![w_2]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}}$. QED

The following result states that denotation is well defined.

Proposition 4.7 Given an interpretation structure (Σ, I), if $(w^\ddagger, u^\ddagger)$ is in $\Delta^\ddagger$ then $[\![w]\!]^{I\rho} = [\![u]\!]^{I\rho}$ for any assignment ρ over I.

Proof: The proof follows by induction on $\Delta^\ddagger$:

- $(w^\ddagger, u^\ddagger)$ is such that w is $\mathsf{p}_i^{v_1\ldots v_n}\langle w_1,\ldots,w_n\rangle$ and u is w_i. Then $[\![w]\!]^{I\rho} = [\![\mathsf{p}_i^{v_1\ldots v_n}\langle w_1,\ldots,w_n\rangle]\!]^{I\rho} = ([\![\langle w_1,\ldots,w_n\rangle]\!]^{I\rho})_i = ([\![w_1]\!]^{I\rho}\ldots[\![w_n]\!]^{I\rho})_i = [\![w_i]\!]^{I\rho} = [\![u]\!]^{I\rho}$;

- $(w^\ddagger, u^\ddagger)$ is such that u is $\langle u_1,\ldots,u_n\rangle$, $w : s \to v_1\ldots v_n$, $u_i : s \to v_i$ and $((\mathsf{p}_i^{v_1\ldots v_n}w)^\ddagger, u_i{}^\ddagger)$ is in $\Delta^\ddagger$ for $i = 1,\ldots,n$. Hence $[\![\mathsf{p}_i^{v_1\ldots v_n}w]\!]^{I\rho} = [\![u_i]\!]^{I\rho}$ by induction hypothesis, for $i = 1,\ldots,n$. So $([\![w]\!]^{I\rho})_i = [\![u_i]\!]^{I\rho}$ for $i = 1,\ldots,n$. Since $[\![w]\!]^{I\rho}$ is a concatenation of basic sets then $[\![w]\!]^{I\rho} = [\![u_1]\!]^{I\rho}\ldots[\![u_n]\!]^{I\rho}$, and so the thesis follows straightforwardly;

- $(w^\ddagger, u^\ddagger)$ is such that w is w_2w_1, u is u_2u_1, and $(w_2{}^\ddagger, u_2{}^\ddagger)$ and $(w_1{}^\ddagger, u_1{}^\ddagger)$ are in $\Delta^\ddagger$. So $[\![w_1]\!]^{I\rho} = [\![u_1]\!]^{I\rho}$ and $[\![w_2]\!]^{I\rho} = [\![u_2]\!]^{I\rho}$ by induction hypothesis for any assignment ρ. Then, by Proposition 4.6, $[\![w]\!]^{I\rho} = [\![w_2w_1]\!]^{I\rho} = [\![w_2]\!]^{I\rho_{s/[\![w_1]\!]^{I\rho}}} = [\![u_2]\!]^{I\rho_{s/[\![u_1]\!]^{I\rho}}} = [\![u_2u_1]\!]^{I\rho} = [\![u]\!]^{I\rho}$;

- $(w^\ddagger, u^\ddagger)$ is such that $w^\ddagger = u^\ddagger$. Then by the uniqueness of the representation $w = v$ and so the result follows straightforwardly;

- $(w^\ddagger, u^\ddagger)$ is such that $(u^\ddagger, w^\ddagger)$ is in $\Delta^\ddagger$. The result follows straightforwardly by induction hypothesis;

- $(w^\ddagger, u^\ddagger)$ is such that $(w^\ddagger, u_0{}^\ddagger)$ and $(u_0{}^\ddagger, u^\ddagger)$ are in $\Delta^\ddagger$. Then the result follows straightforwardly by induction hypothesis. QED

Capitalizing on Proposition 4.7, the *denotation* $[\![\widehat{w}]\!]^{I\rho}$ of a morphism $\widehat{w}$ in G^+ over I and ρ is defined as

$$[\![\widehat{w}]\!]^{I\rho} = [\![w]\!]^{I\rho}.$$

The notions of local and global satisfactions are the usual ones. A schema formula φ is said to be *satisfied by I and ρ*, written as

$$I, \rho \Vdash \varphi$$

whenever $[\![\varphi]\!]^{I\rho}$ is non-empty and is contained in D. Moreover, we say I *satisfies* φ, written as

$$I \Vdash \varphi$$

whenever $I, \rho \Vdash \varphi$ for every assignment ρ over I. Satisfaction is extended to sets of schema formulas as expected: $I, \rho \Vdash \Gamma$ if $I, \rho \Vdash \gamma$ for each $\gamma \in \Gamma$, and similarly for sequences of schema formulas: $I, \rho \Vdash \varphi_1 \ldots \varphi_n$ if $I, \rho \Vdash \varphi_i$ for $i = 1, \ldots, n$.

The definition of denotation given above is the usual for most logics. Examples of logics with a different notion of denotation are the paraconsistent logics referred to in [2]. In the case of these logics, although some of the operations are non deterministic, the denotation of a formula is always a fixed truth value. The definition in our approach of that variant of denotation seems feasible and we intend to explore the details in the future.

Example 4.8 Consider the interpretation structure I_{nd} in Example 2.9. Then

$$I_{\mathrm{nd}} \Vdash q_1 \wedge (\neg\, q_1)$$

since $[\![q_1 \wedge (\neg\, q_1)]\!]^{I_{\mathrm{nd}}\rho} = \{\mathbf{I}, \mathbf{t}\}$ is contained in D, see Example 4.3. Moreover,

$$I_{\mathrm{nd}} \not\Vdash \circ q_1$$

since $[\![\circ\, q_1]\!]^{I_{\mathrm{nd}}\rho} = \{\mathbf{f}\}$ is not contained in D as shown in the same example. Therefore,

$$I_{\mathrm{nd}} \not\Vdash (q_1 \wedge (\neg\, q_1)) \supset (\circ q_1)$$

as expected for logics of formal inconsistency, see [10]. ∇

We are now ready to define semantic entailment. Given an interpretation system $\mathcal{I} = (\Sigma, \mathfrak{I})$ and a set $\Gamma \cup \{\varphi\}$ of schema formulas over Σ, we say that Γ *entails* φ in $\mathcal{I}$, written as

$$\Gamma \vDash_{\mathcal{I}} \varphi,$$

whenever $I \Vdash \Gamma$ implies $I \Vdash \varphi$ for every I in $\mathfrak{I}$. Similarly we define entailment over sequences of schema formulas as follows: $\vec{\gamma} \vDash_{\mathcal{I}} \vec{\varphi}$ whenever $I \Vdash \vec{\gamma}$ implies $I \Vdash \vec{\varphi}$ for every I in $\mathfrak{I}$.

The graph-theoretic semantics developed in this work can be said to subsume algebraic semantics, in the sense that, any logic endowed with an algebraic semantics can be presented in our setting in such a way that satisfaction and entailment are preserved. By a logic with an algebraic semantics we mean a pair composed by a signature and a class of algebras over that signature. Each algebra A is a triple $(A, \cdot, D_A)$ composed by a set A of (truth values) with an operation $c_A : A^n \to A$ for each constructor c of arity n in the signature and a subset D_A contained in A of distinguished values. In this context the denotation $[\![\varphi]\!]^A$ is homomorphic, that is

$$[\![c(\varphi_1, \ldots, \varphi_n)]\!]^A = c_A([\![\varphi_1]\!]^A, \ldots, [\![\varphi_n]\!]^A).$$

A logic L with an algebraic semantics induces an interpretation system $\mathcal{I}(L)$ with the obvious signature and containing, for each algebra A, an interpretation structure $I_A = (G', \alpha, D_A, \blacklozenge)$ defined as follows:

- V' is the set of truth values of the algebra;

- E' is composed, for each n-ary constructor c, by the set of m-edges $c_{a_1,\dots,a_n} : a_1 \dots a_n \to c_A(a_1, \dots, a_n)$ for each $a_1, \dots, a_n \in A$ when $n \geq 1$, or by $c : \blacklozenge \to c_A$ when $n = 0$;

- $\alpha^{\mathsf{v}}(a) = \pi$ for each $a \in A$;

- $\alpha^{\mathsf{e}}(c_{a_1,\dots,a_n}) = c$ for each $a_1, \dots, a_n \in A$ and $\alpha^{\mathsf{e}}(c) = c$.

The graph-theoretic semantics induced by the algebraic semantics coincides exactly in terms of denotation, satisfaction and entailment with the algebraic semantics, as we show now.

Lemma 4.9 Given a logic with an algebraic semantics and an algebra A, then

$$[\![\varphi]\!]^A = [\![\varphi]\!]^{I_A}.$$

Proof: By induction on the structure of φ: φ is $c(\varphi_1, \dots, \varphi_n)$. Therefore $[\![\varphi]\!]^A = [\![c(\varphi_1, \dots, \varphi_n)]\!]^A = c_A([\![\varphi_1]\!]^A, \dots, [\![\varphi_n]\!]^A) = c_A([\![\varphi_1]\!]^{I_A}, \dots, [\![\varphi_n]\!]^{I_A}) = \mathsf{trg}'(c_{[\![\varphi_1]\!]^{I_A},\dots,[\![\varphi_n]\!]^{I_A}}) = \mathsf{trg}'(E'_c([\![\varphi_1]\!]^{I_A} \dots [\![\varphi_n]\!]^{I_A}, -)) = [\![\varphi]\!]^{I_A}$. QED

Lemma 4.10 Given a logic with an algebraic semantics and an algebra A, then

$$A \Vdash \varphi \text{ iff } I_A \Vdash \varphi.$$

Proof: Assume that $A \Vdash \varphi$. Then $[\![\varphi]\!]^A \in D_A$. Hence $[\![\varphi]\!]^{I_A} \in D_A$ by Lemma 4.9. So $I_A \Vdash \varphi$. The other direction follows similarly. QED

Proposition 4.11 Let L be a logic with an algebraic semantics. Then, L and $\mathcal{I}(L)$ share the same entailment.

Proof: Suppose $\Gamma \vDash_L \varphi$ and let I_A be in $\mathcal{I}(L)$ such that $I_A \Vdash \Gamma$. Then $A \Vdash \Gamma$ by Lemma 4.10 and so $A \Vdash \varphi$. Hence also by Lemma 4.10, $I_A \Vdash \varphi$ as we wanted to show. The other direction follows similarly. QED

5 Deductive systems as m-graphs

A deductive system is also described as a m-graph, the *deductive m-graph*. The nodes are formulas and inference rules are m-edges. The sources of each of those m-edges are the premises of the rule and the target is the conclusion. As an example consider the case of the well known Modus Ponens rule as depicted in Figure 13. The intuition behind this rule is as follows: starting with a pair of formulas, we select the first one (with the projection $\widehat{\mathsf{p}}_1^{\pi\pi}$) and consider the formula obtained by their implication (with $\supset$). Then, by MP, we conclude the second formula (with the projection $\widehat{\mathsf{p}}_2^{\pi\pi}$).

