THE CASE AGAINST INFINITY

by

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INTRODUCTION

Gazing into the sublime immensity of the starry night sky, pondering the awesome depths of the past and future, many hold that the Universe must be "infinite" in space and time. Now infinity is certainly a profound notion, holding a powerful emotional appeal across cultures. But while infinity is widely believed in, it has simply been taken for granted that infinity is a logically coherent concept. Such an assumption is mistaken.[1] I will argue that infinity is in fact a *logical absurdity* (that is, a self-contradictory notion) like a square circle or a four-sided triangle.[2] And since logical absurdities cannot refer to anything that actually exists, it follows that there is nothing infinite. I will show why we must conclude that the Universe cannot, therefore, be infinite in either space or time.[3]

My case against infinity as a coherent notion will begin by defining infinity and its related terms (such as infinitude, "the infinite," finitude, finite, indefinite, indefiniteness, etc.). Elucidating these terms will enable us to grasp the traditional understanding of infinity with enough detail to critically assess it. Once all of our terms are defined and understood, I'll then show why the traditional understanding of infinity, based on these definitions, is a self-contradictory notion.[4]

Now if my case against infinity is sound, some radical implications follow for certain fields of inquiry. For instance, even if mathematics is *operationally* consistent in terms of its use of rules for calculation, it cannot be *logically* consistent when it makes use of infinity—at least, not as infinity has been traditionally construed. Consequently, mathematicians should abandon the use of infinity in making calculations in favor of a more logically consistent alternative.

Since I'll show that infinity is not coherent and ought to be dropped as a mathematical value, the question naturally arises as to how mathematics would then conceptualize unfathomably large quantities in calculation. An alternative concept to the traditional notion of infinity must be proposed. Fortunately, such a concept is available to us—a concept called *indefiniteness*. After debunking the traditional notion of infinity, I will explain how indefiniteness serves as a rational alternative. Indefiniteness is not simply conceptual sleight-of-hand; it really is different from infinity in that it avoids the logical contradictions inherent in the notion of infinity but while still retaining the mathematical utility that infinity has. Indeed, to throw out the traditional notion of infinity in favor of indefiniteness would not lose anything significant to the field of mathematics; to the contrary, it would actually make mathematics even more logically consistent than it is now with the notion of infinity still in use. And so I will offer indefiniteness as the new mathematical concept that ought to be used in place of infinity.

In addition, the logical failure of the traditional notion of infinity, and the necessity of replacing infinity as a mathematical value with indefiniteness, carries even more serious implications for physics and cosmology: If infinity as a mathematical value is logically absurd, and if logical absurdities refer to nothing that can actually exist, then infinity as a mathematical value refers to nothing that really exists—at least, not according to the traditional definition of infinity. As a result, measures of space and time cannot really be "infinite" in the usual sense of the term. Thus in showing the self-contradictions involved with the traditional notion of infinity, I will also be presenting reasons why cosmology and physics must hold that neither space nor time can be infinite—no matter how *indefinite* the vastness of space or time may be, the Universe as a whole must still be finite.

DEFINING THE TERMS

We will begin with defining our terms in order to capture some important distinctions. After some explication of the concepts we will be ready for the arguments showing infinity to be a logical absurdity.

INFINITY AND THE INFINITE

Let's start with the term *infinitude*, which means "limitlessness" or the condition of being without limit. Mathematicians and philosophers sometimes make a distinction between two forms of infinitude. One form of infinitude is a limitless sequence known as "infinity."[5] A second form of infinitude is a limitless set sometimes called "the infinite."[6] In both mathematics and academic philosophy there is no clear distinction between the terms infinity and the infinite. In fact, the term infinity sometimes refers to an infinite set while "the infinite" is sometimes synonymous with infinity as a limitless sequence and does *not* refer to an infinite set. For the sake of clarity, I will impose some consistency on these terms by distinguishing between *infinity* and *the infinite* as two different ways of being limitless.

It is important to distinguish between the infinite and infinity. In some cases mathematicians represent the infinite by the symbol $\aleph_0$ (known as aleph-naught or aleph-zero), which designates a *limitless set*. In contrast, they represent infinity by a "lazy eight" symbol called the lemniscate or Mobius strip, denoted as ∞, which designates a *limitless sequence*. Infinity is not the same as the infinite, a limitless set—a collection of things existing *all at once* with no limit to its members. Instead, infinity is a succession or series of things that exist *one after another* in a sequence that has no limit. Both forms of infinitude—infinity as a kind of succession or series and the infinite as a kind of set—are supposed to be "limitless," but not in the sense of having an unknown limit. Rather, both infinity and the infinite have *no limit at all*.

FINITUDE VERSUS INFINITUDE

To have no limit may seem like a straightforward idea, but it is a bit tricky as we'll see. To properly understand the limitlessness of infinity, we first need to consider what it means to have a limit—what it means to be finite.

The word "finite" comes from the Latin words "finitus" (which means to end, bound, or limit) and "finis" (end or limit). In short, if something was regarded as finite, it was thought to have a bound, end, or limit to its measure.[7] The corresponding term "finitude" means simply the quality or condition of being finite. There is nothing illogical about these etymological definitions.

However, there is one caveat: while that which has a boundary or end is indeed finite, something can be finite and yet not have an end or boundary. All bounded things are finite, but not all finite things are bounded. Though that might sound paradoxical at first, a simple example should clear it up: consider a circle. In fact, take any figure, set, series, process, or system in which the sequence of elements within it curves back on itself—such as the circumference of a circle or sphere. The circumference has no end points or boundaries and yet is still finite in the sense of being limited in size or scope as shown by the diameter of the circle. A circumference is thus said to be limited while also being unbounded. So, I take the ordinary meaning of finite as referring to that which is *limited*, but not necessarily having an end or boundary, since there are extents like the circumference of a circle that have no end points or boundaries (edges) and yet are still finite in the sense of having limits in size or scope. In ordinary language then, to be finite is to be limited.

Now let's get a little more technical. In mathematics, to be finite is to be *limited in quantity*. By "quantity" I simply mean to have a positive or negative numerical value. If something is mathematically finite, it is measurable by a positive or negative numerical value that is limited. This is a rather broad way to define finitude, but as we'll see, it is the right definition for mathematics.

But it isn't always the definition used in mathematics. Mathematicians sometimes say that to be finite is to have a limit in reference to the scale of *natural numbers*. Natural numbers are usually identified with positive integers such as 1, 2, 3, 4, 5, etc. Sometimes mathematicians include 0 in the set of natural numbers, in which case the natural numbers refer to all numbers that are non-negative integers. Other mathematicians refer to only the set of positive integers as the natural numbers, while the set of natural numbers together with zero constitutes the set of *whole numbers*. For the sake of clarity, I will take the latter position, identifying natural numbers as the set of positive integers and whole numbers as the set of natural numbers with zero included. With these distinctions in place, one of the usual mathematical definitions of the word "finite" is to have a number of elements (for example, objects) capable of being put into a one-to-one correspondence with a segment of natural numbers.[8]

A "one-to-one correspondence" refers to the ordered matching of members between two sets. The members of one set (which we'll call A) are evenly matched with the members of a second set (we'll call it B). As mathematician J.R. Maddocks explains,

> Evenly matched means that each member of A is paired with one and only one member of B, each member of B is paired with one and only one member of A, and none of the members from either set are left unpaired. The result is that every member of A is paired with exactly one member of B and every member of B is paired with exactly one member of A. In terms of ordered pairs (*a*, *b*), where *a* is a member of A and *b* is a member of B, no two ordered pairs created by this matching process

> have the same first element and no two have the same second element. When this type of matching can be shown to exist, mathematicians say that "a one-to-one correspondence exists between the sets A and B."[9]

As a simple example, take a set of five apples that can be numbered completely with the segment of natural numbers 1 through 5. By counting the apples—one, two, three, four, five apples—we put the apples and the positive integers (which, in this case, represent a segment of natural numbers) into a one-to-one correspondence. To be "finite" in this sense, then, is to be able to be put into just such a correspondence. In this sense of the term, to be finite is for a quantity or set to be either (a) *countable* using a terminating sequence of natural numbers or (b) accurately *designated* as belonging to a terminating sequence of natural numbers.[10] The example of counting the apples is a case of the former and just labeling the apples one through five would be a case of the latter.

These mathematical definitions of what it means to be finite imply that the scale of natural numbers *contains* sequences of numbers that terminate when they are selected as subsets taken from the entire scale, but the definitions leave vague whether or not the scale itself has a definite end and so is limited. If the scale itself terminates, then it is finite. If the scale as such has no end, then it may not be finite—it could be *infinite*.

Let's consider the first option—that the scale of natural numbers has a definite end. Such cannot be true. After all, for any number in the scale, you can (at least in principle) always calculate one number higher. If you can always go higher, then you can go higher *without end* for all time. The scale of natural numbers has "no end" in this sense. But does that mean it is infinite?

That brings us to the second option: it might be assumed that since the scale of natural numbers is without end, it surely has no limit at all and so *must* be infinite. Ah, but not so fast! Perhaps the scale of natural numbers has no end, but that doesn't mean the scale of natural number has *no limit*.

There is a difference between being endless and being limitless. As we've seen, a circle's circumference is endless (there are no end points) but it is also limited, not limitless—one circle can fit inside another like the rings on a target, and the circumference of a circle can be expressed by a rational number. Hence, to be unending is not necessarily the same thing as to be without limit, and so to be unending is not necessarily the same as being *infinite*. As a circle's circumference is endless but not limitless, so too the natural number scale may be endless but not limitless. Therefore, the scale of natural numbers may be endless, but that does not necessarily mean it is "endless" in the sense of being *infinite*.

FINITE—COMPLETE AND INCOMPLETE

On the other hand, while the natural number scale could be endless without also being limitless or "infinite," it also does not need to be endless by being unbounded like a circle. Instead, perhaps the natural number scale should be thought of as "endless" merely in the sense of being always *incomplete*.

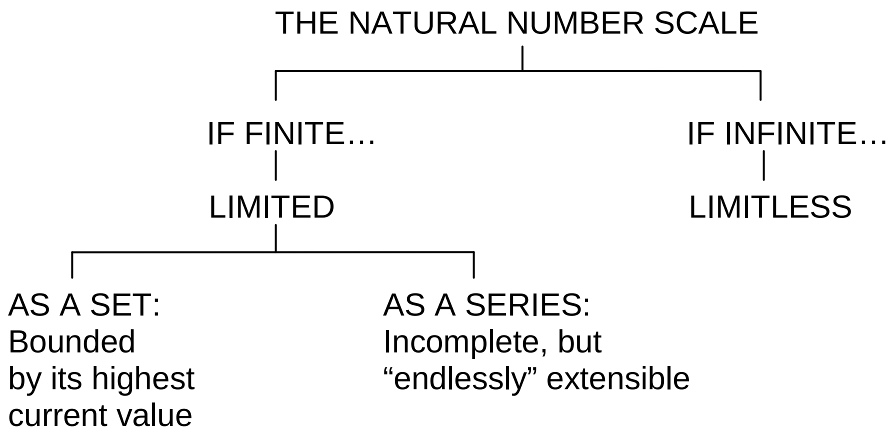

Figure 1: The natural number scale interpreted as either finite or infinite.

The best way to grasp this view of an endless-but-limited natural number scale is by an analogy that distinguishes between *sets* and *series*. A set is a collection of distinct things that exist all at the same time (like the set of inner planets: Mercury, Venus, Earth, and Mars) while a series is the sum of a sequence of things that have existed over time (such as a series denoting the sequence of all species that have ever lived on Earth). With this distinction in mind (also see **Figure 1**), we're ready for an analogy that expresses why the natural number scale can be “endless” but also limited or finite:

Imagine a cylindrical tower built of bricks. The bricks spiral upward from the base to the top of the tower. If the top of the tower is capped by a rooftop, we'll call the tower *complete*—it cannot be made any taller. The capped tower then has an end (and so a limit) to its height; it is finite. But suppose the tower has no roof and instead has merely an unfinished sequence of brickwork at the top. If construction has ceased, the tower stands as an *incomplete* set of bricks but is finite nonetheless. Considered just as a set of bricks, the tower can be complete or incomplete—limited in either case.

Now let's consider the tower as a series rather than as just a set. Suppose that construction has not stopped and will never stop. For as long as there is time the tower will always be made ever taller. In that case, because it is perpetually under construction, being made ever taller, the tower always remains incomplete as well—it is never finished. It is an ongoing process, a series of bricks being laid “without end.” But even though the top of the tower is a sequence of brickwork that continually grows higher, the tower always remains finite because it always has a limited number of bricks from its base to its top at any time. No matter how many bricks are added, no matter how high the tower becomes, there is always a countable number of steps to the top.

By analogy, the sequence of numbers in the scale of natural numbers is like the increasing brickwork in the perpetually heightening tower. Since there is no “roof” to the scale of natural numbers—no highest number beyond which we cannot continue to extrapolate—the scale of natural numbers is never complete. Instead, the scale of natural numbers is like the tower that is never finished—there is always one more brick that could be laid, always one more number that could be calculated, but never a last one. Just as with the growing tower that is always incomplete, so too the natural number scale can be continually extended but will always be incomplete. And because both the fictional tower and natural number scale are inherently incomplete, they are both "endless" processes under construction.

Moreover, although they are both endless as processes with series of steps, the tower and the scale of natural numbers are always finite at any given time. There are a limited number of steps to the highest brick in the tower, and there are a limited number of values to the highest number in the scale of natural numbers at any given moment. Whatever the distance the highest brick is to the base of the tower or highest number in the scale of natural numbers is to the number 1, there are a finite number of steps to the top at any time; no matter how long we go on building the tower or inventing new numerical values, the steps are always countable and there is a highest definite number. Both are processes containing series of steps "without end," but both are also limited as sets of elements, and so finite. As a perpetually constructing tower remains always unfinished but nevertheless finite, so the scale of natural numbers remains always incomplete but finite.

Put another way, the natural number scale could be construed as "terminating" wherever the sequence of natural numbers currently leaves off—it would be finite in that regard since there is always a highest number in the set that has been or that can be calculated for it at any given time. And the scale would also be "endless" as a series because its limit can always be *extended* as long as there is time to do so and no final values have been or will be calculated. Thus, having a limit that is "endlessly" extensible in this way merely means that the natural number scale is inherently and always *incomplete*, not that it is infinite.

If this analysis is correct, then finitude could be a property describing the entire scale of natural numbers. To be finite would not, as traditional mathematics has it, to be able to be put into a one-to-one correspondence with just a *segment* of natural numbers. No, finitude would also be a property of the scale of natural numbers taken as a set at any given time and as an incomplete series. That is not to say, of course, that segments of the natural number scale are not finite—far from it! A segment of five numbers is definitely finite. It is merely to say that the term "finite" should not be defined in too narrow of a sense as corresponding *only* to segments of natural numbers; it can also be applied to the scale of natural numbers itself if we take that scale as being an incomplete product under construction.

