On graph-theoretic fibring of logics

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Abstract

A graph-theoretic account of fibring of logics is developed, capitalizing on the interleaving characteristics of fibring at the linguistic, semantic and proof levels. Fibring of two signatures is seen as an m-graph where the nodes and the m-edges include the sorts and the constructors of the signatures at hand. Fibring of two models is an m-graph where the nodes and the m-edges are the values and the operations in the models, respectively. Fibring of two deductive systems is an m-graph whose nodes are language expressions and the m-edges represent the inference rules of the two original systems. The sobriety of the approach is confirmed by proving that all the fibring notions are universal constructions. This graph-theoretic view is general enough to accommodate very different fibrings of propositional based logics encompassing logics with non-deterministic semantics, logics with an algebraic semantics, logics with partial semantics, and substructural logics, among others. Soundness and weak completeness are proved to be preserved under very general conditions. Strong completeness is also shown to be preserved under tighter conditions. In this setting, the collapsing problem appearing in several combinations of logic systems can be avoided.

1 Introduction

The activity of combining logics offers an important mechanism for modularity, which is an essential ingredient in many applications where logic plays a role. The significance of the combination can be accessed by the preservation of properties that the original logics may have. For instance, it is important to know whether the logic resulting from the combination of complete logics is still complete. Proving preservation results constitute a theoretical challenge and, in many cases, they are obtained only by imposing conditions on the original logics. It is interesting to observe that combination mechanisms start to play an essential role in contemporary applications, like argumentation theory, spatial and temporal reasoning and information security [14, 6, 15]. In some of these topics, the logical context has to be set-up appropriately. It is also worthwhile to refer that several combination mechanisms have been developed in the context of modal logic. Examples are fusion [9] and product [11, 12]. Interestingly, even in the specific case of fusion, preservation results are not so easy to prove as can be seen in [9].

Fibring is a very general mechanism for combining logics that was proposed in [8]. At the language level fibring is an interleaving in the sense that the constructors of the two logics can be interleaved in the language expressions. The same applies to deduction. In a fibring deduction, we can interleave the application of inference rules of the original logics [19, 4]. When developing techniques for fibring one of the main objectives is not imposing too much requirements on the logics being considered. For instance, it would be very interesting to be able to fiber modal logic with a paraconsistent logic getting a new logic suitable for paraconsistent modal reasoning.

A starting point for defining fibring is to set-up the notion of logic system [1]. That is, it has to be specified, in a general way, what is a signature, an interpretation structure and a deductive system, so that a large class of logics can be expressed in this context. Herein, we follow [20], where signatures, interpretation structures and deductive systems are defined based on the concept of multi-graph (m-graph). This graph-theoretic approach goes very well with the interleaving aspects of fibring and offers a very general, intuitive and uniform way of looking into fibring of logic systems and its components.

An interesting aspect of graph-theoretic fibring is that, semantically, it is explicitly defined for each pair of interpretation structures. That is, each interpretation structure in the fibring results from a clear and understandable construction applied to an interpretation structure of each component. This counts as a positive aspect *per se*, but also since it makes possible to have in the fibring a representative of all the models of the original logics (this feature was not present for instance in [23]). It is also worthwhile to mention that fibring is seen as the same universal construction at all levels: at the signature level, at the interpretation structure level and at the deduction level. This means that the logic system resulting from fibring is somehow minimal among the universe of logic systems considered.

Preservation of soundness and weak completeness are proved herein under very general conditions, in contrast with strong completeness that is shown to be preserved but under tighter conditions. These preservation results allow us to conclude that the graph-theoretic semantics we propose for fibring goes hand in hand with the usual fibred deduction. It is worthwhile to mention that, to our knowledge, other approaches to fibring were not successful in proving preservation of weak completeness.

The collapsing problem appearing in several combinations of logic systems can be avoided in this graph-theoretic approach to fibring. Namely, the well known collapse when combining classical propositional logic with intuitionistic propositional logics does not appear [7, 21, 3]. This happens since in our setting the fibred semantics is in a sense an exhaustive interleaving of the semantics of the components, and so the characteristics of the semantics of the components are present in the fibring.

The structure of the paper is as follows. In Section 2, the graph-theoretic fibring of signatures is defined. Section 3 is dedicated to fibring of interpretation structures, and Section 4 concentrates on fibring of deductive systems. These sections start with a motivating example illustrating the main notions to be introduced in the section. Fibring is then defined and illustrated. Afterwards a universal perspective of fibring is given, showing that the constructions are minimal. Finally, the motivating example is revisited on the light of the notions introduced. In Section 5, fibring of logic systems is defined by putting together signatures, interpretation systems and deductive systems. In Section 6, we discuss preservation of soundness. Finally, in Section 7, we investigate preservation of both weak and strong completeness. Throughout the paper we use the fibring of classical and intuitionist logics as the running example. We assume a very moderate knowledge of category theory (the interested reader can consult [16]).

2 Fibring signatures

The motivation is that a formula over the signature resulting from the fibring of two signatures is in some sense a path resulting from the interleaving of the constructors in both signatures. For instance, the path $\neg_i \supset_c \langle q_i, q_c \rangle$ in Figure 1 corresponds to the formula $\neg_i(q_i \supset_c q_c)$ over the signature resulting

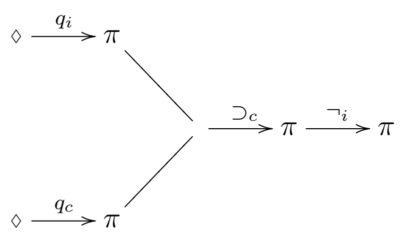

Figure 1: Formula $\neg_i(q_i \supset_c q_c)$ as a path.

from the fibring of the signature for classical propositional logic including the binary constructor $\supset_c$ and the propositional symbol q_c, with the signature for intuitionistic propositional logic including the unary constructor $\neg_i$ and the propositional symbol q_i.

Putting signatures together

By a *multi-graph* (in short a *m-graph*) we mean a tuple

$$G = (V, E, \mathsf{src}, \mathsf{trg})$$

where V is a set (of vertexes or nodes), E is a set (of m-edges), and $\mathsf{src} : E \to V^+$ and $\mathsf{trg} : E \to V$ are maps. A *language signature* or, simply, a *signature* is a tuple

$$\Sigma = (G, \pi, \diamond)$$

where G is a m-graph, and π and $\diamond$ are in V. Nodes in a signature play the role of language *sorts*. Node π is the *propositions sort* (the sort of schema formulas), and node $\diamond$ is the *concrete sort*, used as the source sort of propositional symbols. The m-edges play the role of *constructors* for building expressions of the available sorts.

Example 2.1 Let Π be a set of propositional symbols. The *classical propositional signature* Σ_Π is a m-graph G with sorts π and $\diamond$ and the following m-edges:

- $q : \diamond \to \pi$ for each q in Π;
- $\neg : \pi \to \pi$;
- $\supset : \pi\pi \to \pi$.

The m-edges $\neg$ and $\supset$ represent the connectives negation and implication, respectively. The m-edge q represents a propositional symbol q. ∇

We now describe the signature used for intuitionistic propositional logic.

Example 2.2 Let Π be a set of propositional symbols. The *intuitionistic propositional signature with conjunction and disjunction* $\Sigma_\Pi^{\wedge,\vee}$ is a m-graph obtained from Σ_Π by adding the m-edges

- $\wedge, \vee : \pi\pi \to \pi$

for representing conjunction $\wedge$ and disjunction $\vee$. ∇

The signature resulting from the fibring of two signatures can now be defined. We start by assuming that both m-graphs underlying each signature have the same nodes. That is, they share the same sorts. The *fibring of signatures* Σ_1 and Σ_2 with the same set V of nodes is the triple

$$\Sigma_1 \uplus \Sigma_2 = ((V, E, \mathsf{src}, \mathsf{trg}), \pi, \diamond)$$

where

- E is the disjoint union of E_1 and E_2 with injections $\mathsf{i}_1^{\mathsf{e}} : E_1 \to E$ and $\mathsf{i}_2^{\mathsf{e}} : E_2 \to E$, respectively;
- $\mathsf{src} \circ \mathsf{i}_1^{\mathsf{e}} = \mathsf{src}_1$ and $\mathsf{src} \circ \mathsf{i}_2^{\mathsf{e}} = \mathsf{src}_2$, and similarly for trg.

Example 2.3 The fibring

$$\Sigma_\Pi \uplus \Sigma_\Pi^{\wedge,\vee}$$

of the signature Σ_Π, described in Example 2.1, with the signature $\Sigma_\Pi^{\wedge,\vee}$, described in Example 2.2, is the signature $((V, E, \mathsf{src}, \mathsf{trg}), \pi, \diamond)$ defined as follows

- $V = \{\pi, \diamond\}$;

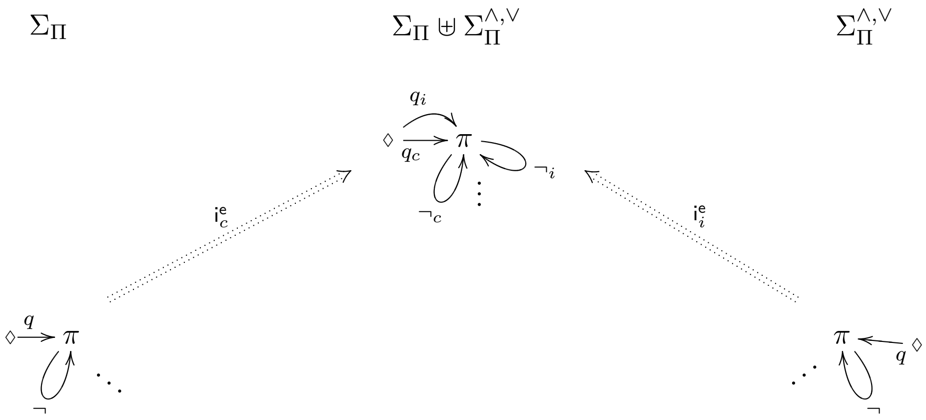

Figure 2: Fibring of the signatures described in Examples 2.1 and 2.2.

- $E(\diamond, \pi) = \Pi_c \cup \Pi_i$ where Π_c and Π_i are disjoint copies of Π;
- $E(\pi, \pi) = \{\neg_c, \neg_i\}$;
- $E(\pi\pi, \pi) = \{\supset_c, \supset_i, \wedge_i, \vee_i\}$;
- $E(s, v) = \emptyset$ otherwise;
- $\mathsf{i}_c^{\mathsf{e}}(\neg) = \neg_c$, $\mathsf{i}_c^{\mathsf{e}}(\supset) = \supset_c$, and $\mathsf{i}_c^{\mathsf{e}}(q) = q_c$ for each $q \in \Pi$;
- $\mathsf{i}_i^{\mathsf{e}}(\neg) = \neg_i$, $\mathsf{i}_i^{\mathsf{e}}(\wedge) = \wedge_i$, $\mathsf{i}_i^{\mathsf{e}}(\vee) = \vee_i$, $\mathsf{i}_i^{\mathsf{e}}(\supset) = \supset_i$, and $\mathsf{i}_i^{\mathsf{e}}(q) = q_i$, for each $q \in \Pi$;

including the classical and the intuitionistic constructors. So, it is possible to consider in its context more expressible formulas than over each component signature. For a diagrammatic description of this example see Figure 2 (where, for the sake of simplicity, only negation connectives as well as one propositional symbol for each component are considered). ∇

As Figure 2 clearly hints, it is possible to relate the signatures Σ_Π and $\Sigma_\Pi^{\wedge,\vee}$ with the signature $\Sigma_\Pi \uplus \Sigma_\Pi^{\wedge,\vee}$ resulting from their fibring. This relationship is established mainly at the level of the m-graphs of the signatures by m-graph morphisms.

An *m-graph morphism* $h : G_1 \to G_2$ is a pair of maps

$$\begin{cases} h^{\mathsf{v}} : V_1 \to V_2 \\ h^{\mathsf{e}} : E_1 \to E_2 \end{cases}$$

such that $\mathsf{src}_2 \circ h^{\mathsf{e}} = h^{\mathsf{v}} \circ \mathsf{src}_1$ and $\mathsf{trg}_2 \circ h^{\mathsf{e}} = h^{\mathsf{v}} \circ \mathsf{trg}_1$. A signature morphism is an m-graph morphism respecting the pointed sorts.

That is, a *signature morphism* from signature Σ_1 to signature Σ_2 both with the same set V of sorts,

$$h : \Sigma_1 \to \Sigma_2$$

is a m-graph morphism $h : G_1 \to G_2$ such that $h(\pi_1) = \pi_2$ and $h(\diamond_1) = \diamond_2$.

Example 2.4 The signature morphisms for the Example 2.3,

$$\mathsf{i}_c : \Sigma_\Pi \to \Sigma_\Pi \uplus \Sigma_\Pi^{\wedge\vee} \text{ and } \mathsf{i}_i : \Sigma_\Pi^{\wedge\vee} \to \Sigma_\Pi \uplus \Sigma_\Pi^{\wedge\vee}$$

are defined as follows:

- i_c is such that $\mathsf{i}_c^{\mathsf{e}}$ is defined in Example 2.3 and $\mathsf{i}_c^{\mathsf{v}}$ is the identity on V;
- i_i is such that $\mathsf{i}_i^{\mathsf{e}}$ is defined in Example 2.3 and $\mathsf{i}_i^{\mathsf{v}}$ is the identity on V. ∇

It remains to discuss the case of fibring of signatures where the set of sorts is not the same. The solution in this case is to enrich both signatures with the appropriate sorts so that after the enrichment they are equal. For instance, assume that Σ_1 and Σ_2 are signature with the sets of sorts V_1 and V_2, respectively. Then we consider enriched signatures Σ_1^+ and Σ_2^+ such that $V_1^+ = V_1 \cup (V_2 \setminus V_1)$, $E_1^+ = E_1$, $V_2^+ = V_2 \cup (V_1 \setminus V_2)$ and $E_2^+ = E_2$. The fibring of Σ_1 and Σ_2 is then defined as $\Sigma_1^+ \uplus \Sigma_2^+$.

Universal construction

The fibring of signatures is a minimal construction in the sense that it contains nothing but information on the components. This fact can be stated by proving a universal property. We denote by **Sig** the category of signatures and their morphisms, and by $\mathbf{Sig}_V$ the subcategory of **Sig** whose objects are signatures with sort set V and whose morphisms h are such that $h^{\mathsf{v}} = \mathsf{id}_V$. In Figure 3, we

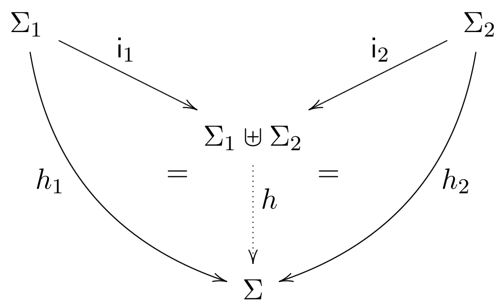

Figure 3: Coproduct of Σ_1 and Σ_2.

describe the main ingredients for showing that the fibring is a coproduct: (1) existence of morphisms i_1 and i_2; (2) universal property: given any morphisms $h_1 : \Sigma_1 \to \Sigma$ and $h_2 : \Sigma_2 \to \Sigma$, there is a unique morphism $h : \Sigma_1 \uplus \Sigma_2 \to \Sigma$ such that $h \circ \mathsf{i}_1 = h_1$ and $h \circ \mathsf{i}_2 = h_2$.

Proposition 2.5 Category $\mathbf{Sig}_V$ has binary coproducts (and so all non-empty finite coproducts).

Proof: Let Σ_1 and Σ_2 be signatures with sort set V. Their coproduct is the triple

$$(\Sigma_1 \uplus \Sigma_2, \mathsf{i}_1, \mathsf{i}_2)$$

where $\Sigma_1 \uplus \Sigma_2$ is the fibring of signatures Σ_1 and Σ_2 and $\mathsf{i}_j = (\mathsf{id}_V, \mathsf{i}_j^{\mathsf{e}})$ for $j = 1, 2$. (1) It is straightforward to check that i_1 are i_2 are signature morphisms; (2) Assume that $h_1 : \Sigma_1 \to \Sigma$ and $h_2 : \Sigma_2 \to \Sigma$ are signature morphisms. Define $h = (\mathsf{id}_V, h^{\mathsf{e}})$ as follows: $h^{\mathsf{e}}(\mathsf{i}_1^{\mathsf{e}}(e_1)) = h_1^{\mathsf{e}}(e_1)$ and $h^{\mathsf{e}}(\mathsf{i}_2^{\mathsf{e}}(e_2)) = h_2^{\mathsf{e}}(e_2)$. It is easy to see that h is a signature morphism, and moreover, that it is the unique morphism that makes the diagrams to commute, QED

Finally, we present fibring as an operator on **Sig**. Let Σ_1 and Σ_2 be signatures in **Sig** not necessarily with the same set of sorts. The first step when defining fibring is to enrich the given signatures so that they have the same set of sorts. To this end, we introduce a $\mathbf{Set}^{\diamond}$-indexed map $\cdot^{(V_1,\pi_1,\diamond_1)}$ on signatures, such that

$$\Sigma_2^{(V_1,\pi_1,\diamond_1)} = ((V, E_2, \mathsf{src}, \mathsf{trg}), \pi, \diamond)$$

where

- $((V, \pi, \diamond), \mathsf{i}_1, \mathsf{i}_2)$ is the coproduct in $\mathbf{Set}^{\diamond}$ of $(V_1, \pi_1, \diamond_1)$ and $(V_2, \pi_2, \diamond_2)$;
- $\mathsf{src} = \mathsf{i}_2 \circ \mathsf{src}_2$ and $\mathsf{trg} = \mathsf{i}_2 \circ \mathsf{trg}_2$;

then, the *fibring* $\mathsf{Fib}_{\mathbf{Sig}}$ of signatures is defined as a map from $|\mathbf{Sig}|^2$ to $|\mathbf{Sig}|$ such that

$$\mathsf{Fib}_{\mathbf{Sig}}(\Sigma_1, \Sigma_2) = \Sigma_1^{(V_2,\pi_2,\diamond_2)} \uplus \Sigma_2^{(V_1,\pi_1,\diamond_1)}.$$

Interleaving of expressions

It is more convenient to work with formulas as morphisms, instead of as paths, in order to capitalize on the additional structure of categories. The formula depicted in Figure 1 as a path, can be seen as the morphism $\neg_i \circ \supset_c \circ \langle q_i, q_c\rangle$ presented in Figure 4. In that diagram, $\pi\pi$ is a sequence of sorts that corre-

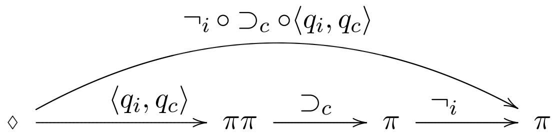

Figure 4: Formula $\neg_i(q_i \supset_c q_c)$ as a morphism.

sponds to the object of the product of π by π. That is, the product of π by π is the triple

$$\pi \times \pi = (\pi\pi, \mathsf{p}_1^{\pi\pi}, \mathsf{p}_2^{\pi\pi})$$

where $\mathsf{p}_1^{\pi\pi} : \pi\pi \to \pi$ and $\mathsf{p}_2^{\pi\pi} : \pi\pi \to \pi$ are projections. The underlying category, G^+, (for more details see [20]), has as objects sequences of sorts in G and as morphisms, besides the ones related to products, tuples and projections, the m-edges and compositions. In this category, for instance, the implication $\supset_c$ is a morphism from the object $\pi\pi$ (of the product of π and π) to the object π. Working in the scope of a category is also very important namely for schema formulas, which are at the heart of the fibring. For instance, the formula

$\neg_i(\xi \supset_c q_c)$ containing the schema variable ξ (which can be instantiated by any other formula) corresponds to the morphism $\neg_i \circ \supset_c \circ \langle \mathsf{p}_1^{\pi\diamond}, q_c \circ \mathsf{p}_2^{\pi\diamond} \rangle$ depicted in Figure 5.