But, in a deductive system it is also necessary to consider an abstraction map to a deductive signature in order to abstract, to explain, the components of the deductive m-graph. So, herein, a deductive system is composed of three parts: the deductive signature, the deductive m-graph and the abstraction map.

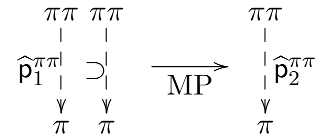

Figure 13: Modus Ponens as an m-edge.

The deductive signature is a language signature enriched with new m-edges for representing inference rules and new m-edges for axioms.

By a *deductive signature* or, simply, a *meta-signature* we mean a tuple

$$\Phi = (\Sigma, \top, \mathsf{R})$$

where $\Sigma = (G, \pi, \diamond)$ is a language signature such that

$$G^\Phi = (V^\Phi, E^\Phi, \mathsf{src}^\Phi, \mathsf{trg}^\Phi)$$

is a m-graph extending G with

- $V^\Phi = V$;
- $E^\Phi = E \cup \mathsf{R}$ where $\mathsf{R} = \{\mathsf{R}_n : \overbrace{\pi \ldots \pi}^{n} \to \pi\}_{n>0}$;

and $\top$ is a set $\{\top^s : s \to \pi\}_{s \in V^+}$. As an example consider the enriched m-graph in Figure 14 for a deductive signature for propositional logic.

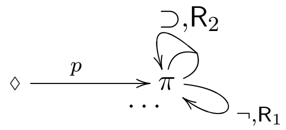

Figure 14: Enriched m-graph G^Φ for the deductive signature of propositional logic.

Each R_n is a symbolic expression for representing inference rules with n premises. Each $\top^s$ is called *s-verum* and is important to represent, in our setting, axioms. An *axiom* is the target of a unary rule whose antecedent is a verum schema formula.

The next step is to define deductive system. A deductive system over a deductive signature is a m-graph where the nodes are language expressions, that is, morphisms of the category generated by the deductive m-graph enriched with the verum edges, and the m-edges include, besides the language constructors (ensuring the commutativity of diagrams), the given inference rules. For instance, the well known Modus Ponens inference rule is seen as a m-edge whose source is the pair composed by the two morphisms corresponding to the premises and whose target is the morphism corresponding to the conclusion, see Figure 13.

Given a deductive signature $(\Sigma, \top, \mathsf{R})$, where Σ is $(G, \pi, \diamond)$, we denote by $G_\top$ the m-graph obtained by enriching G with the m-edges $\top^s : s \to \pi$. We say that a morphism $\widehat{w}$ of $G^+_\top$ *is in* G^+ whenever there is a path u over $G^\dagger$ and $\widehat{u} = \widehat{w}$. We may denote a schema formula of $G^+_\top$ not in G^+ as a *verum schema formula*. Given morphisms $\widehat{w}_1 : s \to s_1$ and $\widehat{w}_2 : s_1 \to s_2$ of $G^+_\top$ in G^+ it is straightforward to see that $\widehat{w}_2 \circ \widehat{w}_1$ is also in G^+. Moreover given the morphism $\widehat{\top^s} : s \to \pi$ of $G^+_\top$ it is straightforward to see that for any $\widehat{u} : s \to s_1$ in $G^+_\top$ the morphism $\widehat{\top^s} \circ \widehat{u}$ is also not in G^+.

By a *deductive system over a meta-signature* Φ we mean a basis (G'', β) over G^Φ where $G'' = (V'', E'', \mathsf{src}'', \mathsf{trg}'')$ is such that

- V'' is the class of morphisms of $G^+_\top$ whose target is in V;
- $E''(\widehat{w}_1 : s \to v_1 \dots \widehat{w}_n : s \to v_n, \widehat{w} : s \to v)$, for $\widehat{w}$ in G^+, contains, among others, the m-edges $e : v_1 \dots v_n \to v$ of E such that $\widehat{w} = e \circ \langle \widehat{w}_1, \dots, \widehat{w}_n \rangle$ in G^+;
- $E''(\widehat{w}_1 : s_1 \to v_1 \dots \widehat{w}_n : s_n \to v_n, \widehat{w} : s \to v) = \emptyset$ whenever $\widehat{w}$ is not in G^+ or $s_i \neq s$ for some $i = 1, \dots, n$, or $\widehat{w}_i$ is not in G^+ and $n \neq 1$;

and β is such that

- $\beta^{\mathsf{v}}(\widehat{w} : s \to v) = v$;
- $\beta^{\mathsf{e}}(e : (\widehat{w}_1 : s \to v_1 \dots \widehat{w}_n : s \to v_n) \to (\widehat{w} : s \to v)) = e$ if e is in E and $\widehat{w} = e \circ \langle \widehat{w}_1, \dots, \widehat{w}_n \rangle$;
- $\beta^{\mathsf{e}}(f') \in \mathsf{R}$ otherwise.

The first condition on E'' imposes that E'' contains the language constructors, as it is usually considered in categorical logic. As imposed in the last condition for β^{e} the other m-edges correspond to inference rules. All the m-edges corresponding to inference rules must have as premises and conclusion, expressions with the same source, and with target π. The same source condition is imposed by the second condition on E'' and is crucial for defining instantiation as we will see below. The target is π by definition of β and of R. The m-edges in $(\beta^{\mathsf{e}})^{-1}(\mathsf{R}_n)$ are called *n-ary inference rules* or simply *n-ary rules.*

By a *deductive system* $\mathcal{D}$ we mean a triple

$$(\Phi, G'', \beta)$$

such that Φ is a meta-signature and (G'', β) is a deductive system over Φ. We now illustrate our notion of deductive system by presenting deductive systems for a variety of logics.

Example 5.1 *Deductive system for classical propositional logic.*
Consider the well known Hilbert axiomatization of classical propositional logic with three axiom schemas and Modus Ponens. This axiomatization can be represented as the deductive system (Φ_Π, G'', β), denoted by $\mathcal{D}^{\mathsf{PL}}_\Pi$, such that:

- Φ_Π is the meta-signature $(\Sigma_\Pi, \top, \mathsf{R})$ where Σ_Π is the propositional signature $(G, \pi, \diamond)$ introduced in Example 2.1;

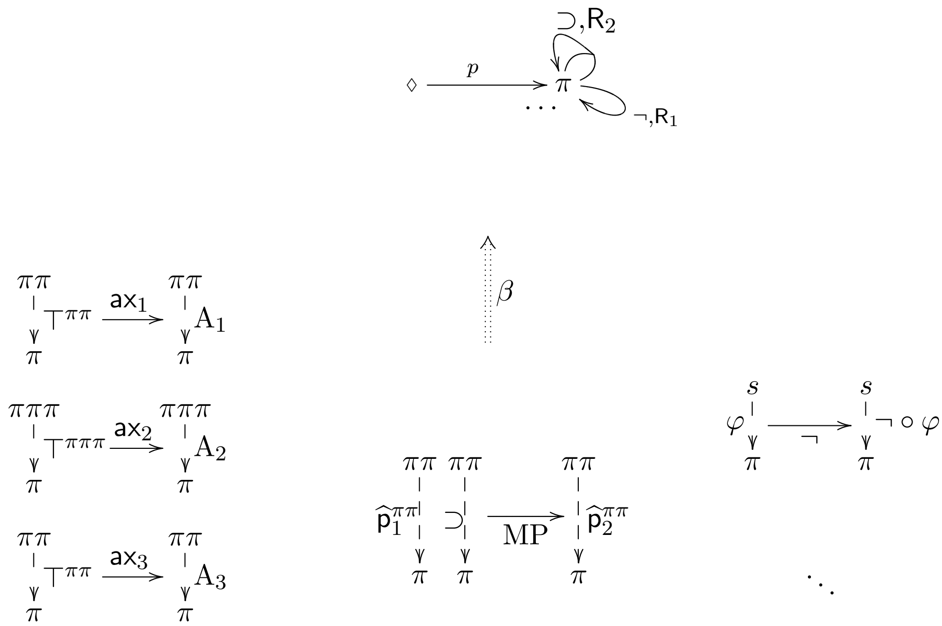

Figure 15: Part of the deductive system of Example 5.1.

- G'' has, besides the mandatory m-edges for connectives, the following ones for rules:
  - m-edge $\mathsf{ax}_1 : \top^{\pi\pi} \to (\xi \supset (\xi' \supset \xi))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_2 : \top^{\pi\pi\pi} \to ((\xi \supset (\xi' \supset \xi'')) \supset ((\xi \supset \xi') \supset (\xi \supset \xi'')))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;
  - m-edge $\mathsf{ax}_3 : \top^{\pi\pi} \to (((\neg\,\xi) \supset (\neg\,\xi')) \supset (\xi' \supset \xi))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathrm{MP} : \widehat{\mathsf{p}}_1^{\pi\pi} \supset \to \widehat{\mathsf{p}}_2^{\pi\pi}$;
- $\beta : G'' \to G^{\Phi_\Pi}$ is such that:
  - $\beta^{\mathsf{e}}(\mathsf{ax}_i) = \mathsf{R}_1$ for $i = 1, 2, 3$;
  - $\beta^{\mathsf{e}}(\mathrm{MP}) = \mathsf{R}_2$.

In the sequel, we can abbreviate the target of the m-edges corresponding to axioms as A. This deductive systems contains the three axiom schemes, represented as unary rules with a verum schema formula as antecedent, and the 2-ary inference rule of MP. Part of the deductive system is depicted in Figure 15. ∇

Example 5.2 *Deductive system for classical propositional modal logic* T. Consider the Hilbert axiomatization of the global consequence relation for the classical propositional modal logic T with three axiom schemas for the propositional part, Modus Ponens, a rule stating that $(\Box\varphi)$ holds for each theorem φ, the normality axiom K and the reflexivity axiom T. This axiomatization can be represented as the deductive system (Φ_Π, G'', β) such that:

- Φ_Π is the meta-signature $(\Sigma_\Pi^\Box, \top, \mathsf{R})$ where $\Sigma_\Pi^\Box$ is the propositional modal signature $(G, \pi, \diamond)$ introduced in Example 2.2;

- G'' has, besides the mandatory m-edges for connectives, the following ones:

  - m-edge $\mathsf{ax}_1 : \top^{\pi\pi} \to (\xi \supset (\xi' \supset \xi))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_2 : \top^{\pi\pi\pi} \to ((\xi \supset (\xi' \supset \xi'')) \supset ((\xi \supset \xi') \supset (\xi \supset \xi'')))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;
  - m-edge $\mathsf{ax}_3 : \top^{\pi\pi} \to (((\neg\,\xi) \supset (\neg\,\xi')) \supset (\xi' \supset \xi))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_\mathrm{K} : \top^{\pi\pi} \to ((\Box(\xi \supset \xi')) \supset ((\Box\xi) \supset (\Box\xi')))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_\mathrm{T} : \top^{\pi\pi} \to ((\Box\xi) \supset \xi)$ where ξ is id_π;
  - m-edge MP : $\widehat{\mathsf{p}}_1^{\pi\pi}\supset \to \widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\Box : \mathsf{id}_\pi \to \Box$;