FINITE—DEFINITE AND INDEFINITE

So far we've seen finite sets and series. We've also seen how they can each be either complete or incomplete. Since there are real finite towers that are roofed and finished while others are left unfinished or in ruins, we can say that the property of finitude contains both complete and incomplete sets. And since scales of numbers are extended by processes of calculation over time in which new values arise in a series, we can also say finitude encompasses complete and incomplete series.

But now completeness and incompleteness also need to be distinguished from a couple of related categories—*definitude* and *indefinitude*. Along with being complete or incomplete then, a finite set or series can also be "definite" or "indefinite."

At this point I should acknowledge that some philosophers distinguish three categories of numeric values or collections: the finite, the infinite, and the indefinite.[11] But I am taking a different approach. The term "indefinite" will designate not a separate category that stands *alongside* the quantitative categories of the finite and the infinite as something different and outside those two categories together. Instead, "indefinite" refers to a subcategory of finite values or collections alongside another subcategory of finite values or collections called "definite."

> The *definite* is that which the form and limits are determined and are apprehended by us. That of which we know not the limits, [fallaciously] comes to be regarded as having none; and hence *indefinite* has been confounded with the *infinite*, though these two must be carefully distinguished. The *infinite* is absolute; it is that which has and can have no limit; the *indefinite* is that of which the limits are not known to us. You can suppose it enlarged or diminished, but still it is finite.[12]

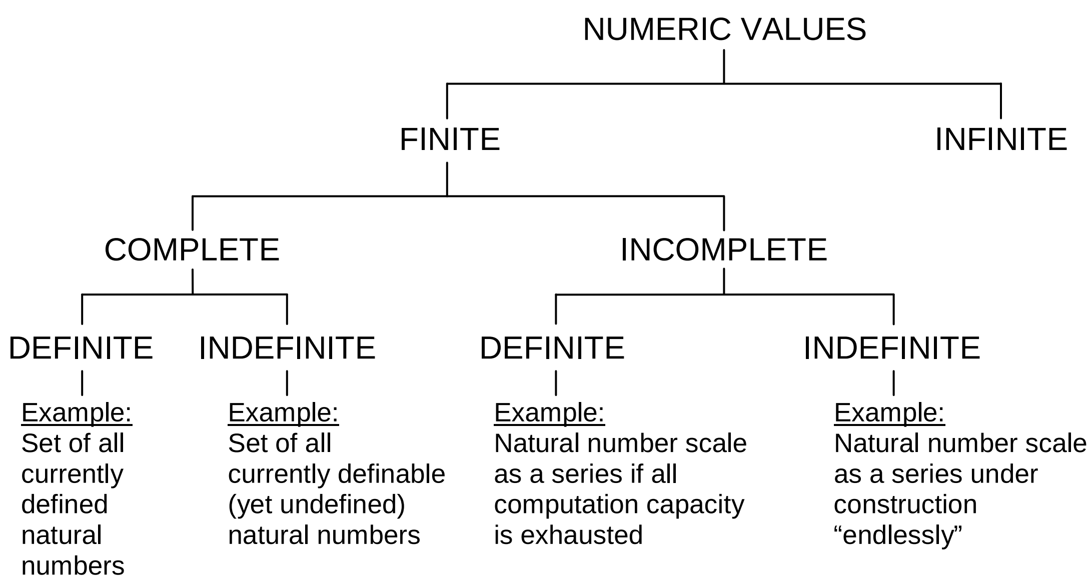

Figure 2: Numeric values categorized by reference to limits.

Along with the complete/incomplete distinctions then, the finite also contains two additional subcategories: definite and indefinite (**Figure 2**). Further, just as *infinitude* can be divided into "the infinite" and "infinity," so too *indefinitude* can be divided into two related concepts: "the indefinite" and "indefiniteness." In some passages I may use the terms "indefiniteness" and "the indefinite" interchangeably. Other passages will clearly draw a distinction between these terms, in which "indefiniteness" indicates an undefined value or sequence of values beyond the defined values in a *series* while "the indefinite" refers to a *set* of undefined values. But I will endeavor to make clear the manner in which I use these terms in the context of each passage.

These distinctions become important when we analyze the scale of natural numbers. Any known segment of the natural number scale is composed of finite values that are definite (defined or determined). But suppose the natural number scale also contains unknown finite values that are indefinite (undefined or undetermined).

There are finite numbers no one has ever calculated and that cannot currently be calculated because their values are off any known scale of natural numbers. That is, there are some finite numbers that have not been invented since they either (1) have values vast beyond the largest finite numbers *so far* conceived or (2) also have values beyond the largest finite numbers that can *in practice* be defined, and yet have finite values that are nevertheless definable *in principle*.

Indefinite numeric values are those values off the defined and known scale of natural numbers, but such numbers while not necessarily definable in actual practice are still definable "in principle"—if only we just had the means (say, a more powerful computer), we could define these higher finite numbers in the set until if/when we have exhausted all our current computing ability to go higher.[13] Such numbers are still finite rather than "infinite" because their values would have limits if they could be calculated; it's simply that those limits would be higher than any value anyone can now *actually* calculate. You can think of these unknown values as potentials latent in the scale of natural numbers—they are reachable in principle even if they cannot be reached in practice. Because these numbers are still finite, it isn't the term "infinite" that describes them; instead, they are best referred to as simply *indefinite*.[14]

To get a clearer idea of what a finite-but-indefinite value would be, consider an example from the classic novel *Watership Down*, by Richard Adams. In the novel, rabbits have their own culture and language. They also have a very primitive mathematics; the rabbits can only count up to the number four. In a footnote, the author explains that to the rabbits any number above four is *hrair*—"a lot."[15] In contrast to Adams's rabbits, we know that four is followed by five and a whole series of higher, but nevertheless finite, numbers. And yet, even our scale of natural numbers, no matter how high the numbers in the scale are calculated, has a *highest* number ever defined up to the present. Perhaps somewhere in the Universe there is an extraterrestrial species with a computing system that can count so fantastically higher than we that our number scales would seem as puny to theirs as the number scale of Adams's rabbits do to ours. Even so, whatever number that anyone, or any being in the Universe, shall ever define as the highest number calculated, such a number and the scale to which it belongs would still be finite wherever the scale leaves off. All the numbers that would lie beyond what anyone knows are simply "a lot"—they go undefined even though they too would be finite if they could be counted.

Now it is true that despite what anyone will ever come up with by calculation or extrapolation (mentally or mechanically) as the highest number conceived, there is nevertheless a logical possibility that, if one only had more computing power, yet another finite number could be defined. But this means only that it is logically possible for there to be numbers that, if they *could* be invented, *would* nevertheless have terminating values and so still be finite. And because this implies limits, albeit extended limits, the scale of natural numbers would remain finite at any step of the way. The natural number scale would have only a finite number of values actually *defined* at any time no matter how far we calculate or extrapolate it.

Looking at the scale of natural numbers this way, we see that there is more than one way for a set of anything to be finite. One way is for the set to be limited because it is both complete and its elements are entirely defined—which does not describe the natural number scale because, while it certainly contains defined elements, not *all* of its elements are defined. Another way for a set to be finite is to contain not only a subset of defined elements, but also a subset of *definable* but currently undefined elements. Since those undefined elements are in principle definable, that also means the subset of defined elements could be increased, which in turn means the superset as a whole (containing both defined and undefined elements) must *also* be considered as a *series*. Suppose the currently undefined elements in that series later become defined, but as a result new undefined elements are generated with the potential to be defined. That would make the series inherently incomplete—a work in ongoing progress in which there are always new definable values that can later become defined. And being incomplete, the series would "terminate" or stop only where it leaves off at any time according to its defined and definable values. So, the series as such would be "indefinite." This describes the scale of natural numbers.

The natural number scale as a set, then, contains defined values and undefined ("indefinite") values at any given time, while the scale itself is also "indefinite" as a series. When the currently undefined values become defined later on, yet more undefined values are generated as new *potentials* to be defined. And so the undefined values currently in the natural number scale that lie latent by virtue of being logically possible (but either not invented yet or not conceivable in actual practice) are only undefined or "indefinite" temporarily, and potentially new ones are always created as the defined subset of numbers in the scale increases in length. (This ongoing process is simplified in **Figure 3**, as I have no idea what the highest value calculated to date is on the natural number scale.)

Since this view fits our experience of working with natural numbers, I am proposing that finitude should be regarded as characterizing the scale of natural numbers in this way—as a set and as a series.

INDEFINITE BUT STILL LIMITED

To briefly sum up on the subject of indefinite sets and series:

The indefinite applies to finite sets, but only to those portions of which are currently undefined. As a subset of the finite natural number scale, the indefinite contains those higher values that are undefined in practice, but are potentially definable in principle. That is, on the scale there are always potential values that are undefined or "indefinite" since they are never reached in actual practice. The scale of natural numbers also contains a *highest* known and defined value at the present time as well as a highest unknown but potentially definable value at the present time (hence these current limits also make the scale finite).

THE NATURAL NUMBER SCALE
AS AN INCOMPLETE FINITE SERIES

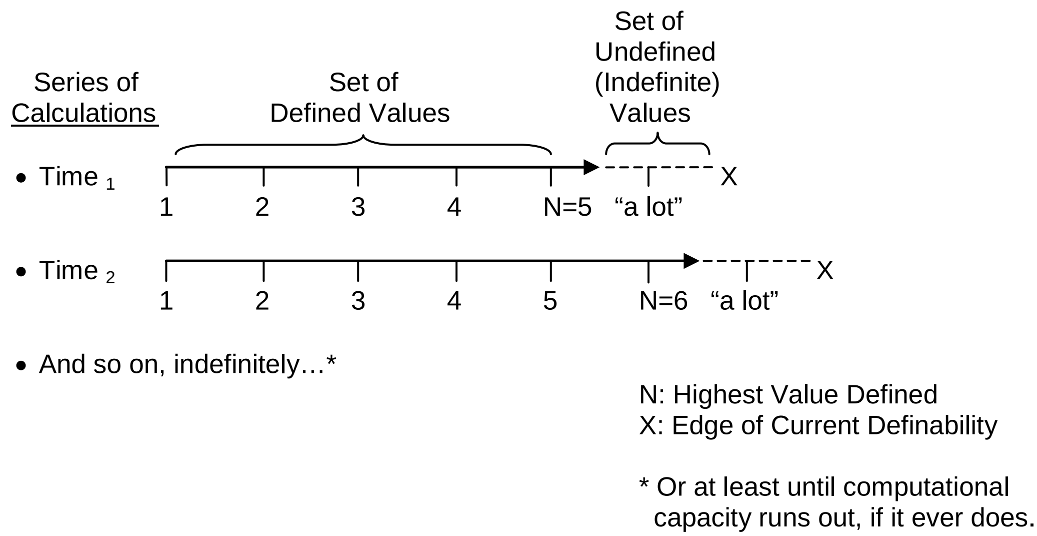

Figure 3: The natural number scale extends as new values are invented during calculation. (The scale is greatly simplified here for convenience.)

So, while at any given time a set of numbers will have a highest value, some sets can be extended in magnitude. For example, the set of all natural numbers may in principle be definable up to a highest value possible right now, but that value could be surpassed when computing power increases and a new possible highest value becomes a potential. Just as Olympic records were meant to be broken, so too the highest values possible now are intended to be overcome by higher possible values later. As computation proceeds from one occasion to the next, topping the last highest number able to be calculated, we see that indefiniteness must apply to finite *series*. As already suggested, the natural number scale may be one of these series. If so, defining finite numbers and values for the natural number scale is an ongoing process that is worked out in a series of steps that is never complete, making the series itself indefinite. The term "indefiniteness" designates a sequence or series that is both non-zero and has "no end" in the sense that the scale, though finite in sequence at any time, can at least in principle be perpetually added onto with more, limited, values that have not yet been defined.

This assumes, of course, that we never run out of time and computing power, which is disputable. Some computer scientists believe that it may not be physically possible to compute the information content for real numbers beyond 2^{512} bytes of computer memory storage. To get an idea of how large a value that is, 2^{32} bytes may only be 4 gigabytes but 2^{1000} is larger than the number of electrons in the perceivable universe. So, rational numbers having integers composed of 2^{512} digits are outside the range of computable numbers. Even computing the value of a real number equal to 2^{50} bytes worth of storage may not be feasible. If true, these limits would make the scale of natural numbers also explicitly bounded in terms of how many digits a value can have.[16] Nevertheless, this still leaves a lot of room for defining higher values of natural numbers for the foreseeable future.

Regardless, indefinitude can be either a set or a series—a set that contains undefined numbers and values or a series that is itself incomplete—or it may be considered as both a set in one sense and a series in another. Recall that the tower analogy distinguished between sets and series, but the perpetually unfinished tower of natural numbers was described as indefinite in both senses. The natural number scale could be limited as a set containing indefinite values, while simultaneously able to have more values defined for it without any foreseeable end—hence growing "indefinitely" as a series.

Once again we see that there is no need for the natural number scale to be necessarily *infinite*. The natural numbers may be always finite and merely indefinite in scale. Because of this possibility, the natural number scale as such does not reveal the distinction between finitude (or, more specifically, indefinitude) and infinitude.

INDEFINITE AND INFINITE: BOTH ARE INCALCULABLE

There is yet another common difficulty with distinguishing infinitude from indefinitude that must be resolved. The term "infinite" has been defined by mathematicians as a quantity that is limitless because its value is *greater* than any terminating sequence of natural numbers.[17] And this mathematical definition in turn implies that infinity is a value that cannot be counted or "a value greater than any computable value."[18] Though these definitions are meant to distinguish infinitude from indefinitude, they still have a few problems making that distinction.

An "indefinite value" could be taken to designate a value in a set that would, in principle, be defined if anyone had the means to reach it even when in actuality they don't. It *would* fall in any known natural number scale if it could be counted, but because it cannot be counted its finite value is off the scale. A value can also be "indefinite" in a series if it corresponds to the "next number" off the (incomplete) scale of natural numbers—a value too big for anyone to think up or any machine to calculate because you cannot reach the "next number" beyond what can be actually computed in the present. In either set or series then, "the indefinite" designates numeric value that is also "greater than any terminating sequence of natural numbers" in the sense that, though finite, the value would lie just beyond where the *known* set of natural numbers actually does leave off ("terminate") at any time—so, such value is indefinite.