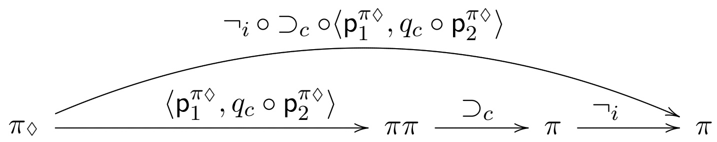

Figure 5: The schema formula $\neg_i(\xi \supset_c q_c)$ as a morphism.

Observe that the schema variable corresponds to the projection $\mathsf{p}_1^{\pi\diamond}$. The formula $\neg_i(q_i \supset_c q_c)$ can be seen as an instantiation of formula $\neg_i(\xi \supset_c q_c)$ by $\langle q_i, \mathsf{id}_\diamond \rangle$ as can be seen in Figure 6. Observe that $(\neg_i \circ \supset_c \circ \langle \mathsf{p}_1^{\pi\diamond}, q_c \circ \mathsf{p}_2^{\pi\diamond} \rangle) \circ \langle q_i, \mathsf{id}_\diamond \rangle = \neg_i \circ \supset_c \circ \langle \mathsf{p}_1^{\pi\diamond} \circ \langle q_i, \mathsf{id}_\diamond \rangle, q_c \circ \mathsf{p}_2^{\pi\diamond} \circ \langle q_i, \mathsf{id}_\diamond \rangle \rangle = \neg_i \circ \supset_c \circ \langle q_i, q_c \circ \mathsf{id}_\diamond \rangle = \neg_i \circ \supset_c \circ \langle q_i, q_c \rangle$. Instantiation corresponds to composition in this categorial setting.

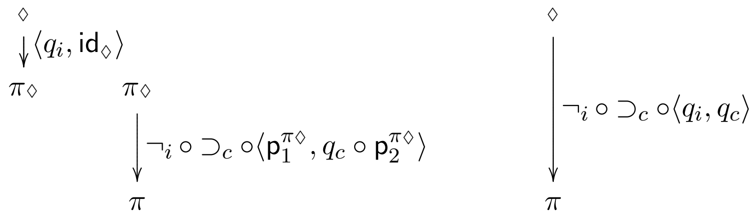

Figure 6: Formula $\neg_i(q_i \supset_c q_c)$ as an instantiation.

3 Fibring interpretation structures

Consider the formula in Figure 1. Intuitively speaking, a denotation for that formula is based on a denotation path as the one in Figure 7. The nodes should be truth-values and the edges are operations in some interpretation structure, say I, for the signature resulting from the fibring of the signature of propositional logic and the signature of intuitionistic logic. It is clear that the operations denoting q_c and to $\supset_c$ should be operations that evaluate the constructors $q, \supset$ of the propositional signature in some interpretation structure, say I_c. Moreover, the operations corresponding to q_i and $\neg_i$ should be operations that evaluate the constructors q and $\supset$ of the intuitionistic signature in some interpretation structure, say I_i. Thus, operations from I_c and I_i should be interleaved in order to get the denotation of a formula over the signature resulting from the fibring of the signatures for classical and intuitionistic logic. The only question is related to the nodes of I. Precisely because of the interleaving of operations the nodes should contain information about truth values from both I_c and I_i.

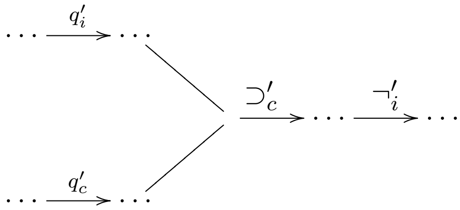

Figure 7: Denotation of $\neg_i(q_i \supset_c q_c)$ as a path.

Thus, in a nutshell and omitting the details, an interpretation structure I for the fibring of signatures Σ_1 and Σ_2 can be obtained from interpretation structures I_1 for Σ_1 and I_2 for Σ_2 in the following way. A truth value in I is a pair (v_1', v_2') where v_1' is a truth value in I_1 and v_2' is a truth value in I_2 either both designated or both non-designated. An operation $o' : (v_{11}', v_{21}') \ldots (v_{1n}', v_{2n}') \to (v_1', v_2')$ should be in I providing that a corresponding operation $o : v_{1n}' \ldots v_{1n}' \to v_1'$ is in I_1, and similarly for operations in I_2.

Putting interpretation structures together

As motivated above, an interpretation structure is an m-graph where the nodes correspond to truth values and m-edges to operations on values. Another key ingredient is that each value should be assigned to a sort and each operation to a constructor in the signature. That is, interpretation structures and signatures should be related by morphisms.

So, an *interpretation structure I over a signature* $(G, \pi, \diamond)$ is a triple

$$(G', \alpha, D, \blacklozenge)$$

where G' is an m-graph, $\alpha : G' \to G$ is an m-graph morphism, $D \subseteq (\alpha^{\mathsf{v}})^{-1}(\pi)$ is a non-empty set and $\blacklozenge \in (\alpha^{\mathsf{v}})^{-1}(\diamond)$.

Observe that V' is partitioned by α: we denote by V_v' the domain $(\alpha^{\mathsf{v}})^{-1}(v)$ of values for each v in V. The elements of V_π' are the truth values and the elements of $V_\diamond'$ are the concrete values. We assume that there is at least a concrete value.

An *interpretation structure* is a pair

$$(\Sigma, I)$$

where Σ is a signature and I is an interpretation structure over Σ.

We now present interpretation structures for classical propositional logic and intuitionistic propositional logic.

Example 3.1 Let $v : \{q_1, q_2, q_3\} \to \{0, 1\}$ be a classical valuation such that $v(q_1) = 1$ and $v(q_2) = v(q_3) = 0$. The interpretation structure $I_c = (G', \alpha, D, \blacklozenge)$ over signature Σ_Π introduced in Example 2.1, where $\Pi = \{q_1, q_2, q_3\}$, corresponding to v, is defined as follows:

- $G' = (V', E', \mathsf{src}', \mathsf{trg}')$ is such that[1]:

[1] Using module 2 arithmetical operations within V'.

$V' = \{0,1\} \cup \{\blacklozenge\}$;
$E' = \{q'_1, q'_2, q'_3, \neg_0, \neg_1, \supset_{00}, \supset_{01}, \supset_{10}, \supset_{11}\}$;
src' and trg' are such that:

$q'_1 : \blacklozenge \to 1$;
$q'_k : \blacklozenge \to 0$ for $k = 2, 3$;
$\neg_{v'} : v' \to (1 - v')$ for each v' in $\{0,1\}$;
$\supset_{v'_1 v'_2} : v'_1\, v'_2 \to ((1 - v'_1) + v'_2)$ for each v'_1 and v'_2 in $\{0,1\}$.

- $\alpha : G' \to G$ is such that:

$\alpha^{\mathsf{v}}(0) = \pi$;
$\alpha^{\mathsf{v}}(1) = \pi$;
$\alpha^{\mathsf{v}}(\blacklozenge) = \diamond$;
$\alpha^{\mathsf{e}}(q'_k) = q_k$ for $k = 1, 2, 3$;
$\alpha^{\mathsf{e}}(\neg_{v'}) = \neg$ for each v' in V'_π;
$\alpha^{\mathsf{e}}(\supset_{v'_1 v'_2}) = \supset$ for each v'_1 and v'_2 in V'_π.

- $D = \{1\}$. ∇

Example 3.2 Let (W, R, v) be the intuitionistic Kripke structure where $W = \{u_1, u_2\}$, $R = \{(u_1, u_1), (u_1, u_2), (u_2, u_2)\}$, $v(q_1) = \{u_2\}$, $v(q_2) = \{u_1, u_2\}$, and $v(q_3) = \emptyset$, for $\Pi = \{q_1, q_2, q_3\}$. By simplicity we will denote by u_2 and u_1u_2 the sets $\{u_2\}$ and $\{u_1, u_2\}$ respectively. The interpretation structure $I_i = (G', \alpha, D, \blacklozenge)$ over the signature $\Sigma_\Pi^{\wedge,\vee}$ introduced in Example 2.2, corresponding to the Kripke structure is defined as follows:

- $G' = (V', E', \mathsf{src}', \mathsf{trg}')$ is such that:

$V' = \{\emptyset, u_2, u_1u_2\} \cup \{\blacklozenge\}$;
$E' = \{q'_1, q'_2, q'_3, \neg_\emptyset, \neg_{u_2}, \neg_{u_1u_2}\} \cup \{\supset_{v'_1 v'_2} : v'_1, v'_2 \in V'_\pi\} \cup \{\wedge_{v'_1 v'_2} : v'_1, v'_2 \in V'_\pi\} \cup \{\vee_{v'_1 v'_2} : v'_1, v'_2 \in V'_\pi\}$;
src' and trg' are such that:

$q'_1 : \blacklozenge \to u_2$;
$q'_2 : \blacklozenge \to u_1u_2$;
$q'_3 : \blacklozenge \to \emptyset$;
$\neg_\emptyset : \emptyset \to u_1u_2$;
$\neg_{u_2} : u_2 \to \emptyset$;
$\neg_{u_1u_2} : u_1u_2 \to \emptyset$;
$\supset_{v'_1 v'_2} : v'_1\, v'_2 \to u_1u_2$ whenever $v'_1 \subseteq v'_2$ for each v'_1 and v'_2 in V'_π;
$\supset_{v'_1 v'_2} : v'_1\, v'_2 \to v'_2$ whenever $v'_1 \not\subseteq v'_2$ for each v'_1 and v'_2 in V'_π;
$\wedge_{v'_1 v'_2} : v'_1\, v'_2 \to v'_1 \cap v'_2$ for each v'_1 and v'_2 in V'_π;
$\vee_{v'_1 v'_2} : v'_1\, v'_2 \to v'_1 \cup v'_2$ for each v'_1 and v'_2 in V'_π.

- $\alpha : G' \to G$ is such that:

$\alpha^{\mathsf{v}}(b) = \pi$ for each $b \in \{\emptyset, u_2, u_1 u_2\}$;
$\alpha^{\mathsf{v}}(\blacklozenge) = \diamond$;
$\alpha^{\mathsf{e}}(q'_k) = q_k$ for $k = 1, 2, 3$;
$\alpha^{\mathsf{e}}(\neg_{v'}) = \neg$ for each $v' \in V'_\pi$;
$\alpha^{\mathsf{e}}(\supset_{v'_1 v'_2}) = \supset$ for each v'_1 and v'_2 in V'_π;
$\alpha^{\mathsf{e}}(\wedge_{v'_1 v'_2}) = \wedge$ for each v'_1 and v'_2 in V'_π;
$\alpha^{\mathsf{e}}(\vee_{v'_1 v'_2}) = \vee$ for each v'_1 and v'_2 in V'_π.

- $D = \{u_1 u_2\}$. ∇

The *fibring of interpretation structures* (Σ_1, I_1) and (Σ_2, I_2), with the same set V of sorts, denoted by

$$(\Sigma_1, I_1) \uplus (\Sigma_2, I_2)$$

is the interpretation structure (Σ, I) where

- Σ is the object of the coproduct in $\mathbf{Sig}_V$ of Σ_1 and Σ_2 with injections i_1 and i_2;
- $I = ((V', E', \mathsf{src}', \mathsf{trg}'), \alpha, D, \blacklozenge)$ is defined as follows:
  - V' is such that:
    - $V'_\pi = (D_1 \times D_2) \cup ((V'_{1\pi} \setminus D_1) \times (V'_{2\pi} \setminus D_2))$;
    - $V'_v = V'_{1v} \times V'_{2v}$ for each $v \in V \setminus \{\pi\}$;
  - $E'(s', t')$ is, for each s' and t' in V'^+, the object of the coproduct in $\mathbf{Set}$ of $E'_1((s')_1, (t')_1)$ and $E'_2((s')_2, (t')_2)$ with injections $(\tau_1)^{\mathsf{e}}_{s't'}$ and $(\tau_2)^{\mathsf{e}}_{s't'}$, respectively;
  - $\mathsf{src}'((\tau_j)^{\mathsf{e}}_{s't'}(e'_j)) = s'$ and $\mathsf{trg}'((\tau_j)^{\mathsf{e}}_{s't'}(e'_j)) = t'$ for $j = 1, 2$;
  - α is such that
    - $\alpha^{\mathsf{v}}((v'_1, v'_2)) = \alpha^{\mathsf{v}}_1(v'_1)$;
    - $\alpha^{\mathsf{e}}((\tau_j)^{\mathsf{e}}_{s't'}(e'_j)) = \mathsf{i}^{\mathsf{e}}_j(\alpha^{\mathsf{e}}_j(e'_j))$ for $j = 1, 2$;
  - $D = D_1 \times D_2$;
  - $\blacklozenge = (\blacklozenge_1, \blacklozenge_2)$;

Observe that the truth values in the fibring are either pairs of distinguished elements or pairs of non distinguished elements.

For the sake of illustration we now describe the interpretation structure resulting from the fibring of the interpretation structure for classical logic, introduced in Example 3.1, with the interpretation structure for intuitionistic logic, introduced in Example 3.2.

Example 3.3 The fibring of the interpretation structures for classical logic, (Σ_Π, I_c), introduced in Example 3.1, and for intuitionistic logic, $(\Sigma_\Pi^{\wedge,\vee}, I_i)$, introduced in Example 3.2, denoted by

$$(\Sigma, I_{c+i})$$

is as follows:

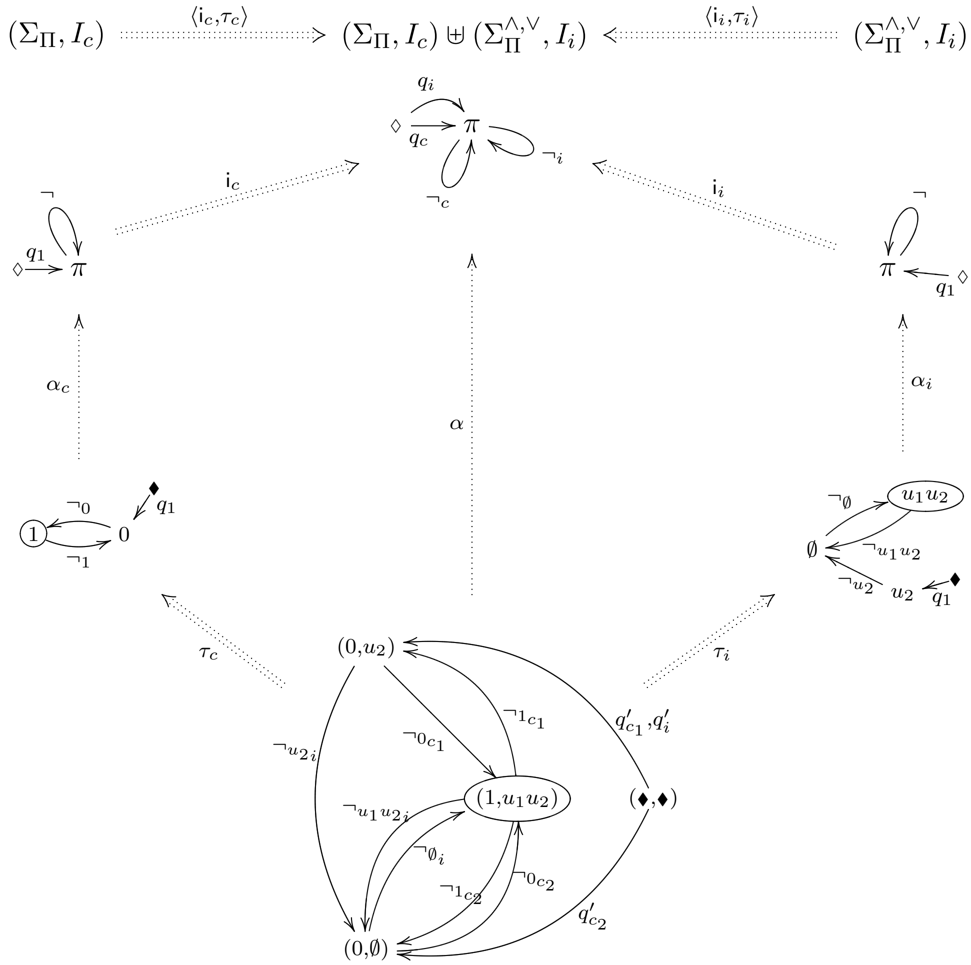

Figure 8: Fibring of interpretation structures for classical and intuitionistic logics (partial representation).

- $\Sigma = ((V, E, \mathsf{src}, \mathsf{trg}), \pi, \diamond)$ where:
  - $V = \{\pi, \diamond\}$;
  - $E(\pi, \pi) = \{\neg_c, \neg_i\}$;
  - $E(\pi\pi, \pi) = \{\supset_c, \supset_i, \wedge_i, \vee_i\}$;
  - $E(\diamond, \pi) = \{q_c, q_{2c}, q_{3c}, q_i, q_{2i}, q_{3i}\}$;
  - all the other components are empty;
- $I_{c+i} = ((V', E', \mathsf{src}', \mathsf{trg}'), \alpha, D, (\blacklozenge, \blacklozenge))$ where:
  - $V' = \{(0, \emptyset), (0, u_2), (1, u_1u_2)\}$;
  - $E'((0, u_2), (1, u_1u_2)) = \{\neg_{0_{c_1}}\}$;
  - $E'((0, u_2), (0, \emptyset)) = \{\neg_{u_{2i}}\}$;
  - $E'((1, u_1u_2), (0, u_2)) = \{\neg_{1_{c_1}}\}$;
  - $E'((0, \emptyset), (1, u_1u_2)) = \{\neg_{\emptyset_i}, \neg_{0_{c_2}}\}$;

- $E'((1,u_1u_2),(0,\emptyset)) = \{\neg_{u_1u_{2i}}, \neg_{1_{c_2}}\}$;
- $E'((\blacklozenge,\blacklozenge),(0,\emptyset)) = \{q'_{c_2}\}$;
- $E'((\blacklozenge,\blacklozenge),(0,u_2)) = \{q'_{c_1}, q'_i\}$;
- $\alpha^{\mathsf{v}}((0,\emptyset)) = \alpha^{\mathsf{v}}((0,u_2)) = \alpha^{\mathsf{v}}((1,u_1u_2)) = \pi$;
- $\alpha^{\mathsf{e}}(q'_{c_1}) = \alpha^{\mathsf{e}}(q'_{c_2}) = q_c$;
- $\alpha^{\mathsf{e}}(q'_i) = q_i$;
- $\alpha^{\mathsf{e}}(\neg_{0_{c_1}}) = \alpha^{\mathsf{e}}(\neg_{0_{c_2}}) = \alpha^{\mathsf{e}}(\neg_{1_{c_1}}) = \alpha^{\mathsf{e}}(\neg_{1_{c_2}}) = \neg_c$;
- $\alpha^{\mathsf{e}}(\neg_{\emptyset_i}) = \alpha^{\mathsf{e}}(\neg_{u_{2i}}) = \alpha^{\mathsf{e}}(\neg_{u_1u_{2i}}) = \neg_i$;
- D is $\{(1,u_1u_2)\}$.
- similarly for the implications, disjunction, conjunction and the remaining propositional symbols;

A graphical description of part of the interpretation structure (Σ, I_{c+i}) (without the implications, disjunction, conjunction and with only a classical and a intuitionistic propositional symbol) can be seen in Figure 8. ∇

By looking into Figure 8, we get the impression that there is a relationship between the interpretation structure in $(\Sigma_\Pi, I_c) \uplus (\Sigma_\Pi^{\wedge,\vee}, I_i)$ to the interpretation structures in both (Σ_Π, I_c) and $(\Sigma_\Pi^{\wedge,\vee}, I_i)$. That is, a contravariant relationship between the interpretation structures besides the covariant relationship between signatures. In order to define such a contravariant relationship we introduce the notion of m-graph transformation.