- $\beta : G'' \to G^{\Phi_\Pi}$ is such that:

  - $\beta^\mathsf{e}(\mathsf{ax}_i) = \mathsf{R}_1$ for $i = 1, 2, 3, K, T$;
  - $\beta^\mathsf{e}(\mathrm{MP}) = \mathsf{R}_2$;
  - $\beta^\mathsf{e}(\Box) = \mathsf{R}_1$. ∇

Example 5.3 *Deductive system for intuitionistic propositional logic.*
Consider the well known Hilbert axiomatization of intuitionistic propositional logic with axiom schemas and Modus Ponens. This axiomatization can be represented as the deductive system (Φ_Π, G'', β) such that:

- Φ_Π is the meta-signature $(\Sigma_\Pi^{\wedge,\vee}, \top, \mathsf{R})$ where $\Sigma_\Pi^{\wedge,\vee}$ is the intuitionistic propositional signature $(G, \pi, \diamond)$ introduced in Example 2.3;

- G'' has, besides the mandatory m-edges for connectives, the following ones:

  - m-edge $\mathsf{ax}_1 : \top^{\pi\pi} \to (\xi \supset (\xi' \supset \xi))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_2 : \top^{\pi\pi\pi} \to ((\xi \supset (\xi' \supset \xi'')) \supset ((\xi \supset \xi') \supset (\xi \supset \xi'')))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;
  - m-edge $\mathsf{ax}_3 : \top^{\pi\pi} \to (\xi \supset (\xi' \supset (\xi \wedge \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_4 : \top^{\pi\pi} \to ((\xi \wedge \xi') \supset \xi)$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_5 : \top^{\pi\pi} \to ((\xi \wedge \xi') \supset \xi')$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_6 : \top^{\pi\pi} \to (\xi \supset (\xi \vee \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_7 : \top^{\pi\pi} \to (\xi' \supset (\xi \vee \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_8 : \top^{\pi\pi\pi} \to ((\xi \supset \xi'') \supset ((\xi' \supset \xi'') \supset ((\xi \vee \xi') \supset \xi'')))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;

– m-edge $\mathsf{ax}_9 : \top^{\pi\pi} \to ((\xi \supset \xi') \supset ((\xi \supset (\neg\,\xi')) \supset (\neg\,\xi)))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;

– m-edge $\mathsf{ax}_{10} : \top^{\pi\pi} \to (\xi \supset ((\neg\,\xi) \supset \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;

– m-edge MP : $\widehat{\mathsf{p}}_1^{\pi\pi}\supset \to \widehat{\mathsf{p}}_2^{\pi\pi}$;

- $\beta : G'' \to G^{\Phi_\Pi}$ is such that:
  - $\beta^{\mathsf{e}}(\mathsf{ax}_i) = \mathsf{R}_1$ for $i = 1, \ldots, 10$;
  - $\beta^{\mathsf{e}}(\mathrm{MP}) = \mathsf{R}_2$. ∇

Example 5.4 *Deductive system for propositional relevance logic* **R**.
Consider the Hilbert axiomatization of relevance logic **R** with axiom schemas, MP and AR. This axiomatization can be represented as the deductive system (Φ_Π, G'', β) such that:

- Φ_Π is the meta-signature $(\Sigma_\Pi^{\wedge,\vee}, \top, \mathsf{R})$ where $\Sigma_\Pi^{\wedge,\vee}$ is the intuitionistic propositional signature $(G, \pi, \diamond)$ introduced in Example 2.3;
- G'' has, besides the mandatory m-edges for connectives, the following ones:
  - m-edge $\mathsf{ax}_1 : \top^{\pi\pi} \to (\xi \supset \xi)$ where ξ is id_π;
  - m-edge $\mathsf{ax}_2 : \top^{\pi\pi\pi} \to ((\xi \supset \xi') \supset ((\xi'' \supset \xi) \supset (\xi'' \supset \xi')))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;
  - m-edge $\mathsf{ax}_3 : \top^{\pi\pi} \to ((\xi \supset (\xi \supset \xi')) \supset (\xi \supset \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_4 : \top^{\pi\pi\pi} \to ((\xi \supset (\xi' \supset \xi'')) \supset (\xi' \supset (\xi \supset \xi'')))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;
  - m-edge $\mathsf{ax}_5 : \top^{\pi\pi} \to ((\xi \wedge \xi') \supset \xi)$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_6 : \top^{\pi\pi} \to ((\xi \wedge \xi') \supset \xi')$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_7 : \top^{\pi\pi\pi} \to (((\xi \supset \xi') \wedge (\xi \supset \xi'')) \supset (\xi \supset (\xi' \wedge \xi'')))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;
  - m-edge $\mathsf{ax}_8 : \top^{\pi\pi} \to (\xi \supset (\xi \vee \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_9 : \top^{\pi\pi} \to (\xi' \supset (\xi \vee \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_{10} : \top^{\pi\pi\pi} \to (((\xi \supset \xi'') \wedge (\xi' \supset \xi'')) \supset ((\xi \vee \xi') \supset \xi''))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;
  - m-edge $\mathsf{ax}_{11} : \top^{\pi\pi\pi} \to ((\xi \wedge (\xi' \vee \xi'')) \supset ((\xi \wedge \xi') \vee \xi''))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;
  - m-edge $\mathsf{ax}_{12} : \top^{\pi} \to ((\xi \supset (\neg\,\xi)) \supset (\neg\,\xi))$ where ξ is id_π;
  - m-edge $\mathsf{ax}_{13} : \top^{\pi\pi} \to ((\xi \supset (\neg\,\xi')) \supset (\xi' \supset (\neg\,\xi)))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge $\mathsf{ax}_{14} : \top^{\pi} \to ((\neg(\neg\,\xi)) \supset \xi)$ where ξ is id_π;
  - m-edge MP : $\widehat{\mathsf{p}}_1^{\pi\pi}\supset \to \widehat{\mathsf{p}}_2^{\pi\pi}$;
  - m-edge AR : $\widehat{\mathsf{p}}_1^{\pi\pi}\widehat{\mathsf{p}}_2^{\pi\pi} \to (\wedge \circ \langle \widehat{\mathsf{p}}_1^{\pi\pi}, \widehat{\mathsf{p}}_2^{\pi\pi} \rangle)$;

- $\beta : G'' \to G^{\Phi_\Pi}$ is such that:
  - $\beta^{\mathsf{e}}(\mathsf{ax}_i) = \mathsf{R}_1$ for $i = 1, \dots, 14$;
  - $\beta^{\mathsf{e}}(\mathrm{MP}) = \mathsf{R}_2$;
  - $\beta^{\mathsf{e}}(\mathrm{AR}) = \mathsf{R}_2$. ∇

Example 5.5 *Deductive system for (one-sorted) equational logic.*
Consider the Hilbert axiomatization of equational logic with one axiom schema and four inference rules. This axiomatization can be represented as the deductive system (Φ_Π, G'', β) such that:

- Φ_Π is the meta-signature $(\Sigma_F^{\mathrm{EQ}}, \top, \mathsf{R})$ where Σ_F^{EQ} is the equational signature $(G, \pi, \diamond)$ introduced in Example 2.5;
- G'' has, besides the mandatory m-edges for connectives, the following ones:
  - $\mathsf{ax} : \top^\theta \to \approx \circ \langle \mathsf{id}_\theta, \mathsf{id}_\theta \rangle$;
  - $\mathrm{SYM} : \approx \to \approx \circ \langle \widehat{\mathsf{p}}_2^{\theta\theta}, \widehat{\mathsf{p}}_1^{\theta\theta} \rangle$;
  - $\mathrm{TRANS} : (\approx \circ \langle \widehat{\mathsf{p}}_1^{\theta\theta\theta}, \widehat{\mathsf{p}}_2^{\theta\theta\theta} \rangle)(\approx \circ \langle \widehat{\mathsf{p}}_2^{\theta\theta\theta}, \widehat{\mathsf{p}}_3^{\theta\theta\theta} \rangle) \to (\approx \circ \langle \widehat{\mathsf{p}}_1^{\theta\theta\theta}, \widehat{\mathsf{p}}_3^{\theta\theta\theta} \rangle)$;
  - $\mathrm{CONG}_f : (\approx \circ \langle \widehat{\mathsf{p}}_1^{\theta\dots\theta}, \widehat{\mathsf{p}}_{n+1}^{\theta\dots\theta} \rangle) \dots (\approx \circ \langle \widehat{\mathsf{p}}_n^{\theta\dots\theta}, \widehat{\mathsf{p}}_{2n}^{\theta\dots\theta} \rangle) \to$
    $(\approx \circ \langle f \circ \langle \widehat{\mathsf{p}}_1^{\theta\dots\theta}, \dots, \widehat{\mathsf{p}}_n^{\theta\dots\theta} \rangle, f \circ \langle \widehat{\mathsf{p}}_{n+1}^{\theta\dots\theta}, \dots, \widehat{\mathsf{p}}_{2n}^{\theta\dots\theta} \rangle \rangle)$;
  - $\mathrm{SUB}_{t',t'',t} : \approx \circ \langle t', t'' \rangle \to \approx \circ \langle t', t'' \rangle \circ t$ for each $t', t'' : s \to \theta$ and $t : s_1 \to s$ morphisms of G^+;
- $\beta : G'' \to G^{\Phi_\Pi}$ is such that:
  - $\beta^{\mathsf{e}}(\mathsf{ax}) = \mathsf{R}_1$;
  - $\beta^{\mathsf{e}}(\mathrm{SYM}) = \mathsf{R}_1$;
  - $\beta^{\mathsf{e}}(\mathrm{TRANS}) = \mathsf{R}_2$;
  - $\beta^{\mathsf{e}}(\mathrm{CONG}_f) = \mathsf{R}_n$ whenever f is in F_n;
  - $\beta^{\mathsf{e}}(\mathrm{SUB}_{t',t'',t}) = \mathsf{R}_1$. ∇

6 Derivation as a path

The next step is to define derivation in the context of a deductive system. The basic ingredient is instantiation of rules. The instantiation of a rule r is accomplished by enriching G''^+ with new morphisms $\widehat{r} \odot \widehat{u}$, denoting the result of the instantiation of r by $\widehat{u}$ (see Figure 16).
We will also denote by $\odot$ the simultaneous instantiation of several rules.