As we saw earlier, a good example is to suppose no one can count higher than 4. In that case, the sequence of natural numbers "terminates" at 4 since no one is able to count higher. The next number higher than 4 that could be counted if anyone had the ability to count higher (and in reality we know it is a finite number) would have to be labeled as "indefinite" by everyone since they wouldn't be able to count higher. Whatever sequence of natural numbers that could ever in practice be invented, there is always in principle a number or value that is finite (just a little greater), a number that could be invented if only the time or resources existed—that value is not "infinite" since it would have a limit; it is simply undefined or "indefinite" since no one is able to invent it.

Another way to designate indefiniteness is to represent the set of natural numbers as 1, 2, 3, 4…*n* where *n* is equal to a finite number that is indefinite (not defined) because it is above the highest number actually defined thus far. Hence, *n* is "greater" than a set of "terminating natural numbers" in the sense that *n* is greater than any number anyone can in practice define or compute. But whatever value *n* could have, it is still finite even if its limit is not defined (given a specified value) as the limits of values calculated from the former numbers in the scale are defined.

Notice that indefiniteness, as the undefined greatness of a quantity, fits the very definition of what is supposed to be infinity; indefiniteness, like infinity, is also "greater than any computable value" because indefinite values can also be too vast to count in actual practice. If indefinite values can be defined as greater than any computable value, then this makes what was supposed to be infinite indistinguishable from what is actually indefinite. Indefiniteness and infinity become conflated.

This conflation is a problem because infinity is not supposed to be the same thing as indefiniteness: to be infinite is to be *not finite*; to be indefinite is to be *undefined* in value but still finite in principle. Unlike indefiniteness, infinity doesn't have a limit that is beyond what is actually calculable; rather, infinity is *without* limit. That's supposed to be the difference. And this difference indicates that we cannot mathematically define infinity as we did in terms of being simply incomputable or as being greater than any terminating sequence of defined natural numbers. If we do, we have no way of distinguishing infinity and indefiniteness since both have values that are off any actually defined scale.

BUT INFINITUDE HAS NO LIMIT

What distinguishes infinitude from indefinitude is not the incalculability of the scales, but the very idea of having or not having limits to those scales. To be indefinite is to have an *unknown limit* in quantity, but to be infinite is to have a non-zero, positive or negative, quantity with *no limit*—to be infinite is to have *limitless quantity*.

It is misleading then, when mathematicians sometimes define infinity in more technical terms, as being a kind of limit. Mathematicians sometimes say, for example, that infinity is "the limit that a function f is said to approach at $x = a$ when $f(x)$ is larger than any preassigned number for all x sufficiently near a."[19] Put in simpler terms, the function of dividing a value, such as the number 1, in half is repeated on each successive result in order to get closer to equaling another number, such as 0. The function allows the process of division to continue from 1 to $^1/_2$, $^1/_4$, $^1/_8$, and so on toward 0. Thus, the limit of the function is the limit of the division process as each value, starting with 1, is divided closer towards 0. But this recursive process of *approximating* zero never actually reaches zero, no matter how long it is carried out. So, while mathematicians use the term "infinity" to indicate the limit of the function, this is actually misleading both because the function is assumed to have no limit and because the term "infinity" literally means to be limitless.[20] Hence, the mathematical definition of infinity as a kind of limit implies a limitless limit, which is self-contradictory.

One way around this contradiction may be to contend that saying infinity is the "limit" of the function f is really just a metaphorical way of saying the function has no limit at all—division can be carried on without limit. That avoids the contradiction, because there is really no sense in which you can "approach" or draw "near" infinity, let alone reach it, since infinity is supposed to be a state that is just as far away no matter how long you make progress. Indeed, what the function "approaches" in the example is zero, not infinity.

However, even using the term "infinity" as a metaphor for the limit of the mathematical function is misleading because the function f may actually be a process of recursive approximation that has an *indefinite* limit rather than *no* limit. The term "indefinite" means to have an undefined limit. The limit may be undefined because it is unknown (so great or minute that it is off any known scale) or it may be undefined because the magnitude or process in question is always left incomplete no matter how much gets added to it. Either way, the indefinite refers to a limited amount that simply goes undefined. Because the indefinite is limited, it is finite and so stands in contrast to the infinite. The infinite does not have an unknown limit; rather, the infinite has absolutely no limit. In terms of function f, perhaps the function is one that one could carry on *indefinitely* (without a defined limit) but not *infinitely* (without any limit).

If, for example, computing power must eventually be exhausted, then the function of division cannot be carried on without limit. In that case, the approximation can only come so close and no closer; x could then only "approximate" the value a up to the point where computability is exhausted. However, we may not be able to define exactly how much computing power we can have before we can compute no further, and therefore we may not be to tell how far the division process can proceed, so it would have to be considered indefinite. Now, as far as we can tell, the limit of the function can always be *extended* (rather than "approached") given more computing power and time to compute, but again the function is not necessarily "limitless" in capacity as there may be an inherent limit to computing power. To say, then, the limit of the function f is "indefinite" rather than infinite would just be to say that the function's limit (such as the limit to the process of actually *dividing* closer to zero) is unknown rather than necessarily nonexistent.

Regardless of which interpretation of the function—either unlimited or indefinitely limited—is correct, to be infinite is not to be indefinite. It would therefore be more accurate to say that infinity, according to its traditional conception, is a *limitless succession or series*—a sequence that has no limit because it is inexhaustible—rather than a limit that can be "approached."

THE INFINITE AS LIMITLESS AND COMPLETE

Both the infinite and infinity are limitlessness in quantity; the former is to be limitless as an entire set of things that exist all at once and the latter is to be limitless as a sequence. Now by presenting a case against the traditional notion of infinity, I will actually be arguing against *both* kinds of infinitude—infinity and the infinite. That is, I will be arguing both against infinity as a limitless sequence and against the infinite as a limitless quantitative set. It's my contention that neither notion of infinitude really makes logical sense.

I will address both kinds of infinitude, showing why they are not logically coherent. Before debunking infinity, I'll start with the other kind of infinitude, showing the self-contradictions involved with "the infinite."

The contradictions in the traditional understanding of the infinite have to do with the concept of sets. A set is a collection of distinct elements, such as numbers or objects, classed together. A traditional notion of the infinite is that it is supposed to be a non-zero set (a set with a positive or negative number of members) that also has "no limit" in the sense of having no highest definable value even in principle. Consider again that if the scale of natural numbers were infinite, then even if you had the means to calculate higher values on the number scale than those that can be defined now or ever in actual practice, you still would never reach a value you could call the highest value of all. (Further, infinity, as a limitless succession of steps in the set, cannot be reached because the infinite—the set as a whole—is inexhaustible.)

But the infinite is not just limitless; the term "infinite" is also meant to describe any set that is *complete* as well as limitless. If an infinite set were *incomplete*, it wouldn't really be infinite—it would merely be indefinite. If something has an unknown or unknowable limit because it is inherently incomplete, then it is indefinite; by contrast, the infinite is not supposed to have an incomplete limit. Rather, the infinite is without any limit while still being complete.

As we shall see, however, the very notion of the infinite as both complete and limitless makes the infinite an inherently self-contradictory notion. A complete-limitless thing is an oxymoron. I'll show that this is exactly the case with a demonstration of how assuming that the infinite is both a complete set and simultaneously a limitless set creates logical contradictions.

TRANSFINITE CALCULATIONS

If the infinite is a complete-limitless set, then one should be able to conceive of the possible existence of various complete sets of things that are limitless in quantity. For example, if we can conceive of an infinite number of things, then we should be able to conceive of an infinite number of integers (numbers that are not fractions) and there should be conceivable various infinite sets of integers or numbers. One could have an infinite set of natural numbers (1, 2, 3, 4, 5, 6, 7, 8, 9…) and also an infinite set of odd natural numbers (1, 3, 5, 7, 9…). But here a contradiction creeps in.

ONE TO ONE

Because in the first segment of all natural numbers there are more integers between 1 and 9 (namely, the integers 2, 3, 4, 5, 6, 7, 8) than in the second segment of odd natural numbers (which contains just 3, 5, and 7), it would seem that the set of all natural numbers must contain more integers than the set of all *odd* natural numbers—no matter how large the two sets are calculated. Taking a look at these two sets of numbers, the entire set of natural numbers appears to be bigger than the entire set of odd natural numbers because the set of odd natural numbers is just a subset of the natural numbers. It looks like if the two sets are infinite, one infinite set should be, somehow, bigger than the other. This suggests that two infinite sets can come in different "sizes." Now that is the problem, because if two sets are really infinite, then they are both limitless. That makes them equal in size; one set can't be "more limitless" in the same way and in the same sense than another limitless set. But here it seems that one set must be larger than the other because the subset of odd numbers is only half the entire set of natural numbers. Equal but different in size is a contradiction.

And as Galileo demonstrated in 1638, the contradiction can be exposed with even more precision.[21] Put in modern terms, Galileo showed that the set of all natural numbers and a given subset of the natural numbers (such as odd natural numbers) can be brought into a "one-to-one correspondence" with each other. This correspondence is made by a mapping that matches each value in the first set to a single value in the other set. So, for instance, in the case of matching all natural numbers to all odd natural numbers, the mapping would be as follows:

1	2	3	4…
↕	↕	↕	↕
1	3	5	7…

Galileo indicated that this kind of one-to-one correspondence means that two paired-off sets, such as the set of natural numbers and the set of odd natural numbers, when extended infinitely are "the same size"—equally limitless. The set of all natural numbers has no more members than a subset of them, like the set of all the odd numbers only. But his thesis also shows that the set of all natural numbers must still be bigger than the subset of odd numbers alone because the set of all natural numbers contains the even numbers as well as the odd numbers. Thus Galileo saw a contradiction: the whole set cannot both have more numbers in it, and so be larger, and yet simultaneously be the same size as one of its subsets.[22]

But suppose this is merely a paradox (the illusion of a contradiction) and not an actual contradiction. One possible way to resolve the paradox is to propose that it is true a given infinite set cannot actually be larger than another infinite set in the sense of one somehow being "more limitless" than the other, so the two sets must have merely the *appearance* of being different in size. Such a position has already been offered in the field of mathematics. In the 1870s, mathematician Georg Cantor proposed that the contradiction exposed by Galileo is merely an illusion, a paradox that can be resolved logically. According to Cantor, even though comparing *finite* segments taken from the two sets seems to suggest that the infinite set of all natural numbers is larger than an infinite subset of odd or even natural numbers, this is merely an illusion. The one-to-one correspondence of the set and subset, when extended *infinitely*, means the two sets are actually the same size. Because the two sets are *limitless*, the whole really can be equal to the part. It is the nature of being limitless that makes the whole no greater than the part, the set no larger than one of its subsets.[23]

Yet this answer is also problematic, for we can avoid the contradictions in a much simpler way. Supposing we concede it is merely an illusion that the two sets must be of different sizes and that both are therefore actually the "same size" when paired up, we still need not conclude that the two sets must be the same size because they go on *infinitely*. Saying at all that the two sets of numbers are infinite is to assume that the two sets are also *complete* sets that somehow exist apart from anyone doing the corresponding between them. It is to say that the two sequences of numbers already exist complete, all at once, in a Platonic world awaiting discovery. But that is a big assumption, and one we need not make. Instead, we could just as easily suppose that the two sets of numbers have no existence outside of calculation and record. They are created in the act of calculation as a pair of *series*. At any given slice of time in an unfolding series, the numbers constitute a finite, definite set. As the pair of series is developed over time, they can be made to go on *indefinitely* as new values are calculated for each of them. So, we need not assume that sets of numbers exist already complete as "infinite" sets in an abstract world of their own. Calculating novel values is a process of extrapolating, inventing, or generating a series—creating the values where they did not exist before instead of discovering values that are already, somehow, "there." That a one-to-one correspondence of two sets can be extended by a process of pairing off new values in a series means at any time that the sets in the series are not necessarily the same size by being "infinite" (complete and yet limitless). Instead, it could be that two sets of numbers are merely "the same size" at any given time only as long as new values for both are extrapolated, tit-for-tat, in an ongoing series of numeric calculations that makes them match up in one-to-one correspondence. Just claiming the one-to-one correspondence between two sets of numbers, then, does not demonstrate infinitude. At the very most, it shows the two sets are at least indefinite as series and only possibly infinite (assuming, of course, that the infinite is coherent to begin with).

There is also another problem with the supposed infinitude of the one-to-one correspondence between the sets. The whole idea behind the one-to-one correspondence of two sets with finite segments of different measure was to show that they are really "the same size" when extended to infinity. And yet, some sets of numbers are also supposed to be infinite even though they can't be paired up by a one-to-one correspondence. For example, Cantor also showed that irrational numbers (numbers with non-repeating decimals of indeterminate length, such as pi, π, or the square root of two, 2√) cannot be put into a one-to-one correspondence with natural numbers without leaving some numbers out.[24] Any segment of numbers in a scale of irrational numbers has more numbers to it than a segment of the natural number scale. When the two scales are extended without limit that once again suggests two infinite sets of numbers can indeed come in different sizes because one has more numbers than the other. The infinite scale of irrational numbers appears "larger" than the infinite scale of natural numbers. And yet, scales of infinite numbers are supposed to be equally limitless. This again appears contradictory.

But Cantor embraced the idea that infinite sets can come in different "sizes." In Cantor's view, infinite sets can be bigger or smaller than other infinite sets in the "density" or "power" of their numbers.[25] That is, one infinite set can be "more infinite" or "more limitless" than another.

Cantor created a system to distinguish the sizes or powers of these infinite sets. He called the set of natural numbers, and any set of numbers that can be put into a one-to-one correspondence with them, "enumerable" or "countable."[26] Since Cantor believed that the natural numbers exist in a set that is already complete and goes on limitlessly, he held that any set of numbers able to be put into a one-to-one correspondence with them are "countably infinite" and all countably infinite sets are really the "same size," merely having the illusion of being different in size.

The sets of numbers that can't be put in one-to-one correspondence with the natural numbers, like irrational numbers, are referred to by mathematicians as "innumerable" or "uncountable." The uncountable number sets are also assumed to go on limitlessly and so represent examples of "uncountably infinite" sets.[27] Because the uncountably infinite sets have "greater power" than the countably infinite sets, they are "bigger" in size or "higher" than sets of countable numbers.

Cantor further proposed a system of *infinite cardinals* (all symbolized by the aleph, $\aleph$) to designate the apparent size, or power, of infinite sets ranging from the least infinite sets (the countably infinite sets represented as $\aleph_0$) up through a sequence of higher, uncountably infinite sets, represented by alephs with a series of subscript integers as $\aleph_1$, $\aleph_2$, $\aleph_3$, etc.[28]

If the idea of infinite sets with various sizes or powers is coherent, then one could also conceive of adding up different infinite sets to get different powers of the infinite, and that would make the infinite computable. In contrast to definitions of infinity as incomputable, Cantor aimed to make the infinite computable with his invention of *transfinite mathematics*. According to transfinite mathematics, infinite sets can be measured or calculated based on their relative sizes or powers, and they can also be added, multiplied, and raised in power.[29] It will be argued later that having various "sizes" of infinite sets is not a logically coherent idea, which entails that transfinite mathematics is not conceptually sound.