A *m-graph transformation* $\tau : G_2 \to G_1$ is a pair

$$\begin{cases} \tau^{\mathsf{v}} : V_2 \to V_1 \\ \{\tau^{\mathsf{e}}_{st} : G_1(\tau^{\mathsf{v}}(s), \tau^{\mathsf{v}}(t)) \to G_2(s,t)\}_{s,t \in V_2^+}. \end{cases}$$

So, an *interpretation structure morphism* between (Σ_1, I_1) and (Σ_2, I_2), is a pair

$$(h, \tau) : (\Sigma_1, I_1) \to (\Sigma_2, I_2)$$

where $h : \Sigma_1 \to \Sigma_2$ is a signature morphism and $\tau : G'_2 \to G'_1$ is a m-graph transformation such that:

- $h^{\mathsf{v}} \circ \alpha_1^{\mathsf{v}} \circ \tau^{\mathsf{v}} = \alpha_2^{\mathsf{v}}$;
- $h^{\mathsf{e}}(\alpha_1^{\mathsf{e}}(e'_1)) = \alpha_2^{\mathsf{e}}(\tau^{\mathsf{e}}_{s'_2t'_2}(e'_1))$ for $e'_1 \in E'_1(\tau^{\mathsf{v}}(s'_2), \tau^{\mathsf{v}}(t'_2))$ and $s'_2, t'_2 \in {V'}_2^+$;
- $(\tau^{\mathsf{v}})^{-1}(D_1) \subseteq D_2$;
- $\tau^{\mathsf{v}}((V'_2)_\pi \setminus D_2) \subseteq ((V'_1)_\pi \setminus D_1)$.

As expected the m-graph transformation is contravariant with the signature morphism. Moreover, the truth value map τ^{v} is contravariant with the signature morphism. Finally, note that each operation in G'_1 can be mapped to several operations in G'_2, and note that the third condition in the definition of interpretation structure morphism is equivalent to saying that if $v'_1 = \tau^{\mathsf{v}}(v'_2)$ and $v'_1 \in D_1$ then $v'_2 \in D_2$.

We now briefly describe the interpretation structure morphisms $(\mathsf{i}_c, \tau_c) : (\Sigma_\Pi, I_c) \to (\Sigma, I_{c+i})$ and $(\mathsf{i}_i, \tau_i) : (\Sigma_\Pi^{\wedge,\vee}, I_i) \to (\Sigma, I_i)$.

Example 3.4 Consider the fibring of interpretation structures in Example 3.3. The interpretation structure morphisms $(\mathsf{i}_c, \tau_c) : (\Sigma_\Pi, I_c) \to (\Sigma, I_{c+i})$ and $(\mathsf{i}_i, \tau_i) : (\Sigma_\Pi^{\wedge,\vee}, I_i) \to (\Sigma, I_i)$ are such that:

- $\tau_c^{\mathsf{v}}((0, \emptyset)) = 0$;
- $\tau_c^{\mathsf{v}}((1, u_1 u_2)) = 1$;
- $\tau_i^{\mathsf{v}}((0, \emptyset)) = \emptyset$;
- $\tau_i^{\mathsf{v}}(1, u_1 u_2) = u_1 u_2$;
- $(\tau_c)^{\mathsf{e}}_{(0,\emptyset)(1,u_1 u_2)}(\neg_0) = \neg_{0_{c_2}}$;

where the other cases are omitted since they are defined similarly. ∇

Universal construction

The objective now is to prove that the fibring of interpretation structures is a minimal construction by showing that it satisfies a universal property. We denote by **Int** the category of interpretation structures and their morphisms, and by $\mathbf{Int}_V$ the subcategory of **Int** composed by all interpretation structures with sort set V and morphisms $(h, \tau) : (\Sigma_1, I_1) \to (\Sigma_2, I_2)$ such that $h^{\mathsf{v}} = \mathsf{id}_V$. Moreover, given a sequence s of pairs, we denote by $(s)_j$ the sequence of all j-components of the pairs in s, for $j = 1, 2$.

Lemma 3.5 Given interpretation structures (Σ_1, I_1) and (Σ_2, I_2) with the same set of sorts, the pair (i_j, τ_j) such that

- i_j is the injection from Σ_j to the coproduct in $\mathbf{Sig}_V$ of Σ_1 with Σ_2;
- $\tau_j^{\mathsf{v}}((v'_1, v'_2)) = v'_j$;
- $(\tau_j)^{\mathsf{e}} = \{(\tau_j)^{\mathsf{e}}_{s't'}\}_{s',t' \in V'^+}$ is such that $(\tau_j)^{\mathsf{e}}_{s't'}$ is an injection in **Set** from $E'_j((s')_j, (t')_j)$ to the coproduct of $E'_1((s')_1, (t')_1)$ and $E'_2((s')_2, (t')_2)$;

is an interpretation system morphism from (Σ_j, I_j) to $(\Sigma_1, I_1) \uplus (\Sigma_2, I_2)$, for $j = 1, 2$.

Proof: It is straightforward to show that τ_1 and τ_2 are m-graph transformations. Note that i_1 and i_2 are signature morphisms by definition. Furthermore, the tuple (i_j, τ_j) is an interpretation structure morphism for $j = 1, 2$ since

- $\mathsf{i}_j^{\mathsf{v}}(\alpha_j^{\mathsf{v}}(\tau_j^{\mathsf{v}}((v'_1, v'_2)))) = \mathsf{i}_j^{\mathsf{v}}(\alpha_j^{\mathsf{v}}(v'_j)) = \alpha_j^{\mathsf{v}}(v'_j) = \alpha^{\mathsf{v}}((v'_1, v'_2))$ by definition of α^{v} using also the fact that $\mathsf{i}_j^{\mathsf{v}}$ is the identity;
- $\mathsf{i}_j^{\mathsf{e}}(\alpha_j^{\mathsf{e}}(e'_j)) = \alpha^{\mathsf{e}}((\tau_j)^{\mathsf{e}}_{s't'}(e'_j))$ for $e'_j \in E'_j((\tau_j)^{\mathsf{v}}(s'), (\tau_j)^{\mathsf{v}}(t'))$ by definition of α^{e};
- $(\tau_1^{\mathsf{v}})^{-1}(D_1) \subseteq D$. Let $v' = (v'_1, v'_2) \in (\tau_1^{\mathsf{v}})^{-1}(D_1)$. Then $\tau_1^{\mathsf{v}}(v') = v'_1 \in D_1$ and so, by definition of D, $v' \in D$. Similarly for $(\tau_2^{\mathsf{v}})^{-1}(D_2) \subseteq D$;

- $\tau_1^{\mathsf{v}}(V'_\pi \setminus D) \subseteq V'_{1\pi} \setminus D_1$. Let $(v'_1, v'_2) \in V'_\pi \setminus D$. By definition of D, we know that $v'_1 \notin D_1$. Similarly for $\tau_2^{\mathsf{v}}(V'_\pi \setminus D) \subseteq V'_{2\pi} \setminus D_2$.

QED

Proposition 3.6 Category $\mathbf{Int}_V$ has binary coproducts.

Proof: Let (Σ_1, I_1) and (Σ_2, I_2) be interpretation structures with sort set V. Their coproduct is

$$((\Sigma_1, I_1) \uplus (\Sigma_2, I_2), (\mathsf{i}_1, \tau_1), (\mathsf{i}_2, \tau_2))$$

where $((\Sigma_1, I_1) \uplus (\Sigma_2, I_2)$ is the fibring of interpretation structures (Σ_1, I_1) and (Σ_2, I_2), and (i_j, τ_j) for $j = 1, 2$ are as in Lemma 3.5.

It is now shown the universal property of the coproduct. Let (Σ_3, I_3) be a interpretation structure in $\mathbf{Int}_V$ and $(g_1, \sigma_1) : (\Sigma_1, I_1) \to (\Sigma_3, I_3)$ and $(g_2, \sigma_2) : (\Sigma_2, I_2) \to (\Sigma_3, I_3)$ interpretation structure morphisms. Consider the tuple (g, σ) such that:

- $g = (\mathsf{id}_V, g^{\mathsf{e}})$ where g^{e} is the unique morphism in **Set** such that $g_1^{\mathsf{e}} = g^{\mathsf{e}} \circ \mathsf{i}_1^{\mathsf{e}}$ and $g_2^{\mathsf{e}} = g^{\mathsf{e}} \circ \mathsf{i}_2^{\mathsf{e}}$;
- $\sigma = (\sigma^{\mathsf{v}}, \sigma^{\mathsf{e}})$ where $\sigma^{\mathsf{v}}(v'_3) = (\sigma_1^{\mathsf{v}}(v'_3), \sigma_2^{\mathsf{v}}(v'_3))$ and

  $$\sigma^{\mathsf{e}}_{s'_3 t'_3}((\tau_j)^{\mathsf{e}}_{\sigma^{\mathsf{v}}(s'_3)\sigma^{\mathsf{v}}(t'_3)}(e'_j)) = (\sigma_j^{\mathsf{e}})_{s'_3 t'_3}(e'_j)$$

  for every $e'_j \in E'_j((\tau_j)^{\mathsf{v}}(\sigma^{\mathsf{v}}(s'_3)), (\tau_j)^{\mathsf{v}}(\sigma^{\mathsf{v}}(t'_3)))$.

1. (g, σ) is an interpretation structure morphism from (Σ, I) to (Σ_3, I_3).

(a) $(\sigma_1^{\mathsf{v}}(v'_3), \sigma_2^{\mathsf{v}}(v'_3)) \in V'$. For instance, assume that $\alpha_3^{\mathsf{v}}(v'_3) = \pi$ and $v'_3 \in (V'_3 \setminus D_3)$. Since $\alpha_1^{\mathsf{v}} \circ \sigma_1^{\mathsf{v}} = \alpha_3^{\mathsf{v}}$, then $\sigma_1^{\mathsf{v}}(v'_3) \in (V'_1)_\pi$. Moreover, $\sigma_1^{\mathsf{v}}(v'_3) \in (V'_1)_\pi \setminus D_1$, since $\sigma_1^{\mathsf{v}}((V'_3)_\pi \setminus D_3) \subseteq (V'_1)_\pi \setminus D_1$.

(b) $\tau_j^{\mathsf{v}} \circ \sigma^{\mathsf{v}} = \sigma_j^{\mathsf{v}}$ for $j = 1, 2$. Indeed $\tau_j^{\mathsf{v}}(\sigma^{\mathsf{v}}(v'_3)) = \tau_j^{\mathsf{v}}((\sigma_1^{\mathsf{v}}(v'_3), \sigma_2^{\mathsf{v}}(v'_3))) = \sigma_j^{\mathsf{v}}(v'_3)$.

(c) σ^{e} is a family of well defined maps. Let $s'_3, t'_3 \in V_3'^{+}$. Assume that $e' \in E'(\sigma^{\mathsf{v}}(s'_3), \sigma^{\mathsf{v}}(t'_3))$. Then $e' = (\tau_j)^{\mathsf{e}}_{\sigma^{\mathsf{v}}(s'_3), \sigma^{\mathsf{v}}(t'_3)}(e'_j)$ for some $j = 1, 2$. We now show that $\mathsf{src}'_j(e'_j) = \sigma_j^{\mathsf{v}}(s'_3)$ and $\mathsf{trg}'_j(e'_j) = \sigma_j^{\mathsf{v}}(t'_3)$. Indeed, $\mathsf{src}'_j(e'_j) = \tau_j^{\mathsf{v}}(\sigma^{\mathsf{v}}(s'_3))$. By (b), $\tau_j^{\mathsf{v}}(\sigma^{\mathsf{v}}(s'_3)) = \sigma_j^{\mathsf{v}}(s'_3)$ as we wanted to show. Similarly for trg'_j.

(d) $g^{\mathsf{v}} \circ \alpha^{\mathsf{v}} \circ \sigma^{\mathsf{v}} = \alpha_3^{\mathsf{v}}$. Indeed, $g^{\mathsf{v}}(\alpha^{\mathsf{v}}(\sigma^{\mathsf{v}}(v'_3))) = g^{\mathsf{v}}(\alpha^{\mathsf{v}}((\sigma_1^{\mathsf{v}}(v'_3), \sigma_2^{\mathsf{v}}(v'_3))))$ by definition of σ^{v}, which is equal to $\alpha^{\mathsf{v}}((\sigma_1^{\mathsf{v}}(v'_3), \sigma_2^{\mathsf{v}}(v'_3)))$ since $g^{\mathsf{v}} = \mathsf{id}_V$. Since $\alpha^{\mathsf{v}}((\sigma_1^{\mathsf{v}}(v'_3), \sigma_2^{\mathsf{v}}(v'_3))) = \alpha_j^{\mathsf{v}}(\sigma_j^{\mathsf{v}}(v'_3))$ for $j = 1, 2$ and $\alpha_j^{\mathsf{v}}(\sigma_j^{\mathsf{v}}(v'_3)) = \alpha_3^{\mathsf{v}}(v'_3)$ (due to the fact that σ_j is a m-graph transformation), the thesis follows.

(e) $g^{\mathsf{e}}(\alpha^{\mathsf{e}}(e')) = \alpha_3^{\mathsf{e}}(\sigma^{\mathsf{e}}_{s'_3 t'_3}(e'))$ for $e' \in E'(\sigma^{\mathsf{v}}(s'_3), \sigma^{\mathsf{v}}(t'_3))$ for $s'_3, t'_3 \in V_3'^{+}$. Assume that $e' = (\tau_j)^{\mathsf{e}}_{\sigma^{\mathsf{v}}(s'_3)\sigma^{\mathsf{v}}(t'_3)}(e'_j)$ with $e'_j \in E'_j(\tau_j^{\mathsf{v}}(\sigma^{\mathsf{v}}(s'_3)), \tau_j^{\mathsf{v}}(\sigma^{\mathsf{v}}(t'_3)))$. Then $\alpha_3^{\mathsf{e}}(\sigma^{\mathsf{e}}_{s'_3 t'_3}(e')) = \alpha_3^{\mathsf{e}}((\sigma_j^{\mathsf{e}})_{\sigma_j^{\mathsf{v}}(s'_3)\sigma_j^{\mathsf{v}}(t'_3)}(e'_j))$, by definition of σ^{e}, which is equal to $g_j^{\mathsf{e}}(\alpha_j^{\mathsf{e}}(e'_j))$ since $\langle g_j, \sigma_j \rangle$ is a m-graph transformation. On the other hand, by definition of g^{e}, $g_j^{\mathsf{e}}(\alpha_j^{\mathsf{e}}(e'_j)) = g^{\mathsf{e}}(\mathsf{i}_j^{\mathsf{e}}(\alpha_j^{\mathsf{e}}(e'_j)))$. Finally, the thesis follows, since

$\mathsf{i}^{\mathsf{e}}_j(\alpha^{\mathsf{e}}_j(e'_j)) = \alpha^{\mathsf{e}}((\tau_j)^{\mathsf{e}}_{\sigma^{\mathsf{v}}(s'_3)\sigma^{\mathsf{v}}(t'_3)}(e'_j))$.

(f) $(\sigma^{\mathsf{v}})^{-1}(D) \subseteq D_3$. Let $v'_3 \in (\sigma^{\mathsf{v}})^{-1}(D)$. Then $\sigma^{\mathsf{v}}(v'_3) = (\sigma^{\mathsf{v}}_1(v'_3), \sigma^{\mathsf{v}}_2(v'_3)) \in D$. Without loss of generality, assume that $\sigma^{\mathsf{v}}_1(v'_3) \in D_1$. Hence $v'_3 \in (\sigma^{\mathsf{v}}_1)^{-1}(D)$ and so $v'_3 \in D_3$.

(g) $\sigma^{\mathsf{v}}(V'_{3\pi} \setminus D_3) \subseteq V'_\pi \setminus D$. Let $v'_3 \in V'_{3\pi} \setminus D_3$. Then $\sigma^{\mathsf{v}}_1(v'_3) \in V'_{1\pi} \setminus D_1$ and $\sigma^{\mathsf{v}}_2(v'_3) \in V'_{2\pi} \setminus D_2$ and so $(\sigma^{\mathsf{v}}_1(v'_3), \sigma^{\mathsf{v}}_2(v'_3)) = \sigma^{\mathsf{v}}(v'_3) \in V'_\pi \setminus D$.

2. $(g, \sigma) \circ (\mathsf{i}_j, \tau_j) = (g_j, \sigma_j)$ for $j = 1, 2$.

(a) $g \circ \mathsf{i}_j = g_j$. That is, $g^{\mathsf{v}} \circ \mathsf{i}^{\mathsf{v}}_j = g^{\mathsf{v}}_j$ and $g^{\mathsf{e}} \circ \mathsf{i}^{\mathsf{e}}_j = g^{\mathsf{e}}_j$. The first equality follows directly since all the maps are identities on V. The second equality follows by the universal property of the coproduct.

(b) $\sigma^{\mathsf{v}}_j = \tau^{\mathsf{v}}_j \circ \sigma^{\mathsf{v}}$. This follows by 1(b) above.

(c) $(\sigma^{\mathsf{e}}_j)_{s'_3t'_3} = \sigma^{\mathsf{e}}_{s'_3t'_3} \circ \tau^{\mathsf{e}}_{\sigma^{\mathsf{v}}(s'_3)\sigma^{\mathsf{v}}(t'_3)}$. The thesis follows since $E'_j(\sigma^{\mathsf{v}}_j(s'_3), \sigma^{\mathsf{v}}_j(t'_3))$ is $E'_j((\tau_j)^{\mathsf{v}}(\sigma^{\mathsf{v}}(s'_3)), (\tau_j)^{\mathsf{v}}(\sigma^{\mathsf{v}}(t'_3)))$ by (c) above.

3. Uniqueness of (g, σ). Assume that there is a m-graph transformation $(\hat{g}, \hat{\sigma})$ such that $(\hat{g}, \hat{\sigma}) \circ (\mathsf{i}_j, \tau_j) = (g_j, \sigma_j)$ for $j = 1, 2$.

(a) $\hat{g}^{\mathsf{v}} = g^{\mathsf{v}}$ since both are the identity on V.

(b) $\hat{g}^{\mathsf{e}} = g^{\mathsf{e}}$. This follows from the universal property.

(c) $\hat{\sigma}^{\mathsf{v}} = \sigma^{\mathsf{v}}$. Let $v'_3 \in V'_3$ and $\hat{\sigma}^{\mathsf{v}}(v'_3) = (u'_1, u'_2)$. Then $\sigma^{\mathsf{v}}(v'_3) = (\sigma^{\mathsf{v}}_1(v'_3), \sigma^{\mathsf{v}}_2(v'_3))$ using the definition of σ^{v}. So $(\sigma^{\mathsf{v}}_1(v'_3), \sigma^{\mathsf{v}}_2(v'_3)) = (\tau^{\mathsf{v}}_1(\hat{\sigma}^{\mathsf{v}}(v'_3)), \tau^{\mathsf{v}}_2(\hat{\sigma}^{\mathsf{v}}(v'_3)))$ by commutativity. By definition of τ^{v}_j, $(\tau^{\mathsf{v}}_1(\hat{\sigma}^{\mathsf{v}}(v'_3)), \tau^{\mathsf{v}}_2(\hat{\sigma}^{\mathsf{v}}(v'_3))) = (u'_1, u'_2)$ and hence $\sigma^{\mathsf{v}}(v'_3) = \hat{\sigma}^{\mathsf{v}}(v'_3)$.