Example 6.1 In order to understand better instantiation of rules in our setting, we make the parallel with the traditional view. Assume that MP is a schema rule of the form

$$\frac{\xi_1 \quad (\xi_1 \supset \xi_2)}{\xi_2}.$$

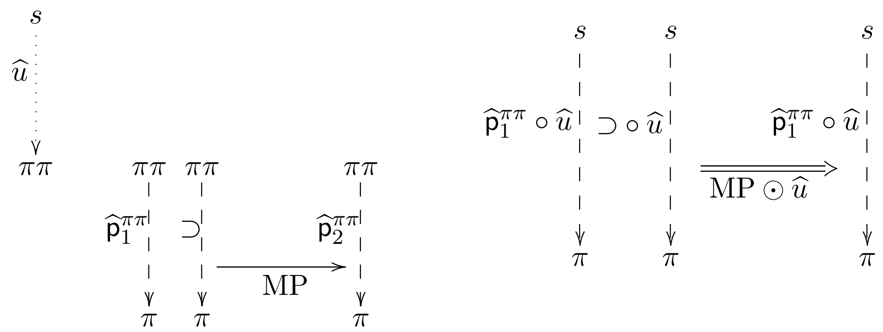

Figure 16: Instantiation of MP by $\widehat{u}$.

By instantiating $\xi_1 \mapsto q_1$ and $\xi_2 \mapsto (q_3 \supset q_2)$ we get the following inference:

$$\frac{q_1 \quad (q_1 \supset (q_3 \supset q_2))}{q_3 \supset q_2}.$$

This can be shortly written as

$$\mathrm{MP}[\xi_1/q_1, \xi_2/(q_3 \supset q_2)]$$

corresponding to the morphism MP $\odot \langle q_1, q_3 \supset q_2 \rangle$. This example is illustrated in Figure 17. ∇

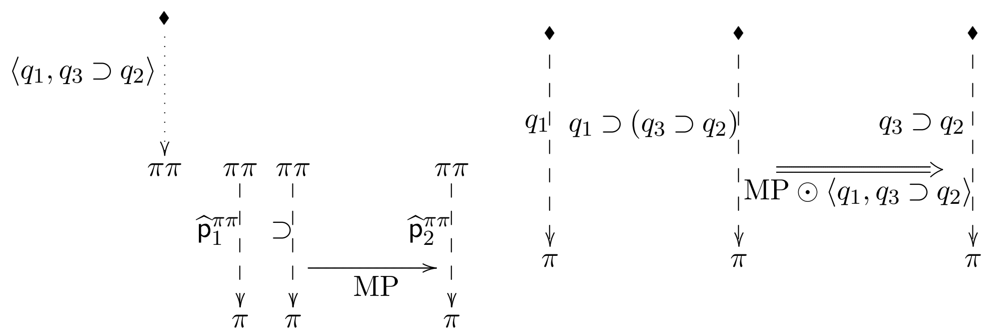

Figure 17: Instantiation of MP as described in Example 6.1.

Intuitively, derivations are seen as a sequence of *derivation steps*, also called *derivation levels*, see Figure 18 and Figure 22, where in each level one or several rules may be applied to different schema formulas coming from the preceding level. The morphism $\mathsf{id}_{\mathsf{id}_\pi}$ is applied in a level to a schema formula when no rule is applied to it in that level. Note that axioms are seen as unary rules whose antecedent is a verum schema formula. So, in order to define derivations, besides the operation $\odot$, which denotes the instantiation of a derivation level by a substitution, we need to consider a new operation $\otimes$ for defining a derivation level. That operation interacts appropriately with $\odot$.

Before defining those operations we introduce some convenient notation. Given $i = 1, \ldots, n$, $s_i = v_{i1} \ldots v_{im_i}$ in V^+ where $v_{i1}, \ldots, v_{im_i}$ are in V, we denote by $\mathsf{p}^{s_1\dots s_n}_{s_i}$ the tuple $\langle \mathsf{p}^{s_1\dots s_n}_{m_1+\dots+m_{i-1}+1}, \dots, \mathsf{p}^{s_1\dots s_n}_{m_1+\dots+m_i}\rangle$. Moreover, given $\widehat{a}_i : s \to v_i$ in G^+ for $i = 1,\dots,n$ we denote by $(\widehat{a}_1 \dots \widehat{a}_n) \circ \widehat{u}$ the sequence $\widehat{a}_1 \circ \widehat{u} \dots \widehat{a}_n \circ \widehat{u}$.

So, in order to define derivations we consider a new category, $G''^{\star}$, which is a smallest category with non empty finite products obtained from G''^{+} by adding the morphisms

- $f_1 \otimes \dots \otimes f_n$:
  $(\widehat{a}_{11}\dots \widehat{a}_{1m_1}) \circ \widehat{\mathsf{p}}^{s_1\dots s_n}_{s_1} \dots (\widehat{a}_{n1}\dots \widehat{a}_{nm_n}) \circ \widehat{\mathsf{p}}^{s_1\dots s_n}_{s_n} \to (\widehat{c}_1 \circ \widehat{\mathsf{p}}^{s_1\dots s_n}_{s_1} \dots \widehat{c}_n \circ \widehat{\mathsf{p}}^{s_1\dots s_n}_{s_n})$
  where $f_i : \widehat{a}_{i1} \dots \widehat{a}_{im_i} \to \widehat{c}_i$ is $\mathsf{id}_{\mathsf{id}_\pi}$ or is in $(\beta^{\mathsf{e}})^{-1}(\mathsf{R})$ and $\mathsf{src}(c_i) = s_i$;
- $\ell \odot \widehat{u} : (\widehat{a}_1 \dots \widehat{a}_m) \circ \widehat{u} \to (\widehat{c}_1 \dots \widehat{c}_n) \circ \widehat{u}$ whenever $\widehat{u}$ in G^+ is composable with $\widehat{c}_1$ and $\ell : \widehat{a}_1 \dots \widehat{a}_m \to \widehat{c}_1 \dots \widehat{c}_n$ is of the form $f_1 \otimes \dots \otimes f_n$;

while imposing:

- $\mathsf{id}_{\mathsf{id}_\pi} \odot \widehat{u} = \mathsf{id}_{\widehat{u}}$;
- $\ell \odot \mathsf{id}_s = \ell$;
- $(\ell \odot \widehat{u}_2) \odot \widehat{u}_1 = \ell \odot (\widehat{u}_2 \circ \widehat{u}_1)$;
- $(f_1 \otimes \dots \otimes f_n) \odot \widehat{u} = (f_1 \odot (\widehat{\mathsf{p}}^{s_1\dots s_n}_{s_1} \circ \widehat{u})) \otimes \dots \otimes (f_n \odot (\widehat{\mathsf{p}}^{s_1\dots s_n}_{s_n} \circ \widehat{u}))$.

Given a morphism $f_1 \otimes \dots \otimes f_n$ named ℓ in $G''^{\star}$, denoting a derivation step, we denote by $\mathsf{CONC}(\ell)$ the target of ℓ and by $\mathsf{ANT}(\ell)$ the source of ℓ. When presenting derivations it is more convenient not indicate explicitly the substitutions used, but instead the rule or axiom resulting from the instantiation by that substitution. For this purpose we write $\ell \star \vec{\varphi}$ whenever there is a substitution $\widehat{u}$ (a morphism in G^+) with $\vec{\varphi} = \mathsf{ANT}(\ell) \circ \widehat{u}$ and such that $\ell \star \vec{\varphi} = \ell \odot \widehat{u}$. For instance, in Example 6.1, $\vec{\varphi}$ is $q_1, q_1 \supset (q_3 \supset q_2)$ and $\widehat{u}$ is $\langle q_1, q_3 \supset q_2\rangle$. Note that, by definition, a substitution $\widehat{u}$ never involves verum schema formulas since $\widehat{u}$ is a morphism in G^+. In the sequel we may use commas to separate elements in a sequence of formulas. We are now ready to define derivations. But first we give a bit of motivation.

Example 6.2 Consider the following derivation in the Hilbert calculus for classical logic stating that $p \supset q$ follows from q:

1. q Hyp
2. $q \supset (p \supset q)$ A_1
3. $p \supset q$ MP 1, 2

which is represented graphically in Figure 18. So $p \supset q$ is obtained by an application of MP:

$$\frac{q \quad q \supset (p \supset q)}{p \supset q}$$

where only q is an hypothesis since the other premise $q \supset (p \supset q)$ is an axiom. In more detail, the derivation can be seen as consisting of two steps, the first

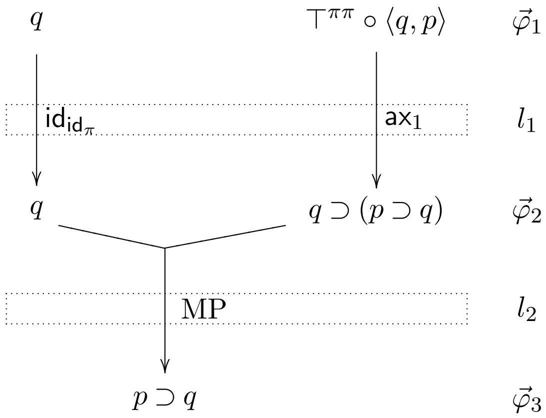

Figure 18: Graphical representation of the derivation in Example 6.2 and in Example 6.3.

one for concluding the axiom $q \supset (p \supset q)$, and the second step consisting of an application of MP with substitution $\xi_1 \mapsto q$ and $\xi_2 \mapsto p \supset q$. This second step, is represented, in our setting, by the morphism

$$\mathrm{MP} \odot \langle q, p \supset q \rangle$$

denoted, more conveniently, by

$$\mathrm{MP} \star q, q \supset (p \supset q).$$

The first step of the derivation is represented, in our setting, by the morphism

$$(\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathsf{ax}_1) \odot \langle q, q, p \rangle$$

which can be denoted also by

$$(\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathsf{ax}_1) \star q, \top^{\pi\pi} \circ \langle q, p \rangle$$

(see Figure 18). ∇

Let $\mathcal{D}$ be a deductive system. A *derivation step* in $\mathcal{D}$ is a morphism of the form $f_1 \otimes \ldots \otimes f_m$ where f_i is either $\mathsf{id}_{\mathsf{id}_\pi}$ or is an element of $(\beta^{\mathsf{e}})^{-1}(\mathsf{R})$, for $i = 1, \ldots, m$ and $m > 0$. An illustration of a derivation step is presented in Figure 20.