But for the moment let us suppose Cantor is correct that the notion of infinite sets is coherent, that infinite sets can come in different sizes and that they can be represented in such a way as to make them computable. Even if we begin with those assumptions, the ability to compute infinite sets consistently still breaks down in certain operations that result in logical contradictions, indicating that the starting assumption of an infinite set being a coherent idea is actually mistaken.

CONTRADICTIONS IN INFINITE SETS

Philosopher William Lane Craig uses an analogy to show the mathematical contradictions involved with making certain kinds of calculations with infinite sets.[30] Craig asks us to imagine that he has an infinite number of marbles in his possession, and that he wants to give you some of them. Suppose, in fact, that he wants to give you an infinite number of marbles.

One way he could do that would be to give you the entire set of marbles. In that case he would have zero marbles left for himself. However, another way he could do it would be to give you all the odd numbered marbles. Then he would still have an infinite number left over for himself, and you would have an infinite set too. You'd have just as many as he would—in fact, each of you would have just as many as Craig originally had before the marbles were divided into odd and even.[31]

These illustrations demonstrate that performing simple calculations involving an infinite number of things leads to mathematical contradictions. For the first case in which Craig handed out all the marbles, an infinite set minus an infinite set is equal to zero ($\aleph_0 - \aleph_0 = 0$); for the second case in which he handed out all the odd numbered marbles, an infinite set minus an infinite set is still infinite ($\aleph_0 - \aleph_0 = \aleph_0$). In each case, the *identical value* was subtracted from the identical value ($\aleph_0 - \aleph_0$) but with contradictory results (0 and $\aleph_0$). Since dividing and subtracting sets of equal amounts should not produce contradictory results, the contradictions involved with calculating infinite sets casts doubt on the infinite as a coherent notion.

In rebuttal, some philosophers concede that subtracting or dividing infinite sets results in *mathematical* contradiction, but they still maintain that this says nothing about the *logical* status of infinite sets in the real world per se. They say that just because the operation of *subtraction* with infinite sets yields contradictory results using transfinite numbers, that doesn't mean *removal* of infinite subsets from infinite sets of objects in the real world would yield contradictions. Philosopher Wes Morriston, for instance, argues, "Addition and subtraction of numbers is one thing; constructing a new set by adding in new members or removing old ones is quite a different thing. Operations of the second sort may be possible even when operations of the first sort make no sense or are undefined."[32] Similarly, philosopher Arnold Guminski states, "There is a manifestly clear difference between the removal of an infinite subset from an infinite set of real or abstract entities and the attempted subtraction of one transfinite cardinal number from another."[33] Both of these philosophers agree that, although subtracting infinite sets in transfinite mathematics normally results in absurdities (self-contradictions), "removal" of one infinite subset from another does not in the real world. And it should be admitted that the two situations in Craig's example seem, at least off hand, to be physically different—giving away all the odd numbered marbles is different from giving the entire set away.

This distinction between subtraction and removal, however, is flawed and so fails to dissolve the contradictions inherent in the infinite. To see why, first consider a finite example in which the operation of subtraction can simply be another way of stating a real world instance of removal. The equation 5 - 3 = 2 can easily be applied to sets of real objects as when I have a set of five apples and I am subtracting, or "removing," three from the set by eating them in order to create a new set of two apples remaining from the original set of five. Using the numbers to label the members of the sets and symbolizing the removal of members with the operation of subtraction is an application of classical mathematics to a real world scenario. And because the results of such calculations in classical mathematics are definite (in this case, the result is 2 rather than both 2 and some other number), we know in this case that two and only two apples will be left even before I eat three out of the five. Thus, in classical mathematics the operation of subtraction on natural numbers yields definite answers, and so instances of subtraction can be grounded in real world examples of removal. The act of "removing" a subset of objects from a set of objects is just an instance of applying mathematical subtraction or division to physical collections in the real world.

There is nothing in transfinite mathematics implying that mathematical operations on infinite sets cannot be applied to logically possible infinite collections in the real world. So, if we are able to consistently subtract or divide infinite sets in transfinite mathematics, we should then without contradiction be able to carry out the removal of infinite subsets from infinite sets of real objects as well. Subtracting and dividing infinite sets should show what would happen in the real world if we could go about "removing" infinite subsets from infinite sets of physical objects. On the other hand, if we would get *mathematical* nonsense by performing inverse operations in transfinite mathematics, then we would also get *logical* nonsense when trying to "remove" an infinite subset of real objects from an infinite set of them. Such a removal would then not be able to be performed in the real world, which does not permit logically contradictory states of affairs to occur. The application of inverse operations in transfinite mathematics to real world instances of removing infinite subsets then, is actually a test of the logical validity of infinite sets. If the math breaks down as we've seen, so does the logic of infinite sets in the real world.

Now let's return to Craig's marble analogy. Craig stated a situation in which he wants to give you all the odd numbered marbles and keep the even numbered ones for himself. He thus wants to give you half the set of marbles—the odd numbered half. Since the set of marbles is a *complete* set of marbles numbered with the complete set of natural numbers, he should be able to *divide* the marbles into odd and even numbers and hand you one or the other half. Because there is an equal number of both odd and even marbles, each half is infinite in number. So, $\aleph_0 - \aleph_0 = \aleph_0$ just as Craig stated.

However, notice again that since the number of marbles in the set is also *limitless*, there are just as many odd numbered marbles in the set (limitless) as there are *total* marbles in the same set (limitless) by the one-to-one correspondence of all the odd numbers with all the odd *and* even numbers. That is, the amount of odd numbered marbles in the set is *equal* to the total number of marbles in the set. To give away just the odd numbered marbles is, therefore, to give away the *same number* of marbles as *all* the marbles. So, to hand you all the odd numbered marbles is just to hand you all the marbles! The limitlessness of the set of marbles entails that the set *cannot* be divided into odd and even with *only* one or the other half being subtracted or "removed." Because "half" an infinite set is equal to the whole infinite set, you cannot divide an infinite set to begin with. So, once again $\aleph_0$ - $\aleph_0 = 0$ just as in Craig's analogy when he wanted to hand over the entire set on purpose.

The situation is therefore even more contradictory than Craig described: by virtue of being *complete*, the infinite set of marbles can be divided up leaving two infinite sets with one for you just as in the equation $\aleph_0$ - $\aleph_0 = \aleph_0$. However, by virtue of being *limitless*, the infinite set of marbles cannot be divided up, and so one person is left without any marbles just as the equation $\aleph_0$ - $\aleph_0 = 0$ states. The mathematical contradiction in the quantity left over is a result of the logical contradiction of an infinite set being both "divisible" and yet "not divisible." Conversely, the mathematical contradictions in dividing or subtracting infinite sets also imply logical contradictions in "removing" infinite subsets from infinite sets of objects, like marbles, in the real world. Subtracting or "removing" the odd numbered subset of marbles from the total set of marbles ($\aleph_0$ - $\aleph_0$) cannot leave a remainder that is both infinite ($\aleph_0$) and not infinite (0) simultaneously, which is both a mathematical *and* a logical contradiction. Since logical contradictions cannot manifest in reality, there can be no infinite set of marbles or an infinite set of anything else in the real world.

INFINITE SETS AND SQUARE CIRCLES

Some philosophers try to dissolve these contradictions and save the notion of the infinite from logical refutation by asserting that if calculating infinites can be made mathematically coherent, then infinite sets and processes involving them can be logically coherent and therefore could still exist in the real world. The question then becomes, can it be done?

One way to make calculating infinite sets consistent is the approach Cantor used in transfinite mathematics: invent rules for adding, multiplying, and raising the power of infinite sets, but deny that subtraction and division of infinite sets can take place *precisely because* the rules used for the former are not consistent with the latter operations. In transfinite math, you can add but not subtract, multiply but not divide because otherwise you get contradictions just as we've seen. Rather than conclude the mathematical contradictions of inverse operations show a problem with the internal logic of transfinite mathematics and the concept of an infinite set (a reductio ad absurdum), Cantor simply concluded that transfinite mathematics is logically coherent alright, you just get no *particular* result when trying to subtract or divide infinite sets, so the real world doesn't allow such actions either. With this interpretation in place, the mathematical contradictions resulting from subtracting or dividing infinites can be ignored by relying on a simple rule of not allowing any operations that would produce "indeterminate" results.

Now if it is conceded that "removing" an infinite subset from an infinite set of something in the real world is just an instance of applying subtraction or division in transfinite math to real collections, then to say that subtraction and division are not allowed in transfinite math because they produce indeterminate results implies that removing infinite subsets of objects from infinite sets of objects in the real world would also get indeterminate results, and so cannot occur. But that is really just a clever way of avoiding the admission that the "indeterminate" results stem from a logical contradiction (divisible yet not divisible) inherent in the notion of infinite sets that is exposed when the notion of infinite sets is applied to real world situations. Consider that in the real world, you can't prevent someone from trying to give half their marbles away. If the only reason they can't do so is because it would result in both an infinite and a zero amount left over simultaneously, then that just shows that infinite sets in the real world can't exist because the very notion necessarily *implies* logical contradictions. It may not be arbitrary to forbid subtraction or division in transfinite math if the goal is simply to make a consistent mathematical system, but it is merely a trick to prohibit corresponding "removals" of infinite collections if you know that allowing them would result in logical contradictions in the real world.

Claiming that infinite sets of objects can exist because the rules for calculating them remain consistent as long as you don't allow subtraction and division is like saying square circles can exist because we could create a formula that allows them to be used, provided some qualifications are put in place that don't allow us to expose the contradictions resulting from attempting to calculate a round square or a circle with corners.

What would a square circle formula look like? In principle we could create a formula using hyperbolic spaces that will allow us to combine squares and circles into "square circles," or at least transform the transcendental numbers used to calculate the area of a circle into the algebraic numbers used to calculate the area of a square, thereby converting the area of a circle into the area of a square. Suppose we could get such a formula to work consistently but we also have to stipulate that it only works with square circles and that certain other operations in algebra and transcendental math break down in attempts to use them with square circles. The result would be that square circles are not any less self-contradictory; all that we would end up proving in the construction of such a formula is that a coherent game of square circle calculation can be made as long as the rules are limited in an ad hoc fashion so that the illogic of square circles is not allowed to be exposed by taking the concept to its, well, logical end. Transfinite mathematics is in the same boat—the game is coherent only because we won't allow logic to proceed down its natural path so that the self-contradictory nature of its subject would be exposed. This allows the idea of computable infinities in the real world to retain the illusion of being logically coherent.

AD HOC MANEUVERS

If transfinite mathematics really is logically coherent in denying such operations to occur, then there should be no need to modify Cantor's system. It should stand on its own without need for extra rules and distinctions to allow for subtraction and division of infinite sets. And yet, philosopher Blake T. Ostler reports, "Graham Oppy has pointed out that there are a number of other theories of transfinite numbers that have been developed that deal with inverse operations with transfinite numbers without contradiction."[34] This is actually an admission that Cantor's transfinite mathematics needs revision to become consistent with logically possible scenarios in the real world.

Revising transfinite mathematics with new "theories" looks suspiciously ad hoc, a bit of conceptual maneuvering designed to preserve an inherently illogical notion (the infinite) from refutation. But just to make sure Oppy's examples are not dismissed out of hand, they should be examined to see if they make the infinite any more logically coherent for the real world.[35] Oppy pointed specifically to three examples of inverse operations that can be applied to transfinite mathematics.[36] However, he admitted that two of his examples are not, strictly speaking, transfinite. Indeed, they deal with what might be called *indefinite numbers* that are nevertheless finite (limited). His third example from the work of J. H. Conway, on the other hand, does show that Cantor's transfinite mathematics can be modified to carry out inverse operations, at least with transfinite *ordinal* numbers.[37]

Despite this partial success though, there is still a conceptual problem with Conway's modifications: some transfinite ordinals are equivalent to transfinite cardinals. In transfinite mathematics, the least infinite ordinal ω is identified with the infinite cardinal $\aleph_0$. So, the consistent mathematical operations Conway created for transfinite ordinals disappear if the same set ω (which is a "countable infinite") is treated as the transfinite cardinal $\aleph_0$ (also a countable infinite). That means Conway's modifications may merely be mathematical sleight of hand, which should not be surprising since his book is entitled *On Numbers and Games*.[38]

It is therefore not clear from Oppy's last example that the infinite is a *logically* coherent notion and not just a nonsensical notion that can, in some contexts, be used in a *mathematically* consistent way. It is certainly *necessary* that the rules of inverse calculations for infinite sets must be consistent, but the real test of any system calculating infinites is if the system can show how infinite sets can be realized as logically possible for scenarios in the actual world. So, modifying Cantor's mathematics to consistently carry out inverse calculations with transfinite ordinals is not *sufficient* in itself show that such calculations can be consistently applied to logically possible sets of objects in the real world like Craig's set of marbles. It isn't merely the possibility of inventing a coherent system of rules that would make the infinite logical since we might even manage to do the same with nonsensical objects like square circles; rather, it's the ability for such a system of rules to show how the infinite sets in abstract mathematics can refer to real world infinite collections without logical contradictions resulting. Hence, the ability to do inverse calculations must be shown to work consistently with logically possible real-world type collections, and not just with numeric abstractions, otherwise the infinite is not shown to be coherent; we would be back to consistently calculating with symbols that refer to logical absurdities like shuffling supposed square circles. None of Oppy's examples, including Conway's modifications, makes it clear that transfinite mathematics can be logically applied to the real world.

To make up for this limitation, Guminski proposed the creation of an alternative version of the application of transfinite mathematics to the real world.[39] His alternative version, which he simply calls "AV," basically denies that infinite sets in the real world have the same number of members that their infinite subsets have. In contrast, transfinite mathematics holds that "countably infinite" sets are equivalent (or "equipollent" as Guminski says) to their proper subsets. So, transfinite mathematics cannot be applied to infinite collections in the real world without bringing in some new rules of calculation that allow for quantitative differences between infinite sets and their infinite subsets.

Guminski's solution sounds clever, but it rests on another logical contradiction. Guminski holds that in the real world an infinite set is not equivalent to one of its infinite subsets, which is really just to say that one infinite set is more limitless in its number of elements than another infinite set. This is the same claim Cantor made in his attempts to show that "uncountably infinite" sets (like $\aleph_1$) can be more limitless or "more infinite" than countably infinite sets ($\aleph_0$). The distinctions Cantor made between the so-called "sizes" or "densities" or "degrees" or "powers" of infinite sets employed in transfinite mathematics imply that some infinite sets have more members than other infinite sets and so are "more limitless." But the phrases "more limitless" and "more infinite" are oxymoronic. To say a set is "more limitless" or "more infinite" than another is no more logically consistent than saying one thing can literally be "more perfect" than another perfect thing.[40] Something is perfect if it can't get any better, so to say something can be "more perfect" is an oxymoron. So too, if something is limitless, then nothing can have more members than it has; "more limitless" (or "more infinite") is also a contradiction in terms.

