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Is the Liar Paradox Never Strictly Classical?*

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【Abstract】 The present paper investigates whether strictly classical inferences contribute to the formalization of (genuine) paradoxes within natural deduction. Tennant's criterion for paradoxicality relies on the generation of an infinite reduction sequence, which distinguishes genuine paradoxes from mere inconsistencies. His methodological conjecture posits that genuine paradoxes are never strictly classical and can be derived without classical inferences such as the *Law of Excluded Middle*, *Dilemma*, *Classical Reductio*, and *Double Negation Elimination*.

It appears that there were two reasons for Tennant's proposal of the methodological conjecture. The one is that strictly classical inferences hinder the generation of an infinite reduction sequence and the other is that strictly classical inferences have no role in the formalization of genuine paradoxes.

This paper raises questions about these two reasons. Focusing on the liar paradox, it will be argued that strictly classical inferences do not interfere with the generation of an infinite reduction sequence and that the liar sentence may implicitly entail strictly classical inferences. Should this analysis hold, it would call into question not only Tennant's motivation for advancing the methodological conjecture, but also challenge his contention that genuine paradoxes—exemplified by the liar paradox—are never constructed with reliance on strictly classical inferences.

【Key Words】 Liar paradox, Genuine paradoxes, Strictly classical inferences, Intuitionistic relevant logic, Neil Tennant.

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1 Introduction

Philosophical investigations often center on the extent to which classical inference contributes significantly to the derivation of contradictions through paradoxes. Dummett (1993: p. 454) proposes that the emergence of paradoxes is intertwined with the indefinitely extensible concept, defined as those whose extensions are inherently indeterminable, allowing for any object set under such a concept to be expanded. It is frequently observed that his view forms the basis of an argument challenging classical logic, suggesting that a logic capable of expressing indefinitely extensible concepts would be non-classical, characterized by the indeterminate truth value of its statements.

In discussing the disconnection between semantic paradoxes and the idea of indefinitely extensible concepts, Williamson (1998: p. 2) highlights the limitations of intuitionistic logic in blocking paradoxes, stating:

> From a purely technical perspective, intuitionistic logic presents no obvious advantage. In the simplest paradoxes, a plausible general principle turns out to have a substitution instance of the form [$\varphi \leftrightarrow \neg\varphi$], which is inconsistent in both intuitionistic and classical logic. Adopting intuitionistic logic would not enable us to retain the plausible general principle while blocking the inference to a contradiction.

In contrast to Williamson, Field (2008: p. 8) outlines an approach for deriving the explicit contradiction $\varphi \wedge \neg\varphi$ from $\varphi \leftrightarrow \neg\varphi$, termed the Central Argument. He states, "Without [excluded middle], it isn't immediately obvious why [$\varphi \leftrightarrow \neg\varphi$] should be contradictory." He then points out that the absence of excluded middle in intuitionistic logic leads to the invalidation of the Central Argument regarding the contradiction of $\varphi \leftrightarrow \neg\varphi$. His analysis primarily focuses on the excluded middle, suggesting its critical role in paradoxes. Field (2008: p. 15) writes:

> ... we should seriously consider restricting the law of excluded middle (though not in the way intuitionists propose) ... I take excluded middle to be clearly suspect only for certain sentences that have a kind of "inherent circularity" because they contain predicates like 'true' ..

In a noteworthy discourse, Tennant (2017: Ch. 11) recognized as an intuitionistic relevant logician or a core logician, argues that the law of excluded middle is not essential in deriving the explicit contradiction $\varphi \wedge \neg\varphi$ from $\varphi \leftrightarrow \neg\varphi$. Additionally, Tennant (2017: p. 282) proposes a methodological principle:

> One is dealing with a case of genuine semantic paradox only if it can be derived relevantly, and without using excluded middle (or any of its intuitionistic equivalents).

This principle stems from what he earlier termed the methodological conjecture in Tennant (2015a: p. 589) that "paradoxes are never strictly classical."

Tennant's strictly classical inferences of negation include the law of excluded middle, dilemma, classical reductio, and double negation elimination, in addition to any inferences that demand an appeal to any of these specific rules (cf. Tennant (2015a: p. 4)). His methodological conjecture, or principle, posits that the set-theoretic paradoxes, as well as the semantic paradoxes relating to 'true' and 'true-of,' emerge from reasoning that is fundamentally constructive and adheres to relevantist principles. It claims that strictly classical inferences should not be held responsible for the emergence of such paradoxes.

Tennant (2015a: p. 589) introduces the methodological conjecture that genuine paradoxes are never strictly classical and claims that some troubles in formalizing paradoxes in natural deduction are specifically caused by strictly classical inferences, as opposed to those in constructive reasoning.

The acceptability of classical inferences within constructivist reasoning remains a contentious issue, with numerous constructivists, including Tennant (1996; 2002; 2015a), not accepting them as constructive. Nevertheless, the "troubles" associated with the methodological conjecture diverge slightly from issues surrounding the constructiveness of the classical reductio. The primary concern lies in the troubles arising from the classical reductio in the exploration of the proof-theoretic structure of paradoxes. Specifically, deriving the liar paradox through classical reductio fails to generate the anticipated fundamental outcome in paradoxical reasoning: an infinite reduction sequence. As a potential resolution, the methodological conjecture is put forth.

The present paper explores two reasons put forth by Tennant in support of the methodological conjecture: (1) classical inferences impede the generation of an infinite reduction sequence, and (2) classical inferences have no role in the formalization of genuine paradoxes. The analysis of (1) will demonstrate that using classical reductio in the liar paradox derivation can lead to an infinite reduction sequence. The argument will be made that the generation of an infinite reduction sequence hinges on the selected reduction procedures and rules. For (2), the analysis will concentrate on the liar paradox, which Tennant deems a genuine paradox, showing that its formalization in natural deduction may require strictly classical inferences. As an intuitionistic relevant logician who refers to himself as a 'core logician', Tennant should consider the liar sentence as lacking a proof. However, the standard natural deduction system he utilizes fails to adequately express prooflessness. Consequently, a sentence leading to a contradiction, such as the liar sentence, becomes derivable in natural deduction, which in turn derives forms of strictly classical inferences.

Section 2 will present foundational terminology and the rules of natural deduction, subsequently introducing Tennant's criteria for Genuine paradoxes and offering a proposed solution to these paradoxes. The discussion in Section 3 revolves around Tennant's methodological conjecture, aimed at defending the proof-theoretic solution to paradoxes. This section will expound on two key motivations underlying this conjecture: the prevention of the generation of an infinite reduction sequence, and the lack of any substantive role for classical inferences in the process of formalizing genuine paradoxes in natural deduction.

Section 4 refutes the first reason. Specifically, it will be argued that an infinite reduction sequence can be generated even when employing strictly classical inferences such as *Classical Reductio*. Furthermore, it will be discussed that generating an infinite reduction sequence is determined by which reduction procedures are selected. Section 5 will contend that while the formalization of paradoxes in natural deduction may not overtly employ strictly classical inferences, it does permit the form of classical inference for certain statements. This phenomenon is due to the limitations of the standard natural deduction system in expressing the concept of prooflessness as used for the liar sentence by intuitionists such as Tennant, and it will be argued that this weakens Tennant's second reason for proposing the methodological conjecture, thus providing grounds for its reconsideration.