By a *derivation* in $\mathcal{D}$ we mean a pair

$$d = \ell_1, \ldots, \ell_n; \vec{\varphi}_1$$

where each ℓ_i is a derivation step and $\vec{\varphi}_1$ is a sequence of morphisms in V'' such that the sequence given by $\vec{\varphi}_{i+1} = \mathsf{CONC}(\ell_i \star \vec{\varphi}_i)$, for $i = 1, \ldots, n$, is well defined, and so there exists the composite morphism

$$(\ell_n \star \vec{\varphi}_n) \circ \ldots \circ (\ell_1 \star \vec{\varphi}_1)$$

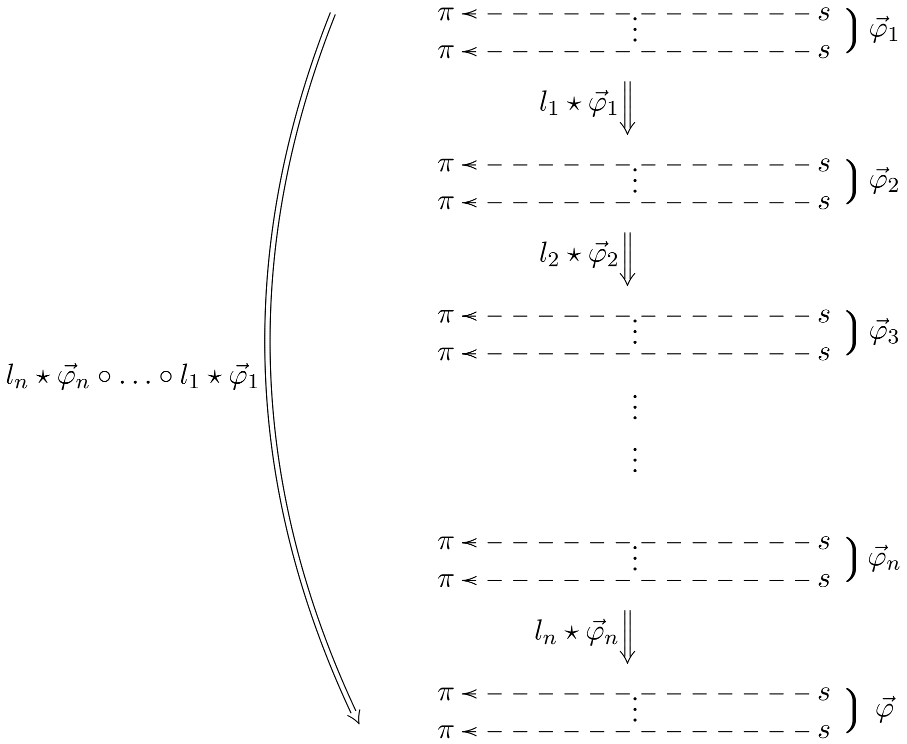

Figure 19: (Effective) derivation as a composite morphism in $G''^{\star}$.

in $G''^{\star}$ (see Figure 19). The morphism above is called the *effective derivation* associated with the derivation $\ell_1, \ldots, \ell_n; \vec{\varphi}_1$. When there is no ambiguity we may use the term *derivation* to refer also to the effective derivation. In the sequel we will denote by d_i the morphism $(\ell_i \star \vec{\varphi}_i) \circ \ldots \circ (\ell_1 \star \vec{\varphi}_1)$.

The *set of hypothesis* $\mathsf{HYP}(d)$ of the derivation $d = \ell_1, \ldots, \ell_n; \vec{\varphi}_1$, where $\vec{\varphi}_1$ is of the form $\varphi_{11} \ldots \varphi_{1m_1}$ for $m_1 > 0$, is the set of the φ_{1i}'s that are in G^{+}. As usual we write

$$\Gamma \vdash_{\mathcal{D}} \vec{\varphi}$$

if there is a derivation d in $\mathcal{D}$ such that $\mathsf{CONC}(d_n) = \vec{\varphi}$ and $\mathsf{HYP}(d) \subseteq \Gamma$, where $\vec{\varphi}$ is a sequence and Γ a set of schema formulas of G^{+}.

The definition of consequence deserves some comments. Firstly, observe that $\vec{\varphi}$ is a sequence of formulas possibly containing more than one formula. So multi-conclusion derivations can be naturally defined. Secondly, a set of hypothesis was considered instead of a sequence. This classical perspective intends to reflect derivations in standard Hilbert systems, as it is treated in this work. However, instead of Γ itself, we could pick the subsequence of $\vec{\varphi}_1$ of hypothesis. This would provide us with more information about the effective number and even about the order of the premises used in the derivation, in line with some substructural logics of resources. Thirdly, note that it is possible to use several rules in parallel by means of the $\otimes$ operator in each step of the derivation. This could open the possibility of considering parallel reasoning.

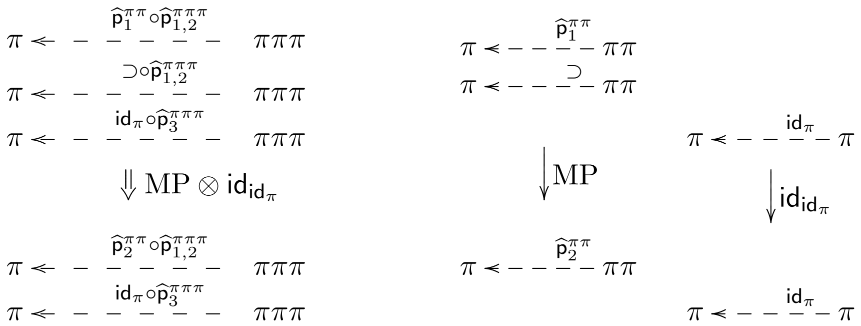

Figure 20: Graphical representation of the first derivation step in Example 6.4.

Finally, observe that schema formulas not in G^+, that is, morphisms involving verum schema formulas, can only appear as antecedents of the first step of a derivation, since: 1. the conclusion of a deductive rule, by definition, is in G^+, 2. substitutions are morphisms in G^+. So, axiom rules can only be used in the first step of a derivation.

Example 6.3 The derivation in Example 6.2 depicted in Figure 18 can be expressed in our setting as the derivation d in $\mathcal{D}^{\mathsf{PL}}_{\{p,q\}}$ given by

$$\mathrm{MP}, (\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathsf{ax}_1); \vec{\varphi}_1$$

stating that

$$q \vdash_{\mathcal{D}^{\mathsf{PL}}_{\{p,q\}}} p \supset q$$

where $\vec{\varphi}_1$ is the sequence $q, \top^{\pi\pi} \circ \langle q, p\rangle$. In fact, $\vec{\varphi}_1$ is $\mathsf{ANT}(\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathsf{ax}_1) \circ \widehat{u}_1$ for $\widehat{u}_1 = \langle q, q, p\rangle$. So, $\vec{\varphi}_2 = q, q \supset (p \supset q)$ and $\vec{\varphi}_2 = \mathsf{ANT}(\mathrm{MP}) \circ \widehat{u}_2$ for $\widehat{u}_2 = \langle q, p \supset q\rangle$. Hence $\vec{\varphi}_3 = p \supset q$ since $\vec{\varphi}_3 = \mathsf{CONC}(\mathrm{MP}) \circ \widehat{u}_2$. The set $\mathsf{HYP}(d)$ is $\{q\}$ since $\top^{\pi\pi} \circ \langle q, p\rangle$, an instance of the $\pi\pi$-verum, is a schema formula not in G^+. ∇

Example 6.4 The derivation d given by

$$\mathrm{MP}, (\mathrm{MP} \otimes \mathsf{id}_{\mathsf{id}_\pi}); \xi_1, \xi_1 \supset \xi_2, \xi_2 \supset \xi_3$$

where ξ_1, ξ_2 and ξ_3 are the projections $\widehat{\mathsf{p}}^{\pi\pi\pi}_1$, $\widehat{\mathsf{p}}^{\pi\pi\pi}_2$ and $\widehat{\mathsf{p}}^{\pi\pi\pi}_3$ respectively, states that

$$\xi_1, (\xi_1 \supset \xi_2), (\xi_2 \supset \xi_3) \vdash_{\mathcal{D}^{\mathsf{PL}}_{\Pi}} \xi_3.$$

In fact, the morphism $\widehat{u}_1 = \langle \xi_1, \xi_2, \xi_2 \supset \xi_3\rangle$ is such that

$$\vec{\varphi}_1 = \mathsf{ANT}(\mathrm{MP} \otimes \mathsf{id}_{\mathsf{id}_\pi}) \circ \widehat{u}_1$$

since $\mathsf{ANT}(\mathrm{MP} \otimes \mathsf{id}_{\mathsf{id}_\pi})$ is the sequence $\widehat{\mathsf{p}}^{\pi\pi}_1 \circ \widehat{\mathsf{p}}^{\pi\pi\pi}_{1,2}, \supset \circ\, \widehat{\mathsf{p}}^{\pi\pi\pi}_{1,2}, \mathsf{id}_\pi \circ \widehat{\mathsf{p}}^{\pi\pi\pi}_3$. Thus $(\mathrm{MP} \otimes \mathsf{id}_{\mathsf{id}_\pi}) \star \vec{\varphi}_1 : \vec{\varphi}_1 \to \vec{\varphi}_2$ is a morphism in $G''^{\star}$ where

$$\vec{\varphi}_2 = (\widehat{\mathsf{p}}^{\pi\pi}_2 \circ \widehat{\mathsf{p}}^{\pi\pi\pi}_{1,2}, \mathsf{id}_\pi \circ \widehat{\mathsf{p}}^{\pi\pi\pi}_3) \circ \widehat{u}_1 = \xi_2, \xi_2 \supset \xi_3.$$

$$
\begin{array}{ll}
\pi \xleftarrow{\xi_1=\widehat{\mathsf{p}}_1^{\pi\pi}\circ\widehat{\mathsf{p}}_{1,2}^{\pi\pi\pi}\circ\widehat{u}_1} \pi\pi\pi & \\
\pi \xleftarrow{\xi_1\supset\xi_2=\supset\circ\widehat{\mathsf{p}}_{1,2}^{\pi\pi\pi}\circ\widehat{u}_1} \pi\pi\pi & \\
\pi \xleftarrow{\xi_2\supset\xi_3=\mathsf{id}_\pi\circ\widehat{\mathsf{p}}_3^{\pi\pi\pi}\circ\widehat{u}_1} \pi\pi\pi & \pi\pi\pi \\
\Big\Downarrow (\mathrm{MP}\otimes\mathsf{id}_{\mathsf{id}_\pi})\star\vec{\varphi}_1=(\mathrm{MP}\otimes\mathsf{id}_{\mathsf{id}_\pi})\odot\widehat{u}_1 & \Big\downarrow \widehat{u}_1=\langle\xi_1,\xi_2,\xi_2\supset\xi_3\rangle \\
 & \pi\pi\pi \\
\pi \xleftarrow{\xi_2=\widehat{\mathsf{p}}_2^{\pi\pi}\circ\widehat{\mathsf{p}}_{1,2}^{\pi\pi\pi}\circ\widehat{u}_1=\widehat{\mathsf{p}}_1^{\pi\pi}\circ\widehat{u}_2} \pi\pi\pi & \\
\pi \xleftarrow{\xi_2\supset\xi_3=\mathsf{id}_\pi\circ\widehat{\mathsf{p}}_3^{\pi\pi\pi}\circ\widehat{u}_1=\supset\circ\widehat{u}_2} \pi\pi\pi & \pi\pi\pi \\
\Big\Downarrow \mathrm{MP}\star\vec{\varphi}_2=\mathrm{MP}\odot\widehat{u}_2 & \Big\downarrow \widehat{u}_2=\langle\xi_2,\xi_3\rangle \\
 & \pi\pi\pi \\
\pi \xleftarrow{\xi_3=\widehat{\mathsf{p}}_2^{\pi\pi}\circ\widehat{u}_2} \pi\pi\pi &
\end{array}
$$

Figure 21: Graphical representation of the derivation in Example 6.4.