(d) $\hat{\sigma}^{\mathsf{e}}_{s'_3t'_3} = \sigma^{\mathsf{e}}_{s'_3t'_3}$. Note that $\hat{\sigma}^{\mathsf{v}} = \sigma^{\mathsf{v}}$. Let $e' \in E'(\sigma^{\mathsf{v}}(s'_3), \sigma^{\mathsf{v}}(t'_3))$. Then $e' = (\tau_j)^{\mathsf{e}}_{\sigma^{\mathsf{v}}(s'_3)\sigma^{\mathsf{v}}(t'_3)}(e'_j)$ with $e'_j \in E'_j(\tau^{\mathsf{v}}_j(\sigma^{\mathsf{v}}(s'_3)), \tau^{\mathsf{v}}_j(\sigma^{\mathsf{v}}(t'_3)))$. Then $\sigma^{\mathsf{e}}_{s'_3t'_3}(e') = (\sigma^{\mathsf{e}}_j)_{\sigma^{\mathsf{v}}_j(s'_3)\sigma^{\mathsf{v}}_j(t'_3)}(e'_j) = \hat{\sigma}^{\mathsf{e}}_{s'_3t'_3}(e')$. QED

In order to be able to define fibring for any pair of interpretation structures, and not only for structures with the same set of sorts, we define a $\mathbf{Set}^\diamond$ indexed map $\cdot^{(V_1,\pi_1,\diamond_1)}$ that enriches each interpretation structure with the sorts only in the other structure, in such a way that satisfaction is maintained:

- $(\Sigma_2, I_2)^{(V_1,\pi_1,\diamond_1)} = (\Sigma_2^{(V_1,\pi_1,\diamond_1)}, I_2^{(V_1,\pi_1,\diamond_1)})$ where:
  - $((V, \pi, \diamond), \mathsf{i}_1, \mathsf{i}_2)$ is the coproduct in $\mathbf{Set}^\diamond$ of $(V_1, \pi_1, \diamond_1)$ and $(V_2, \pi_2, \diamond_2)$;
  - $I_2^{(V_1,\pi_1,\diamond_1)} = ((V', E'_2, \mathsf{src}', \mathsf{trg}'), (\alpha^{\mathsf{v}}, \alpha^{\mathsf{e}}_2), D_2, \blacklozenge_2)$ is such that:
    * $V' = V'_2 \cup \{{v_1}' : v_1 \in V_1 \setminus V_2\}$;
    * $\alpha^{\mathsf{v}} : V' \to V$ is such that
      - $\alpha^{\mathsf{v}}(v') = \mathsf{i}_2(\alpha^{\mathsf{v}}_2(v'))$ for v' in V'_2;
      - $\alpha^{\mathsf{v}}({v_1}') = \mathsf{i}_1(v_1)$ otherwise;
    * $\mathsf{src}' : E'_2 \to V'^+$ and $\mathsf{trg}' : E'_2 \to V'$ coincide with src'_2 and trg'_2;

Proposition 3.7 The interpretation structure (Σ_2, I_2) and $(\Sigma_2, I_2)^{(V_1,\pi_1,\diamond_1)}$ satisfy the same formulas.

We omit the proof of the proposition above since it follows straightforwardly.

So, the fibring $\mathsf{Fib}_{\mathbf{Int}}$ of interpretation structures is defined as a map from $|\mathbf{Int}|^2$ to $|\mathbf{Int}|$ such that:

$$\mathsf{Fib}_{\mathbf{Int}}((\Sigma_1, I_1), (\Sigma_2, I_2)) = (\Sigma_1, I_1)^{(V_2,\pi_2,\diamond_2)} \uplus (\Sigma_2, I_2)^{(V_1,\pi_1,\diamond_1)}.$$

Interleaving of denotations

Consider the interpretation structure (Σ, I_{c+i}) in Example 3.3 depicted in Figure 8 resulting from the fibring of the interpretation structure (Σ_Π, I_c) for classical logic and of the interpretation structure $(\Sigma_\Pi^{\wedge,\vee}, I_i)$ for intuitionistic logic. The denotation

$$[\![\neg_i \supset_c \langle q_i, q_c\rangle]\!]^{I_{c+i}}$$

of the path $\neg_i \supset_c \langle q_i, q_c\rangle$ corresponding to the formula $\neg_i(q_i \supset_c q_c)$ should be based on the denotation of q_c and of q_i and on the operations assigned to the connectives $\neg_i$ and $\supset_c$. In general, for evaluating ew_1 over I, we start by evaluating w_1 and getting a set of values. Then, for each value s' we pick up all the m-edges in G' with source s' and which are mapped into e. Finally, the envisaged denotation is obtained by taking the collection of targets of such edges. In the case at hand, starting by the denotation of q_c,

$$\begin{aligned}[\![q_c]\!]^{I_{c+i}} &= \mathsf{trg}'(\{q'_{c_1} : (\blacklozenge,\blacklozenge) \to (0,u_2), q'_{c_2} : (\blacklozenge,\blacklozenge) \to (0,\emptyset)\}) \\ &= \{(0,u_2),(0,\emptyset)\}.\end{aligned}$$

Observe that there are two operations, $q'_{c_1} : \blacklozenge \to (0,u_2)$ and $q'_{c_2} : \blacklozenge \to (0,\emptyset)$, denoting q_c in the interpretation structure I_{c+i}, that is, operations that are assigned by α^{e} to q_c. So the denotation of q_c is the set of truth values corresponding to the target of those operations. Similarly for the denotation of q_i:

$$\begin{aligned}[\![q_i]\!]^{I_{c+i}} &= \mathsf{trg}'(\{q'_i : (\blacklozenge,\blacklozenge) \to (0,u_2)\}) \\ &= \{(0,u_2)\}\end{aligned}$$

The denotation of $[\![\langle q_i, q_c\rangle]\!]^{I_{c+i}}$ is the concatenation of the denotation of each formula in the tuple.

$$\begin{aligned}[\![\langle q_i, q_c\rangle]\!]^{I_{c+i}} &= [\![\langle q\rangle_i]\!]^{I_{c+i}}[\![\langle q\rangle_c]\!]^{I_{c+i}} \\ &= \{(0,u_2)\}\{(0,u_2),(0,\emptyset)\} \\ &= \{(0,u_2)(0,u_2),(0,u_2)(0,\emptyset)\}\end{aligned}$$

Respecting the structure of the path, the denotation of $[\![\supset_c \langle q_i, q_c\rangle]\!]^{I_{c+i}}$ is obtained from the denotation of $[\![\langle q_i, q_c\rangle]\!]^{I_{c+i}}$ by applying the operations corresponding to $\supset_c$:

$$\begin{aligned}[\![\supset_c \langle q_i, q_c\rangle]\!]^{I_{c+i}} &= \mathsf{trg}'(E'_{\supset_c}([\![\langle q_i, q_c\rangle]\!]^{I_{c+i}}, -)) \\ &= \mathsf{trg}'(E'_{\supset_c}((0,u_2)(0,u_2), -)) \cup \mathsf{trg}'(E'_{\supset_c}((0,u_2)(0,\emptyset), -)) \\ &= \{(1, u_1u_2)\}\end{aligned}$$

Similarly,

$$\begin{array}{rcl}
[\![\neg_i \supset_c \langle q_i, q_c\rangle]\!]^{I_{c+i}} & = & \mathsf{trg}'(E'_{\neg_i}([\![\supset_c \langle q_i, q_c\rangle]\!]^{I_{c+i}}, -)) \\
& = & \mathsf{trg}'(E'_{\neg_i}((1, u_1u_2), -)) \\
& = & \mathsf{trg}'(\{\neg_{u_1u_{2i}} : (1, u_1u_2) \to (0, \emptyset)\}) \\
& = & \{(0, \emptyset)\}.
\end{array}$$

The *denotation of a concrete formula* corresponding to the concrete path $w : \diamond \to t$ over $G^\dagger$ at I, represented by

$$[\![w]\!]^I$$

is a concatenation of basic sets contained in $V_t'^+$, inductively defined on the complexity of the path w as follows:

- $[\![\epsilon_\diamond]\!]^I$ is $\{\blacklozenge\}$;
- $[\![\mathsf{p}_i^{v_1\ldots v_m} w_1]\!]^I$ is $([\![w_1]\!]^I)_i$ where $v_1, \ldots, v_m$ are in V;
- $[\![\langle w_1, \ldots, w_n\rangle w_0]\!]^I$ is $[\![w_1 w_0]\!]^I \ldots [\![w_n w_0]\!]^I$;
- $[\![e w_1]\!]^I$ is the union of $\mathsf{trg}'(E'_e(v', -))$ for each v' in $[\![w_1]\!]^I$, when e is in E;

where a subset S of $V'^+_{v_1\ldots v_n}$ is a concatenation of basic sets whenever there exist $S_1 \subseteq V'_{v_1}, \ldots, S_n \subseteq V'_{v_n}$ such that S is $S_1 \ldots S_n$.

Consider now the denotation of a schema formula. In this case the denotation of the schema variables should be given by an assignment, which must be also a component in the denotation process. More precisely, an *assignment*

$$\rho$$

for an interpretation structure I over a signature Σ is a family $\{\rho_s\}_{s\in V^+}$ such that ρ_s is $[\![w_s]\!]^I$ for some concrete path $w_s : \diamond \to s$.

The *denotation of a formula* corresponding to the path $w : s \to t$ over $G^\dagger$ at I and ρ, denoted by

$$[\![w]\!]^{I\rho}$$

is inductively defined on the complexity of the path w similarly to the denotation of a concrete path with the exception that $[\![\epsilon_s]\!]^{I\rho}$ is ρ_s.

A formula φ is said to be *satisfied by I and ρ*, written as

$$I, \rho \Vdash \varphi$$

whenever $[\![\varphi]\!]^{I\rho}$ is non-empty and is contained in D. Moreover, we say I *satisfies* φ, written as

$$I \Vdash \varphi$$

whenever $I, \rho \Vdash \varphi$ for every assignment ρ over I. Satisfaction is extended to sets of schema formulas as expected: $I, \rho \Vdash \Gamma$ if $I, \rho \Vdash \gamma$ for each $\gamma \in \Gamma$, and similarly for sequences of schema formulas: $I, \rho \Vdash \varphi_1 \ldots \varphi_n$ if $I, \rho \Vdash \varphi_i$ for $i = 1, \ldots, n$.

Example 3.8 Observe that

$$I_{c+i}, \rho \not\Vdash (\neg_i (\neg_i \xi)) \supset_i \xi$$

where ρ is such that $\rho_\pi = [\![q_i]\!]^{I_{c+i}}$ since $[\![\supset_i \langle \neg_i \neg_i \xi, \xi \rangle]\!]^{I\rho} = \{(0, u_2)\}$ is not contained in D. Indeed,

$$\begin{array}{rcl} [\![\epsilon_\pi]\!]^{I_{c+i}\rho} & = & \rho_\pi \\ & = & [\![q_i]\!]^{I_{c+i}} \\ & = & \mathsf{trg}'(\{q'_i : (\blacklozenge, \blacklozenge) \to (0, u_2)\}) \\ & = & \{(0, u_2)\} \end{array}$$

$$\begin{array}{rcl} [\![\xi]\!]^{I_{c+i}\rho} & = & [\![\mathsf{p}_1^\pi]\!]^{I_{c+i}\rho} \\ & = & [\![\mathsf{p}_1^\pi \epsilon_\pi]\!]^{I_{c+i}\rho} \\ & = & ([\![\epsilon_\pi]\!]^{I_{c+i}\rho})_1 \\ & = & [\![\epsilon_\pi]\!]^{I_{c+i}\rho} \\ & = & \{(0, u_2)\} \end{array}$$

$$\begin{array}{rcl} [\![\neg_i \xi]\!]^{I_{c+i}\rho} & = & \mathsf{trg}'(E'_{\neg_i}([\![\xi]\!]^{I_{c+i}\rho}, -)) \\ & = & \mathsf{trg}'(E'_{\neg_i}((0, u_2), -)) \\ & = & \mathsf{trg}'(\{\neg_{u_2 i} : (0, u_2) \to (0, \emptyset)\}) \\ & = & \{(0, \emptyset)\} \end{array}$$

$$\begin{array}{rcl} [\![\neg_i \neg_i \xi]\!]^{I_{c+i}\rho} & = & \mathsf{trg}'(E'_{\neg_i}([\![\neg_i \xi]\!]^{I_{c+i}\rho}, -)) \\ & = & \mathsf{trg}'(E'_{\neg_i}((0, \emptyset), -)) \\ & = & \mathsf{trg}'(\{\neg_{\emptyset_i} : (0, \emptyset) \to (1, u_1 u_2)\} \\ & = & \{(1, u_1 u_2)\} \end{array}$$

$$\begin{array}{rcl} [\![\langle \neg_i \neg_i \xi, \xi \rangle]\!]^{I_{c+i}\rho} & = & [\![\neg_i \neg_i \xi]\!]^{I_{c+i}\rho} [\![\xi]\!]^{I_{c+i}\rho} \\ & = & \{(1, u_1 u_2)\}\{(0, u_2)\} \\ & = & \{(1, u_1 u_2)(0, u_2)\} \end{array}$$

$$\begin{array}{rcl} [\![\supset_i \langle \neg_i \neg_i \xi, \xi \rangle]\!]^{I_{c+i}\rho} & = & \mathsf{trg}'(E'_{\supset_i}([\![\langle \neg_i \neg_i \xi, \xi \rangle]\!]^{I_{c+i}\rho}, -)) \\ & = & \mathsf{trg}'(E'_{\supset_i}((1, u_1 u_2)(0, u_2), -)) \\ & = & \{(0, u_2)\}. \end{array}$$

This example shows that the intuitionistic implication *does not collapse* into the classical implication. Therefore, the graph-theoretic fibring of interpretation structures retains the intuitionistic character of the component. ∇

Thus, the example above shows that the fibring of classical propositional logic with intuitionistic propositional logic does not collapse into classical logic, since otherwise the interpretation structure (Σ, I_{c+i}) would satisfy $((\neg_i(\neg_i \xi)) \supset_i \xi)$ showing that intuitionistic connectives behave like classical connectives.

4 Fibring deductive systems

Consider again the fibring of classical and intuitionistic logics. In Figure 9 we present what should be a derivation of $\neg_i \neg_i q_c$ from $\{\neg_i(q_i \supset_c q_c), (\neg_i(q_i \supset_c q_c)) \supset_i q_c, q_c \supset_c (\neg_i \neg_i q_c)\}$ in the fibring of classical and intuitionistic logic. The derivation should proceed by applying first MP_i (the Modus Ponens rule in intuitionistic logic) and then MP_c (the Modus Ponens rule in classical logic). From an intuitive point of view a derivation in the deductive system resulting from the fibring should result from the interleaving of rules and axioms from both deductive systems.

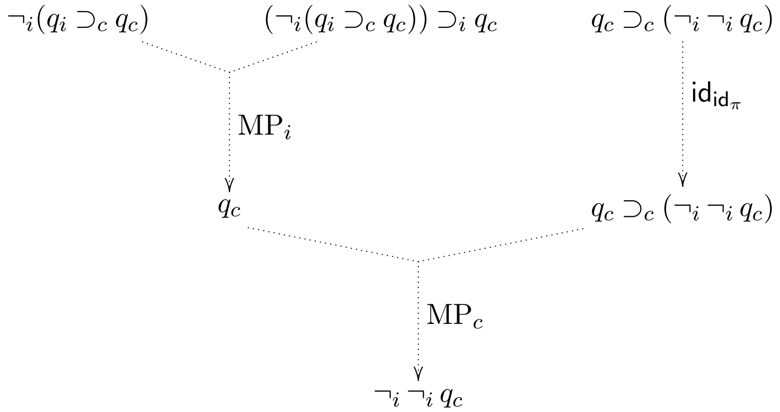

Figure 9: Derivation in the fibring of classical and intuitionistic logics

In order to define derivations we now introduce the notion of deductive system, again as an m-graph but now with formulas as nodes and inference rules as m-edges. Axioms are seen as inference rules as we shall see below.

Putting deductive systems together

According to what we said before we start by enriching a signature with symbols to represent both axioms and inference rules. A *meta-signature* is a tuple

$$\Phi = (\Sigma, \top, \mathsf{R})$$

where $\Sigma = (G, \pi, \diamond)$ is a language signature such that $G^\Phi = (V^\Phi, E^\Phi, \mathsf{src}^\Phi, \mathsf{trg}^\Phi)$ is a m-graph extending G and

- $V^\Phi = V$;
- $E^\Phi = E \cup \mathsf{R}$ where $\mathsf{R} = \{\mathsf{R}_n : \overbrace{\pi \ldots \pi}^{n} \to \pi\}_{n>0}$;

and $\top$ is a set $\{\top^s : s \to \pi\}_{s \in V^+}$. Each R_n is a symbol for representing inference rules with n premises. The edge $\top^s$ is called *s-verum* and is important to represent, in our setting, axioms. An *axiom* is the target of a unary rule whose antecedent is a verum schema formula. We denote by $G_\top$ the m-graph obtained by enriching G with the m-edges $\top^s : s \to \pi$.

A *morphism between meta-signatures* h from $\Phi_1 = (\Sigma_1, \top_1, \mathsf{R}_1)$ to $\Phi_2 = (\Sigma_2, \top_2, \mathsf{R}_2)$ is a m-graph morphism $h : G_\top^{\Phi_1} \to G_\top^{\Phi_2}$ such that its restriction h_{G_1} to G_1 is a signature morphism from Σ_1 to Σ_2, $h^{\mathsf{e}}(\top^s) = \top^{h^{\mathsf{v}}(s)}$ and $h^{\mathsf{e}}((\mathsf{R}_1)_n) = (\mathsf{R}_2)_n$ for each $n > 0$. As seen for interpretation structures we assume that, for fibring, the signatures have the same set V of sorts. In this case a meta-signature morphism h is equivalent to a signature morphism, and in this situation we may confuse h with h_{G_1} and with $h_{G_{1\top}}$.

Our objective now is to say what is a deductive system based on a m-graph. According to the intuition, the nodes are language expressions and there are two kinds of m-edges to consider. One kind is for inference rules and axioms as illustrated in Figure 10 and in Figure 11, respectively.

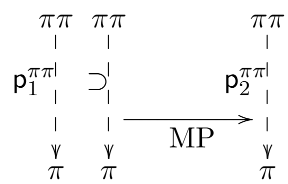

Figure 10: M-edge for Modus Ponens.

As it is well known Modus Ponens (MP) is a binary inference rule. The intuition for the graph description of MP is based on the fact that MP is applied to an implication formula and its antecedent (the first argument of the implication), and allows to conclude the consequent of the implication (its second argument). Thus, MP is an m-edge whose source is a sequence of two morphisms, one for the implication and the other for the first projection. The target of MP is the second projection. In the same vein, an axiom is an m-edge whose source is an appropriate special morphism $\top^s$ and the target is the morphism corresponding to the axiom formula. That is, an axiom is seen as a special case of unary rule (see Figure 11). The second kind of m-edges in a

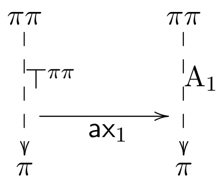

Figure 11: M-edge for axiom ax_1.

deductive system corresponds to the constructors that make formula morphisms to commute, see Figure 12.