2 Tennant's Genuine Paradoxes and Their Resolutions

Following the exploration by Prawitz (1965: pp. 94–95) into the derivation of $\perp$ originating from the set-theoretic paradox, which generates an infinite reduction sequence, Tennant (1982; 2016) identified the infinite reduction sequence as the key inferential characteristic separating a mere inconsistency from what he termed a "genuine paradox."

Tennant (1982: p. 283) initially introduced a proof-theoretic criterion for paradoxicality, stating that a paradoxical derivation, employing a specific type of *id est* inferences, leads to $\perp$ (or an unacceptable conclusion) while also producing an infinite reduction sequence. Tennant (2016: p. 598) interpreted these infinite reduction sequences as the proof-theorist's depiction of the inherent circularity (or helicality) in paradoxes. The criterion is recognized as a measure for identifying infinite reduction sequences that arise from the derivation of $\perp$ pertinent to the paradoxes under consideration.

Criticism soon arose against the early version of the proof-theoretic criterion for paradoxicality, notably from Schroeder-Heister and Tranchini (2017; 2018), who challenged its broad scope. This critique was substantiated using an Ekman case from Ekman (1998) as a representative counterexample, showing the criterion's tendency to be overly inclusive. This particular case demonstrated that the criterion could erroneously classify intuitively non-paradoxical derivations as paradoxical. To rectify this issue of overgeneration, Tennant (2016; 2017) refined the criterion, adding a new condition requiring the generalized form of all elimination rules in paradoxical derivations.

The preliminary clarification in Section 2.1 lays the groundwork for a comprehensive understanding of Tennant's concept of 'genuine paradox', which, along with solutions to genuine paradoxes, is elaborated upon in Section 2.2.

2.1 Preliminaries: Some terminologies and natural deduction rules

In the interest of clarity, our language incorporates the constants $\neg$ for negation and $\perp$ for absurdity. A unary truth predicate $T(x)$ and the corner quotes $\ulcorner\urcorner$ can also be used.[1)] Let φ, ψ, and σ be arbitrary formulas. A *derivation* in natural deduction signifies the process of inferring results from given assumptions or premises via particular inference rules, aligning with the descriptions in "deduction" as outlined by Prawitz (1965: p. 17) and "proof" as detailed by Tennant (2017: p. 17). In addition, the following conventions are adopted: if a derivation $\mathfrak{D}$ ends with a formula φ, it is expressed as shown on the left below, and φ is called an *end-formula.*

There are rules for $\neg$, $T(x)$, and some strictly classical rules for *classical reductio*(*CR*), *dilemma*(*DL*), and *the law of excluded middle*(*LEM*) in the natural deduction with the form of general elimination rules.[2)]

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1) The left and right corner quotes, $\ulcorner\urcorner$, are often used in the truth predicate $T(x)$ to encode formulae into coded expressions. For instance, if φ is a given formula, $\ulcorner\varphi\urcorner$ refers to φ. If $\psi(x)$ is a formula with one free variable x, then $\psi(\ulcorner\varphi\urcorner)$ is a formula describing that φ denoted by $\ulcorner\varphi\urcorner$ is ψ.

2) While Tennant (2016; 2017; 2021) showed a preference for the term "parallelized" over "general," this paper employs the term "general" in reference to general elimination rules. This terminology stems from the initial introduction of general elimination rules by Schroeder-Heister (1984a; 1984b), who developed a *general schema* for the introduction and elimination rules concerning principal operators. Additionally, Tennant (2017) implemented standard (or serial) versions of the $\neg E-$rule in his core logic. In this work, the general $\neg E-$rule is

$$
\dfrac{\begin{array}{c}[\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\neg\varphi}\neg I_{,1}
\qquad
\dfrac{\neg\varphi \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array} \quad \begin{array}{c}[\bot]^1\\ \mathfrak{D}_3\\ \psi\end{array}}{\psi}\neg E_{,1}
\qquad
\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \varphi\end{array}}{T(\ulcorner\varphi\urcorner)}TI
\qquad
\dfrac{T(\ulcorner\varphi\urcorner) \quad \begin{array}{c}[\varphi]^1\\ \mathfrak{D}_2\\ \psi\end{array}}{\psi}TE_{,1}
$$

$$
\dfrac{\begin{array}{c}[\neg\varphi]^1\\ \mathfrak{D}\\ \bot\end{array}}{\varphi}CR_{,1}
\qquad
\dfrac{\begin{array}{c}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{array} \quad \begin{array}{c}[\neg\varphi]^2\\ \mathfrak{D}_2\\ \psi\end{array}}{\psi}DL_{,1,2}
\qquad
\dfrac{}{\varphi\vee\neg\varphi}LEM
$$

The formulas directly above the line in each rule is referred to as the "premise," and the formula directly below the line is the "conclusion." *Assumptions* that can be discharged are in square brackets, for example, $[\varphi]$. Similar to restrictions on Tennant's intuitionistic relevant logic, recently called "core logic," when the $\neg I-$, $CR-$, and $DL-$ rules are applied, vacuous discharge is prohibited. (Cf. Tennant (2015b; 2015c; 2016; 2017; 2021). The *open assumptions* of a derivation are assumptions on which the end formula depends. A derivation is called *closed* if it does not include any open assumptions and is called *open* otherwise. A *major premise* of the elimination rule for an operator is the premise that contains the operator in the elimination rule. For clarity, the major premise of the $E-$rule is placed at the far left of the premises in elimination rules, with all remaining premises labeled as *minor premises*. *Maximum formula occurrence* is the conclusion of an introduction rule simultaneously serving as the major premise of an elimination rule.[3] Standard reduction pro-

adopted to minimize potential disputes, with the findings being underpinned by Tennant's core logic.

3) It is commonly acknowledged that a conclusion of the $CR-$rule, if it is not an atomic formula and also acts as the major premise in an elimination rule, is typically considered a maximum formula occurrence.(Cf. Prawitz (1965)) Nonethe-

cedures for $\neg$ and $T(x)$ are acknowledged and accepted. (Cf. Choi (2019; 2023).)

For any two derivations with the same end formula $\mathfrak{D}_{\mathfrak{a}}$ and $\mathfrak{D}_{\mathfrak{b}}$, an *immediate sub-derivation* of $\mathfrak{D}_{\mathfrak{a}}$ is an initial part of $\mathfrak{D}_{\mathfrak{a}}$ that ends with the premise of the last inference rule in $\mathfrak{D}_{\mathfrak{a}}$. A *sub-derivation* is the reflexive and transitive closure of an immediate sub-derivation. Further, $\mathfrak{D}_{\mathfrak{a}} \rhd \mathfrak{D}_{\mathfrak{b}}$ means that $\mathfrak{D}_{\mathfrak{a}}$ *reduces* to $\mathfrak{D}_{\mathfrak{b}}$ by applying a single reduction procedure to a sub-derivation of $\mathfrak{D}_{\mathfrak{a}}$. Then, "$\mathfrak{D}_{\mathfrak{a}} \rhd_{T(x)} \mathfrak{D}_{\mathfrak{b}}$" means that $\mathfrak{D}_{\mathfrak{a}}$ reduces to $\mathfrak{D}_{\mathfrak{b}}$. Let $\mathbb{R}$ be a set of reductions. The following definitions are used.[4)]