Hence, not only is Cantor's definition of the infinite as a complete (divisible) and yet limitless (indivisible) set self-contradictory, it contains another self-contradiction in the claim that one limitless set can be more limitless than another limitless set. So too, Guminski's solution just makes the same self-contradiction as Cantor's system of infinite cardinals by applying the incoherent idea of "more limitless" sets to real world collections.

It is true that a lack of one-to-one correspondence between two sets of numbers shows that one set contains more members than the other. However, it should be noticed that the extent of any of the selected sets that can actually be examined are finite in size. There is no need to assume that the finite selections examined are taken from larger sets that exist already complete and limitless in a Platonic world of their own. Rather, the sets examined could be two finite portions selected out of two *incomplete series* of numbers that may go on *indefinitely* as they are being calculated. That is, the selections may show a difference in size due to a lack of one-to-one correspondence, but since both sets are simply finite portions of two series being compared with one another as they are being generated (invented), there is no need to say that they are "infinite" at all; the series may just be incomplete or indefinite. This is a far more logically consistent view of sets lacking one-to-one correspondence than the idea of one infinite set being somehow more limitless than another infinite set.

Ultimately, Ostler, Oppy, Guminski, and others attempting to salvage transfinite mathematics all fail because their solutions are based on the same contradiction inherent in Cantor's definition of the infinite: the contradiction of being both complete and limitless simultaneously.

COMPLETE AND LIMITLESS: THE ROOT OF THE PROBLEM

To be complete is to be a whole that lacks no elements or members. To be complete assumes that there are more than zero members in the set and that the members in the set constitute an entire collection. And to have elements or members in completion implies that the set of those members is divisible, at least in principle. Whether I have a complete set of marbles or a complete set of 1-100 integers, the complete set can be divided—black and white marbles, odd and even integers—so that it can be made incomplete as the act of division creates two new complete sets. Now I don't mean that to be complete is actually to be divided; rather, I simply mean that completeness implies the set is *able* to be divided; to be complete is to be *divisible*.

Now take limitlessness: for a set or series to be "limitless" implies that it can't be divided because the same amount always remains. Either you can't take away from the limitless, or no matter how much you do take away you never end up with less than you originally had. That which is limitless is that which is also *indivisible*.

The infinite is therefore something that is complete, thus divisible, and yet limitless, therefore indivisible. Since being a divisible-yet-indivisible thing is a contradiction in terms, so too the traditional notion of the infinite is self-contradictory.

WHAT ABOUT INFINITE OBJECTS?

It might be countered that the infinite must be coherent because irrational numbers are infinite—complete and limitless—as demonstrated by their sequences of numerals which have no final values. For example, pi (π), the ratio of a circle's circumference to its diameter, never has a final value; pi is an "irrational number"—a number that cannot be written as a simple fraction and that has no final value when written in decimal form. Pi is equal to 3.14159...where further decimal places continue to give an ever more "precise" or explicit value, but a value that has no final or "definite" value. So far pi has been calculated to billions of decimal places. And yet, even with the use of supercomputers, there will never be a terminating value for pi because we can always calculate further, getting new "more precise" values for pi. It is often assumed that the values for such irrational numbers must be infinite—existing as a complete set that is uncovered or discovered in acts of calculation as an archeologist uncovers chambers of ruins buried under the earth.

But we do not need to assume that the future values for pi are "there" (somewhere in a metaphysical world) waiting to be discovered as one might discover a lost city or an uncharted island. Rather, we could equally assume that values for pi are "discovered" only in the sense that they are *calculated, extrapolated,* or *invented* using the given mathematical rules. And since new calculations can always be carried on, provided there is time to do so, new decimal places for pi can always be computed so long as a human or machine is around to do the computing. Thus, the value of pi is not necessarily infinite; it could be simply indefinite. Anytime your calculator or computer runs out of computing power for an irrational number, it has indicated that the next value that would have been computed is off the charts and so is undefined or "indefinite." The limit of the computable can always be pushed back by increasing computing power, but there will always be undefined values left over so long as there is time to do calculations—the value of pi will always remain indefinite.

A similar example is the square root of two (√2), also sometimes considered to be an "infinite object" because its value is equal to a number having an unending series of decimal places (1.4142136…). The key word here is "series" in designating the decimal places of the expression: like pi, the square root of two does not necessarily have to exist as a set of decimal places already complete in some Platonic world of forms awaiting discovery by mathematicians. Instead, it is better to consider the square root of two as having a *series* of decimal places, rather than a complete set of decimal places, that can "endlessly" be extended because the number of places in the series' decimal value is always *incomplete*. Since no one has calculated a terminating decimal place for √2, we could simply consider its value to be just *the highest value that anyone has calculated so far* for it and so further consider this irrational number to be actually finite but simply *indefinite* in extent since it is always incomplete.

Supposedly "infinite objects" can thus be regarded instead as *indefinite objects* or incomplete objects. Whether we are talking about the value of pi, the square root of two, or some other irrational or transcendental value, such values are best understood as indefinite instead of infinite. This naturally follows given that the logical contradiction of divisible indivisibility inherent in the traditional notion of the infinite has not been resolved.

But showing the logical problems involved with abstract sets and irrational numbers may not seem persuasive to those who believe that space and time are infinite. Even so, as we shall see, the very logical problems raised with transfinite mathematics have a direct bearing on whether or not space and time can be "infinite."

ON INFINITE SPACE

Because assuming the natural number scale to be *infinite* results in logical contradictions, we must conclude that the natural number scale, even when extended toward no clearly defined end (or "indefinitely"), is really finite in the sense that the scale terminates where we leave it *incomplete*. Consequently, any set of objects accurately designated with values from the finite scale of numbers must also be finite—no matter how vast or extended how far beyond our ability to compute. Otherwise we again end up with the internal contradictions of the infinite. So, the number of stars and planets in the Universe, for example, must be finite even if they are in number vast beyond any scale of computation that could in actual practice be defined (i.e., indefinite). No matter how large the number of stars and planets is in the Universe, that number must be finite since all sets, to be without logical contradiction, must be finite.

This naturally entails that the *space* containing those celestial bodies, no matter how *indefinitely* vast it actually is, must be finite rather than infinite.

That this is so is revealed by further logical problems manifesting when space is assumed not to be finite. For instance, the idea that space can be infinite suggests that any given amount of space can be measured in terms of sets, as in sets of infinite places or sets of *infinitesimal* "points" in space. But if, as we've seen, infinite sets are logically incoherent, then so too must be infinite sets of places or points.

Consider any given line segment. A line segment is often assumed to be made up of an infinite set of infinitesimal points. That is, space is supposedly composed or made up of points that are "infinitely small" or "approaching" zero in size or increment but not equal to zero. However, assuming points to be real extents of space that are infinitesimal brings back the same logical problems we faced with infinite sets: if any given amount of space is composed of a set of infinitesimal points, then this implies an *infinite set* of infinitesimal points making up the given space. That means we can once again add an infinite set of points in one space (like the infinite number of points making up the length of one book shelf) to infinite sets of points in other spaces (such as the infinite number of points running the lengths of the other shelves in the bookcase) and so we are back to transfinite mathematics and the same logical problems we've already seen with calculating infinite sets: If I can *add* the infinite number of points in one space to the infinite number of points in another space (as with adding the infinite sets of infinitesimal points extending along two bookshelves), there is no plausible reason why I can't *subtract* one infinite set of points from another or *divide* an infinite set of points (such as when I cut the length of a bookshelf in half and end up with two sets of infinitesimal points), but subtracting and dividing infinite sets is denied by transfinite mathematics to avoid the contradictions we saw before. So once again, the arbitrariness inherent in the notion of the infinite is shown—the operations are forbidden so that the illusion of coherence can be preserved in relation to the infinite.

Still, some may desire to maintain that dividing an infinite extent *does* make sense because we can always divide a given space into smaller spaces, on down to the infinitesimal, so infinitesimal points *must* exist and therefore the infinite is coherent and must exist too. In response, we cannot in actual *practice* divide anything up into infinitely small parts. And even if the process of division is carried out in mere conception alone, we still run into the above problems with the infinite.

We can avoid the troubles with infinite sets, though, if we suppose that points of space are neither infinitesimal nor have any extension at all.[41] A given space is not a set of an infinite number of points; instead, a mathematical point is simply a geometrical unit with *no size*—no extension—at all and space is not "composed" of points in the way that a necklace may be made of pearls. Points do not "make up" a given line, area, or volume like atoms making up a material body. Instead, the line, area, or volume is what is basic and points are arbitrary divisions of these fundamental wholes.

Space is a continuum that is only *measured* geometrically by given points where continuous lines, planes, or volumes intersect or are divided in order to make distinct relations. Points do not make up space like a set of bricks forming a wall or a series of dots added together to compose a line. Instead, imagine that the line is what is taken as basic; while a person measuring the line can artificially divide the line up into a set of dots, the line prior to any such division is fundamentally a seamless continuum made of no discrete parts. Of course, in reality if the mark of a line were drawn with some material medium like a fountain pen, the material line would be composed of atoms in the ink that are themselves discrete—a row of dots forming a line—but the material mark of the line is not important; an imaginary line could also be used to designate just a region of the immaterial continuum of space itself. So too, points are not entities in and of themselves, combined contiguously to form the whole of space. To the contrary, space is itself a continuum—a whole without independent parts. It is only when we use geometry to measure space by dimensions that we divide space up into given sets of points. Points are then just imaginary units of division employed by geometry to *describe* space; points are not discrete entities with their own existence. As P.O. Johnson put it:

> A line might be divisible at any arbitrarily chosen point, but this does not mean it is constituted by discrete points…The line is not a complex of which points are simple constituents. Moreover, if it was such a complex, and did contain or consist of points, then these could not be infinitely numerous, for this would imply that the series of points was unending, and therefore that they could not form a complex or constitute a whole.[42]

Space should thus be regarded as a "fluid" continuum without subunits or parts; it's a continuum that we only *measure* according to "dimensions" when we divide space up conceptually for convenience. These dimensions also have no inherent subunits or parts but can in turn be divided into sets of points with each point being a mathematical *division* of space having no size and only serving to demarcate one extent from another. So, there is no problem with the space of a line having dimension while points do not, for the points of the line represent only conceptual divisions of what is really a continuum.

This is not to belittle the idea of points: the idea of spatial points is of indispensable aid in making measurements. Even if there are no actual physical points composing space like building blocks, we can still consider points to be handy concepts for dividing up the continuum of space into quantities in order to make measurements of distance. Points are useful heuristic devises or geometrical tools that are superimposed by imagination on space for making accurate references to *portions* of the continuum of space. So, even if the continuum of space can only be divided up artificially into points, points nevertheless can refer to definite regions of space where we choose to map space with mathematical lines and planes that intersect. It is simply that, as heuristic devices, points need not be "infinite" in number or even have any size at all.

If space is not composed of infinitely small points, we again come to the conclusion that actual infinite collections need not exist, not even in empty space.

Indeed, from the former arguments above, infinitesimal points make no sense at all because the concept of the infinite is self-contradictory.

And yet, even if one concedes these arguments, one might want to object again by stating that it has only been proven that *infinitesimals* do not exist—it has not been shown that infinite spans of space altogether do not exist. It might be argued that there exist infinitely many sets of places (finite extents of space). So, space could still actually *extend infinitely* in all directions—the Universe could be infinitely big. After all, many are given to saying that space is “endless” or “boundless.”

This objection won’t hold, however, for a couple of reasons. First, as noted previously, to be “endless” or “boundless” can still mean to be finite. A line segment is a bounded thing because it has ends, but circles have no ends and so are “boundless” along their circumference while their limited diameters and “closed” geometry still prove them to be finite. A two-dimensional plane can also curve back upon itself forming a three-dimensional sphere; the surface of the sphere has no edges around its circumference either—it is also “endless” and “boundless” while being finite nonetheless. Circles and spheres are both examples of things boundless or endless, then, but they are both finite. So to say space is boundless or endless does not in itself imply infinitude.

Second, even if we more narrowly use terms like “boundless” and “endless” to specifically mean “limitless,” we encounter problems: if the notion of the infinite is self-contradictory because it implies divisible indivisibility, then assertions of space being infinite are not logically sound. The objection that space is boundless or endless fails then, because it does not withstand the logical difficulties already raised with actual infinite *sets* of things, be those sets of numbers, objects, points of space, or other increments of space. An infinite set of finite elements, such as places, still implies the same logical contradictions we saw earlier when a set is both complete and limitless. We must conclude that neither in abstract sets, nor in actual things, nor in numbers, nor in space are actual infinites to be found.

INFINITY AND INFINITE TIME

The logical problems with the infinite in both sets and extents of space had to be revealed before delving into the arguments against *infinity*, to which we’ll now turn.

Just as what is called “the infinite” is supposedly, according to its traditional conception, a complete-limitless set (a limitless collection of things existing all at once), so too what is called “infinity” has traditionally been taken to mean a complete-limitless sequence (things existing or events taking place, one after another, in a complete succession or series without limit). Although infinity is about limitless successions or series rather than limitless sets, the same mathematical problems that were found with the infinite also pertain to infinity. If, for instance, we attempt to subtract from or divide infinities, again there is a contradiction in the results ($\infty - \infty = 0$ and $\infty - \infty = \infty$) just as there is when subtracting one infinite set from another or in dividing infinite sets.

This mathematical contradiction entails a logical contradiction when we attempt to apply infinite subtraction or division to real world types of situations. Suppose, for example, that you are travelling at infinite speed. You then decide to slow down by an infinite amount. How fast do you end up going? On the one hand, we know that one infinity minus another infinity yields an infinity. On the other hand, we know that an infinity minus an infinity is zero. So which is it—are you still travelling at an infinite speed after slowing down by an infinite amount, or have you stopped altogether? The answer is both simultaneously because subtracting infinities yields contradictions when applied to real world instances of removal.

The reason such a contradiction occurs is because the notion of infinite succession actually assumes the validity of infinite sets. That is, if a succession of steps goes on "infinitely" in the traditional sense of the term, or to such a state called infinity, then presumably the succession of steps can be carried out *infinitely* because there already exists a complete-limitless set, an "infinite" set, of steps that can actually be taken. But the logical contradictions inherent in the traditional notion of infinite sets (as revealed when attempting to apply them to real world situations) entails that the infinite is not a logically coherent concept. Since logical self-contradictions cannot exist in the real world, there cannot actually be a sequence of steps that constitutes an infinite set. And since no complete set of limitless steps can exist to be taken, then no succession of steps can be carried out "infinitely." If there are no infinite sets, then there can be no infinite succession taking place in any of those sets—no "infinity." And so if the notion of infinite sets is not logically valid, then neither is the notion of infinite succession, of "infinity." According to its traditional conception, infinity as a succession must therefore be just as logically incoherent and physically nonexistent as an infinite set.