A *deductive system over* over a meta-signature Φ is a pair (G'', β) where G'' is such that

- V'' is the set of all expressions over $G_\top$ (that is, the morphisms of $G_\top^+$);

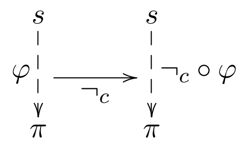

Figure 12: M-edges for a constructor in the deductive system for the fibring of classical and intuitionistic logics.

- $E''(\varphi_1 : s \to v_1 \dots \varphi_n : s \to v_n, \varphi : s \to v)$, for φ in G^+, contains, among others, the m-edges $e : v_1 \dots v_n \to v$ of E such that $\varphi = e \circ \langle \varphi_1, \dots, \varphi_n \rangle$ is in G^+;
- $E''(\varphi_1 : s_1 \to v_1 \dots \varphi_n : s_n \to v_n, \varphi : s \to v) = \emptyset$ whenever φ is not in G^+ or $s_k \neq s$ for some $k = 1, \dots, n$, or φ_k is not in G^+ and $n \neq 1$;

and β is a signature morphism from G'' to G^Φ such that

- $\beta^{\mathsf{v}}(\varphi : s \to v) = v$;
- $\beta^{\mathsf{e}}(e : (\varphi_1 : s \to v_1 \dots \varphi_n : s \to v_n) \to (\varphi : s \to v)) = e$ if e is in E and $\varphi = e \circ \langle \varphi_1, \dots, \varphi_n \rangle$;
- $\beta^{\mathsf{e}}(f') \in \mathsf{R}$ otherwise.

The first clause on E'' imposes the inclusion of the constructor morphisms that when composed with the source expression coincides with the target expression, as exemplified in Figure 12. As stated in the last clause of β^{e} all the other m-edges correspond to inference rules. The m-edges corresponding to inference rules must have as premises and conclusion, expressions with the same source, and with target π. The same source condition is imposed by the second clause of the definition of E'' and is crucial for instantiating a rule in a derivation.

A *deductive system* $\mathcal{D}$ is a triple

$$(\Phi, G'', \beta)$$

such that Φ is a meta-signature and (G'', β) is a deductive system over Φ.

Example 4.1 The deductive system $\mathcal{D}_c = (\Phi_\Pi, G'', \beta)$ for classical propositional logic is as follows:

- Φ_Π is the meta-signature $(\Sigma_\Pi, \top, \mathsf{R})$ where Σ_Π is the propositional signature $(G, \pi, \diamond)$ introduced in Example 2.1;
- G'' has, besides the mandatory m-edges for connectives, the following ones for rules:
  - m-edge $\mathsf{ax}_1 : \top^{\pi\pi} \to (\xi \supset (\xi' \supset \xi))$ where ξ is $\mathsf{p}_1^{\pi\pi}$ and ξ' is $\mathsf{p}_2^{\pi\pi}$, see Figure 11;
  - m-edge $\mathsf{ax}_2 : \top^{\pi\pi\pi} \to ((\xi \supset (\xi' \supset \xi'')) \supset ((\xi \supset \xi') \supset (\xi \supset \xi'')))$ where ξ is $\mathsf{p}_1^{\pi\pi\pi}$, ξ' is $\mathsf{p}_2^{\pi\pi\pi}$ and ξ'' is $\mathsf{p}_3^{\pi\pi\pi}$;

– m-edge $\mathsf{ax}_3 : \top^{\pi\pi} \to (((\neg\,\xi) \supset (\neg\,\xi')) \supset (\xi' \supset \xi))$ where ξ is $\mathsf{p}_1^{\pi\pi}$ and ξ' is $\mathsf{p}_2^{\pi\pi}$;
– m-edge $\mathrm{MP} : \mathsf{p}_1^{\pi\pi}\supset \to \mathsf{p}_2^{\pi\pi}$, see Figure 10;

- $\beta : G'' \to G^{\Phi_\Pi}$ is such that:
  – $\beta^{\mathsf{e}}(\mathsf{ax}_k) = \mathsf{R}_1$ for $k = 1, 2, 3$;
  – $\beta^{\mathsf{e}}(\mathrm{MP}) = \mathsf{R}_2$.

In the sequel, we can denote the target of ax by A. The target of ax_i is the usual axiom of an Hilbert calculus for classical propositional logic but written with the schema variables ξ, ξ' and ξ'', so that they can be instantiated with formulas in the fibred language when applied in a derivation. ∇

Example 4.2 The deductive system $\mathcal{D}_i = (\Phi_\Pi, G'', \beta)$ for intuitionistic propositional logic is as follows:

- Φ_Π is the meta-signature $(\Sigma_\Pi^{\wedge,\vee}, \top, \mathsf{R})$ where $\Sigma_\Pi^{\wedge,\vee}$ is the intuitionistic propositional signature $(G, \pi, \diamond)$ introduced in Example 2.2;
- G'' has, besides the mandatory m-edges for connectives, the following ones:
  – m-edge $\mathsf{ax}_1 : \top^{\pi\pi} \to (\xi \supset (\xi' \supset \xi))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$, see Figure 11;
  – m-edge $\mathsf{ax}_2 : \top^{\pi\pi\pi} \to ((\xi \supset (\xi' \supset \xi'')) \supset ((\xi \supset \xi') \supset (\xi \supset \xi'')))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;
  – m-edge $\mathsf{ax}_3 : \top^{\pi\pi} \to (\xi \supset (\xi' \supset (\xi \wedge \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  – m-edge $\mathsf{ax}_4 : \top^{\pi\pi} \to ((\xi \wedge \xi') \supset \xi)$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  – m-edge $\mathsf{ax}_5 : \top^{\pi\pi} \to ((\xi \wedge \xi') \supset \xi')$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  – m-edge $\mathsf{ax}_6 : \top^{\pi\pi} \to (\xi \supset (\xi \vee \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  – m-edge $\mathsf{ax}_7 : \top^{\pi\pi} \to (\xi' \supset (\xi \vee \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  – m-edge $\mathsf{ax}_8 : \top^{\pi\pi\pi} \to ((\xi \supset \xi'') \supset ((\xi' \supset \xi'') \supset ((\xi \vee \xi') \supset \xi'')))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi\pi}$, ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi\pi}$ and ξ'' is $\widehat{\mathsf{p}}_3^{\pi\pi\pi}$;
  – m-edge $\mathsf{ax}_9 : \top^{\pi\pi} \to ((\xi \supset \xi') \supset ((\xi \supset (\neg\,\xi')) \supset (\neg\,\xi)))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  – m-edge $\mathsf{ax}_{10} : \top^{\pi\pi} \to (\xi \supset ((\neg\,\xi) \supset \xi'))$ where ξ is $\widehat{\mathsf{p}}_1^{\pi\pi}$ and ξ' is $\widehat{\mathsf{p}}_2^{\pi\pi}$;
  – m-edge $\mathrm{MP} : \widehat{\mathsf{p}}_1^{\pi\pi}\supset \to \widehat{\mathsf{p}}_2^{\pi\pi}$, see Figure 10;
- $\beta : G'' \to G^{\Phi_\Pi}$ is such that:
  – $\beta^{\mathsf{e}}(\mathsf{ax}_k) = \mathsf{R}_1$ for $k = 1, \ldots, 10$;
  – $\beta^{\mathsf{e}}(\mathrm{MP}) = \mathsf{R}_2$. ∇

The *fibring of deductive systems* $\mathcal{D}_1 = (\Phi_1, G''_1, \beta_1)$ and $\mathcal{D}_2 = (\Phi_2, G''_2, \beta_2)$ with the same set V of sorts, the same $\top$ and the same R is the deductive system

$$\mathcal{D}_1 \uplus \mathcal{D}_2 = ((\Sigma, \top, \mathsf{R}), G'', \beta)$$

such that

- $\Sigma = (G, \pi, \diamond)$ is the coproduct $\Sigma_1 \uplus \Sigma_2$ in $\mathbf{Sig}_V$ of Σ_1 and Σ_2 with injections i_1 and i_2);
- $G'' = (V'', E'', \mathsf{src}'', \mathsf{trg}'')$ is such that,
  - V'' is the class of morphism of $G^+_\top$ whose target is in V;
  - E'' contains:
    - $e : (\varphi_1 : s \to v_1 \dots \varphi_n : s \to v_n) \to (e \circ \langle \varphi_1, \dots, \varphi_n \rangle : s \to v)$ whenever $e : v_1 \dots v_n \to v_0$ is in E and $\varphi_1, \dots, \varphi_n$ are in G^+;
    - the disjoint union of $(\beta_1{}^{\mathsf{e}})^{-1}(\mathsf{R})$ and $(\beta_2{}^{\mathsf{e}})^{-1}(\mathsf{R})$ such that $\mathsf{src}''(r) = (\mathsf{i}^{\mathsf{e}}_k)^+(\mathsf{src}''_1(r_k))$ and $\mathsf{trg}''(r) = (\mathsf{i}^{\mathsf{e}}_k)^+(\mathsf{trg}''_k(r_k))$, whenever r is the image of $r_k \in (\beta_k{}^{\mathsf{e}})^{-1}(\mathsf{R})$ in the disjoint union, for $k = 1, 2$;
- $\beta^{\mathsf{v}}(\varphi : s \to v) = v$;
- $\beta^{\mathsf{e}}(e : (\varphi_1 : s \to v_1 \dots \varphi_n : s \to v_n) \to (e \circ \langle \varphi_1, \dots, \varphi_n \rangle : s \to v)) = e$ if e is in E and $\varphi_1, \dots, \varphi_n$ are in G^+;
- $\beta^{\mathsf{e}}(r) = \mathsf{R}_n$ if r is the image of $r_k \in (\beta_k{}^{\mathsf{e}})^{-1}(\mathsf{R}_n)$ in the disjoint union, for some $k = 1, 2$.

Example 4.3 The deductive system

$$\mathcal{D}_{c+i}$$

resulting from the fibring of the deductive system $\mathcal{D}_c$ for classical logic introduced in Example 4.1 and the deductive system $\mathcal{D}_i$ for intuitionistic logic introduced in Example 4.2 is the deductive system

$$\mathcal{D}_c \uplus \mathcal{D}_i = (\Phi, G'', \beta)$$

defined as follows:

- Φ is the meta-signature $(\Sigma_c \uplus \Sigma_i, \top, \mathsf{R})$ with injections i_c and i_i;
- G'' has
  - the mandatory m-edges for constructors, renamed according to the appropriate signature morphisms i_c and i_i;
  - m-edge ax_{kc} corresponding to the axiom ax_k in $\mathcal{D}_c$ by appropriately renaming its source and target according to the signature morphism i_c, for $k = 1, 2, 3$;

- m-edge ax_{ki} corresponding to the axiom ax_k in $\mathcal{D}_i$ by appropriately renaming its source and target according to the signature morphism i_i, for $k = 1, \ldots, 8$;
- m-edge MP_c for MP in $\mathcal{D}_c$, where $\supset$ is renamed to $\supset_c$;
- m-edge MP_i for MP in $\mathcal{D}_i$, where $\supset$ is renamed to $\supset_i$.

- $\beta : G'' \to G^\Phi$ is such that:
  - $\beta^{\mathsf{e}}(\mathsf{ax}_{kc}) = \mathsf{R}_1$ for $k = 1, 2, 3$;
  - $\beta^{\mathsf{e}}(\mathsf{ax}_{ki}) = \mathsf{R}_1$ for $k = 1, \ldots, 8$;
  - $\beta^{\mathsf{e}}(\mathrm{MP}_c) = \beta^{\mathsf{e}}(\mathrm{MP}_i) = \mathsf{R}_2$.

A graphical representation of part of this deductive system can be seen in Figure 13. ∇

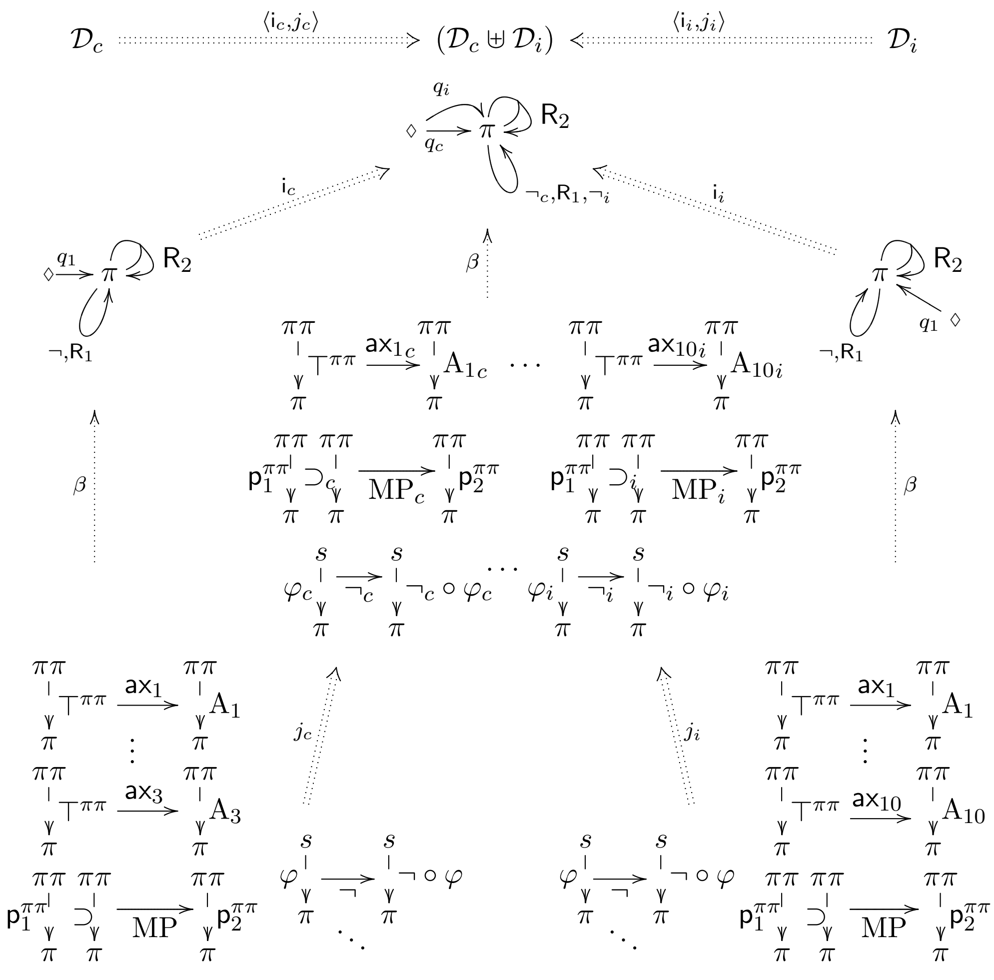

Figure 13: Fibring of deductive systems for classical and intuitionistic logic.

It is clear that the fibring of two deductive systems is related with the components via a morphim.

A *deductive system morphism* from the deductive system (Φ_1, G_1'', β_1) to the deductive system (Φ_2, G_2'', β_2) is a pair (h, g) where h is a meta-signature morphism from Φ_1 to Φ_2 and g is a m-graph morphism from G_1'' to G_2'' such that:

- $g^{\mathsf{v}}(\varphi_1)$ is $(h^{\mathsf{e}})^+(\varphi_1)$ for each node φ_1 of V_1'';
- $h^{\mathsf{v}} \circ \beta_1^{\mathsf{v}} = \beta_2^{\mathsf{v}} \circ g^{\mathsf{v}}$;
- $h^{\mathsf{e}} \circ \beta_1^{\mathsf{e}} = \beta_2^{\mathsf{e}} \circ g^{\mathsf{e}}$.

We will prove later on, on Proposition 4.5, that there are deductive system morphisms $(\mathsf{i}_1, j_1) : \mathcal{D}_1 \to \mathcal{D}_1 \uplus \mathcal{D}_2$ and $(\mathsf{i}_2, j_2) : \mathcal{D}_2 \to \mathcal{D}_1 \uplus \mathcal{D}_2$ relating the deductive systems to the deductive system resulting from their fibring.

Universal constructions

We denote by $\mathbf{MSig}_V$ the category of meta-signatures with the same set V os sorts and their morphisms. We now show that meta-signatures with the same set of sorts can be composed.

Proposition 4.4 Category $\mathbf{MSig}_V$ has binary coproducts.

Proof: Let $(\Sigma_1, \top_1, \mathsf{R}_1)$ and $(\Sigma_2, \top_2, \mathsf{R}_2)$ be meta-signatures denoted by Φ_1 and Φ_2 respectively, with the same set V of sorts. Their coproduct is the triple

$$\Phi_1 \uplus \Phi_2 = ((\Sigma_1 \uplus \Sigma_2, \top, \mathsf{R}), \mathsf{i}_1, \mathsf{i}_2)$$

where $(\Sigma_1 \uplus \Sigma_2, \mathsf{i}_{1G_1}, \mathsf{i}_{2G_2})$ is a coproduct in $\mathbf{Sig}_V$ of Σ_1 and Σ_2. It is straightforward to check that the triple is indeed a coproduct. QED

We say that a formula φ of $G^+_\top$ *is in* G^+ whenever there is a path over $G^\dagger$ such that the corresponding formula is equal to φ. We may denote a schema formula of $G^+_\top$ not in G^+ as a verum schema formula. Given morphisms $\varphi_1 : s \to s_1$ and $\varphi_2 : s_1 \to s_2$ of $G^+_\top$ in G^+, it is straightforward to see that $\varphi_2 \circ \varphi_1$ is also in G^+. Moreover given the morphism $\top^s : s \to \pi$ of $G^+_\top$ it is straightforward to see that for any $\varphi : s \to s_1$ in $G^+_\top$ the morphism $\top^s \circ \varphi$ is also not in G^+.

Let **Ded** be the category of deductive systems and their morphisms and $\mathbf{Ded}_V$ the full subcategory of **Ded** composed by all deductive systems with the same set V of sorts.

Proposition 4.5 Category $\mathbf{Ded}_V$ has coproducts.

Proof: The coproduct of deductive systems $\mathcal{D}_1 = (\Phi_1, G_1'', \beta_1)$ and $\mathcal{D}_2 = (\Phi_2, G_2'', \beta_2)$ in $\mathbf{Ded}_V$ is

$$(\mathcal{D}_1 \uplus \mathcal{D}_2, (\mathsf{i}_1, j_1), (\mathsf{i}_2, j_2))$$

where

- $\mathcal{D}_1 \uplus \mathcal{D}_2$ is the fibring of $\mathcal{D}_1$ and $\mathcal{D}_2$;

- $((\Sigma, \top, \mathsf{R}), \mathsf{i}_1, \mathsf{i}_2)$ is a coproduct in $\mathbf{MSig}_V$ of Φ_1 and Φ_2;
- $j_1^{\mathsf{v}}(\varphi) = (\mathsf{i}_k^{\mathsf{e}})^+(\varphi)$ for each φ in V_k'', for $k = 1, 2$;
- ${j_k}^{\mathsf{e}}(e) = {\mathsf{i}_k}^{\mathsf{e}}(e)$ if e is in E_k, for $k = 1, 2$;
- $({j_k}^{\mathsf{e}})$ restricted to $({\beta_k}^{\mathsf{e}})^{-1}(\mathsf{R})$ is an injection to the coproduct in **Set** of $({\beta_1}^{\mathsf{e}})^{-1}(\mathsf{R})$ and $({\beta_2}^{\mathsf{e}})^{-1}(\mathsf{R})$, for $k = 1, 2$.

It is straightforward to show that (i_1, j_1) and (i_2, j_2) are deductive system morphisms.