Definition 2.1. A sequence $< \mathfrak{D}_{\mathfrak{1}}, ..., \mathfrak{D}_{\mathfrak{i}}, \mathfrak{D}_{\mathfrak{i+1}}, ... >$ is a *reduction sequence* relative to $\mathbb{R}$ iff $\mathfrak{D}_{\mathfrak{i}} \rhd \mathfrak{D}_{\mathfrak{i+1}}$ relative to $\mathbb{R}$, where $1 \leqslant i$ for any natural number i. A derivation $\mathfrak{D}_{\mathfrak{1}}$ is *reducible* to $\mathfrak{D}_{\mathfrak{i}}$ ($\mathfrak{D}_{\mathfrak{1}} \succ \mathfrak{D}_{\mathfrak{i}}$) relative to $\mathbb{R}$ iff there is a sequence $< \mathfrak{D}_{\mathfrak{1}}, \mathfrak{D}_{\mathfrak{2}}, ..., \mathfrak{D}_{\mathfrak{i}} >$ relative to $\mathbb{R}$ where for each $j < i$, $\mathfrak{D}_{\mathfrak{j}} \rhd \mathfrak{D}_{\mathfrak{j+1}}$; $\mathfrak{D}_{\mathfrak{1}}$ is *irreducible* relative to $\mathbb{R}$ iff there is no derivation $\mathfrak{D}'$ to which $\mathfrak{D}_{\mathfrak{1}} \rhd \mathfrak{D}'$ relative to $\mathbb{R}$ except $\mathfrak{D}_{\mathfrak{1}}$ itself.

Definition 2.2. The derivation $\mathfrak{D}$ is *normal* (or in *normal form*) relative to $\mathbb{R}$ iff $\mathfrak{D}$ has no maximum formula occurrence and is irreducible to $\mathbb{R}$. A reduction sequence *terminates* iff it has a finite num-

less, instances exist where an atomic formula, serving as both the conclusion of the $CR-$rule and the major premise of an elimination rule, is deemed a maximum formula. Attention should be given to the fact that the definitions of "maximum formula occurrence" and "normal form" can vary based on the author or the intent of the proof. In the present discourse, if an atomic formula is the conclusion of the $CR-$rule and the major premise of an elimination rule, it will not be treated as a maximum formula.

4) In Definition 2.1, for any term x and y, let $x \leq y$ mean that x is less than or equal to y. In the context of Definition 2.2, the term "relative to $\mathbb{R}$" is excluded for ease of presentation, assuming the concise descriptions maintain their clarity and precision.

ber of derivations and its last derivation is in normal form. A derivation $\mathfrak{D}$ is *normalizable* relative to $\mathbb{R}$ iff there is a terminating reduction sequence relative to $\mathbb{R}$ starting from $\mathfrak{D}$.

Non-terminating reduction sequences bifurcate into two principal types. The first type is characterized by a derivation incapable of being reduced to a normal form by any reduction procedure, while the second type comprises a derivation capable of generating an infinite reduction sequence. To elaborate, for any derivation $\mathfrak{D}$, $\mathfrak{D}$ *generates an infinite reduction sequence* iff there is a derivation $\mathfrak{D}'$ such that $\mathfrak{D} \succ \mathfrak{D}'$ but its reduction sequence does not terminate. It is thus evident that when $\mathfrak{D}$ generates an infinite reduction sequence, $\mathfrak{D}$ has a non-terminating reduction sequence and is not normalizable.

Following this section, an examination will be conducted of Tennant's derivation of the liar paradox as presented in Tennant (2016; 2017). The purpose is to elucidate his concept of genuine paradoxes and the resolutions he proposes for them.

2.2 Tennant's Solution to Genuine Paradoxes

Let S_L be a natural deduction system with rules for $\neg$, $T(x)$, and the following Tennant's rules for the liar sentence Φ used in Tennant (2017: pp. 298 – 302). As presented by Tennant (2017: p. 299), the reduction procedure for Φ is as follows.[5)]

[5)] Tennant (2017: p. 299) illustrates the reduction procedure for $T(x)$ through a separate graphical representation. Nonetheless, the fundamental concept underpinning this reduction remains congruent with the ongoing discussion, thus maintaining a perspective compatible with the prevailing discourse.

$$
\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\Phi}\,\Phi I_{,1}
\qquad
\dfrac{\Phi \qquad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_2\\ \varphi\end{array}}{\varphi}\,\Phi E_{,1}
\qquad
\dfrac{\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\Phi}\,\Phi I_{,1} \qquad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^2\\ \mathfrak{D}_2\\ \varphi\end{array}}{\varphi}\,\Phi E_{,2}
\quad \rhd_\Phi \quad
\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \dfrac{\bot}{\neg T(\ulcorner\Phi\urcorner)}\,\neg I_{,1}\\ \mathfrak{D}_2\\ \varphi\end{array}
$$

S_L also incorporates a set $\mathbb{R}_L$ of reduction procedures, which involve reductions for $\neg$, $T(x)$, and Φ. Then, it can be easily shown that there is a closed derivation of $\bot$ in S_L relative to $\mathbb{R}_L$ that generates an infinite reduction sequence and thus is not normalizable. (Cf. Choi (2023: Sec. 2.2) and Tennant (2017: Sec. 11.5).)

The infinite reduction sequence from the liar paradox has been characterized by Tennant (1982: pp. 270–271) as a *falling into a looping reduction sequence*, and the completeness conjecture on paradoxicality was put forth as the proof-theoretic criterion for identifying paradoxical derivations.

> The completeness conjecture is then that [a] set of sentences is paradoxical ... iff there is some proof of [$\bot$] ... , involving those sentences in *id est* inferences that [have] a looping reduction sequence. Tennant (1982: p. 283)

The concept of *id est* inferences is used to describe cases where a formula can be interchanged with its negation or predication. Serving as the *id est* rules for the liar sentence Φ are the $\Phi I-$ and $\Phi E-$rules. Moreover, the infinite reduction sequence has been identified as the quintessential element of paradoxical derivations, a proof-theoretic criterion expounded upon in Tennant (2016).[6] :

6) The term "non-terminating reduction sequence" used by Tennant (1982; 1995;

> Tennant (1982) proposed a proof-theoretic criterion, or test, for paradoxicality—that of [an *infinite*] *reduction sequence* initiated by the "proofs of ⊥" associated with the paradoxes in question (p. 271).

The initial criterion established that a derivation is deemed *paradoxical* if it effectively deduces ⊥, relies on *id est* inferences, and generates an infinite reduction sequence. Building further on the initial criterion, Tennant (2016; 2017) suggested a further condition that mandates all elimination rules to be presented in a generalized form, aiming to sidestep the problem presented by Schroeder-Heister and Tranchini (2017). While Schroeder-Heister and Tranchini (2017) and Choi (2019) have criticized the supplementary condition for its inadequacy in addressing the issue, this critique does not substantially hinder the investigation into the role of classical inference within paradoxical derivations, which remains the central theme of our discussion. A summary of Tennnat's criterion for paradoxicality presents itself thus[7)] :

Tennant's Criterion for Paradoxicality(*TCP*): Let $\mathfrak{D}$ be any derivation

2015a; 2016; 2017) is synonymous with the infinite reduction sequence in this context. Both looping and spiral reduction sequences are not only instances of non-terminating reduction sequences but also exemplify infinite reduction sequences.