INFINITE TIME AND A SERIES OF TROUBLES

The logical problems with infinity as a complete-limitless series or succession result from the incoherence of the infinite as a complete-limitless set. Notice as well that a series or succession is really a sequence of *time*. Time is often assumed to be infinite, but just as the notion of infinite space was shown to be logically incoherent, so too infinite time turns out to be logically absurd as well.

Suppose we can take some period of time and see if an infinite series can be found within it; we might assume that time has an infinite series of moments or events making it up—an infinite past and/or an infinite future. And maybe any finite length of time is made up of an infinite series of temporal moments or instants. The question of infinity then becomes, can there actually be an infinite series—a *complete-limitless* series—of things such as instants, moments, or events in time? Some philosophers have argued that the notion of an infinite series or an infinite amount of time could exist. To the contrary, I'll show that time cannot be an infinite series.

For a series to be infinite means that the members of the series each come into existence, one *after* another, in *succession* until an infinite number exists or has existed. In other words, an infinite series is not a series that never reaches completion, for that would just be an *indefinite* though finite series. No, an infinite series is a series that has or will *complete* an infinite number of steps. The question is this: can an infinite number of steps ever be completed and does such an idea even make sense?

One attempt to argue for an infinite series or number of steps being completed is to suppose that any finite period of time contains an infinite number of infinitely short moments or "instants" of time that are completed from one period of time to the next. Any period of time is made up of smaller periods of time. An hour is made up of minutes. A minute is made up of seconds. A second is divided further into milliseconds (each millisecond is one thousandth of a second), into microseconds (each being a millionth of a second), into nanoseconds (each a billionth of a second), into picoseconds (each a trillionth of a second), into femtoseconds (each a quadrillionth of a second), and so on. Since a single period of time can always be divided into smaller units of time, it could still be objected that an "instant" of time must exist as a temporal period so short in duration that it is *infinitely* below the scale of femtoseconds. An "instant" of time in this view is the temporal equivalent of an "infinitesimal" point of space. Since no one wants to deny that instants of time exist, and time continues to pass for any given duration, a collection of temporal instants can exist as an actual, infinite set of temporal periods strung together into whatever duration you want to consider, be it a second, a minute, an hour, etc. Any finite period or duration of time then, contains an actual infinite set of instants—infinitely short moments of time. And since we move from one second, one minute, etc., to the next, then we must move from one complete infinite series of instants in time to the next. So, infinite amounts of time do exist.

But this argument is flawed because the same problems with spatial infinitesimals also apply to temporal infinitesimals: an infinitely small number of instants cannot add up like temporal atoms to make a given duration of time since infinities cannot be calculated or completed by addition; infinity remains an incomputable value. And even if we make infinities computable by drawing upon transfinite mathematics, we still run into the same arbitrariness that we had before with transfinite mathematics allowing some operations but not others to conceal contradictions.

The proper conclusion is that instants of time are not "infinitely short" periods of time that together make up seconds and minutes like links make up a chain. Nor are instants of time periods so short in duration as to be finite but nevertheless indefinite in principle as well as in practice. Instead, instants have *no duration at all*; instants of time are just moments of time *without duration*.[43] They are like points of space—instants have no temporal extension and so are not building blocks of time. Rather, instants of time are only our own arbitrary or artificial *divisions* or "slices" taken from what is fundamentally an unbroken, fluid, and continuous process of change in space. It is the process that is basic, not the instants of it. Thus, the whole of a given period of time can be arbitrarily divided into instants for heuristic purposes of measurement, but time itself has no "infinitesimal" periods that make it up like a string of pearls. Moreover, instants of time do not constitute an actual *set* of temporal moments that somehow exist "all at once," side-by-side, in a block of *spacetime* either. Rather, instants of time exist only as a sequence or *series* of our own artificial divisions of a seamless process of change and this series is simply a convenient mathematical mapping of moments used to measure the order of events that take place in the continuum of change we call "time." If instants of time, like spatial points, are just heuristic tools of measurement used for conceptually dividing up what is in reality a continuous process of change, then there are again no infinitesimals and there need be no actual infinities making up any finite length of time.

In an attempt to counter that infinite series and infinite amounts of time still make sense, some philosophers have argued that an infinite number of steps can be completed by going through the sequence of steps infinitely fast.[44] So, it is argued, while it is true that if any *finite* number is "taken from" an infinite series other members will remain, there is no reason that an *infinite* number of steps cannot be taken from an infinite sequence by means such as *infinite acceleration*.[45] If so, then an infinite sequence can exist as a complete infinite set. Therefore, it is argued, if a complete infinite sequence of steps can exist, then an infinite amount of time could also exist as a completed series of instants, moments, or events (at least, relative to some infinitely accelerated frame of reference).

Such an argument is fallacious for trying to prove that an actual infinite amount of time or temporal, infinitesimal "instants" can exist, though, for there is still a mathematical problem with trying to "take an infinite number from" an infinite set of steps by infinite acceleration. In fact, it's the same problem we saw in the former example of trying to infinitely decelerate: Either taking an infinite number away from the infinite sequence would still leave an infinite amount of steps left to complete (infinity minus infinity is infinity) or it would complete the sequence all in one jump (infinity minus infinity is zero). Those proposing infinite acceleration deny the former and affirm the latter. But even if we allow this denial to get them out of mathematical contradiction, completing an infinite number of things all at once by infinite acceleration simply amounts to confusing infinite acceleration with *instantaneity*.

Moving instantaneously from here to there takes *no* time, not an "infinitely short" period of time; even infinite acceleration is not really an instantaneous "all at once" jump. That is, it still takes time, even if it is an infinitely short amount of time, to get up to an infinite amount of speed. So, an infinite sequence can't be crossed "all at once" by infinite acceleration. That means infinity minus infinity, at least in this case, is still infinity: an actual infinite has not been *completed*, for an infinite number of steps yet remain.

Further, infinite acceleration means that there is another infinite set to cross: because we have to start at speed zero and then work up to an infinite speed by accelerating, that means another infinite number of steps have to be completed—namely the infinite number of *speeds* that must be crossed to reach *infinite speed*. But then wasn't the whole point of proposing infinite acceleration in the first place to find out if an infinite series could be completed? Proposing infinite acceleration begs the question because infinite acceleration (going faster and faster) involves traversing an infinite series of steps from one speed to another in an infinite sequence of speeds, and *completing* an infinite sequence is what we were questioning to begin with.

Consequently, it simply isn't clear that an actual infinite series could be completed. And if that is not clear, then it isn't clear that an infinite amount of temporal instants could ever add up to make a complete infinite set of instants within any finite period of time. So the question still remains if infinite time is even a coherent notion at all.

MOMENTS DON'T ADD UP INTO INFINITE SETS

We've seen the problem with trying to complete an infinite series. This entails that the future is not infinite in the sense that there will ever be an actual infinite amount of time that has taken place. To see why the future is not actually infinite, consider the following logical problem:

Tristram Shandy lives forever and writes one day of his autobiography each year. Though he falls further behind every year, he can still write about everyday of his life if he lives forever. How can this be; how can he fall further and further behind and yet write about every day he will ever live? This is really only a conundrum if one thinks both that Shandy must forever be falling behind in his task and yet at some point of time infinitely in the future will *complete* his autobiography. There is no such point of time in the future, however. Shandy will never complete his autobiography, but he will write about everyday of his life.[46] This means only that though the whole of the autobiography will never be written, no part of the autobiography will remain unwritten forever. In other words, for any day of Shandy's life, Shandy will write about that day at some point in the future.

But now notice that we don't have an infinite future that is ever complete or realized. This "infinite" future is not actually infinite since the whole autobiography will never be written; at most you could consider the future to be a sequence or series of *potential* moments that do not yet actually exist in relation to the present moment. This idea is itself dubious, but if it were consistent, then the future could be "potentially" infinite even if it can't be *actually* infinite.

What about the past? Has an infinite amount of time already taken place? One could conceivably represent the infinite past with a straight timeline arrow pointing indefinitely far away, into an endless past. Since the past contains only events that have already, actually, occurred and our representation of the past as infinite entails that the Universe had no beginning, then have we found an actual, complete infinity? That all depends on what is meant by an "infinite" past.

In conceiving the past to be infinite, we must not assert that the Universe has in its past a beginning infinitely far removed from the present. To have an infinite past does not mean that the Universe did have a beginning or "first moment," which can be represented as time *t*, but that the beginning *t* is simply a moment infinitely removed from the present moment. In other words, to say the past is infinite is not to assert that the Universe began at some infinitely distant moment of time *t* in the past such that the Universe had to pass through infinite time to get from *t* to the present moment. So, for example, we must not try to imagine that there is a beginning of time infinitely in the past from which one could have begun counting until one had reached the present. Rather, to have an infinite past implies that the Universe had *no beginning at all*, not a beginning infinitely in the past.

So far, so good. It is true that you can't say time really had a beginning but that beginning is infinitely in the past, for an infinite past implies *no beginning*. In support of this point, suppose, similar to Tristram Shandy's case, that one had an infinite life span and that for each day of one's life one could think back to the events of an entire year before the last year one had contemplated the day previous. One would eventually think back to the year in which one was born. Let's say one then begins pondering the years prior to one's birth, again contemplating each day one year previous to the year contemplated the day before. One would then be thinking back through historical events in the past before one's own birth, back through the centuries, millennia, and beyond. If in sequentially contemplating prior moments without end, one were never able, even in principle, to arrive at a beginning, the past would be infinite. As the events of the past are thought of, the thinker could not even in principle exhaust the events of the infinite past. Hence, it is impossible to traverse an infinite past by going successively backward through time, even if such traversal is attempted only in imagination.[47] Nor is the number of events traversed in time a function of the direction of movement: one cannot cross an actual infinite by counting *forward*, as shown before, anymore than one can cross an infinite by going backward in time. This much is consistent enough.

But this qualification of what an infinite past means is also misleading, for it is here that another problem with time creeps in: In order to arrive at the present, all events in the past had to occur *in succession*. If any sequence of prior events did not actually occur, then the events succeeding those events could not have occurred. For example, if any sequence of events comprising the year 1492 CE did not occur, then neither did the years 1493, 1494, etc. occur. Thus, the events in years succeeding those years could not have occurred to bring us to the present and so the present moment would not exist, which it must. There must have been events in the past that were succeeded by a sequence of events to bring us to the present. But now consider an infinite past. If events stretch infinitely into the past, then there must be an event or a sequence of events that lie *infinitely* removed from the present. Therefore, even if time is "without beginning" in the sense of having an infinite past, there must still be events that are *infinitely* in the past, that have occurred, and that have been succeeded by other events to get us to the present moment.

So, while one couldn't start counting from a supposed "beginning of time" that lies infinitely in the past and come to the present, one should nevertheless have been able, given an immortal life span, to count from *some* past event *infinitely removed* from the present and yet still complete the infinite sequence to reach the present moment. And that is precisely the problem, because just as Tristram Shandy never reaches an infinite future event, so too one in the infinite past cannot reach the present, which for that individual always lies infinitely far ahead.

Therefore, the contradiction remains: if the past is infinite, then there must be an event or events infinitely in the past, but there can't be because the present could never be reached from it and so we wouldn't exist now. The only way out of this contradiction is to assert that the past must therefore be finite.

Suppose, though, that we try to resolve this contradiction by changing our assumptions. If might be first conceded that because infinity is not an actual number, but only a mathematical property, you can't start counting 1, 2, 3, etc. and then—poof—suddenly arrive at ∞ somewhere down the road, reaching infinity as if it is the endpoint of a process. And it then might be supposed that if there is no endpoint that is infinite, there can be no starting point that is infinite either. So, perhaps it is meaningless to speak of "starting" or "arriving" at a moment or event that is infinitely removed from the present. We may then dismiss the idea that an infinite future implies some event "infinitely" in the future and we may also dismiss the notion that an infinite past implies some moment or event "infinitely" in the past. In this view, there is no event or moment lying infinitely in the past that one could use as a starting point to begin counting forward in time.[48] Instead, there are no particular moments or events that are "infinitely" in the past or future at all; every event lays only a finite amount of time away from every other event, no matter how long the time is between them. Hence, although you can never get to infinity by successive addition, the past or future—indeed, time itself—could nevertheless be "infinite" as a whole.

Does that solve the problem? Actually, it doesn't, for two reasons: First, if the past is infinite, then it is wrong to say that *no* event exists *infinitely* far in the past. If there is no event infinitely in the past, then the past is not really infinite. If any and every event *e* in the past is only finitely removed from the present, then the past is finite no matter how far back we go to reach *e*. For every event in the past to be finite and yet for time to have no beginning only implies that time is a finite unbounded circuit or loop as described earlier; not an infinite sequence.

Second, even if it is conceded that time is somehow infinite "as a whole" but that no event *in* time is "infinitely" removed from the present, another problem persists. An infinite past still implies an infinite *number of events* in the past. Now take the infinite number of events from the present to the endless past as a whole—just as you can't start counting by regress and reach an event lying infinitely in the past, so too you can't invert the direction of movement along *that same number of events* and have the present be reachable either. As philosopher J.P. Moreland said,

> [If the past were infinite], then reaching the present moment would be like counting to zero from negative infinity (*n*,…, -3, -2, -1). Counting to positive infinity from zero (0, 1, 2, 3,…, *n*) involves the same number of events as does counting to zero (the present moment) from negative infinity. The number of events traversed is not a function of direction, and the latter task is as problematic as the former—not because both allegedly involve starting from some point (the former has such a point, the latter does not), but because of the impossibility of traversing an actual infinite by successive addition.[49]

The problems with trying to get to infinity by successive addition have already been shown. But those problems are compounded when time is brought into the picture, for events in the past had to have occurred, *in succession*, in order for time to proceed to the present. That means if the past is infinite, then the present couldn't be reached and so it doesn't exist. And yet, here we are. Moreland elucidates:

> Further, coming to the present moment by traversing an infinite past is worse than counting to positive infinity from zero, because the former cannot even get started. It is like trying to jump out of a bottomless pit. The whole idea of getting a foothold in the series in order to make progress is unintelligible. Take any specifiable event in the past. In order to reach that event, one would already have to traverse an actual infinite, and the problem is perfectly iterative—it applies to each point in the past.
>
> By the way, this was a major point of several of Zeno's paradoxes. He was not merely trying to show that one could not finish crossing an actual infinite, but that one could not even begin moving, because the problem of crossing was perfectly general and iterative.[50]

Moreland has correctly concluded that if the past were infinite, then the Universe would have had to have undergone an actual infinite *series* of steps to get to the present, which is impossible. And since one cannot cross an actual infinite, then the past must have been finite.