Let $\mathcal{D}_3$ be a deductive system in $\mathbf{Ded}_V$ and $(h_1, g_1) : \mathcal{D}_1 \to \mathcal{D}_3$ and $(h_2, g_2) : \mathcal{D}_2 \to \mathcal{D}_3$ deductive system morphisms. Consider the pair (h, g) such that:

- $h = (\mathsf{id}_V, h^{\mathsf{e}})$ is the unique morphism in **MSig** such that $h_1 = h \circ \mathsf{i}_1$ and $h_2 = h \circ \mathsf{i}_2$;
- $g^{\mathsf{v}}(\widehat{w}) = (h^{\mathsf{e}})^+(\widehat{w})$ for each vertex $\widehat{w}$ of V'';
- $g^{\mathsf{e}}({\mathsf{i}_1}^{\mathsf{e}}(e_1)) = {g_1}^{\mathsf{e}}(e_1)$ and $g^{\mathsf{e}}({\mathsf{i}_2}^{\mathsf{e}}(e_2)) = {g_2}^{\mathsf{e}}(e_2)$;
- $g^{\mathsf{e}}({j_1}^{\mathsf{e}}(f_1)) = {g_1}^{\mathsf{e}}(f_1)$ when $f_1 \in ({\beta_1}^{\mathsf{e}})^{-1}(\mathsf{R}_1)$;
- $g^{\mathsf{e}}({j_2}^{\mathsf{e}}(f_2)) = {g_2}^{\mathsf{e}}(f_2)$ for $f_2 \in ({\beta_2}^{\mathsf{e}})^{-1}(\mathsf{R}_2)$.

(1) h is a meta-signature morphism from Φ to Φ_3 by definition.

(2) g is a m-graph morphism from G'' to G_3'' since:

(a) $g^{\mathsf{v}}(\widehat{w}) = (h^{\mathsf{e}})^+(\widehat{w})$ for each object $\widehat{w}$ of G''.

(b) $h^{\mathsf{v}} \circ \beta^{\mathsf{v}} = \beta_3^{\mathsf{v}} \circ g^{\mathsf{v}}$. The thesis follows since $h^{\mathsf{v}}(\beta^{\mathsf{v}}(\widehat{w} : s \to v)) = h^{\mathsf{v}}(v)$ and $\beta_3^{\mathsf{v}}(g^{\mathsf{v}}(\widehat{w})) = \beta_3^{\mathsf{v}}((h^{\mathsf{e}})^+(\widehat{w}) : (h^{\mathsf{v}})^+(s) \to h^{\mathsf{v}}(v)) = h^{\mathsf{v}}(v)$.

(c) $h^{\mathsf{e}} \circ \beta^{\mathsf{e}} = \beta_3^{\mathsf{e}} \circ g^{\mathsf{e}}$. There are two cases to consider: (i) $h^{\mathsf{e}}(\beta^{\mathsf{e}}({\mathsf{i}_1}^{\mathsf{e}}(e_1))) = h^{\mathsf{e}}({\mathsf{i}_1}^{\mathsf{e}}(e_1))$ and $\beta_3^{\mathsf{e}}(g^{\mathsf{e}}({\mathsf{i}_1}^{\mathsf{e}}(e_1))) = \beta_3^{\mathsf{e}}({g_1}^{\mathsf{e}}(e_1)) = {h_1}^{\mathsf{e}}(\beta_1^{\mathsf{e}}(e_1)) = {h_1}^{\mathsf{e}}(e_1)$ and so the result follows by definition of h. Similarly for e_2 in E_2. (ii) Let $r_1 \in ({\beta_1}^{\mathsf{e}})^{-1}(\mathsf{R}_{1n})$. Then $h^{\mathsf{e}}(\beta^{\mathsf{e}}({j_1}^{\mathsf{e}}(r_1))) = h^{\mathsf{e}}(\mathsf{R}_n) = (\mathsf{R}_3)_n$ and $\beta_3^{\mathsf{e}}(g^{\mathsf{e}}({j_1}^{\mathsf{e}}(r_1))) = \beta_3^{\mathsf{e}}({g_1}^{\mathsf{e}}(r_1)) = h_1^{\mathsf{e}}(\beta_1^{\mathsf{e}}(r_1)) = h_1^{\mathsf{e}}(\mathsf{R}_{1n}) = (\mathsf{R}_3)_n$.

(3) $(h, g) \circ (\mathsf{i}_1, j_1) = (h_1, g_1)$. Commutativity for the first argument holds by definition. On the other hand, $g^{\mathsf{v}}({j_1}^{\mathsf{v}}(\widehat{w}_1)) = (h^{\mathsf{e}})^+(j_1^{\mathsf{v}}(\widehat{w}_1)) = (h^{\mathsf{e}})^+((\mathsf{i}_1^{\mathsf{e}})^+(\widehat{w}_1)) = (h_1^{\mathsf{e}})^+(\widehat{w}_1) = g_1^{\mathsf{v}}(\widehat{w}_1)$. Moreover, (i) given e_1 in E_1, $g^{\mathsf{e}}(j_1^{\mathsf{e}}(e_1)) = g^{\mathsf{e}}(\mathsf{i}_1^{\mathsf{e}}(e_1)) = g_1^{\mathsf{e}}(e_1)$; and (ii) given $f_1 \in ({\beta_1}^{\mathsf{e}})^{-1}(\mathsf{R}_1)$, $g^{\mathsf{e}}(j_1^{\mathsf{e}}(f_1)) = {g_1}^{\mathsf{e}}(f_1)$ by definition.

(4) $(h, g) \circ (\mathsf{i}_2, j_2) = (h_2, g_2)$. Similar to case (3).

(5) Uniqueness of (h, g). Let (h', g') be a deductive system morphism from $\mathcal{D}_1 \cup \mathcal{D}_2$ to $\mathcal{D}_3$ such that $(h', g') \circ (\mathsf{i}_1, j_1) = (h_1, g_1)$ and $(h', g') \circ (\mathsf{i}_2, j_2) = \langle h_2, g_1 \rangle$. Then $h'^{\mathsf{v}} = \mathsf{id}_V$. Since $h'^{\mathsf{e}} \circ \mathsf{i}_1 = h^{\mathsf{e}}$ then $h'^{\mathsf{e}} = h^{\mathsf{e}}$. Moreover $g'^{\mathsf{v}}(\widehat{w}) = g^{\mathsf{v}}(\widehat{w})$ since $g'^{\mathsf{v}}(\widehat{w}) = (h'^{\mathsf{e}})^+(\widehat{w}) = (h^{\mathsf{e}})^+(\widehat{w}) = g^{\mathsf{v}}(\widehat{w})$ by definition of deductive system morphism. Let e_1 be in E_1. Then $g'^{\mathsf{e}}(j_1^{\mathsf{e}}(e_1)) = g_1^{\mathsf{e}}(e_1) = g^{\mathsf{e}}({\mathsf{i}_1}^{\mathsf{e}}(e_1)) = g^{\mathsf{e}}({j_1}^{\mathsf{e}}(e_1))$. Let $f_1 \in ({\beta_1}^{\mathsf{e}})^{-1}(\mathsf{R}_1)$. Then $g'^{\mathsf{e}}({j_1}^{\mathsf{e}}(f_1)) = {g_1}^{\mathsf{e}}(f_1) = g^{\mathsf{e}}({j_1}^{\mathsf{e}}(f_1))$. QED

As we did for signatures and interpretation structures, we only consider the coproduct of deductive systems over signatures with the same set of sorts. So, in order to define the unconstrained fibring of deductive systems over signatures with different sets of sorts, we have to enrich each deductive system with the sorts of the other system. For that we define a $\mathbf{Set}^{\Diamond}$ indexed map $\cdot^{(V_1,\pi_1,\Diamond_1)}$ on deductive systems as follows:

- $((\Sigma_2, \top_2, \mathsf{R}_2), G''_2, \beta_2)^{(V_1,\pi_1,\Diamond_1)} = ((\Sigma_2{}^{(V_1,\pi_1,\Diamond_1)}, \top_2, \mathsf{R}_2), G'', (\beta^{\mathsf{v}}, \beta^{\mathsf{e}}_2))$ with:
  - $\Sigma_2^{(V_1,\pi_1,\Diamond_1)} = (G, \pi, \diamond)$;
  - $G'' = (V'', E''_2, \mathsf{src}'', \mathsf{trg}'')$ is such that
    - V'' is the class of morphisms of $G^+_\top$ whose target is in V;
    - $\mathsf{src}'' : E''_2 \to V''^{+}$, $\mathsf{trg}'' : E''_2 \to V''$ coincide with src''_2 and trg''_2, respectively;
    - $\beta^{\mathsf{v}} : V'' \to V$ is such that $\beta^{\mathsf{v}}(\widehat{w} : s \to v) = v$.

The fibring $\mathsf{Fib}_{\mathbf{Ded}}$ of deductive systems is a map from $|\mathbf{Ded}|^2$ to $|\mathbf{Ded}|$ such that:

$$\mathsf{Fib}_{\mathbf{Ded}}(\mathcal{D}_1, \mathcal{D}_2) = \mathcal{D}_1{}^{(V_2,\pi_2,\Diamond_2)} \uplus \mathcal{D}_2{}^{(V_1,\pi_1,\Diamond_1)}.$$

Interleaving of derivations

Revisiting the motivating example of this section, assume that we want to show that $\neg_i \neg_i q_c$ follows from $\{\neg_i(q_i \supset_c q_c), (\neg_i(q_i \supset_c q_c) \supset_i q_c, q_c \supset_c (\neg_i \neg_i q_c)\}$ in $\mathcal{D}_{c+i}$. This means that we have to provide a derivation whose conclusion is $\neg_i \neg_i q_c$ and whose hypotheses are elements of the set $\{\neg_i(q_i \supset_c q_c), (\neg_i(q_i \supset_c q_c) \supset_i q_c, q_c \supset_c (\neg_i \neg_i q_c)\}$. Herein a derivation is described by two sequences separated by a semi-colon. A sequence of steps where each step corresponds to the application of either one inference rule or several rules (putted together by the operator $\otimes$) to different formulas, and the sequence of the formulas used but not generated in the derivation. For instance, the derivation depicted in Figure 9 is described by:

$$\mathrm{MP}_c, (\mathrm{MP}_i \otimes \mathsf{id}_{\mathsf{id}_\pi})\,;\, \neg_i(q_i \supset_c q_c),\ (\neg_i(q_i \supset_c q_c) \supset_i q_c,\ q_c \supset_c (\neg_i \neg_i q_c),$$

and consists of two steps, MP_c and $(\mathrm{MP}_i \otimes \mathsf{id}_{\mathsf{id}_\pi})$, applied to the sequence of formulas $\neg_i(q_i \supset_c q_c)$, $(\neg_i(q_i \supset_c q_c) \supset_i q_c$, $q_c \supset_c (\neg_i \neg_i q_c)$. Observe that when a formula is not used by an inference rule in a step the $\mathsf{id}_{\mathsf{id}_\pi}$ morphism is used instead.

The formulas used but not generated in a derivation can be either hypotheses or axioms. For instance,

$$\mathrm{MP}_c,\ (\mathsf{id}_{\mathsf{id}_\pi} \otimes \mathsf{ax}_{1c})\,;\, q_i, q_i \supset_c (q_c \supset_c q_i)$$

is a derivation in $\mathcal{D}_{c+i}$, depicted in Figure 14, for $q_c \supset_c q_i$ from $\{q_i\}$. Recall that axioms are seen as unary inference rules having as source a verum morphism, that is, a morphism involving the $\top$ morphism.

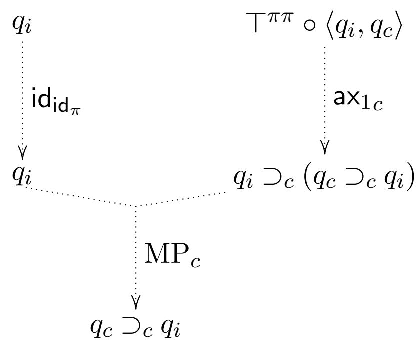

Figure 14: Derivation of $q_c \supset_c q_i$ from $\{q_i\}$.

For more details on the notion of derivation in a graph-theoretic account of logics the reader can consult [20] (namely for notions like instantiation of rules, instantiation of derivation steps, derivation as a morphism, and the categorical context in which derivations are set up).

Given a set Γ of formulas and a sequence of formulas $\vec{\varphi}$, we say that $\vec{\varphi}$ *is derived from* Γ in the deductive system $\mathcal{D}$, written

$$\Gamma \vdash_{\mathcal{D}} \vec{\varphi}$$

whenever there is a derivation $l_1, \ldots, l_n; \vec{\psi}$ of $\vec{\varphi}$ such that the formulas in $\vec{\psi}$ that are not premises of axioms are in Γ. For instance, from the examples above we can conclude that

$$\{\neg_i(q_i \supset_c q_c), (\neg_i(q_i \supset_c q_c)) \supset_i q_c, q_c \supset_c (\neg_i\neg_i q_c)\} \vdash_{\mathcal{D}_{c+i}} \neg_i\neg_i q_c$$

and

$$\{q_i\} \vdash_{\mathcal{D}_{c+i}} q_c \supset_c q_i.$$

5 Fibring of logic systems

A logic system has three components: the signature, the interpretation system and the deduction system. All of them are defined in terms of m-graphs. The language and derivation are described in the induced categories. A *logic system* is a triple $\mathcal{L} = (\Sigma, \mathcal{I}, \mathcal{D})$ such that $\mathcal{I} = (\Sigma, \mathfrak{I})$ is an interpretation system and $\mathcal{D} = (\Phi, G'', \beta)$ is a deductive system; where Φ is a meta-signature over Σ. Given an interpretation system $\mathcal{I} = (\Sigma, \mathfrak{I})$ and a set $\Gamma \cup \{\varphi\}$ of schema formulas over Σ, we say that Γ *entails* φ in $\mathcal{I}$, written as

$$\Gamma \vDash_{\mathcal{I}} \varphi,$$

whenever $I \Vdash \Gamma$ implies $I \Vdash \varphi$ for every I in $\mathfrak{I}$. Similarly, entailment over sequences of schema formulas can be defined. The logic system $\mathcal{L}$ is said to be *sound* if

$$\Gamma \vDash_{\mathcal{I}} \varphi \text{ whenever } \Gamma \vdash_{\mathcal{D}} \varphi$$

for a set $\Gamma \cup \{\varphi\}$ of formulas and is said to be *strong complete* if the converse holds. The logic system $\mathcal{L}$ is said to be *weak complete* if

$$\vdash_{\mathcal{I}} \varphi \text{ whenever } \vDash_{\mathcal{D}} \varphi.$$

Observe that a strong complete logic system is also weak complete but not the other way around in general. Finally, an interpretation structure I in $\mathfrak{I}$ is said to be *sound for a rule r in $\mathcal{D}$* if

$$I, \rho \Vdash \mathsf{CONC}(r) \text{ whenever } I, \rho \Vdash \mathsf{proper}(\mathsf{ANT}(r))$$

for every assignment ρ over I, where the map $\mathsf{proper}(\cdot)$ when applied to a sequence $\vec{\varphi}$ of schema formulas in $G^+_\top$ returns the subsequence of schema formulas that are in G^+. When I is sound for all the rules in $\mathcal{D}$ we say that I is *sound for $\mathcal{D}$*. These schema formulas are called proper. The logic system $\mathcal{L}$ is said to be *sound for a deductive rule* r in $\mathcal{D}$, if all its interpretation structures over its signature are sound for r.

Example 5.1 The tuple $(\Sigma_\Pi, \mathcal{I}_c, \mathcal{D}_c)$ where

- Σ_Π is the signature for classical logic introduced in Example 2.1;
- $\mathfrak{I}_c$ is the class of all interpretation structures over Σ_Π sound for $\mathcal{D}_c$;
- $\mathcal{D}_c$ is the deductive system for classical logic introduced in Example 4.1;

is a logic system, denoted by $\mathcal{L}_c$, for classical propositional logic. Note that the interpretation structure I_c for classical logic introduced in Example 3.1 is sound for $\mathcal{D}_c$ as it is straightforward to show. So I_c is in $\mathfrak{I}_c$. ∇

Example 5.2 The tuple $(\Sigma_\Pi^{\wedge,\vee}, \mathcal{I}_i, \mathcal{D}_i)$ where

- $\Sigma_\Pi^{\wedge,\vee}$ is the signature intuitionistic logic introduced in Example 2.2;
- $\mathfrak{I}_i$ is the class of all interpretation structures over $\Sigma_\Pi^{\wedge,\vee}$ sound for $\mathcal{D}_i$;
- $\mathcal{D}_i$ is the intuitionistic deductive system for introduced in Example 4.2

is a logic system, denoted by $\mathcal{L}_i$, for intuitionistic propositional logic. Note that the interpretation structure I_i for intuitionistic logic introduced in Example 3.2 is sound for $\mathcal{D}_i$ as it is straightforward to show. So I_i is in $\mathfrak{I}_i$. ∇

We denote by **Log** the class of all logic systems. The fibring Fib of logic systems is a map from $\mathbf{Log}^2$ to $\mathbf{Log}$ such that:

$$\mathsf{Fib}(\mathcal{L}_1, \mathcal{L}_2) = (\Sigma, \mathcal{I}, \mathcal{D})$$

where

- $\Sigma = \mathsf{Fib}_{\mathbf{Sig}}(\Sigma_1, \Sigma_2)$;
- $\mathfrak{I}$ is composed by the interpretation structures I over Σ such that the pair $(\Sigma, I) = \mathsf{Fib}_{\mathbf{Int}}((\Sigma_1, I_1), (\Sigma_2, I_2))$ for each $I_1 \in \mathfrak{I}_1$ and $I_2 \in \mathfrak{I}_2$;

- $\mathcal{D} = \mathsf{Fib}_{\mathbf{Ded}}(\mathcal{D}_1, \mathcal{D}_2)$.

We now explain why there is no collapse, semantically, in the logic system resulting from the graph-theoretic fibring of the logic systems for classical and intuitionistic logics presented above.

Example 5.3 The fibring of the logic system $\mathcal{L}_c$, described in Example 5.1 and the logic system $\mathcal{L}_i$, described in Example 5.2, is the logic system $\mathcal{L}_{c+i} = (\Sigma_{c+i}, \mathcal{I}_{c+i}, \mathcal{D}_{c+i})$ where

- $\Sigma_{c+i} = \mathsf{Fib}_{\mathbf{Sig}}(\Sigma_\Pi, \Sigma_\Pi^{\wedge,\vee})$;
- $\mathfrak{I}_{c+i}$ is composed by the interpretation structures I over Σ_{c+i} such that $(\Sigma_{c+i}, I) = \mathsf{Fib}_{\mathbf{Int}}((\Sigma_\Pi, I_1), (\Sigma_\Pi^{\wedge,\vee}, I_2))$ for each $I_1 \in \mathfrak{I}_c$ and $I_2 \in \mathfrak{I}_i$;
- $\mathcal{D}_{c+i} = \mathsf{Fib}_{\mathbf{Ded}}(\mathcal{D}_c, \mathcal{D}_i)$;

and so the interpretation structure described in Example 8 resulting from the fibring of interpretation structures for classical and intuitionistic logic is also in $\mathfrak{I}_{c+i}$. Hence, taking into account Example 3.8, $\not\models_{\mathcal{I}_{c+i}} (\neg_i(\neg_i\, q_i)) \supset_i q_i$. ∇

6 Soundness preservation

We start by investigating a property of the interpretation structure morphism that is important in the proof of the preservation of soundness. In particular, the injection morphisms from the components to the fibring enjoy this property.