7) Certain elements demand contemplation from a philosophical standpoint. In relation to condition (i), it is observed that ⊥ is not the sole conclusion emanating from paradoxical derivations. During the formalization of Curry's paradox, the usage of a propositional variable *p* is considered viable. (Refer to Tennant (1982)). Concerning condition (ii), it is acknowledged that infinite reduction sequences are bifurcated into loops and spirals. As Tennant (1995: p. 207) suggests, self-referential paradoxes are primarily characterized by looping reduction sequences, whereas non-self-referential paradoxes are inclined towards spiraling reduction sequences. Choi (2021) offers an expanded discussion on Tennant's conjecture regarding self-referential paradoxes. The focus of the present paper is exclusively on self-referential paradoxes.

of a natural deduction system S and $\mathbb{R}$ be a set of reduction procedures of S. $\mathfrak{D}$ is *paradoxical* iff

(i) $\mathfrak{D}$ is a (open/closed) derivation of $\bot$,

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) $\mathfrak{D}$ generates an infinite reduction sequence, and

(iv) all elimination rules in $\mathfrak{D}$ are stated in general form.

The criterion, as delineated by Tennant (1982: p. 285), manifests the completeness conjecture pertaining to genuine paradoxes, leading to a subsequent summarization.

The Completeness Conjecture for Genuine Paradoxes: For any derivation $\mathfrak{D}$ in a natural deduction, $\mathfrak{D}$ formalizes a *genuine paradox* iff $\mathfrak{D}$ is paradoxical.

It is discernible from Tennant (2017: pp. 286–287) that the completeness conjecture bears a relationship to the resolution of genuine paradoxes.

> How ... are we to solve the paradoxes? It is not from this study to venture any new suggestions beyond those of Tennant (1982) and Tennant (1995). Those works provided ... proofs, formalized as natural deductions, for all the major paradoxes ... . They showed that all these ... proofs ... cannot be converted into normal form. The original proof-theoretic thesis stands:
>
> > Genuine paradoxes are those whose associated *proofs of absurdity*, when formalized as natural deductions, cannot be converted into normal form.
>
> This conjecture provides a proof-theoretic criterion for the identification of genuine paradoxes ... .

If the conjecture is correct, then any derivation associated with genuine paradoxes, particularly those identified as paradoxical derivations, will initiate an infinite reduction sequence, rendering it non-normalizable. This conjecture, if true, would present a proof-theoretic resolution to paradoxes.

As a proponent of anti-realism and constructivism, Tennant (2015a: p. 578) asserts that "every truth is knowable, and its truth consists in the existence of a(n in principle) surveyable truthmaker, also called a (canonical) proof." In addition, he contends that a (constructive) proof should be capable of conversion into normal form, leading him to propose a proof-theoretic principle for constructive mathematics.

> The following principle is a cornerstone of proof-theoretic foundations for constructive mathematics:
>
> For every proof Π that we may provide for a mathematical theorem φ, it must be possible, in principle, to transform Π, *via* a finite sequence of applicable reduction procedures, into a *canonical* proof of φ, that is, a proof of φ that is *in normal form*, so that none of the reduction procedure is applicable to it. Tennant (2015a: p. 579)

While advocating the proof-theoretic principle for *constructive mathematics*, it's noted that the principle has applicability in a general case. The belief is that any derivation representing a proof of a true statement should, in principle, be capable of being transformed into normal form. Moreover, in proposing his earlier criterion, Tennant (1982) emphasizes the significance of normalizability, stating:

> The general loss of normalizability, confined as it is according to [*TCP*] to just the paradoxical part of the semantically closed language, is a small price to pay for the protection it

> gives against paradox itself. Logic plays its role as an instrument of knowledge only insofar as it keeps proofs in sharp focus, through the lens of normality. Normali[z]ability, in the context of semantically closed languages, is not to be pressed as a general pre-condition for the very possibility of talking sense; rather, normality of proof is to be pressed as a general pre-condition for the very possibility of telling the truth. Tennant (1982: p. 284)

Under the assumption that each derivation representing a proof of a true statement, ought in principle to be reducible to a normal derivation, such a requirement could act as a barrier to paradoxical derivations, thereby serving as a proof-theoretic resolution to paradoxes. While Tennant did not explicitly put forward the requirement as a proof-theoretic solution, from *TCP*, it is reasonable to infer the following principle as a plausible proof-theoretic answer to paradoxes.

The Requirement of a Normal Derivation(*RND*): For any derivation $\mathfrak{D}$ in natural deduction, $\mathfrak{D}$ is acceptable iff $\mathfrak{D}$ is (in principle) convertible into a normal derivation.

In his analysis contrasting his natural deduction system for naive set-theory with that of Fitch's, Prawitz (1965: p. 95) proposes a requirement akin to *RND*, articulating that "the set-theoretical paradoxes ruled out by the requirement that the [derivations] shall be normal." Furthermore, Prawitz (1965: p. 96) contends that this requirement is less *ad hoc* compared to the simple/special restrictions Fitch introduced, as noted in Fitch (1952: Sec. 18 and 20). Although neither Prawitz nor Tennant explicitly state that *RND* (or a similar requirement) constitutes a solution to paradoxes, their perspectives suggest the possibility that they considered *RND* as a solution.

3 Tennant's Methodological Conjecture

As identified by Tennant (2016: pp. 12-16), the liar paradox stands as a significant example of a genuine paradox. Tennant (2015a: pp. 588-589) intriguingly suggests a derivation for the liar paradox, applying the *classical reductio* rule in a manner that avoids the generation of an infinite reduction sequence. A similar derivation of the liar paradox can be envisaged through a corresponding method.

Proposition 3.1. *Let S_{LC} be an extension of S_L by adding the CR−rule. $\mathbb{R}_{LC}$ is an extension of $\mathbb{R}_L$ by adding admissible reductions related to the CR−rule.*[8] *There is a closed normal derivation of $\bot$ in S_{LC} relative to $\mathbb{R}_{LC}$.*

Proof. Two claims verify this result.

Claim 1. There is a closed derivation Σ_2 of $\bot$ in S_{LC}.

First, an open derivation Σ_1 of $\bot$ from $[\neg\Psi]$ is obtained, as shown below.

$$\dfrac{[\neg\Phi]^1 \qquad \dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^3 \qquad \dfrac{[\neg\Phi]^1 \quad [\Phi]^2 \quad [\bot]^4}{\bot}\,\neg E_{,4}}{\bot}\,TE_{,2}}{\Phi}\,\Phi I_{,3} \qquad [\bot]^5}{\bot}\,\neg E_{,5}$$

8) Let S and S' constitute any natural deduction system. S' is an *extension* of S iff S' is S itself or results from S by adding further rules. Let $\mathbb{R}$ and $\mathbb{R}'$ be any set of reduction procedures. A set $\mathbb{R}'$ is an *extension* of $\mathbb{R}$ iff $\mathbb{R}'$ is $\mathbb{R}$ itself or results from $\mathbb{R}$ by adding further reductions.

Subsequently, a closed derivation Σ_2 of $\bot$ in S_{LC} is demonstrated below.

$$\dfrac{\dfrac{\begin{matrix}[\neg\Phi]^6\\ \Sigma_1\\ \bot\end{matrix}}{\Phi}\,CR_{,6} \qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^7 \quad \dfrac{\dfrac{\begin{matrix}[\neg\Phi]^1\\ \Sigma_1\\ \bot\end{matrix}}{\Phi}\,CR_{,1}}{T(\ulcorner\Phi\urcorner)}\,TI \quad [\bot]^8}{\bot}\,\neg E_{,8}}{\bot}\,\Phi E_{,7}$$

Claim 2. Σ_2 is in normal form.