The second derivation step is as follows: the morphism $\widehat{u}_2 = \langle \xi_2, \xi_3 \rangle$ is taken such that $\vec{\varphi}_2 = (\widehat{\mathsf{p}}_1^{\pi\pi}, \supset) \circ \widehat{u}_2$ where the sequence $\widehat{\mathsf{p}}_1^{\pi\pi}, \supset$ is $\mathsf{ANT}(\mathrm{MP})$. Hence, $\mathrm{MP} \star \vec{\varphi}_2 : \vec{\varphi}_2 \to \vec{\varphi}_3$ in $G''^{\star}$, where $\vec{\varphi}_3 = \widehat{\mathsf{p}}_2^{\pi\pi} \circ \widehat{u}_2 = \xi_3$. This derivation is graphically represented in Figure 21. ∇

Example 6.5 In the Hilbert calculus for classical logic we can derive ξ_1 from ξ_2 and $\neg\,\xi_2$, as follows:

1.	ξ_2	Hyp
2.	$\neg\,\xi_2$	Hyp
3.	$(\neg\,\xi_2) \supset ((\neg\,\xi_1) \supset (\neg\,\xi_2))$	A_1
4.	$(\neg\,\xi_1) \supset (\neg\,\xi_2)$	MP $2, 3$
5.	$((\neg\,\xi_1) \supset (\neg\,\xi_2)) \supset (\xi_2 \supset \xi_1)$	A_3
6.	$\xi_2 \supset \xi_1$	MP $4, 5$
7.	ξ_1	MP $1, 6$

In order to understand how derivations are expressed in our setting we should distinguish between the assumed formulas (hypothesis or axioms) and the formulas derived during the process. In the above derivation the formulas in steps 1., 2., 3. and 5. are assumed and the others are derived. Intuitively speaking, in our setting the assumed formulas are putted altogether in the initial sequence $\vec{\varphi}_1$. The intuition behind the derivation is depicted in Figure 22. Formally we can consider the derivation d in $\mathcal{D}_{\Pi}^{\mathsf{PL}}$ given by

$$\mathrm{MP}, (\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathrm{MP}), (\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathrm{MP} \otimes \mathsf{id}_{\mathsf{id}_\pi}), (\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathsf{id}_{\mathsf{id}_\pi} \otimes \mathsf{ax}_1 \otimes \mathsf{ax}_3); \vec{\varphi}_1$$

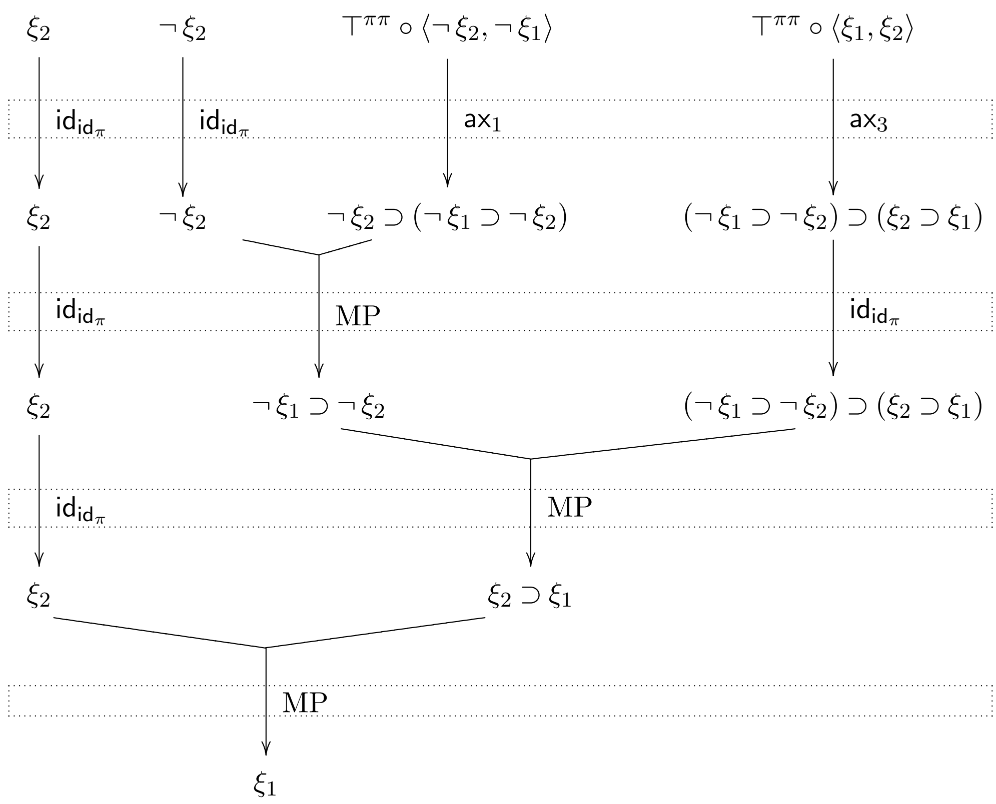

Figure 22: Deduction steps of the derivation of Examples 6.5.

where $\vec{\varphi}_1$ is the sequence $\xi_2, \neg\,\xi_2, \top^{\pi\pi} \circ \langle \neg\,\xi_2, \neg\,\xi_1 \rangle, \top^{\pi\pi} \circ \langle \xi_1, \xi_2 \rangle$, and ξ_1 and ξ_2 are $\mathsf{p}_1^{\pi\pi}$ and $\mathsf{p}_2^{\pi\pi}$, respectively, stating that

$$\xi_2, \neg\xi_2 \vdash_{\mathcal{D}_\Pi^{\mathsf{PL}}} \xi_1.$$

In fact, taking $\widehat{u}_1 = \langle \xi_2, \neg\,\xi_2, \neg\,\xi_2, \neg\,\xi_1, \xi_1, \xi_2 \rangle$ we get $\vec{\varphi}_2$ equal to the sequence $\xi_2, \neg\xi_2, \neg\xi_2 \supset (\neg\xi_1 \supset \neg\xi_2), (\neg\xi_1 \supset \neg\xi_2) \supset (\xi_2 \supset \xi_1)$. Moreover taking $\widehat{u}_2 = \langle \xi_2, \neg\,\xi_2, \neg\,\xi_1 \supset \neg\,\xi_2, (\neg\,\xi_1 \supset \neg\,\xi_2) \supset (\xi_2 \supset \xi_1) \rangle$ we get $\vec{\varphi}_3 = \xi_2, \neg\,\xi_1 \supset \neg\,\xi_2, (\neg\,\xi_1 \supset \neg\,\xi_2) \supset (\xi_2 \supset \xi_1)$ by applying the second derivation step. Now, by taking $\widehat{u}_3 = \langle \xi_2, \neg\,\xi_1 \supset \neg\,\xi_2, \xi_2 \supset \xi_1 \rangle$ we get $\vec{\varphi}_4 = \xi_2, \xi_2 \supset \xi_1$ by the third derivation step. Finally, $\widehat{u}_4 = \langle \xi_2, \xi_1 \rangle$ allows to conclude $\vec{\varphi}_5 = \xi_1$ by the last inference step. This derivation can be visualized in Figure 22. ∇

The notion of relevant deduction can be expressed in our setting with minor adjustments by defining $\vec{\gamma} \vdash_{\mathcal{D}} \vec{\varphi}$ whenever there is a derivation $d = \ell_1, \ldots, \ell_n; \vec{\varphi}_1$ such that $\mathsf{CONC}(d_n) = \vec{\varphi}$ and $\vec{\gamma}$ is $\vec{\varphi}_1$ without the schema formulas not in G^+.

7 Putting semantics and deduction together

We start by defining logic system, obtained by putting together a signature, a interpretation system and a deduction system. As seen in previous sections all of these components are defined in terms of m-graphs. More rigorously, a *logic system* is a triple

$$\mathcal{L} = (\Sigma, \mathcal{I}, \mathcal{D})$$

such that:

- $\mathcal{I} = (\Sigma, \mathfrak{I})$ is an interpretation system;
- $\mathcal{D} = (\Phi, G'', \beta)$ is a deductive system where Φ is a meta-signature over Σ.

The logic system $\mathcal{L}$ is said to be *sound* if $\Gamma \vDash_{\mathcal{I}} \varphi$ whenever $\Gamma \vdash_{\mathcal{D}} \varphi$, where φ is a formula and Γ is a set of formulas of G^+, and is said to be *complete* if the converse holds. A logic system is said to be *weakly complete* if $\vdash_{\mathcal{D}} \varphi$ whenever $\vDash_{\mathcal{I}} \varphi$, for each formula φ of G^+.

7.1 Soundness

Given a logic system $\mathcal{L}$, I in $\mathfrak{I}$ is said to be *sound for a deductive rule* r in $\mathcal{D}$, if $I, \rho \Vdash \mathsf{CONC}(r)$ whenever $I, \rho \Vdash \mathsf{proper}(\mathsf{ANT}(r))$ for every assignment ρ over I, where the map $\mathsf{proper}(\cdot)$ when applied to a sequence $\vec{\varphi}$ of schema formulas in $G^+_\top$ returns the subsequence of schema formulas that are in G^+. These schema formulas are called *proper*. The logic system $\mathcal{L}$ is said to be *sound for a deductive rule* r in $\mathcal{D}$, if all its interpretation structures over its signature are sound for r.

We now prove two propositions useful to establish the soundness theorem.

Proposition 7.1 A logic system $\mathcal{L}$ sound for a deductive rule r is such that $I, \rho \Vdash \mathsf{CONC}(r) \circ \widehat{u}$ whenever $I, \rho \Vdash \mathsf{proper}(\mathsf{ANT}(r)) \circ \widehat{u}$ for I in $\mathfrak{I}$, assignment ρ over I and morphism $\widehat{u}$ in G^+ composable with the schema formulas in r.