An interpretation structure morphism $(h,\tau) : (\Sigma_1, I_1) \to (\Sigma_2, I_2)$ is said to be *non-creative* whenever for each e_1 in E_1 and $e'_2 : s'_2 \to t'_2$ in $E'_{2h(e_1)}$ there is e'_1 in E'_{1e_1} such that $\tau^{\mathsf{e}}_{s'_2 t'_2}(e'_1) = e'_2$. Moreover, given an interpretation structure morphism $(h,\tau) : (\Sigma_1, I_1) \to (\Sigma_2, I_2)$ and an assignment ρ_2 over I_2, we denote by

$$\rho_2{}^{(h,\tau)}$$

the assignment over I_1 such that $(\rho_2{}^{(h,\tau)})_{s_1} = \tau^{\mathsf{v}}((\rho_2)_{h^{\mathsf{v}}(s_1)})$.

The next result relates denotations in different interpretation structures related by a non-creative interpretation structure morphism.

Lemma 6.1 Given a non-creative interpretation structure morphism $(h,\tau) : (\Sigma_1, I_1) \to (\Sigma_2, I_2)$, a morphism $\widehat{w}_1$ of $G_1{}^+$, and an assignment ρ_2 over I_2,

$$[\![\widehat{w}_1]\!]^{I_1\rho_2{}^{(h,\tau)}} = \tau^{\mathsf{v}}([\![h^+(\widehat{w}_1)]\!]^{I_2\rho_2}).$$

Proof: We show equivalently that $[\![w_1]\!]^{I_1\rho_2{}^{(h,\tau)}} = \tau^{\mathsf{v}}([\![h^\dagger(w_1)]\!]^{I_2\rho_2})$ by induction on the complexity of w_1:

- w_1 is ϵ_{s_1}. Then $[\![w_1]\!]^{I_1\rho_2{}^{(h,\tau)}} = (\rho_2{}^{(h,\tau)})_{s_1} = \tau^{\mathsf{v}}((\rho_2)_{h^{\mathsf{v}}(s_1)}) = \tau^{\mathsf{v}}([\![\epsilon_{h^{\mathsf{v}}(s_1)}]\!]^{I_2\rho_2}) = \tau^{\mathsf{v}}([\![h^\dagger(w_1)]\!]^{I_2\rho_2})$.

- w_1 is $\mathsf{p}_i^{v_1\dots v_n} w_{10}$. Then $[\![w_1]\!]^{I_1\rho_2{}^{(h,\tau)}} = [\![\mathsf{p}_i^{v_1\dots v_n} w_{10}]\!]^{I_1\rho_2{}^{(h,\tau)}} = ([\![w_{10}]\!]^{I_1\rho_2{}^{(h,\tau)}})_i =$

$(\tau^{\mathsf{v}}([\![h^\dagger(w_{10})]\!]^{I_2\rho_2}))_i = \tau^{\mathsf{v}}(([\![h^\dagger(w_{10})]\!]^{I_2\rho_2})_i) = \tau^{\mathsf{v}}([\![\mathsf{p}_i^{h^{\mathsf{v}}(v_1)\ldots h^{\mathsf{v}}(v_n)} h^\dagger(w_{10})]\!]^{I_2\rho_2}) = \tau^{\mathsf{v}}(\ [\![h^\dagger(\mathsf{p}_i^{v_1\ldots v_n} w_{10})]\!]^{I_2\rho_2})$;

- w_1 is $\langle w_{11}, \ldots, w_{1n}\rangle w_{10}$. Then $[\![w_1]\!]^{I_1\rho_2{}^{(h,\tau)}} = [\![\langle w_{11}, \ldots, w_{1n}\rangle w_{10}]\!]^{I_1\rho_2{}^{(h,\tau)}} = [\![w_{11}w_{10}]\!]^{I_1\rho_2{}^{(h,\tau)}} \ldots [\![w_{1n}w_{10}]\!]^{I_1\rho_2{}^{(h,\tau)}}$ which, by induction hypothesis, is equal to $\tau^{\mathsf{v}}([\![h^\dagger(w_{11}w_{10})]\!]^{I_2\rho_2}) \ldots \tau^{\mathsf{v}}([\![h^\dagger(w_{1n}w_{10})]\!]^{I_2\rho_2})$ equal to the set $\tau^{\mathsf{v}}([\![h^\dagger(w_{11}w_{10}]\!]^{I_2\rho_2} \ldots [\![h^\dagger(w_{1n}w_{10})]\!]^{I_2\rho_2}) = \tau^{\mathsf{v}}([\![\langle h^\dagger(w_{11}), \ldots, h^\dagger(w_{1n})\rangle h^\dagger(w_{10})]\!]^{I_2\rho_2})$ as we wanted to show;

- w_1 is ew_{10}. So $[\![w_1]\!]^{I_1\rho_2{}^{(h,\tau)}} = [\![ew_{10}]\!]^{I_1\rho_2{}^{(h,\tau)}} = \mathsf{trg}_1'(E_{1e}'([\![w_{10}]\!]^{I_1\rho_2{}^{(h,\tau)}}, -)) = \mathsf{trg}_1'(E_{1e}'(\tau^{\mathsf{v}}([\![h^\dagger(w_{10})]\!]^{I_2\rho_2}), -)) = \tau^{\mathsf{v}}(\mathsf{trg}_2'(E_{2h(e)}'([\![h^\dagger(w_{10})]\!]^{I_2\rho_2}, -)))$ since (h, τ) is non-creative, which is $\tau^{\mathsf{v}}([\![h^\dagger(ew_{10})]\!]^{I_2\rho_2})$. QED

The relationship established for denotations can be extended to satisfaction in a similar way.

Proposition 6.2 Given a non-creative interpretation structure morphism $(h, \tau) : (\Sigma_1, I_1) \to (\Sigma_2, I_2)$, a morphism φ_1 in $G_1{}^+$, and an assignment ρ_2 over I_2,

$$I_2, \rho_2 \Vdash h^+(\varphi_1) \text{ whenever } I_1, \rho_2{}^{(h,\tau)} \Vdash \varphi_1.$$

Proof: Assume that $I_1, \rho_2{}^{(h,\tau)} \Vdash \varphi_1$. Henceforth $[\![\varphi_1]\!]^{I_1\rho_2{}^{(h,\tau)}} \subseteq D_1$. Then $(\tau^{\mathsf{v}})^{-1}([\![\varphi_1]\!]^{I_1\rho_2{}^{(h,\tau)}}) \subseteq D_2$ since (h, τ) is an interpretation structure morphism and so $(\tau^{\mathsf{v}})^{-1}(D_1) \subseteq D_2$. So, by Lemma 6.1, $(\tau^{\mathsf{v}})^{-1}(\tau^{\mathsf{v}}([\![h^+(\varphi_1)]\!]^{I_2\rho_2})) \subseteq D_2$. Hence $[\![h^+(\varphi_1)]\!]^{I_2\rho_2} \subseteq D_2$ and so $I_2, \rho_2 \Vdash h^+(\varphi_1)$. QED

The next result states the converse of the previous proposition under different conditions.

Proposition 6.3 Given a interpretation structure morphism $(h, \tau) : (\Sigma_1, I_1) \to (\Sigma_2, I_2)$ such that $\tau^{\mathsf{v}}(D_2) \subseteq D_1$, a schema formula φ_1 in $G_1{}^+$, and an assignment ρ_2 over I_2,

$$I_1, \rho_2{}^{(h,\tau)} \Vdash \varphi_1 \text{ whenever } I_2, \rho_2 \Vdash h^+(\varphi_1).$$

Proof: Assume that $I_2, \rho_2 \Vdash h^+(\varphi_1)$. Henceforth $[\![h^+(\varphi_1)]\!]^{I_2\rho_2} \subseteq D_2$. Then $\tau^{\mathsf{v}}([\![h^+(\varphi_1)]\!]^{I_2\rho_2}) \subseteq D_1$ by hypothesis. Hence $[\![\varphi_1]\!]^{I_1\rho_2{}^{(h,\tau)}} \subseteq D_1$ by Lemma 6.1, and so $I_1, \rho_2{}^{(h,\tau)} \Vdash \varphi_1$. QED

Recall from [20] the result stating that: *A logic system is sound if it is sound for its deductive rules.*

Theorem 6.4 Soundness is preserved by the fibring of logic systems sound for its rules.

Proof: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be logic systems sound for the rules. Note that the rules of $\mathsf{Fib}_{\mathbf{Ded}}(\mathcal{D}_1, \mathcal{D}_2)$ are the image by the respective signature morphisms of the rules in $\mathcal{D}_1$ and in $\mathcal{D}_2$, respectively. We show that $\mathsf{Fib}(\mathcal{L}_1, \mathcal{L}_2)$ is sound for its inference rules. Let $(\Sigma, I) = (\Sigma_1, I_1)^{(V_2,\pi_2,\diamond_2)} \uplus (\Sigma_2, I_2)^{(V_1,\pi_1,\diamond_1)}$ be an interpretation structure in the fibring and ρ an assignment over I. Let r_1 be a rule in $\mathcal{D}_1$. Assume that $I, \rho \Vdash \mathsf{i_1}^+(\mathsf{proper}(\mathsf{ANT}(r_1)))$. Then, by Proposition 6.3, ${I_1}^{(V_2,\pi_2,\Diamond_2)}, \rho^{(\mathsf{i}_1,\tau_1)} \Vdash \mathsf{proper}(\mathsf{ANT}(r_1))$. Hence, $I_1, \rho^{(\mathsf{i}_1,\tau_1)} \Vdash \mathsf{proper}(\mathsf{ANT}(r_1))$ by Proposition 3.7 and so, since $\mathcal{L}_1$ is sound for its rules, $I_1, \rho^{(\mathsf{i}_1,\tau_1)} \Vdash \mathsf{CONC}(r_1)$. Again by Proposition 3.7, ${I_1}^{(V_2,\pi_2,\Diamond_2)}, \rho^{(\mathsf{i}_1,\tau_1)} \Vdash \mathsf{CONC}(r_1)$. Finally, by Proposition 6.2, $I, \rho \Vdash \mathsf{i_1}^+(\mathsf{CONC}(r_1))$ since the morphism from $(\Sigma_1, I_1)^{(V_2,\pi_2,\Diamond_2)}$ to (Σ, I) is non-creative. The proof for a rule in $\mathcal{D}_2$ follows a similar reasoning so we omit it. QED

Basically, Theorem 6.4 says that soundness is in almost all cases preserved by fibring, since normally a logic system does not contain interpretation structures that are not sound for its rules. Soundness is useful for establishing the non-collapse of intuitionistic into classic connectives, deductively, in $\mathcal{L}_{c+i}$. Recall that in Example 5.3 we already concluded that there is not, semantically, such a collapse in $\mathcal{L}_{c+i}$.

Example 6.5 The logic system $\mathcal{L}_{c+i}$ presented in Example 5.3 resulting from the fibring of the logic system $\mathcal{L}_c$ for classical propositional logic and of the logic system $\mathcal{L}_i$ for intuitionistic propositional logic is sound. So, $\not\vdash_{\mathcal{D}_{c+i}} (\neg_i(\neg_i\, q_i)) \supset_i q_i$ since $\not\vDash_{\mathcal{I}_{c+i}} (\neg_i(\neg_i\, q_i)) \supset_i q_i$. ∇

Taking into account Theorem 6.4, it is straightforward to see that the fibring of the logic systems composed by the deductive systems described in [20] and with all the interpretation structures sound for the rules, is sound (the case of the relevance logic has to be worked further due to the different notion of derivation). Moreover, it is also straightforward to see that there is also no collapse of the entailment and of the consequence relations of one system into another.

7 Completeness preservation

The goal of this section is to investigate preservation of completeness. Preservation of completeness means that the logic system resulting from the fibring is complete whenever the component logic systems are complete. Usually completeness is not preserved unless we impose more properties on the components of the logic systems at hand. That is, in most cases, it is not possible to prove completeness preservation for all possible complete component logic systems. We start by defining the canonical interpretation structure induced by a deductive system.

Canonical structure

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be deductive systems with the same set of sorts, and Γ a set of formulas in G^+ where G is $G_1 \uplus G_2$. The *canonical structure* $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$ *generated by* $\mathcal{D}_1$ *over* $\mathcal{D}_2$ *and* Γ, is such that:

- $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) = (\Sigma_1, (G'_1, \alpha_1, D_1, \blacklozenge_1))$ where

– $G'_1 = \langle V'_1, E'_1, \mathsf{src}'_1, \mathsf{trg}'_1 \rangle$ is such that
  * V'_1 are the morphisms of G^+ whose target is an element of V;
  * $E'_1(\widehat{w}_1 \ldots \widehat{w}_n, \widehat{w})$ is composed by all the m-edges e in E_1 such that $\widehat{w} = \widehat{e} \circ \langle \widehat{w}_1, \ldots, \widehat{w}_n \rangle$ is in G^+;

– $\alpha_1^{\mathsf{v}}(e : s \to v) = v$ and $\alpha_1^{\mathsf{e}}(c) = c$;

– $D_1 = \{\widehat{w} \in V'_1 : \Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \widehat{w}\}$;

– $\circ_1$ is the morphism $\mathsf{id}_\diamond$ in G^+.

In the sequel we will write $S_{\mathcal{D}_2}(\mathcal{D}_1)$ for $S^{\emptyset}_{\mathcal{D}_2}(\mathcal{D}_1)$, and will assume that G means the m-graph $G_1 \uplus G_2$ (the disjoint union of the two m-graphs). We now show that the $S_{\mathcal{D}_2}(\mathcal{D}_1)$ is sound for the deductive rules in $\mathcal{D}_1$.

Lemma 7.1 Given deductive systems $\mathcal{D}_1$ and $\mathcal{D}_2$ with the same set of sorts, a set Γ of formulas in G^+, a path $w : s \to t$ over $G^\dagger$, and an assignment ρ over $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$, $[\![w]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho} = \widehat{w} \circ \rho_s$.

Proof: The proof follows by induction on the complexity of w:

- w is ϵ_s. Then $[\![w]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho} = \rho_s = \mathsf{id}_s \circ \rho_s = \widehat{\epsilon}_s \circ \rho_s = \widehat{w} \circ \rho_s$;

- w is $\mathsf{p}_i^{s_1} w_1$. Then $[\![w]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho} = [\![\mathsf{p}_i^{s_1} w_1]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho} = ([\![w_1]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho})_i = (\widehat{w}_1 \circ \rho_s)_i = \widehat{\mathsf{p}}_i^{s_1} \circ \widehat{w_1} \circ \rho_s = \widehat{w} \circ \rho_s$;

- w is $\langle w_1, \ldots, w_n \rangle w_0$. Hence $[\![w]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho} = [\![w_1 w_0]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho} \ldots [\![w_n w_0]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho} = (\widehat{w}_1 \circ \widehat{w}_0 \circ \rho_s) \ldots (\widehat{w}_n \circ \widehat{w}_0 \circ \rho_s) = \widehat{\langle w_1, \ldots, w_n \rangle} \circ \widehat{w}_0 \circ \widehat{\rho}_s$ as we wanted to show;

- w is $e w_1$. Therefore $[\![w]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho} = \mathsf{trg}'(E'_e([\![w_1]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho}, -)) = \mathsf{trg}'(E'_e(\widehat{w}_1 \circ \rho_s, -)) = \widehat{e} \circ \widehat{w}_1 \circ \rho_s = \widehat{w} \circ \rho_s$. QED

The importance of the canonical structure is that in the context of this structure it is possible to relate satisfaction with derivation.

Lemma 7.2 Given deductive systems $\mathcal{D}_1$ and $\mathcal{D}_2$ with the same set of sorts, and a set $\Gamma \cup \{\varphi : s \to \pi\}$ of schema formulas in G^+, $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \varphi \circ \rho_s$ if and only if $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1), \rho \Vdash \varphi$, for every assignment ρ over $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$.

Proof: Let ρ be an assignment over $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$. Then $\Gamma \vdash_{\mathcal{D}} \varphi \circ \rho_s$ if and only if, by Lemma 7.1, $\Gamma \vdash_{\mathcal{D}} [\![\varphi]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho}$ iff $[\![\varphi]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1),\rho} \subseteq D$ iff $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1), \rho \Vdash \varphi$. QED

We now state a useful lemma that when the proper premises of a rule are derivable, then the conclusion of the rule is also derivable. We omit its proof since it constitutes a particular case of Lemma 6.6 in [20].

Lemma 7.3 For every deductive rule r in $\mathcal{D}_1$, set of formulas Γ and expression $\widehat{u}$ in G^+, $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \mathsf{CONC}(r) \circ \widehat{u}$ whenever $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \mathsf{proper}(\mathsf{ANT}(r)) \circ \widehat{u}$.

In the following two propositions we show that $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$ is sound for the rules in $\mathcal{D}_1$.

Proposition 7.4 For every deductive rule r in $\mathcal{D}_1$, set of formulas Γ in G^+, and assignment ρ over $S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1)$, $S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1), \rho \Vdash \mathsf{CONC}(r)$ whenever $S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1), \rho \Vdash \mathsf{proper}(\mathsf{ANT}(r))$.

Proof: Let $\mathsf{proper}(\mathsf{ANT}(r))$ be $\varphi_1 \dots \varphi_n$. Assume $S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1), \rho \Vdash \mathsf{proper}(\mathsf{ANT}(r))$. Then $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \varphi_i \circ \rho_s$, by Lemma 7.2, for $i = 1, \dots, n$. As a consequence, $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \mathsf{CONC}(r) \circ \rho_s$, by Lemma 7.3, and so, $S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1), \rho \Vdash \mathsf{CONC}(r)$, by Lemma 7.2. QED

Weak completeness

We start by investigating preservation of weak completeness (that is, no hypotheses are present). With this purpose in mind, we need to better understand the denotation of a formula in the fibring of two canonical structures.

Lemma 7.5 Given deductive systems $\mathcal{D}_1$ and $\mathcal{D}_2$ sharing the same set of sorts V, a set Γ of formulas in G^+ and a concrete path w over $G^\dagger$, $(\widehat{w}, \widehat{w}) \in [\![w]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)}$ if $\mathsf{trg}^\dagger(w)$ is an element of V.

Proof: The proof follows by induction on the complexity of w:

- w is $\epsilon_\Diamond$. The thesis follows since $(\widehat{w}, \widehat{w})$ is $(\mathsf{id}_\Diamond, \mathsf{id}_\Diamond)$ and $[\![w]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)} = \rho_\Diamond = \{\blacklozenge\} = \{(\blacklozenge_1, \blacklozenge_2)\} = \{(\mathsf{id}_\Diamond, \mathsf{id}_\Diamond)\}$;

- w is $\mathsf{p}^{s_1}_i w_1$. Note that w_1 is not $\epsilon_\Diamond$, since otherwise s_1 is $\diamond$, which is not possible since the source of a projection has length at least 2. Hence, the path w_1 ends in a tuple as it is straightforward to see. So, assume that w_1 is $\langle w_{11}, \dots, w_{1n}\rangle w_0$. Then $(\widehat{w}, \widehat{w}) = (\widehat{\mathsf{p}}^{s_1}_i \circ \widehat{w}_1, \widehat{\mathsf{p}}^{s_1}_i \circ \widehat{w}_1) = (\widehat{\mathsf{p}}^{s_1}_i \circ \widehat{\langle w_{11}, \dots, w_{1n}\rangle} \circ \widehat{w}_0, \widehat{\mathsf{p}}^{s_1}_i \circ \widehat{\langle w_{11}, \dots, w_{1n}\rangle} \circ \widehat{w}_0) = (\widehat{w}_{1i} \circ \widehat{w}_0, \widehat{w}_{1i} \circ \widehat{w}_0)$ and $[\![w]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)} = [\![\mathsf{p}^{s_1}_i w_1]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)} = ([\![w_1]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)})_i = ([\![w_{11} w_0]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)} \dots [\![w_{1n} w_0]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)})_i = [\![w_{1i} w_0]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)}$ and so the thesis follows by induction hypothesis;

- w is $e w_1$. Observe that $[\![w]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)} = \mathsf{trg}'(E'_e([\![w_1]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)}, -))$. The thesis follows since $\mathsf{trg}'(E'_e((\widehat{w}_1, \widehat{w}_1), -)) \subseteq \mathsf{trg}'(E'_e([\![w_1]\!]^{S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^\Gamma_{\mathcal{D}_1}(\mathcal{D}_2)}, -))$ by induction hypothesis, and $(\widehat{w}, \widehat{w}) = (\widehat{e} \circ \widehat{w}_1, \widehat{e} \circ \widehat{w}_1) \in \mathsf{trg}'(E'_e((\widehat{w}_1, \widehat{w}_1), -))$. QED

Recall from [20], the notion of representative which plays an important role in the proof of completeness for a single logic system. A logic system contains *a representative of the canonical structure over a set* Γ when it contains an interpretation structure I_Γ such that

- $I_\Gamma \Vdash \varphi$ implies $S^\Gamma(\mathcal{D}) \Vdash \varphi$;
- $I_\Gamma \Vdash \Gamma$;

for every formula φ and set of formulas Γ in G^+.