Φ in the $\Phi E-$rule of Σ_2 is not a maximum formula because Φ is an atomic. There is no reduction procedure in $\mathbb{R}_{LC}$ applicable to Σ_2. Therefore, Σ_2 is in normal form. □

No reduction in $\mathbb{R}_{LC}$ applies to the liar sentence Φ when it is the conclusion of the $CR-$rule, due to its nature as an atomic formula in S_{LC}. It is essential to recognize that if an atomic formula, concluding the $CR-$rule, is also the major premise of the $\Phi E-$ rule, then Σ_2 does not qualify as a normal derivation. Utilizing the DL rule for *dilemma*, the ensuing conclusion can be drawn:

Proposition 3.2. *Let S_{LD} be an extension of S_L by adding the $DL-$rule. Then, there is a closed normal derivation of $\bot$ in S_{LD} relative to $\mathbb{R}_L$.*

Proof. Two claims verify this result.

Claim 1. There is a closed derivation Σ_4 of $\bot$ in S_{LD}.

Initially, a closed derivation Σ_3 of $\bot$ is established, as below.

$$\dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^2 \qquad \dfrac{[\Phi]^3 \qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^1 \quad [T(\ulcorner\Phi\urcorner)]^2 \quad [\bot]^4}{\bot}\,\neg E_{,4}}{\bot}\,\Phi E_{,1}}{\bot}\,TE_{,3}}{\Phi}\,\Phi I_{,2}$$

Then, by using $DL-$rule, a closed derivation Σ_4 of $\bot$ is obtained.

$$
\dfrac{\dfrac{[\neg\Phi]^5 \quad \begin{matrix}\Sigma_3\\ \Phi\end{matrix} \quad [\bot]^7}{\bot}\,\neg E_{,7} \qquad \dfrac{[\Phi]^8 \qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^6 \quad \dfrac{[\Phi]^8}{T(\ulcorner\Phi\urcorner)}\,TI \quad [\bot]^9}{\bot}\,\neg E_{,9}}{\bot}\,\Phi E_{,6}}{\bot}\,DL_{,5,8}
$$

Claim 2. Σ_4 is in normal form Given that Σ_4 lacks a maximum formula and is devoid of any applicable reduction procedures within $\mathbb{R}_L$, it follows that Σ_4 is in its normal form. □

In the context of Proposition 3.1, Σ_2 and Σ_4 of Proposition 3.2 constitute normal derivations of $\bot$. However, their inability to meet the requirements of *TCP* precludes them from being classified as derivations of a genuine paradox, as per the standards of the completeness conjecture. A notable issue is the perspective of Tennant (2016), who considered the liar paradox as a genuine paradox. Another concern is the ineffectiveness of *RND* as a solution to the liar paradox when Σ_2 and Σ_4 are taken as accurate formalizations of the liar. The scope of this paper, however, is centered on the latter issue with the assumption that the liar paradox is a genuine one.[9)]

The acceptance of the liar paradox as a genuine paradox, to sidestep needless controversies, highlights a significant point: the derivations Σ_2 and Σ_4 do not meet the requirements of *TCP*. Therefore, it becomes evident that *RND* does not offer a viable solution to the liar paradox when considering Σ_2 and Σ_4. In tackling the challenge, Tennant (2015a: p. 289) suggested that certain classical inferences, including *CR* and *DL*, introduce difficulties in the derivations associated with the liar paradox. He termed this issue as a *classical rub*.

9) For an exploration of the first problem, readers may refer to Choi (2023).

> Now here's the classical rub: this proof *appears to be in normal form*. The use of classical *reductio* has masked the real defect that lies at the heart of paradoxical reasoning (according to my account) – the abnormality that makes itself evident only when one hews to a constructivist line ... .

His view is that classical inference introduces the trouble absent in constructive reasoning. While there is a debate over the constructivist's endorsement of *CR* and *DL*, and many constructivists challenge their constructiveness, his emphasis is different. His concern revolves around the role of *CR* and *DL* in scrutinizing the proof-theoretic structure of paradoxes. There are two key issues: first, the liar paradox derivations utilizing *CR* and *DL*, such as Σ_2 and Σ_4, do not lead to the central aspect of paradoxical derivation, the infinite reduction sequence. Second, the application of classical inferences in formalizing the liar paradox is unnecessary. Tennant (2015a: pp. 589) puts forward the methodological conjecture as a response to the trouble, diverging from the perspective that the difficulty is limited to instances involving classical inferences.

> Paradoxes are never strictly classical. The kind of conceptual trouble that a paradox reveals will afflict the intuitionist just as seriously as it does the classicist. Therefore, attempted solutions to the paradoxes, if they are to be genuine solutions, must be available to the intuitionist. Nothing about an attempted solution to a paradox should imply that the trouble it reveals lies with strictly classical moves of reasoning.

Based on his methodological conjecture, which is also described as a methodological principle in Tennant (2017: p. 282), coupled with *RND*, there is an implication that avoiding classical inference in the construction of genuine paradoxes might resolve the issue that their derivations do not lead to an infinite reduction sequence.

Should the application of classical inference be independent of generating an infinite reduction sequence, then *RND* may be unsuitable as a solution for Tennant's genuine paradox. Section 4 addresses this issue, while the subsequent Section 5 explores whether strictly classical inference is truly unnecessary in formalizing the liar paradox, or if there is a possibility that the application of the liar sentence itself suggests an implicit use of classical inference. In such a case, Tennant's methodological conjecture would also be undermined.

4 *Classical Reductio* Does Not Affect the Generation of an Infinite Reduction Sequence.

This section demonstrates that the simple application of the $CR-$rule is not a contributing factor in generating an infinite reduction sequence. It becomes evident in the initial case that when the liar paradox is derived through the use of the $CR-$rule, an infinite reduction sequence is generated. Conversely, the second case reveals a derivation of the liar paradox, where the absence of strictly classical inferences results in the failure to generate an infinite reduction sequence.

Consider first a case where the use of the $CR-$rule leads to the generation of an infinite reduction sequence. The $CR-$rule is often regarded as an elimination rule, due to its role in negating assumptions within a formula. On occasion, the $CR-$rule is thought of as a succinct representation of both the $\neg I-$rule and the double negation elimination rule. Meanwhile, Milne (1994: p. 58) interpreted the $CR-$rule as a rule for introducing the formula φ; that is, the derivation of $\perp$ from the assumption $\neg\varphi$ introduces φ. This gives rise to the set of rules presented on the left. The $CRE-$rule and $\neg E-$rule differ from one another in that they have distinct major premises. The standard reduction procedure for the $CR-$ and the $CRE-$rules is delineated to the right.

$$\dfrac{\begin{array}{c}[\neg\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\varphi}\,CR_{,1} \qquad \dfrac{\varphi \quad \begin{array}{c}\mathfrak{D}_2\\ \neg\varphi\end{array} \quad \begin{array}{c}[\bot]^1\\ \mathfrak{D}_3\\ \psi\end{array}}{\psi}\,CRE_{,1} \qquad \dfrac{\dfrac{\begin{array}{c}[\neg\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\varphi}\,CR_{,1} \quad \begin{array}{c}\mathfrak{D}_2\\ \neg\varphi\end{array} \quad \begin{array}{c}[\bot]^2\\ \mathfrak{D}_3\\ \psi\end{array}}{\psi}\,CRE_{,2} \quad \rhd_{CRE} \quad \begin{array}{c}\mathfrak{D}_2\\ \neg\varphi\\ \mathfrak{D}_1\\ \bot\\ \mathfrak{D}_3\\ \psi\end{array}$$

We then obtain a system S_{LCE} by adding the $CRE-$rule to S_{LC}. In addition, $\mathbb{R}_{LCE}$ is obtained by supplementing $\rhd_{CRE}$ with $\mathbb{R}_{LC}$.