Proof: Let $r : (\psi_1 : s \to \pi \dots \psi_m : s \to \pi) \to (\varphi : s \to \pi)$ and denote by $\varphi_1, \dots, \varphi_n$ the proper antecedents of r. Assume that $I, \rho \Vdash \mathsf{proper}(\mathsf{ANT}(r)) \circ \widehat{u}$, that is, $I, \rho \Vdash \varphi_i \circ \widehat{u}$ for $i = 1, \dots, n$. Hence, $[\![\varphi_i \circ \widehat{u}]\!]^{I\rho} \subseteq D$, and so, using Proposition 4.6, $[\![\varphi_i]\!]^{I\rho_{s/[\![u]\!]^{I\rho}}} \subseteq D$, for $i = 1, \dots, n$. Since $\mathcal{L}$ is sound for r, then $[\![\varphi]\!]^{I\rho_{s/[\![u]\!]^{I\rho}}} \subseteq D$, and by Proposition 4.6 and definition of denotation, $[\![\varphi \circ \widehat{u}]\!]^{I\rho} \subseteq D$. So $I, \rho \Vdash \mathsf{CONC}(r) \circ \widehat{u}$. QED

Proposition 7.2 Given a logic system $\mathcal{L}$ sound for its rules, a derivation step ℓ, and a morphism $\widehat{u}$ in G^+ such that $\ell \odot \widehat{u}$ is definable, then $I, \rho \Vdash \mathsf{CONC}(\ell \odot \widehat{u})$ whenever $I, \rho \Vdash \mathsf{proper}(\mathsf{ANT}(\ell \odot \widehat{u})$, for every I in $\mathfrak{I}$ and assignment ρ over I.

Proof: Assume $I, \rho \Vdash \mathsf{proper}(\mathsf{ANT}(\ell \odot \widehat{u})$ and let ℓ be $f_1 \otimes \dots \otimes f_n$ where $f_i : \widehat{a}'_{i1} \dots \widehat{a}'_{im'_i} \to \widehat{c}_i$ is $\mathsf{id}_{\mathsf{id}_\pi}$ or is in $(\beta^{\mathsf{e}})^{-1}(\mathsf{R})$, and $i = 1, \dots, n$. Denote by $\widehat{a}_{i1} \dots \widehat{a}_{im_i}$ the subsequence of proper antecedents of f_i. Then $\mathsf{proper}(\mathsf{ANT}(\ell \odot \widehat{u}) = ((\widehat{a}_{11} \dots \widehat{a}_{1m_1}) \circ \widehat{\mathsf{p}}^{s_1 \dots s_n}_{s_1} \dots (\widehat{a}_{n1} \dots \widehat{a}_{nm_n}) \circ \widehat{\mathsf{p}}^{s_1 \dots s_n}_{s_n}) \circ \widehat{u}$ and $\mathsf{CONC}(\ell \odot \widehat{u}) = (\widehat{c}_1 \circ \widehat{\mathsf{p}}^{s_1 \dots s_n}_{s_1} \dots \widehat{c}_n \circ \widehat{\mathsf{p}}^{s_1 \dots s_n}_{s_n}) \circ \widehat{u}$. So $I, \rho \Vdash \mathsf{proper}(\mathsf{ANT}(f_i) \circ (\widehat{\mathsf{p}}^{s_1 \dots s_n}_{s_i} \circ \widehat{u})$ for $i = 1, \dots, n$. We now show that $I, \rho \Vdash \mathsf{CONC}(\ell \odot \widehat{u})$ that is $I, \rho \Vdash \mathsf{CONC}(f_i) \circ (\widehat{\mathsf{p}}^{s_1 \dots s_n}_{s_i} \circ \widehat{u})$ for $i = 1, \dots, n$. Let $i \in \{1, \dots, n\}$. There are two cases to consider: (i) f_i is $\mathsf{id}_{\mathsf{id}_\pi}$. Then $\mathsf{CONC}(f_i) = \mathsf{proper}(\mathsf{ANT}(f_i))$ and so $I, \rho \Vdash \mathsf{CONC}(f_i) \circ (\widehat{\mathsf{p}}^{s_1 \dots s_n}_{s_i} \circ \widehat{u})$ using the hypothesis. (ii) f_1 is in $(\beta^{\mathsf{e}})^{-1}(\mathsf{R})$, and so f_1 is a deductive rule. Then the result follows by Proposition 7.1. QED

The soundness theorem establishes soundness for rules as a sufficient condition for a logic system to be sound.

Theorem 7.3 A logic system is sound if it is sound for its deductive rules.

Proof: Let $\mathcal{L} = (\Sigma, \mathcal{I}, \mathcal{D})$ be a logic system sound for all its deductive rules, and assume that $\Gamma \vdash_{\mathcal{D}} \vec{\varphi}$ for a sequence $\vec{\varphi}$ and a set Γ of formulas of G^+. Let $\ell_1, \ldots, \ell_n; \vec{\varphi}_1$ be a derivation for $\Gamma \vdash_{\mathcal{D}} \vec{\varphi}$, and I in $\mathfrak{I}$ such that $I \Vdash \Gamma$. Denote by $\vec{\varphi}_{1_p}$ the sequence with the proper formulas of $\vec{\varphi}_1$. Since the schema formulas in $\vec{\varphi}_{1_p}$ are in Γ, by definition of derivation, we can conclude that they are concrete formulas and that $I \Vdash \vec{\varphi}_{1_p}$ using the hypothesis. We prove that $I \Vdash \vec{\varphi}$ by induction on n. Let ρ be an assignment over I.

Base ($n = 1$) Note that $\mathsf{proper}(\mathsf{ANT}(\ell_1 \star \vec{\varphi}_1)) = \vec{\varphi}_{1_p}$, and that there is a morphism $\widehat{u}_1$ in G^+ such that $\vec{\varphi}_{1_p} = \mathsf{proper}(\mathsf{ANT}(\ell_1)) \circ \widehat{u}_1$ and $\ell_1 \star \vec{\varphi}_1 = \ell_1 \odot \widehat{u}_1$. Hence $I \Vdash \mathsf{proper}(\mathsf{ANT}(\ell_1 \odot \widehat{u}_1))$ and so, by Proposition 7.2, $I, \rho \Vdash \mathsf{CONC}(\ell_1 \odot \widehat{u}_1)$, that is, $I, \rho \Vdash \mathsf{CONC}(\ell_1 \star \vec{\varphi}_1)$. The thesis follows since $\mathsf{CONC}(\ell_1 \star \vec{\varphi}_1) = \vec{\varphi}$.

Step: Let $\vec{\varphi}_n = \mathsf{CONC}(\ell_{n-1} \star \vec{\varphi}_{n-1} \circ \ldots \circ \ell_1 \star \vec{\varphi}_1)$. Note that the schema formulas in $\vec{\varphi}_n$ are in G^+, that is, they do not involve verum schema formulas. On the other hand, $\ell_1, \ldots, \ell_{n-1}; \vec{\varphi}_1$ is a derivation for $\Gamma \vdash_{\mathcal{D}} \vec{\varphi}_n$. Hence, by the induction hypothesis, $\Gamma \vDash_{\mathcal{I}} \vec{\varphi}_n$. So $I \Vdash \vec{\varphi}_n$ since $I \Vdash \Gamma$. Therefore $I \Vdash \mathsf{ANT}(\ell_n \star \vec{\varphi}_n)$, and there is a morphism $\widehat{u}_n$ in G^+ such that $\vec{\varphi}_n = \mathsf{ANT}(\ell_n) \circ \widehat{u}_n$ and $\ell_n \star \vec{\varphi}_n = \ell_n \odot \widehat{u}_n$. Hence $I \Vdash \mathsf{ANT}(\ell_n \odot \widehat{u}_n)$ and so, by Proposition 7.2, $I, \rho \Vdash \mathsf{CONC}(\ell_n \odot \widehat{u}_n)$, that is, $I, \rho \Vdash \mathsf{CONC}(\ell_n \star \vec{\varphi}_n)$. The thesis follows since $\mathsf{CONC}(\ell_n \star \vec{\varphi}_n) = \vec{\varphi}$. QED

7.2 Completeness

Our completeness result relies on the notion of a canonical interpretation structure generated by a deductive system and a set of formulas. More rigorously, let $\mathcal{D}$ be a deductive system and Γ a set of formulas in G^+. The *canonical interpretation structure* $S^\Gamma(\mathcal{D}) = (\Sigma, (G', \alpha, D, \blacklozenge))$ generated by $\mathcal{D}$ and Γ, is such that:

- $G' = (V', E', \mathsf{src}', \mathsf{trg}')$ where
  - V' are the morphisms of G^+ whose target is an element of V;
  - $E'(\widehat{w}_1 \ldots \widehat{w}_n, \widehat{w})$ is composed by all the m-edges e of E such that $\widehat{w} = \widehat{e} \circ \langle \widehat{w}_1, \ldots, \widehat{w}_n \rangle$ in G^+;
  - the definition of src' and trg' is straightforward from the definition of m-edges;
- $\alpha^{\mathsf{v}}(\widehat{w} : s \to v) = v$ and $\alpha^{\mathsf{e}}(e) = e$;
- $D = \{\widehat{w} \in V' : \Gamma \vdash_{\mathcal{D}} \widehat{w}\}$;
- $\blacklozenge$ is the morphism $\mathsf{id}_\diamond$ in G^+.

We may write $S(\mathcal{D})$ for $S^\emptyset(\mathcal{D})$. Denotation in the canonical structure has a very simple form as we show in the next lemma.

Lemma 7.4 Given a deductive system $\mathcal{D}$, a set Γ of formulas in G^+, a path $w : s \to t$ over $G^\dagger$, and an assignment ρ over $S^\Gamma(\mathcal{D})$, then $[\![w]\!]^{S^\Gamma(\mathcal{D}),\rho} = \widehat{w} \circ \rho_s$.

Proof: The proof follows by induction on the complexity of w:

- w is ϵ_s. Then $[\![w]\!]^{S^\Gamma(\mathcal{D}),\rho} = \rho_s = \mathsf{id}_s \circ \rho_s = \widehat{\epsilon_s} \circ \rho_s = \widehat{w} \circ \rho_s$;

- w is $\mathsf{p}_i^{s_1} w_1$. Then $[\![w]\!]^{S^\Gamma(\mathcal{D}),\rho} = [\![\mathsf{p}_i^{s_1} w_1]\!]^{S^\Gamma(\mathcal{D}),\rho} = ([\![w_1]\!]^{S^\Gamma(\mathcal{D}),\rho})_i = (\widehat{w}_1 \circ \rho_s)_i = \widehat{\mathsf{p}}_i^{s_1} \circ \widehat{w_1} \circ \rho_s = \widehat{w} \circ \rho_s$;

- w is $\langle w_1, \ldots, w_n \rangle w_0$. Hence $[\![w]\!]^{S^\Gamma(\mathcal{D}),\rho} = [\![w_1 w_0]\!]^{S^\Gamma(\mathcal{D}),\rho} \ldots [\![w_n w_0]\!]^{S^\Gamma(\mathcal{D}),\rho} = (\widehat{w}_1 \circ \widehat{w}_0 \circ \rho_s) \ldots (\widehat{w}_n \circ \widehat{w}_0 \circ \rho_s) = \widehat{\langle w_1, \ldots, w_n \rangle} \circ \widehat{w}_0 \circ \widehat{\rho}_s$ as we wanted to show;

- w is $e w_1$. Therefore $[\![w]\!]^{S^\Gamma(\mathcal{D}),\rho} = \mathsf{trg}'(E'_e([\![w_1]\!]^{S^\Gamma(\mathcal{D}),\rho}, -)) = \mathsf{trg}'(E'_e(\widehat{w}_1 \circ \rho_s, -)) = \widehat{e} \circ \widehat{w}_1 \circ \rho_s = \widehat{w} \circ \rho_s$. QED

Capitalizing on the result of denotation in the canonical structure, it is possible to establish an important lemma relating satisfaction in the canonical structure with derivation.