The next result shows that $S_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S_{\mathcal{D}_1}(\mathcal{D}_2)$ is a representative of the canonical structure for the fibring of deductive systems $\mathcal{D}_1$ and $\mathcal{D}_2$.

Proposition 7.6 Given deductive systems $\mathcal{D}_1$ and $\mathcal{D}_2$ sharing the same set of sorts and a formula φ of formulas in $L(\Sigma_1 \uplus \Sigma_2)$, then

$$S_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S_{\mathcal{D}_1}(\mathcal{D}_2) \Vdash \varphi \text{ implies } S(\mathcal{D}_1 \uplus \mathcal{D}_2) \Vdash \varphi.$$

Proof: Assume that $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2) \Vdash \varphi$. Then $[\![\varphi]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)}$ is a set of distinguished truth values in $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$. On the other hand, by Lemma 7.5, $(\varphi, \varphi) \in [\![\varphi]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)}$. Therefore φ is distinguished in both $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$ and $S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$. Hence, $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \varphi$ and so φ is also distinguished in $S^{\Gamma}(\mathcal{D}_1 \uplus \mathcal{D}_2)$. So $S^{\Gamma}(\mathcal{D}_1 \uplus \mathcal{D}_2) \Vdash \varphi$ since $[\![\varphi]\!]^{S^{\Gamma}(\mathcal{D}_1 \uplus \mathcal{D}_2)} = \{\varphi\}$ using Lemma 6.4 of [20]. QED

In order to show preservation of weak completeness, we use the following result proved in Theorem 6.9 of [20]: *A logic system is weakly complete if it contains a representative of the canonical structure over the empty set.*

Theorem 7.7 Weak completeness is preserved by fibring of logic systems, each containing the canonical structure over the other and the empty set.

Proof: Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be such that $S_{\mathcal{D}_2}(\mathcal{D}_1) \in \mathcal{I}_1$ and $S_{\mathcal{D}_1}(\mathcal{D}_2) \in \mathcal{I}_2$. So $S_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S_{\mathcal{D}_1}(\mathcal{D}_2)$ is in $\mathcal{I}$. Using Proposition 7.6, we can conclude that $S_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S_{\mathcal{D}_1}(\mathcal{D}_2)$ is a representative of the canonical structure generated by $\mathcal{D}_1 \uplus \mathcal{D}_2$ and $\emptyset$. So the fibring has a representative of that deductive system for the empty set. Hence, invoking Theorem 6.9 of [20], we can conclude that $\mathcal{L}_1 \uplus \mathcal{L}_2$ is weakly complete. QED

We omit the proof of the next result since it follows straightforwardly by Theorem 7.7 and Proposition 7.4.

Corollary 7.8 Weak completeness is preserved by fibring logic systems containing all the interpretation structures that are sound with respect to the rules.

Corollary 7.8 basically says that in almost all cases the logic system resulting from the fibring is at least weak complete. Similarly to Theorem 6.4, this happens since a logic system, normally, does not contain interpretation structures that are not sound for its rules.

Example 7.9 *Weak completeness for the logic system resulting from the fibring of the logic systems for classical and intuitionistic propositional logics.*
The logic system $\mathcal{L}_{c+i}$ presented in Example 5.3, resulting from the fibring of the logic system $\mathcal{L}_c$ for classical propositional logic and of the logic system $\mathcal{L}_i$ for intuitionistic propositional logic, is weak complete. ∇

Taking into account Corollary 7.8, it is straightforward to see that the fibring of the logic systems composed by the deductive systems described in [20] and with all the interpretation structures sound for the rules, is weak complete (the case of the relevance logic has to be worked further due to the different notion of derivation). So, for instance, the fibring of the logic systems for paraconsistent and modal logic T is weak complete as well as the fibring of the logic systems for paraconsistent and intuitionistic propositional logics, as well as the fibring of the logic systems for equational and paraconsistent propositional logics.

Strong completeness

Preservation of strong completeness is a harder task in finding sufficient conditions for the component logics. A key concept is the notion of a truth consistent interpretation structure.

Given an interpretation structure $I = \langle G', \beta, \blacklozenge \rangle$ over a signature Σ, a set $S \subseteq V'_\pi$ is *truth consistent* whenever $S \subseteq D$ or $S \cap D = \emptyset$. The interpretation structure I is *truth consistent* if for every $e : v_1 \dots v_n \to \pi$ in E and S_i contained in V'_{v_i} such that S_i is truth consistent when v_i is π, for $i = 1, \dots, n$, then $\mathsf{trg}'(E'_e(S_1 \dots S_n, -))$ is truth consistent.

Lemma 7.10 Let I be a truth consistent interpretation structure over a signature Σ, ρ an assignment over I, $w : s \to v_1 \dots v_n$ a path in $G^\dagger$ with $v_1, \dots, v_n$ in V and $[\![w]\!]^{I,\rho} = S_1 \dots S_n$ with S_i contained in V'_{s_i} for $i = 1, \dots, n$. Then S_i is truth consistent when s_i is π, for all $i = 1, \dots, n$.

Proof: The proof follows by induction on the complexity of w:

w is ϵ_s. Then $[\![w]\!]^{I,\rho} = \rho_s$ which satisfies the requirement.

w is $\mathsf{p}_i^{s_1} w_1$. Then $[\![w]\!]^{I,\rho} = ([\![w_1]\!]^{I,\rho})_i$ which is truth consistent by induction hypothesis.

w is $\langle w_1, \dots, w_n \rangle w_0$ and $w_i : s_0 \to s_i$ for $i = 1, \dots, n$ and $w_0 : s \to s_0$. Then $[\![w]\!]^{I,\rho} = [\![w_1 w_0]\!]^{I,\rho} \dots [\![w_n w_0]\!]^{I,\rho}$. So the result follows straightforwardly by induction hypothesis.

w is $e w_1$. Then $[\![w]\!]^{I,\rho}$ is $\mathsf{trg}'(E'_e([\![w_1]\!]^{I,\rho}, -))$. The result follows immediately, using the induction hypothesis on $[\![w_1]\!]^{I,\rho}$ since I is truth consistent. QED

The following notion is the deductive counterpart of the truth consistency presented above. It resembles congruence as was adopted in previous works like [23]. But is that work the component logic systems had to share implication which is not the case herein.

A deductive system $\mathcal{D}$ is said to be *componentwise congruent* for Γ whenever $\Gamma \vdash_\mathcal{D} \varphi_i$ iff $\Gamma \vdash_\mathcal{D} \psi_i$ for $i = 1, \dots, n$ implies

$$\Gamma \vdash_\mathcal{D} c(\varphi_1, \dots, \varphi_n) \text{ iff } \Gamma \vdash_\mathcal{D} c(\psi_1, \dots, \psi_n)$$

for each m-edge c in Σ.

The following result relates componentwise congruence with truth consistency over the canonical structures.

Lemma 7.11 Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be deductive systems. If $\mathcal{D}_1 \uplus \mathcal{D}_2$ is componentwise congruent then $S^\Gamma_{\mathcal{D}_2}(\mathcal{D}_1)$ is truth consistent.

Proof: Assume that $\mathcal{D}_1 \uplus \mathcal{D}_2$ is componentwise congruent. Let $e : v_1 \dots v_n \to \pi$ in E and S_i contained in V'_{v_i} such that S_i is truth consistent when v_i is π, for $i = 1, \dots, n$. Let $\varphi_1, \varphi_2 \in \mathsf{trg}'(E'_e(S_1 \dots S_n, -))$. Assume that φ_1 is a distinguished value. Let $\varphi_{i1}, \dots, \varphi_{in}$ be such that $\varphi_i = e \circ (\varphi_{i1} \dots \varphi_{in})$ for $i = 1, 2$. Observe that, for each $j = 1, \dots, n$, either $\varphi_{1j}, \varphi_{2j}$ are distinguished values or none is. So either $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \varphi_{1j}$ and $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \varphi_{2j}$ or neither holds, for $j = 1, \ldots, n$. Using the fact that $\mathcal{D}_1 \uplus \mathcal{D}_2$ is componentwise congruent, either $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \varphi_1$ and $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \varphi_2$ or neither holds. Hence $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \varphi_2$ and so φ_2 is distinguished as well. QED

The following result establishes the preservation of truth consistency by the fibring of canonical structures for the component logics.

Lemma 7.12 Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be deductive systems. If $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$ and $S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$ are truth consistent then so is $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$.

Proof: Let $e : v_1 \ldots v_n \to \pi$ in E and S_i contained in V'_{v_i} such that S_i is truth consistent when v_i is π, for $i = 1, \ldots, n$. Let $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in \mathsf{trg}'(E'_e(S_1 \ldots S_n, -))$. Assume that (φ_1, φ_2) is a distinguished value. Then φ_1 and φ_2 are distinguished elements in $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$ and $S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$, respectively. Assume, with no loss of generality, that $e \in E_1$. So $\varphi_1 = e \circ (\varphi_{11} \ldots \varphi_{1n})$ and $\psi_1 = e \circ (\psi_{11} \ldots \psi_{1n})$ where $(\varphi_{1i}, \delta_{2i}) \in S_i$ for some δ_{2i} and $(\psi_{1i}, \theta_{2i}) \in S_i$ for some δ_{1i} and θ_{2i}, $i = 1, \ldots, n$. Since S_i is truth consistent then either $(\varphi_{1i}, \delta_{2i})$ and (ψ_{1i}, θ_{2i}) are distinguished values in $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$ or none is. So either φ_{1i} and ψ_{1i} are distinguished values in $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$ or none is. So either φ_1 and ψ_1 are distinguished values in $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$ or none is. Therefore ψ_1 is a distinguished element in $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$ and so (ψ_1, ψ_2) is a distinguished value in $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$. QED

The next result shows that $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$ is a representative of the canonical structure for the fibring of deductive systems $\mathcal{D}_1$ and $\mathcal{D}_2$, providing that the fibring of deductive systems is componentwise congruent.

Proposition 7.13 Given deductive systems $\mathcal{D}_1$ and $\mathcal{D}_2$ sharing the same set of sorts, and a set $\Gamma \cup \{\varphi\}$ of formulas in G^+, then

- $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2) \Vdash \varphi$ implies $S^{\Gamma}(\mathcal{D}_1 \uplus \mathcal{D}_2) \Vdash \varphi$;
- $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2) \Vdash \Gamma$

assuming that $\mathcal{D}_1 \uplus \mathcal{D}_2$ is componentwise congruent for Γ.

Proof:
(1) $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2) \Vdash \varphi$ implies $S^{\Gamma}(\mathcal{D}_1 \uplus \mathcal{D}_2) \Vdash \varphi$. The proof of this implication mimics the proof of Proposition 7.6 so we omit it.

(2) $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2) \Vdash \Gamma$. Let $\gamma \in \Gamma$. Then $\Gamma \vdash_{\mathcal{D}_1 \uplus \mathcal{D}_2} \gamma$. So γ is a distinguished truth value in both $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1)$ and $S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$ and so (γ, γ) is a distinguished truth value in $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$. By Lemma 7.5, $(\gamma, \gamma) \in [\![\gamma]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)}$ and so all the truth values of $[\![\gamma]\!]^{S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)}$ are distinguished, since $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2)$ is truth consistent by Lemma 7.12 and Lemma 7.11. Hence, $S^{\Gamma}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\Gamma}_{\mathcal{D}_1}(\mathcal{D}_2) \Vdash \gamma$. QED

We now introduce the well known notion of a maximal set of formulas. A logic system $\mathcal{L}$ has *maximals* if for each consistent set Γ of formulas and formula φ such that $\Gamma \not\vdash_{\mathcal{D}} \varphi$, there is a consistent set $\overline{\Gamma}$ extending Γ such that $\overline{\Gamma} \not\vdash_{\mathcal{D}} \varphi$ and $\overline{\Gamma} \cup \{\psi\} \vdash_{\mathcal{D}} \varphi$ for every $\psi \notin \Gamma$. We call $\overline{\Gamma}$ a *maximal set* for Γ and φ.

Theorem 7.14 A logic system with maximals containing a representative of the canonical structure over a maximal set for each set of formulas and formula is strong complete.

Proof: Let Γ be a set of formulas and φ a formula. Assume that $\Gamma \not\vdash_{\mathcal{D}} \varphi$. Then there is a maximal set $\overline{\Gamma}$ for Γ and φ. Let $I_{\overline{\Gamma}} \in \mathcal{I}$ be an interpretation structure representing $S^{\overline{\Gamma}}(\mathcal{D})$. Then, $S^{\overline{\Gamma}}(\mathcal{D}) \Vdash \overline{\Gamma}$ and $S^{\overline{\Gamma}}(\mathcal{D}) \not\Vdash \varphi$ and so $I_{\overline{\Gamma}} \not\Vdash \varphi$ and $I_{\overline{\Gamma}} \Vdash \overline{\Gamma}$. Therefore, $\Gamma \not\vDash_{\mathcal{I}} \varphi$ since $I_{\overline{\Gamma}} \not\Vdash \varphi$ and $I_{\overline{\Gamma}} \Vdash \Gamma$ and $I_{\overline{\Gamma}} \in \mathcal{I}$. QED

We are now ready to provide a sufficient condition for the preservation of strong completeness by fibring.

Theorem 7.15 Strong completeness is preserved by fibring of logic systems such that:

- the fibring has maximals, for each set of formulas Γ and formula, enjoying componentwise congruence for $\overline{\Gamma}$;
- each component contains the canonical structure over the other and those maximal sets.

Proof: Let Γ be a set of formulas and φ a formula. Then there is a maximal set $\overline{\Gamma}$ for Γ and φ. Moreover, $\mathcal{D}_1 \uplus \mathcal{D}_2$ is componentwise congruent for $\overline{\Gamma}$. Moreover, by hypothesis, $S^{\overline{\Gamma}}_{\mathcal{D}_2}(\mathcal{D}_1) \in \mathcal{I}_1$ and $S^{\overline{\Gamma}}_{\mathcal{D}_1}(\mathcal{D}_2) \in \mathcal{I}_2$. Using Proposition 7.13, we can conclude that $S^{\overline{\Gamma}}_{\mathcal{D}_2}(\mathcal{D}_1) \uplus S^{\overline{\Gamma}}_{\mathcal{D}_1}(\mathcal{D}_2) \in \mathcal{I}$ is a representative of the canonical structure $S^{\overline{\Gamma}}(\mathcal{D}_1 \uplus \mathcal{D}_2)$. Finally, by Theorem 7.14, we can conclude that the fibring is complete. QED

The conditions in Theorem 7.15 are not so general as the ones we obtained for the preservation of soundness and weak completeness. Therefore, less logic systems can be fibred with preservation of strong completeness. Nevertheless, preservation of strong completeness holds, among others, in the fibring of modal logics at least as stronger as T.

The preservation of soundness and completeness ensures that the graph-theoretic semantics developed herein is adequate with respect to the envisaged deductive system. Moreover, it provides a criterion for checking in a semantic way that other combination approaches are sound and complete with respect to the widely recognized deductive system for fibring. For instance, the combination approaches described in [22, 3]. We can further outline this idea. Consider two logics each one characterized by a class of models M' and M'', respectively. Let $(\Sigma', \mathcal{I}', \mathcal{D}')$ and $(\Sigma'', \mathcal{I}'', \mathcal{D}'')$ be graph-theoretic logic systems for the same logics whose graph-theoretic fibring is sound and complete. Assume that $m' \sqcup m''$ is the model resulting from combining $m' \in M'$ and $m'' \in M''$.

One can ask whether the combined logic characterized by the class of models $M = \{m' \sqcup m'' : m' \in M', m'' \in M''\}$ is sound and complete with respect to the deductive system for fibring. The results in this paper help to provide an answer to this question when the following conditions are satisfied:

- for every $m \in M$ there are $I' \in \mathcal{I}'$ and $I'' \in \mathcal{I}''$ such that $I' \uplus I''$ satisfies the same formulas as m;
- for every $I' \in \mathcal{I}'$ and $I'' \in \mathcal{I}''$ there is $m \in M$ such that m satisfies the same formulas as $I' \uplus I''$.

The first condition guarantees that the combination is sound with respect to the deductive system for fibring. For instance, assume that $\vdash_{\mathcal{D}} \varphi$. Let $m \in M$ be an arbitrary model. Then, by the first condition, there are $I' \in \mathcal{I}'$ and $I'' \in \mathcal{I}''$ such that $I' \uplus I''$ satisfies the same formulas as m. By soundness of the graph-theoretic approach we conclude that $I' \uplus I'' \Vdash \varphi$ and so m satisfies φ as desired.

The second condition guarantees that the combination is complete with respect to the deductive system for fibring. For instance, assume that φ is valid in M. It is enough to show that φ is valid in the graph-theoretic semantics resulting from fibring. Let $I' \in \mathcal{I}'$ and $I'' \in \mathcal{I}''$. Then, by the second condition, there is $m \in M$ such that m and $I' \uplus I''$ satisfies the same formulas. So, $I' \uplus I''$ satisfies φ as wanted.

8 Concluding remarks

In the sequel of [20], we developed a graph-theoretic account of fibring of logic systems. The approach is general enough to cover fibring of a large class of logics, encompassing logics with a non-deterministic semantics as well as substructural logics like relevance logic. Preservation of soundness and weak completeness were proved under very general assumptions. Preservation of strong completeness by fibring requires a tighter context. To the best of our knowledge, results on weak completeness were not proved before in previous work on fibring.

It is also worthwhile to point out that the graph-theoretic setting provides the means to avoid some well known collapses [7] in a very natural way, different from previous works on the topic [21, 3]. For instance, in the case of the modulated fibring, the collapse was avoided by restricting instantiation in derivation which is not the case in the graph-theoretic approach.

Several extensions of this work are worthwhile to pursue. The first step is to analyze preservation results about cut elimination, interpolation, quantifier elimination and decidability, among others. We hope to capitalize on the results obtained in [5, 10, 17] for interpolation and general characterizations of logics with cut elimination [18]. We also believe that the graph-theoretic approach should be explored in the field of fusion of modal logics, namely to cope with modal logics endowed with general semantics and having different valuation sets. We also intend to investigate a graph-theoretic account of combination of theories with the aim of obtaining preservation of interpolation and decidability. Some results on combining theories can be seen in [2, 13].

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