Proposition 4.1. *There is a closed derivation of $\bot$ in S_{LCE} relative to $\mathbb{R}_{LCE}$, which generates an infinite reduction sequence and is therefore not normalizable.*

Proof. Two claims establish this result.

Claim 1. There is a closed derivation Π_3 of $\bot$.

First, we have an open derivation Π_1 of $\bot$ from $[\neg\Phi]$.

$$\dfrac{[\neg\Phi]^2 \qquad \dfrac{\dfrac{[T(\ulcorner(\Phi)\urcorner)]^3 \qquad \dfrac{[\Phi]^1 \quad [\neg\Phi]^2 \quad [\bot]^4}{\bot}\,CRE_{,4}}{\bot}\,TE_{,1}}{\Phi}\,\Phi I_{,3} \qquad [\bot]^5}{\bot}\,\neg E_{,5}$$

With Π_1, there is an open derivation Π_2 of $\bot$ from $[\Phi]$ as follows.

$$\dfrac{[\Phi]^6 \qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^7 \qquad \dfrac{\dfrac{\begin{matrix}[\neg\Phi]^2\\ \Pi_1\\ \bot\end{matrix}}{\Phi}\,CR_{,2}}{T(\ulcorner\Phi\urcorner)}\,TI \qquad [\bot]^8}{\bot}\,\neg E_{,8}}{\bot}\,\Phi E_{,7}$$

Then, we have a closed derivation Π_3 of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[\neg\Phi]^2\\ \Pi_1\\ \bot\end{matrix}}{\Phi}\,CR_{,2} \qquad \dfrac{\begin{matrix}[\Psi]^6\\ \Pi_2\\ \bot\end{matrix}}{\neg\Phi}\,\neg I_{,6} \qquad [\bot]^9}{\bot}\,CRE_{,9}$$

Claim 2. Π_3 generates an infinite reduction sequence and is therefore not normalizable.

Since Φ in the $CRE-$rule of Π_3 is a maximum formula, we apply $\rhd_{CRE}$ to Π_3 to obtain the following derivation:

$$\dfrac{\dfrac{\begin{matrix}[\Phi]^{10}\\ \Pi_2\\ \bot\end{matrix}}{\neg\Phi}\,\neg I_{,10} \qquad \dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^3 \qquad \dfrac{[\Phi]^1 \qquad \dfrac{\begin{matrix}[\Phi]^6\\ \Pi_2\\ \bot\end{matrix}}{\neg\Phi}\,\neg I_{,6} \qquad [\bot]^9}{\bot}\,CRE_{,9}}{\bot}\,TE_{,1}}{\Phi}\,\Phi I_{,3} \qquad [\bot]^{11}}{\bot}\,\neg E_{,11}$$

The maximum formula $\neg\Phi$ persists in the last $\neg E-$ rule. By applying $\rhd_{\neg}$ and $\rhd_{\Phi}$, we obtain the following derivation.

$$\dfrac{\dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^3 \qquad \dfrac{[\Phi]^1 \qquad \dfrac{\begin{matrix}[\Phi]^6\\ \Pi_2\\ \bot\end{matrix}}{\neg\Phi}\,\neg I_{,6} \qquad [\bot]^9}{\bot}\,CRE_{,9}}{\bot}\,TE_{,1}}{\neg T(\ulcorner\Phi\urcorner)}\,\neg I_{,3} \qquad \dfrac{\dfrac{\begin{matrix}[\neg\Phi]^2\\ \Pi_1\\ \bot\end{matrix}}{\Phi}\,CR_{,2}}{T(\ulcorner\Phi\urcorner)}\,TI \qquad [\bot]^{12}}{\bot}\,\neg E_{,12}$$

Then, the application of $\rhd_{\neg}$ and $\rhd_{T(x)}$ yields the same derivation as Π_3. Therefore, Π_3 generates an infinite reduction sequence and is not normalizable. □

Although both the derivations Σ_2 in Proposition 3.1 and Π_3 in Proposition 4.1 use the $CR-$rule, Σ_2 does not satisfy *TCP*, but Π_3 does. Σ_2 causes the classical rub that it formalizes the liar paradox but does not generates an infinite reduction sequence, whereas Π_3 does not.[10)]

Π_3 illustrates that an infinite reduction sequence can still be generated even with the use of the $CR-$ rule.[11)] This raises the question: Could cases of classical rub occur independently of the $CR-$ rule? Should such cases be discovered, it would establish that the utiliza-

10) Milne's view on the $CR-$rule as an introduction may be slightly questionable in this regard. For instance, one may ask, "Can the $CR-$rule serve as a proper introduction?" and "Is it possible to prove the normalization theorem for classical logic using Milne's reduction?" His idea has not yet been systematically investigated. However, whether Milne's idea is acceptable is beyond the scope of this paper. The study primarily focuses on finding the cause that prevents the infinite reduction sequence. We use his idea to explain the fact that when every rule applied in the derivation has a corresponding introduction and elimination rule, the classical rub does not occur.

11) While certain critics may refute the above conclusion due to their rejection of Milne's rule, it remains possible to generate an infinite reduction sequence from Σ_2 in Proposition 3.2 by utilizing Stålmarck (1991)'s reduction for *CR*. This holds true regardless of one's stance on Milne's rule. For a more comprehensive examination of this matter, consult Choi (2021: Sec. 3).

tion of the $CR-$ rule in formalizing the liar paradox is not connected to the generation of an infinite reduction sequence.

In Tennant (2015a: pp. 585-589), he derives the liar paradox, applying the axiom $\ulcorner\Phi\urcorner = \ulcorner\neg T(\ulcorner\Phi\urcorner)\urcorner$ to the liar sentence Φ. The elimination rule for equality, $=$, first put forward by Martin-Löf (1971) and expanded upon by Tennant (2007) and Read (2016), is directly applicable to the axiom. The focus here is on applying the generalized form of the elimination rule for $=$.[12)]

$$\dfrac{t_1 = t_2 \quad \begin{matrix} \mathfrak{D}_1 \\ \varphi(t_1) \end{matrix} \quad \begin{matrix} [\varphi(t_2)]^1 \\ \mathfrak{D}_2 \\ \psi \end{matrix}}{\psi} = E_{1,1} \qquad \dfrac{t_1 = t_2 \quad \begin{matrix} \mathfrak{D}_3 \\ \varphi(t_2) \end{matrix} \quad \begin{matrix} [\varphi(t_1)]^1 \\ \mathfrak{D}_4 \\ \psi \end{matrix}}{\psi} = E_{2,1}$$

Applying $T(x)$ to φ within both $= E_1-$ and $= E_2-$ rules leads particular instances of $= E-$ rules, notably $= E_{T_1}-$ and $= E_{T_2}-$ rules.

$$\dfrac{t_1 = t_2 \quad \begin{matrix} \mathfrak{D}_1 \\ T(t_1) \end{matrix} \quad \begin{matrix} [T(t_2)]^1 \\ \mathfrak{D}_2 \\ \psi \end{matrix}}{\psi} = E_{T_1,1} \qquad \dfrac{t_1 = t_2 \quad \begin{matrix} \mathfrak{D}_3 \\ T(t_2) \end{matrix} \quad \begin{matrix} [T(t_1)]^1 \\ \mathfrak{D}_4 \\ \psi \end{matrix}}{\psi} = E_{T_2,1}$$

Then, a closed normal derivation of $\bot$ is obtained that formalizes the liar paradox without applying the $CR-$rule and does not generate an infinite reduction sequence.