Craig used the analogy of dominoes to make a similar point: "If there were a beginningless series of falling dominoes, would not the number of dominoes fallen prior to today be actually infinite?"[51] Indeed it would. In fact, if an infinite number of dominoes could fall in a series so that an infinite number of falls have occurred before the present, then that means an infinite number of dominoes have already *completed* falling. But then, if the dominoes could complete a sequence of falls, they could not have been falling *without limit*—infinitely. A contradiction. Hence, an infinite past is impossible.

There are some academics who don't agree, of course. The late mathematician and science writer Martin Gardner was one of them. Gardner stated,

> An endless regress is absurd only to someone who finds it ugly or disturbing. There is nothing logically absurd about an infinite regress. We may feel uncomfortable with the infinite set of integers, but who wants to deny that the sequence goes to infinity in both positive and negative directions? Fractions in the sequence 1/2, 1/3, 1/4, 1/5…get smaller and smaller but the sequence never ends with a smallest fraction.[52]

Who would want to deny an *infinite* set of integers or fractions? I do for one and I am not alone, as the famous 20th Century philosopher Ludwig Wittgenstein was also skeptical of infinite sets.[53] Since the traditional notion of infinite sets tends to generate so many apparent contradictions, it is my contention that it should be replaced by a more logically coherent notion, such as the concept of either an *indefinite set* or an *indefinite series*. Instead of supposing that integers and fractions belong to limitless sets that exist already complete in a Platonic world awaiting discovery, we could just as well consider integers and fractions as belonging to *indefinite* sets or series.

An indefinite series may contain both a set of defined values (like 1/2, 1/3, 1/4, 1/5 in Garner's example) and also a set of undefined, or "indefinite" values (indicated by the ellipses…or higher than any fraction we can actually calculate). As higher finite values are calculated those newly defined values represent the definition of values that were previously undefined, values that were but potentials in the series. As those values are defined, yet more undefined values lying beyond those invented or calculated for the series come to exist as new potentials for definition. At any given time, though, both the number of defined and the number of undefined values in the series are finite. So integers, fractions, and the like need not be somehow "already there" in a supposedly infinite set, but could simply constitute a finite set at any instant in the calculation process while new values are produced *indefinitely* in the sequence of values as a series.

Moreover, because actual infinite sets remain logically absurd for all the reasons given earlier, that means infinite *regresses* remain absurd since they imply infinite sets; a regress of steps can't be infinite because if it were, then the number of steps would equal an infinite *set* of steps and we'd run into the same old problems involved with trying to cross an infinite set of things, just as in the Tristram Shandy illustration. Regresses can be carried on indefinitely or "irrationally," but they can't be carried on infinitely. Thus, it isn't that infinite regresses are seen as absurd because they are supposedly ugly or disturbing; rather, they must be rejected due to being logically incoherent.

So far it appears that time, like space, must be finite. But this isn't quite the end of the matter. It's only been concluded that the concept of infinity refers to nothing that can *actually* exist. Even if infinities don't refer to *actually* endless sets or sequences, perhaps infinities could still exist as *potentially* unlimited sets or series. In that case, maybe time is "infinite" in the sense that it has the "potential" to unfold without limit.

POTENTIAL INFINITES

Since infinite sets don't actually exist and infinite series can't actually be completed, perhaps infinities nevertheless exist as series that merely have the "potential" to be unending. In his book, *Achilles in the Quantum Universe, The Definitive History of Infinity*, Richard Morris pointed out that before Cantor's transfinite mathematics became popular, "mathematicians generally denied that it was meaningful to speak of infinite numbers." Instead,

> They did make use of unending "infinite series" of numbers. A simple example of such a series would be the set 1, 2, 3,…that we have already encountered. Another would be the series of fractions 1/2, 1/4, 1/8…But here the basic idea was that such a series, like the Energizer bunny, kept going, and going, and going. Even in a steadily increasing series such as 1, 2, 3,…you never got to infinity, only to progressively larger numbers…[54]

These earlier mathematicians were on the right track, and academics have since lost sight of that when they became awestruck by transfinite mathematics and the idea that the infinite could be made mathematically rigorous. But since the infinite is a self-contradictory notion, no infinite sets of numbers need exist; there are only *indefinite* sets of numbers beyond those defined, and no *series* can exist as a succession summed up into an actually complete infinite set.

This position agrees more or less with that of Aristotle, who seems to have also believed that an actual infinite is supposed to be a complete collection—a set of things that exist all together at once—and no such collection actually exists because infinite collections or sets imply logical contradictions. In contrast, he asserted that a potential infinity or "potential infinite" is not a set but a series that is finite at every stage but can never actually be completed.[55]

Aristotle's position that actual infinites are absurd entails that periods of time, whether instants in a second or events stretching from past to future, can also not be totaled up as actual, infinite sets. An actual infinite amount of time has never existed and will never exist. But even if time cannot exist as a series of events that can be summed into an actual, complete infinite set of events, perhaps time nevertheless exists as a series of events that can be *potentially* carried out in an "infinite" way.

A potential infinite of time, in contrast to an actual infinite of time, Aristotle construed as a series of moments that can increase in number "forever" but is always finite at any given interval, and at no moment in time is a subset of completed tasks in the series equal to the completion of the whole series. For instance, if one were to start counting toward infinity beginning with 1, 2, 3, etc., at no moment in time would one's counting reach infinity—no matter how high a number you reach, the value is always finite even though you can potentially carry the series on without end. At each step in counting the series maintains only a *potential infinite*. In this view, time "never ends" because it is a process that is left always incomplete. Since time is never complete, time remains finite at any given interval and so avoids the problems involved with "actually infinite" sets of moments.

If time is in some regard infinite then, it is only potentially and not actually infinite. But now we must ask if this is really the case—is time potentially infinite and, if so, does this potential infinite lay in the past or in the future, or are both past and future "potentially infinite"?

In terms of the past, moments have already occurred in succession and as moments occur they move from being potential to actual. The past therefore contains all the moments that, relative to the present, have already been realized or *actualized*—made "actual." This in turn implies that the series of moments that have already occurred can be summed up into the set of all actualized moments. So, if the past does stretch back infinitely, it must contain an infinite set of actualized moments—an actual infinite—and not just a *potential* infinite set of moments. But the past can't be an actual infinite, for we've already seen the logical contradictions that result with actual infinite sets. Our only conclusion can be that while the past may contain an actual set of moments, it does not contain an actual *infinite* set of moments, and since its moments have all been actualized, neither can the past be *potentially* infinite. Whether actual or potential, the past must be finite.

So that at leaves the future; is the future potentially infinite? The only way that the future could be infinite is if time has the potential to continue on infinitely. But is even this the case? It's possible, but only if infinity is a logically coherent concept. Only logically coherent concepts—for example, concepts like the concept of a square or the concept of a circle—refer to things that can exist. As pointed out earlier, there are some concepts that appear at first glance to be logical because we are able to use them in grammatically correct ways, but which are nonetheless logically meaningless. Concepts such as "square circles" may be used in grammatically correct sentences, but these concepts have no coherent meaning themselves. Their internal logical incoherence indicates that they can refer to nothing that exists either actually *or* potentially: there cannot even potentially be such a thing as a square circle. Now I have argued that actual infinities are logically incoherent in this manner. If my argument is sound, then time does not even have the possibility of being actually infinite. So it becomes vacuous to say time can be "potentially" infinite because that which has no possibility of being realized—of being made actual—has no real potential at all. Because the future has no *possibility* of being actually infinite then, it has no *potential* to be infinite—no potential to be a complete-limitless set or series. In terms of potentiality, the future as a series of events must therefore be finite, no matter how long that series goes on.

It must be conceded, however, that although time is finite, it could still go on "without end" in some sense. Perhaps time is simply an open, linear process that unfolds indefinitely in the sense of continuing "without end" while remaining actually finite at any step in its progress—forever an incomplete process. This interpretation of "endless" time fits Aristotle's idea of a potential infinite insofar as time can continue indefinitely, but in stating time is "potentially infinite" Aristotle misunderstood and mislabeled *indefiniteness*. Time has no potential to be infinite because actual infinities cannot be realized, but it does not violate logic to hold that the future is always finite while unfolding "indefinitely." Aristotle's "potential infinite" is just a misnomer for indefiniteness.

PLATO VERSUS ARISTOTLE ON THE NATURE OF MATHEMATICS

Historically, the two most influential philosophers of mathematics have been Plato and Aristotle. Whether mathematicians realize it or not, all mathematical thought on infinity has been influenced by those two philosophers. For Plato, mathematical objects exist independently of human conception—they are not invented but discovered, as one would discover a new property of stellar objects. Aristotle, on the other hand, seems to have regarded mathematical objects as abstractions that are invented during the process of calculation. While Plato regarded mathematical objects—numbers, values, geometric figures, etc.—as existing independently in a metaphysical world of "Forms," Aristotle believed that mathematical objects have no metaphysical existence independent of human conception; they exist only as descriptions or deductions. I have taken the Aristotelian approach over the Platonic.

These days most mathematicians and theoretical physicists are Platonists: they believe that nature is inherently mathematical or quantitative in structure and so numbers have a metaphysical reality of their own. As Cambridge Professor of Mathematical Sciences Dr. John D. Barrow states, "…mathematical properties of things are real and intrinsic to them. They are more than mere labels. We discover them; we do not merely invent them."[56] From that point of view, if a numeric property seems paradoxical, then that indicates only our limitations in understanding the nature of numbers. Infinity is supposedly such a property. So, even if infinity is paradoxical, it is nevertheless a genuine feature of certain sets or series. The infinite exists already out there in nature, awaiting "discovery" by mathematicians or physicists when they calculate the world. From this perspective, when infinities arise in calculating acceleration toward light speed, in determining celestial masses, or in extrapolating the number of supposed parallel universes, physicists are finding out something profound about the Universe that our finite minds don't fully comprehend.

In contrast, most philosophers lean toward the Aristotelian view: mathematics is a useful tool for describing certain features of the natural world, as when the Celsius and Fahrenheit scales are used to measure the same heat, but numbers certainly have no independent existence in a Platonic world of Forms. Rather, new numbers are invented or "extrapolated" during the act of calculation and have no prior existence all their own. If they have any other existence, they exist only as logical possibilities of calculation or extrapolation. Being calculable, at least in a logically possible sense, is the defining feature of a number. In this view, mathematics isn't a world that exists apart from the possibilities of calculation or invention. Mathematics isn't an intrinsic part of nature; instead, it's like a strategic game with rules that are invented by the game makers—the mathematicians. And when the rules break down, it's time to revise the rules.

In a modern Aristotelian view, if a numeric property is "paradoxical," then that paradox results from the artificial rules of an invented mathematics. And if the solution of the paradox can't be derived without ad hoc maneuvers, it's a good bet the supposed paradoxical property is pointing to an outright contradiction somewhere in the mathematical system. In other words, there's likely a fallacious assumption somewhere in the system causing the contradiction. If so, then the assumption needs altered or thrown out in favor of a more consistent idea so that the contradiction can be avoided. One such "paradoxical" property is infinity. When the paradoxes of infinity are exposed as genuine contradictions, it can be clearly seen that infinities don't reveal anything about the Universe—instead, they point to bad assumptions about the nature of mathematics and its place in describing the Universe.

Now Aristotle himself thought that only an "actual infinite" was self-contradictory while a "potential infinite" was still a coherent idea, but I go a step further and claim that the traditional understanding of infinitude itself is the problem. In arguing that infinity (and the infinite), as a device for calculation, is a logical absurdity, I have taken a reformed *neo*-Aristotelian approach over the Platonic approach to mathematics. From this perspective, the notion of an actual infinitude generates self-contradictions and potential infinite seems to be a misnomer for indefiniteness, so it is best to abandon the idea of using infinitudes to make mathematical calculations altogether.

A COHERENT MEANING OF INFINITE

To get beyond the self-contradictions of infinitudes requires throwing out the traditional understanding of what the term “infinite” means. If we discard the traditional sense, then what would the term “infinite” refer to?

Most of the time, when folks use a word like “infinite” to describe something they simply mean that it is “uncountable” or “vast beyond comprehension.” Either of these meanings though, could just as easily be attributed to quantities that are finite, but indefinitely large. The lexical definition of infinity employed by the lay person is thus even vaguer than the technical definitions used by logicians and mathematicians. It would be better for us all to use more precise words like “vast” and “indefinite” (or even “endless” and “boundless”) to get our meaning across than to inaccurately use words like “infinite.”

In any case, the term infinite still means to be non-finite—to have “no limit.” For something to have a limit—for it to be finite—implies that it is quantifiable by positive or negative numerical values. In contrast, for something to have no limit—for it to be infinite—implies that it is *unquantifiable* with positive or negative values.

Now notice that this meaning of “no limit” fits the concept of the number zero. Zero has absolutely no positive or negative quantity at all, and so it has no positive or negative limits. Ergo, something that can be said to be “limitless” or “infinite,” at least in certain respects, is the number zero.

Further, to be limitless is to be indivisible, and because zero has no positive or negative limits, it cannot be divided. So again zero has infinite, or non-finite, characteristics. We can therefore think of infinitude as limitlessness by virtue of *lacking any quantity* to be limited, and in this interpretation of what it means to be “infinite,” zero certainly has infinite features.

This is, of course, not how the infinite is traditionally understood, but it may be the most logically consistent way to define the term “infinite” given the logical breakdown of the traditional notion of the infinite as a complete-yet-limitless set. However, there is a caveat: while this nontraditional understanding of the term “infinite” would define it as indicating zero members to a given set, the converse is not true: zero is not necessarily infinite in every sense or context. Zero is “infinite” in some respects, but not in others. Zero and infinity are not synonymous.

Rather, “infinity” is a condition that zero has in the context of zero being equal to a set that is unquantifiable—a set that has no has “no limit” only in the sense of having no quantity and so no positive or negative limits. If there is no quantity to measure, there is certainly no limit to consider. Thus, infinity is “the condition of being infinite” (limitless) and zero has that condition by lying outside the natural number scale; by having no positive or negative value. It would not be correct, however, to say zero is defined as limitless in all respects, in all contexts.

For instance, though zero has no limits in itself, it can nevertheless function as a limit in reference to positive and negative values, as when we say that fractions can be divided ever closer to zero as a limit. Also, while according to this view saying a set is "infinite" would be to say it is unquantifiable, null or zero, it would nevertheless not be correct to say that zero can function mathematically as a value in an identical way to how the traditional idea of infinity is supposed to function as a value. $5 + \infty = \infty$, according to the traditional notion of infinity, but $5 + 0 = 5$. So, zero may be "infinite" in a sense, but it does not mathematically function like infinity has traditionally been proposed to function. The term "infinite" may therefore be better defined as indicating a zero set, but zero is only "infinite" in certain contexts and not according to the traditional mathematical operations for infinity.