Lemma 7.5 Given a deductive system $\mathcal{D}$, a set Γ of formulas and a schema formula $\varphi : s \to \pi$ over the signature of $\mathcal{D}$, $\Gamma \vdash_{\mathcal{D}} \varphi \circ \rho_s$ if and only if $S^\Gamma(\mathcal{D}), \rho \Vdash \varphi$, for every assignment ρ over $S^\Gamma(\mathcal{D})$.

Proof: Let ρ be an assignment over $S^\Gamma(\mathcal{D})$. Then $\Gamma \vdash_{\mathcal{D}} \varphi \circ \rho_s$ if and only if, by Lemma 7.4, $\Gamma \vdash_{\mathcal{D}} [\![\varphi]\!]^{S^\Gamma(\mathcal{D}),\rho}$ iff $[\![\varphi]\!]^{S^\Gamma(\mathcal{D}),\rho} \subseteq D$ iff $S^\Gamma(\mathcal{D}), \rho \Vdash \varphi$. QED

In a subsequent proposition we show that $S^\Gamma(\mathcal{D})$ is sound for the rules in $\mathcal{D}$, but first we show a useful lemma.

Lemma 7.6 For every deductive rule r in $\mathcal{D}$, set of formulas Γ, and expression $\widehat{u}$ in G^+, then $\Gamma \vdash_{\mathcal{D}} \mathsf{CONC}(r) \circ \widehat{u}$ whenever $\Gamma \vdash_{\mathcal{D}} \mathsf{proper}(\mathsf{ANT}(r)) \circ \widehat{u}$.

Proof: Let $\mathsf{ANT}(r)$ be the sequence $\varphi_1 \ldots \varphi_n$. We start by considering the case that all the antecedents of r are in G^+, that is, $\mathsf{proper}(\mathsf{ANT}(r))$ is equal to $\mathsf{ANT}(r)$. Assume that $\Gamma \vdash_{\mathcal{D}} \varphi_i \circ \widehat{u}$ for $i = 1, \ldots, n$. Let $\ell_{i1}, \ldots, \ell_{im_i}; \vec{\varphi}_{i1}$ be a derivation for $\Gamma \vdash_{\mathcal{D}} \varphi_i \circ \widehat{u}$ for $i = 1, \ldots, n$. Let m be the maximum of the m_i, and let $\ell_{im_i+1}, \ldots, \ell_{im}$ denote the morphism $\mathsf{id}_{\mathsf{id}_\pi}$, for $i = 1, \ldots, n$. Moreover, assume that ℓ_{ij} is $f_{ij1} \otimes \ldots \otimes f_{ijm_{ij}}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and denote by ℓ_k the morphism $f_{1k1} \otimes \ldots \otimes f_{1km_{1k}} \otimes f_{nk1} \otimes \ldots \otimes f_{nkm_{nk}}$ for $k = 1, \ldots, m$. Note that $\ell_1, \ldots, \ell_m; \vec{\varphi}_{11} \ldots \vec{\varphi}_{n1}$ is a derivation for $\Gamma \vdash_{\mathcal{D}} \varphi_1 \circ \widehat{u} \ldots \varphi_n \circ \widehat{u}$. So $\ell_1, \ldots, \ell_m, r; \vec{\varphi}_{11} \ldots \vec{\varphi}_{n1}$ is a derivation for $\Gamma \vdash_{\mathcal{D}} \mathsf{CONC}(r) \circ \widehat{u}$ as we wanted to show.

Assume now that r has an antecedent a_1 not in G^+, that is, involving a verum schema formula. Then r has no other antecedent. So $r; (a_1 \circ \widehat{u})$ is a derivation for $\vdash_{\mathcal{D}} \mathsf{CONC}(r) \circ \widehat{u}$ and so for $\Gamma \vdash_{\mathcal{D}} \mathsf{CONC}(r) \circ \widehat{u}$. QED

Proposition 7.7 For every deductive rule r in $\mathcal{D}$, set of formulas Γ, and assignment ρ over $S^\Gamma(\mathcal{D})$, $S^\Gamma(\mathcal{D}), \rho \Vdash \mathsf{CONC}(r)$ if $S^\Gamma(\mathcal{D}), \rho \Vdash \mathsf{proper}(\mathsf{ANT}(r))$.

Proof: Assume that $S^\Gamma(\mathcal{D}), \rho \Vdash \mathsf{proper}(\mathsf{ANT}(r))$ and denote by $\varphi_1 \ldots \varphi_n$ the sequence $\mathsf{proper}(\mathsf{ANT}(r))$. Then $\Gamma \vdash_\mathcal{D} \varphi_i \circ \rho_s$, by Lemma 7.5, for $i = 1, \ldots, n$. Hence $\Gamma \vdash_\mathcal{D} \mathsf{CONC}(r) \circ \rho_s$, by Lemma 7.6, and so, $S^\Gamma(\mathcal{D}), \rho \Vdash \mathsf{CONC}(r)$, by Lemma 7.5. QED

In order for completeness to hold in a logic system it is not necessary to impose as sufficient condition that its interpretation system contains canonical structures, as we show below. It is enough to guarantee that its interpretation system contains structures that share with canonical structures some characteristics. We call these structures, representatives of a canonical structure. A logic system contains *a representative of the canonical structure over a set* Γ when it contains an interpretation structure I_Γ such that

- $I_\Gamma \Vdash \varphi$ implies $S^\Gamma(\mathcal{D}) \Vdash \varphi$;
- $I_\Gamma \Vdash \Gamma$;

for every formula φ and set of formulas Γ in G^+.

Theorem 7.8 A logic system with representatives of the canonical structures over all sets of formulas is complete.

Proof: Let Γ be a set of formulas and φ a formula. Assume that $\Gamma \not\vdash_\mathcal{D} \varphi$. Let $I_\Gamma \in \mathcal{I}$ be the representative of $S^\Gamma(\mathcal{D})$. Then $S^\Gamma(\mathcal{D}) \Vdash \Gamma$ and $S^\Gamma(\mathcal{D}) \not\Vdash \varphi$ by Lemma 7.5. Then $I_\Gamma \Vdash \Gamma$ and $I_\Gamma \not\Vdash \varphi$. So $\Gamma \not\vDash_\mathcal{I} \varphi$ since $I_\Gamma \in \mathcal{I}$. QED

A similar theorem can be established for weak completeness. The proof of the theorem is omitted since it very similar to the proof of Theorem 7.8.

Theorem 7.9 A logic system is weakly complete if it contains a representative of the canonical structure over the empty set.

Corollary 7.10 A logic system is (weakly) complete whenever it contains all the interpretation structures that are sound with respect to the rules.

Proof: Assume that $\mathcal{L}$ contains all the interpretation structures that are sound with respect to the rules. Then $S^\Gamma(\mathcal{D}) \in \mathcal{I}$, for any set Γ, using Proposition 7.7 and so, by Theorem 7.8, we conclude that $\mathcal{L}$ is complete. QED

We now present several cases of logic systems enjoying sufficient conditions for completeness.

Example 7.11 Some logic systems to which Corollary 7.10 apply:

- the logic system for classical propositional logic with the deductive system presented in Example 5.1 and all the interpretation structures sound for MP and the axioms;
- the logic system for classical propositional modal logic T with the deductive system presented in Example 5.2 and all the interpretation structures sound for MP, K, T and the axioms;

- the logic system for intuitionistic propositional logic with the deductive system presented in Example 5.3 and all the interpretation structures sound for MP and the axioms;
- the logic system for relevance logic **R** with the deductive system presented in Example 5.4 and all the interpretation structures sound for MP, AR and the axioms, provided some minor adjustments are made in the definition of consequence in order to accommodate the notion of relevant deduction. In this case we only apply Corollary 7.10 in order to establish weak completeness.
- the logic system for one-sorted equational logic with the deductive system presented in Example 5.5 and all the interpretation structures sound for SYM, TRANS, CONG_f, $\mathrm{SUB}_{t',t'',t}$ and the axiom A. ∇

8 Towards provisos and quantification

The next step extending this work is to investigate how to accommodate quantification and provisos in deduction rules. We now give some preliminary ideas on how to proceed in this direction, using as example the logic in [13] including classical and intuitionistic connectives. More specifically we are interested in its axiom $(\varphi \supset_c (\psi \supset_i \varphi))$ which has the proviso that φ is a persistent formula. A persistent formula is one where every occurrence of classical implication $\supset_c$ and classical negation $\neg_c$ is in the scope of the intuitionistic implication $\supset_i$ or in the scope of the intuitionistic negation $\neg_i$.

In our setting this proviso should be accommodated at all levels: at the signature level, at the semantic level and at the deductive level. At the signature level, a new sort ν and a m-edge $P : \pi \to \nu$ should be introduced. At the semantic level, ν should be interpreted as either true or false and the m-edges mapping to P relate a truth value with true if the proviso is satisfied by all the (schema) formulas that may have as denotation that truth value, and to false otherwise. At the deductive level we should consider $(\varphi \supset_c (\psi \supset_i \varphi))$ as a unary rule having as antecedent $P(\varphi)$ (stating that φ is persistent) and as consequent the axiom. Moreover, we should add specific rules for dealing with persistency. For instance, we should add a rule stating that for every formula φ, we have $P \circ \neg_i \circ \varphi$.

Dealing with quantification is also a challenge. Besides accommodating the first-order provisos we have to deal with the definition in our setting of the substitution of variables and its relationship with quantification. In the presence of quantifiers, the interplay between the variables and term schema should also be clarified.

9 Concluding remarks

We presented a uniform and diagrammatic way of describing logics systems using m-graphs. Signatures, interpretation structures and deductive systems are based on m-graphs. Under this perspective, formulas and derivations are morphisms in the appropriate generated categories. The approach is general enough to represent logics in different guises, namely substructural logics and logics endowed with a nondeterministic semantics. Moreover, it subsumes all logics endowed with an algebraic semantics. It seems worthwhile to explore in our setting the notion of denotation of a formula as a single truth-value even in the presence of non-deterministic operations as in the case of some paraconsistent logics [2]. General soundness and completeness results were proved.

One of the major challenges is to extend the graph-theoretic approach to logics that support provisos and quantification as we already anticipated in Section 8. Another topic of interest is to investigate how to adjust our approach for algebraizable and protoalgebraic logics. On the deductive side, herein we concentrated on Hilbert axiomatizations. We intend to extend it to other kinds of deductive systems, namely sequent calculi. Furthermore, deductive systems over higher-order languages are also worthwhile to explore. General results about cut elimination and interpolation are also envisaged in this extended framework.

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