Let S_T be a natural deduction system containing $TI-$, $TE-$, $\neg I-$, $\neg E-$, and $= E_T-$rules. S_T has a set $\mathbb{R}_T$ of reduction procedures for

12) Read (2016) formulated introduction and elimination rules for $=$ as higher-order rules. He also required that the formula φ be a predicate variable limited to monadic predicate letters, ensuring that $\varphi(t_1)$ and $\varphi(t_2)$ are atomic. Despite the generalized elimination rule for $=$ aligning with his proposal, it is not treated as a higher-order rule in this context. Read's restrictions are not considered essential for implementing the elimination rule for $=$ in here.

$T(x)$ and $\neg$ with the permutation conversions for the general elimination rules. For some formula Ψ, $\ulcorner\Psi\urcorner = \ulcorner\neg T(\ulcorner\Psi\urcorner)\urcorner$ is an axiom of S_T. Then, we have the following result.

Proposition 4.2. *There is a closed normal derivation of* $\bot$ *in* S_T *relative to* $\mathbb{R}_T$.

Proof. Two claims prove this result.

Claim 1. There is a closed derivation Π_5 of $\bot$.

First, an open derivation Π_4 of $\bot$ from $[T(\ulcorner\Psi\urcorner)]$ is established.

$$\dfrac{\dfrac{}{\ulcorner\Psi\urcorner = \ulcorner\neg T(\ulcorner\Psi\urcorner)\urcorner}\,Ax \qquad [T(\ulcorner\Psi\urcorner)]^1 \qquad \dfrac{[T(\ulcorner\neg T(\ulcorner\Psi\urcorner)\urcorner)]^2 \qquad \dfrac{[\neg T(\ulcorner\Psi\urcorner)]^4 \quad [T(\ulcorner\Psi\urcorner)]^1 \quad [\bot]^3}{\bot}\,\neg E_{,3}}{\bot}\,TE_{,4}}{\bot}\,{=}E_{T_1,2}$$

Then, there is a closed derivation Π_5 of $\bot$.

$$\dfrac{\dfrac{}{\ulcorner\Psi\urcorner = \ulcorner\neg T(\ulcorner\Psi\urcorner)\urcorner}\,Ax \qquad \dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Psi\urcorner)]^1\\ \Pi_4\\ \bot\end{matrix}}{\neg T(\ulcorner\Psi\urcorner)}\,\neg I_{,1}}{T(\ulcorner\neg T(\ulcorner\Psi\urcorner)\urcorner)}\,TI \qquad \begin{matrix}[T(\ulcorner\Psi\urcorner)]^3\\ \Pi_4\\ \bot\end{matrix}}{\bot}\,{=}E_{T_2,3}$$

Claim 2. Π_5 is in normal form.

Π_5 has no maximum formula and is irreducible. Hence, we have the result. □

In summary, Π_3 in Proposition 4.1 employs the $CR-$ rule to formalize the liar paradox and subsequently generates an infinite reduction sequence. Conversely, Π_5 in Proposition 4.2, which does not utilize the $CR-$ rule for the same purpose, fails to generate an infinite reduction sequence. This demonstrates that applying the $CR-$ rule does not influence the generation of an infinite reduction sequence.

The following section will discuss whether strictly classical inference is indeed unnecessary for formalizing the liar paradox, or whether the application of the liar sentence itself might imply an implicit use of classical inference.

5 The Liar Paradox May Imply Strictly Classical Inferences.

Tennant (2017: Ch.11) argues against the necessity of strictly classical inferences in the formalization of genuine paradoxes. The primary justification for this stance lies in the observation that paradoxes can be formalized within standard natural deduction systems, without resorting to strictly classical inferences. However, this argument may be undermined if standard natural deduction systems inherently permit strictly classical inferences. Should it be feasible to derive strictly classical inferences without explicit use of strictly classical rules, Tennant's foundation weakens, subsequently diminishing the support for the methodological conjecture.

A primary consideration is the challenge of articulating the intuitionists' notion of "prooflessness" within the framework of standard natural deduction. The liar paradox, for instance, receives an intuitionistic resolution by declaring the liar sentence devoid of truth value due to its lack of proof, thereby precluding the derivation of any contradiction as no consequences can be entertained. Accurate analysis of the liar paradox necessitates the capacity to express "prooflessness." The inability to convey this concept within standard natural deduction might render the system inadequate for properly evaluating the reasoning employed in the liar paradox.

Intuitionists typically attribute no truth value to the liar sentence, given that the intuitionistic interpretation of truth provides three classifications for determining a sentence's truth-value: true, false, and truth-valueless. A sentence φ is classified as true if and only if it possesses a proof, false if it has a disproof, and truth-valueless in the absence of both. According to this interpretation, 'φ is not true' carries three potential meanings. First, it's unknown whether there's a proof for φ. Second, it's known that there's no proof for φ. Third, there exists a disproof of φ. If we follow the first interpretation of the sentence, φ becomes a meaningless statement and thus carries no consequences.

As noted by Choi (2018), the standard natural deduction system lacks the capacity to express the intuitionistic concept of 'prooflessness,' and the systems introduced in this paper similarly lack the means to express a sentence without proof. From an intuitionistic perspective, the statement 'φ is false,' equivalent to '$\neg\varphi$ is true,' implies 'φ is not true,' but the converse does not hold. Notably,'φ is not true' can only be expressed in natural deduction as $\neg T(\ulcorner \varphi \urcorner)$, which is equivalent to $T(\ulcorner \neg\varphi \urcorner)$. (Cf. Choi (2018: Appendix A).) This shows that standard natural deduction systems, however, fail to capture this nuanced distinction.[13)]

Furthermore, in standard natural deduction systems, there exists a normal derivation for the liar sentence Φ, and it is possible to derive strictly classical inferences using this derivation. It has been noted that Tennant's strictly classical negation inferences include *CR*, *DL*,

[13)] The supposition that Tennant might equate 'φ is not true' solely with 'φ is false' could be entertained. However, this perspective fails to address the pivotal concept of 'prooflessness' in intuitionistic (relevance) logic. The absence of a means to articulate this crucial notion persists as a significant limitation. Acknowledgment is due to Professor Eunsuk Yang for bringing this interpretative possibility to light.

LEM, and double negation elimination *DNE*. Utilizing Tennant's rule for the liar sentence Φ, it is possible to derive a normal derivation $\mathfrak{D}_1$ for Φ with ease, on the left below. Subsequently, applying the $\vee$-rule results in the normal derivation $\mathfrak{D}_2$ of $\Phi \vee \neg\Phi$ on the right below.

$$\dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^2 \quad \dfrac{[\Phi]^4 \quad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^1 \quad [T(\ulcorner\Phi\urcorner)]^2 \quad [\bot]^3}{\bot}\,\neg E_{,3}}{\bot}\,\Phi E_{,1}}{\bot}\,TE_{,4}}{\Phi}\,\Phi I_2 \qquad\qquad \dfrac{\begin{matrix}\mathfrak{D}_1\\ \Phi\end{matrix}}{\Phi\vee\neg\Phi}\,\vee I$$

The derivation $\mathfrak{D}_2$ demonstrates that the law of excluded middle holds for the liar sentence Φ. Similarly, by employing the $\rightarrow I$-rule with an undischarged assumption, one can derive $\neg\neg\Phi \rightarrow \Phi$ as a normal derivation. Utilizing this, it can also be shown that *CR*, *DL*, and *LEM* hold specifically for Φ.

Of course, for a core logician(a.k.a an intuitionistic relevant logician) who accepts *RND*, since a normal derivation is a genuine (or a canonical) proof, it can be said that a sentence raising paradoxes like the liar sentence has a proof or a disproof, but not a genuine proof. However, since the liar sentence Φ possesses what he terms a canonical proof, making it a meaningful statement with a genuine proof, it is implausible to regard Φ as a truth-valueless statement, or as one having a genuine proof.

From the perspective of a core logician (a.k.a an intuitionistic relevant logician), who accepts *RND*, a normal derivation constitutes a genuine or canonical proof. Consequently, it can be argued that a sentence inducing paradoxes, such as the liar sentence, possesses a (dis)proof, albeit not a genuine proof. Yet, this necessitates explaining the significance of deriving a normal derivation of the liar sentence Φ through $\mathfrak{D}_1$ above. Φ is deemed to have a canonical proof, which establishes it as a meaningful statement with a genuine proof. This characterization precludes the classification of Φ as either truth-valueless or as having a genuine proof.

Can this stance be interpreted as suggesting that the liar sentence ϕ or the liar's rule for Φ implicitly implies a strictly classical inference? On this matter, Tennant (2021: p. 30) would assert that such a claim is incorrect because rules such as $\neg I-$ and $\rightarrow I-$ do not permit vacuous discharge.

> The systems of Core Logic and Classical Core Logic differ from the three aforementioned orthodox systems in certain crucial ways. ... The tweaks that the Core systems impose, of which there are three, are as follows. First, $\neg I$ does not allow vacuous discharge. This holds for both the Core systems. Likewise, in Classical Core Logic, Classical Reductio does not allow vacuous discharge.

Nevertheless, the restriction on vacuous discharge applies solely within the framework of Tennant's core logic. Given that the argument that classical inference is irrelevant to paradoxes is aimed at classicists, if Tennant's philosophical standpoint is grounded in his core logic, classicists are under no obligation to accept it. Therefore, his belief that classical inferences has no role to formalize the liar paradox in natural deduction holds true only within the context of his core logic.

Although this analysis falls short of demonstrating that strictly classical inferences play a definitive role in formalizing specific paradoxes within natural deduction, the arguments presented in Sections 4 and 5 substantially undermine the two primary reasons for Tennant's methodological conjecture. This weakening of support opens the door for a reevaluation of his conjecture.

6 Conclusion

In the present paper, Tennant's criterion for genuine paradoxes and his methodological conjecture regarding the role of strictly classical inferences in the paradoxical derivation of these paradoxes are explored. Tennant's criterion emphasizes the generation of an infinite reduction sequence, distinguishing genuine paradoxes from mere inconsistencies. He claims that formalizations of genuine paradoxes in natural deduction do not need to use strictly classical inferences, such as *LEM*, *DL*, *CR*, and *DNE*, thus supporting his conjecture that paradoxes are never strictly classical.

Tennant introduces the concept of 'classical rub,' arguing that strictly classical inferences, particularly classical reductio, introduce complications that obstruct the infinite reduction sequence essential for formalizing genuine paradoxes. His 'classical rub' highlights how classical inferences can mask the underlying abnormalities in paradoxical reasoning, further supporting his stance that genuine paradoxes can and should be derived within a framework excluding these classical inferences. He maintains that genuine paradoxes can be adequately addressed using a constructive and relevantist approach, avoiding the pitfalls of classical reasoning. By adhering to intuitionistic relevant logic, Tennant believes it is possible to circumvent issues arising from strictly classical inferences, thereby providing a clearer and more accurate formalization of genuine paradoxes.

Sections 4 and 5 critically examine Tennant's claims by investigating whether classical reductio impacts the generation of an infinite reduction sequence and considering the possibility that the use of the liar sentence inherently suggests classical inferences. Through detailed analysis, Section 4 argues that the use of *Classical Reductio* does not impede the generation of an infinite reduction sequence. This finding challenges his main reason for the methodological conjecture by demonstrating that classical reductio, contrary to Tennant's assertion, does not affect the infinite reduction sequence.

Additionally, Section 5 explores the characteristics of the liar sentence formalized in natural deduction. The analysis suggests that the derivation of the liar sentence leads to have the same form of strictly classical inferences. This is because standard natural deduction systems, including those introduced in the present paper, are irrelevant to expressing an intuitionist's (or core logician's) notion of prooflessness. This result calls into question Tennant's assertion that genuine paradoxes are devoid of any inferential role for strictly classical inferences.

In conclusion, the present paper challenges Tennant's two reasons for suggesting the methodological conjecture. It demonstrates that classical reductio does not hinder the generation of an infinite reduction sequence. Furthermore, it argues that it remains an open question whether strictly classical inferences have no role in formalizing genuine paradoxes in natural deduction. These results suggest that strictly classical inferences might still play a crucial role in the derivation of genuine paradoxes, contrary to Tennant's assertions.

The investigation calls for a reconsideration of the role of strictly classical inferences in the formalization of genuine paradoxes, advocating for a more nuanced understanding of the interplay between different logical systems in the formalization and resolution of genuine paradoxes. This critical perspective aims to advance the ongoing discourse in philosophical logic and contribute to a deeper comprehension of the foundational issues surrounding paradoxes and their derivations.

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거짓말쟁이 역설은 절대 엄격히 고전적이지 않은가?

최 승 락

본 논문은 자연연역 체계에서 (진정한) 역설을 형식화하는 데 순수히 고전적인 추론의 기여 여부를 탐구한다. 테넌트의 역설에 대한 기준은 무한한 환원열을 생성하는지에 의존하며, 이를 통해 그는 단순한 모순과 진정한 역설을 구별한다. 그의 방법론적 가설은 진정한 역설은 결코 엄밀히 고전적인 추론에 의존하지 않으며, 배중률, 딜레마, 고전적 귀류법, 이중 부정 제거와 같은 엄밀히 고전적인 추론 없이도 도출될 수 있다는 주장에 기인한다.

테넌트가 방법론적 가설을 제안한 이유는 두 가지로 보인다. 첫째, 엄밀히 고전적인 추론은 무한한 환원열의 생성을 방해하며, 둘째, 엄밀히 고전적인 추론은 진정한 역설의 형식화에 기여하지 않는다는 것이다.

본 논문은 이 두 가지 이유에 대해 의문을 제기한다. 특히 거짓말쟁이 역설에 초점을 맞추어, 엄밀히 고전적인 추론이 무한한 환원열의 생성을 방해하지 않으며, 거짓말쟁이 문장이 암묵적으로 엄밀히 고전적인 추론을 허용할 수 있음을 논할 것이다. 만약 이 분석이 옳다면, 이는 테넌트의 방법론적 가설 제안 동기뿐만 아니라, 거짓말쟁이 역설과 같은 진정한 역설이 엄밀히 고전적인 추론에 의존하지 않고 구성된다는 그의 주장에 대한 도전이 될 것이다.

주요어: 거짓말쟁이 역설, 진정한 역설, 엄밀히 고전적인 추론, 직관주의 연관 논리, 니일 테넌트.