This interpretation of the meaning of the term "infinite" entails that it is merely an *adjective* designating zero as the state of having no quantity with a limit. To take this position is to solve the logical contradictions inherent in the traditional notions of the infinite and infinity, but it also renders the infinite and infinity impractical and misleading as tools for calculation even if such notions are still useful as predicates for expressing the quality of being zero.

Regardless, given the logical contradictions involved with the traditional notion of infinitudes, this new understanding of the term "infinite" is certainly the one we ought to adopt for logical consistency. The practicality of attempting to change how our culture uses the terms "infinite" and "infinity," however, is another matter entirely.

FINITE MATHEMATICS

Mathematicians have been using infinity (∞) in equations for centuries, and have used the infinite ($\aleph_0$) in set theory since the 19th Century. Suppose the mathematics community were to accept the fact that the traditional understandings of infinity and the infinite are illogical. From exposing the illogic of those conceptions, what becomes of using infinities or infinite sets in mathematics?

The answer depends on what mathematics is about. Insofar as mathematics is simply an intellectual game, then the logical breakdown of infinitudes does not amount to much; mathematicians can continue to play with infinities in classical mathematics and infinite sets in transfinite mathematics just as gamers can continue role-playing in fictional worlds. On the other hand, insofar as mathematical operations are used in attempts to reveal the nature of the Universe, then the logical incoherence of infinitudes means that they reveal nothing of nature since logical contradictions cannot refer to real circumstances.

Many mathematicians have assumed that the infinite really is a property of nature, existing in a Platonic world on its own apart from minds attempting to conceive it, and that Cantor's transfinite mathematics teaches us how this property of nature operates. But that is a big assumption, and one which we are not intellectually compelled to make. It could equally be assumed that infinite sets are mathematical inventions, in which case the rules for manipulating infinite sets do not indicate "how the infinite works" as if those rules are merely descriptions of an infinite set's behavior that mathematicians discover like the principles of atomic motion in a condensed gas are discovered by physicists. That is, it could just as well be supposed that the infinite is an *invented* idea that refers to no existing natural property at all.[57] If that position is correct, the rules for calculating infinite sets are only "discovered" in the same way that a new strategy in chess is discovered. This kind of "discovery" is actually a form of invention; it is the invention of new rules of inference for manipulating concepts in a pattern coherent with the rules previously established for those concepts.

If this is correct, then the rules of transfinite mathematics are not really the discovery of some phenomenon independent of human activity, but simply the invention of a system of inference. Further, because the traditional view of infinite sets contains self-contradictions, that system of inference has no coherent *application* to understanding reality in terms of *measurement*. Infinite sets and transfinite mathematics are better interpreted as elements of a mathematical game rather than a means for understanding the quantitative aspects of nature. Transfinite mathematics is therefore actually misleading about the nature of real sets of things. That is, the "infinite sets" of transfinite mathematics not only do not refer to real sets in nature, but actually lead us astray in understanding the quantifiable aspects of reality. Consequently, Cantor's math ought to be rejected as a tool for investigating reality even if it is saved as a kind of academic game.[58]

Since the infinite game is logically incoherent when applied to reality, mathematics needs a new way of construing what the terms "infinite" and "infinity" could coherently mean. We've seen that the only coherent definition of infinitudes is to define them as "not finite" in the sense of having no quantity at all. If the term "infinite" is to have any logical meaning in set theory, it would actually have to mean "no limit" simply in terms of being a zero quantity. When we refer to sets of objects in nature then, to say the set is "infinite" by this new understanding would be to say that it has zero members. To say, for instance, the set of planets in the Universe is infinite at a given time would be to say that there simply are no planets existing at that time—quite the opposite of an unfathomably large number.

Because the traditional understanding of infinitudes turns out to be illogical and "infinite" as a term can only logically mean zero, then the infinite ($\aleph_0$) as a set would have to simply designate a zero or null set. But then the aleph cardinals in transfinite mathematics would not be needed as we already have conventional means to indicate zero or null sets, and so the alephs can be dropped altogether from set theory as tools of measurement. The revision of mathematics that is needed would, then, remove the infinite from use altogether as a tool of measurement when applying set theory to the real world, rendering transfinite mathematics entirely irrelevant as an investigative tool, while the term "infinite" would only be used to refer to the limitless properties of zero/null sets (if indeed the term would ever be used at all).

Moreover, if the term "infinity" means to be limitless as something zero in quantity, then infinity (∞) as a quantitative value would simply indicate a lack of succession in steps—zero in succession. Consequently, the lemniscate can be dropped from use in ordinary mathematics and the term "infinity" now becomes redundant if it is used simply to indicate zero, since zero as a value suffices on its own. Classical mathematics therefore needs to undergo revision by dropping the traditional idea of infinity in favor of "infinity" as the condition of being zero.

So far then, we can conclude that term "infinite" ought to indicate the limitlessness of just being zero in quantity for sets or steps of succession, calling for a revision of mathematical practice when calculating nature, a revision that drops the traditional notions of infinite sets and infinities. But this conclusion does not yet clarify what concept we must use to substitute for the traditional use of the infinite and infinity in making measurements. How would Cantor's transfinite mathematics be replaced once the infinite is taken to mean a lack of quantity? And, for that matter, how would infinity as a value be replaced in classical mathematics once we understand infinity to be simply a lack of limit by virtue of being zero?

The answer to revising both transfinite mathematics and classical mathematics is to replace the traditional use of infinitude with an alternative concept—*indefinitude*. Just as the traditional idea of infinitude is divided into two categories—the infinite ($\aleph_0$) and infinity (∞)—so too indefinitude can be divided into two categories: *the indefinite* (which will be represented by the Greek letter omega, denoted Ω) and *indefiniteness* (to be arbitrarily designated by the Hebrew letter qof, denoted ק). Indefinitude in the form of either omega or qof indicates our inability to describe reality completely—a representation for the limits of our ability to calculate rather than a description of a quantitative property of reality as is alleged with the traditional idea of infinitude.[59]

If a value is "indefinite," then that means either (A) it would be found to be the terminating value in a set of values if the set could be measured or counted, but is either too minute or too vast to be measured or counted in actual practice or (B) it is finite in the sense of being currently undefined as the next value, or sequence of values, beyond the highest or lowest value that can be actually computed or invented for a series. In A, it is merely our practice of counting or computing that limits us from finding the end and eliminating the value's status as indefinite. In B our number system just trails off (1, 2, 3…), implying more to come, but simply doesn't specify how much more because it is left *incomplete*. If the highest natural number that could actually be invented were ten, then the next number that would otherwise be invented after that is left "indefinite." In either case A or B, a higher finite value is indicated but not defined, so the value is "indefinite."

What will be called "the indefinite" shall refer to any set that has so many members that the highest value(s) in the set is indefinite according to A. As for putting the indefinite into notation for mathematical use, we could take a lesson from Cantor. Just as Cantor designated various "powers" of infinite sets, we can designate various degrees of *indefinite* sets. Using the omega to represent the indefinite, we could then assign subscript numerals to the omega for indicating the various degrees of indefiniteness in a sequence of indefinite sets. So, similar to Cantor's aleph series, we could use omega-naught followed by a sequence of higher degrees of indefinite sets (depicted as Ω_0, Ω_1, Ω_2, Ω_3, etc.), in a series that can go on indefinitely as new indefinite sets are invented.

In fact, taking this position, right down to using the omega for notation, has already been proposed by Dr. Shaughan Lavine, Associate Professor of Philosophy at the University of Arizona. In his article "Finite Mathematics," Lavine used the omega along with the same series of subscripts to represent sets of indefinitely large numbers.[60] Each omega in the series represents a finite, but indefinite, amount of numbers available in the corresponding set.[61] While the value for the numbers in each indefinite set is incalculable or incomputable because the numbers in each set run off any defined scale, the sets can nevertheless be symbolized with omegas and used in a finite arithmetic with one another. Thus, with Lavine's finite mathematics, use of *infinite* sets becomes unnecessary and transfinite mathematics can be dropped without losing anything important for describing the quantitative properties of nature.[62]

Then too, the old concept of infinity can be replaced by the concept of "indefiniteness." Indefiniteness, symbolized as ק, will refer to the state of a succession as being incomplete, implying more to come. Any series of values that previously would have been said to go on to infinity can now be said to fade into indefiniteness because we can always, given the time, calculate new finite values, with the next value in line remaining undefined until the limit is extended further. The next highest value for an irrational or transcendental number, such as pi or the square root of two, is indefinite in that sense. New values for these ratios could always be invented, given whatever time remains for us to do so.

The *indefiniteness symbol* of qof could easily be used in place of the lemniscate in equations without loss of mathematical function (or perhaps I should say "malfunction" since infinities really indicate we've run off the scale anyway). So, for example, take an equation that, traditionally, would use the lemniscate:

$$int\ \{a\} \wedge \{b\}\ f(t)\ dt = \infty$$

This equation is taken to mean that *f*(*t*) does not limit a nonzero area from *a* to *b* as finite. But such an equation can be rewritten as:

$$int\ \{a\} \wedge \{b\}\ f(t)\ dt = \text{ק}$$

In the second version of the equation, although *f*(*t*) means there is no *definite* finite area that can be calculated from *a* to *b*, which means only that we have an instance of a finite area that is *indefinitely* large (off the defined scale) rather than infinite (complete but limitless) in itself. The new version of the equation indicates that any *attempt to calculate* the area goes on indefinitely rather than there being a supposedly infinite area to be calculated. The mathematical function of the indefiniteness symbol thus remains the same as that of the lemniscate, but the *meaning* is different. Now we have a logically meaningful expression without loss of mathematical function.

Functionally speaking then, nothing is lost by replacing infinity with indefiniteness—values that were traditionally thought to go on "infinitely" could just as well be thought of as going on "indefinitely." (And in terms of ordinary discourse, on those occasions in which one would use the adjective "infinite" one could just as well say "indefinite," and the traditional use of the adverb "infinitely" could be replaced with the adverb "indefinitely," as more accurate in reference.)

Drawing a distinction between the indefinite (Ω) and indefiniteness (ק), we could retain Lavine's omega notation to designate indefinite *sets* while adding the qof to Lavine's notation for representing indefiniteness in *series* or *succession*, as when recursive addition goes on "indefinitely." Otherwise, nothing in Lavine's finite mathematics needs to change; it works well for avoiding the illogical implications of the traditional notion of infinities and infinite sets, and so stands as an appropriate revision of classical mathematics. As Lavine states, "Every theorem of ordinary mathematics has a natural counterpart in finite mathematics."[63]

Whether the professional field of mathematics will ever replace infinity (∞) with indefiniteness (ק), and the infinite ($\aleph_0$) with the indefinite (Ω_0), or not remains to be seen. Certainly replacing the old concepts and symbols with these would be the correct course to take. Classical mathematics and set theory would have to be revised with a finite mathematics such as that developed by Lavine, or a version of it as suggested, but doing so would avoid the contradictions of infinitudes and so give us a more accurate means of calculating nature. If adopted, these changes would also greatly impact the systems of measurement used in other fields of inquiry, in particular physics and cosmology, yielding for us a new understanding of the Universe.

THE FINITE UNIVERSE

Throughout physics and cosmology infinitudes are asserted to exist. For example, there has been a number of *Steady State* models proposed in cosmology. According to these models, space is infinite in all directions and time is infinite in both past and future. Individual galaxies, stars, and planets may be born and perish across the infinite expanse of space, but as old matter is destroyed new matter is created in an infinite cycle of generation and annihilation. The Universe thus maintains a "steady state" of new worlds springing into being to replace old worlds.

In contrast, some versions of the currently popular *Big Bang* model hold that space and time have a beginning. Around 13.7 billion years ago space and time sprang into existence out of an infinitesimal spec of space called a *singularity* that lasted for an infinitesimally short amount of time in which the Universe was infinitely hot and dense due to having an infinite amount of energy and mass compressed into the singularity. In some versions of Big Bang cosmology, the Universe will eventually collapse back into the singularity it arose from. Hence, in Big Bang cosmology, the Universe may be finite in expanses of space and/or eras of time, but infinitudes are still asserted to exist in the form of "singularities."

Other versions of Big Bang cosmology hold that the observable universe is just part of a greater universe that exists “alongside” other “parallel universes” making up a complete set of universes called *the Multiverse*. As each individual universe reaches the end of its life and dies, other universes are being born with the entire Multiverse, as the set of all universes, maintaining an infinite number of universes at all times. At any one time there are an infinite number of universes in the Multiverse, so the Big Bang that created our universe is just one in an infinite cycle of big bangs; each big bang creates its own universe, which lives for a time and then dies to be followed by another big bang creation event. The whole process carries on in an infinite cycle of universe creation and destruction events across the entire Multiverse. The Multiverse cosmology is thus a kind of Steady State model on a grander scale.

These cosmological models are just a few examples of the many instances in which infinitudes are assumed. But if the case against infinity (and the infinite) holds, then the infinite magnitudes that are being conjectured in physics and cosmology are absurd; that is, they are self-contradictions signaling either that a theory has taken a wrong turn or at least that the theory must be reinterpreted in terms of indefinite, but nevertheless finite, magnitudes. Just as rejecting the infinite for the indefinite would give us a more coherent mathematics, so too it can give us more accurate science and cosmology.[64]

All that remains is for the theorists to reevaluate the notion of infinity. And reevaluate it they must, for we have seen that the traditional notions of infinity and “the infinite” are self-contradictory. There are no infinite magnitudes. However vast magnitudes of space and time may be, however indefinite our calculations of them are, the Universe must be finite.

NOTES

1. There is much opposition to skepticism of infinity (and "the infinite") from professional mathematicians, scientists, and academic philosophers. For example, philosopher Thomas Ash states that skepticism of infinity constitutes the fallacy of *argument from personal incredulity* since those arguing against infinity allegedly assert that the infinite must not make sense merely because they have a hard time imagining infinite sets and series. He goes on to say that, "our limited human imaginations are poor guides to what properties the universe can have" (Ash 2001).

   There are at least two problems with making that charge of fallacy: First, the charge of committing the argument from personal incredulity may apply to some arguments that have been made against the infinite, but it does not apply to arguments such as mine (or even to the particular argument opposed by the quoted philosopher making the charge). My arguments make the case that there are inherent *logical contradictions* in the traditional notion of infinity. That issue has nothing to do with the limits of human imagination. If something is self-contradictory, it isn't a rational concept—in which case it is little wonder one can't imagine it! Second, to dismiss skepticism of infinity or the infinite by claiming that it only looks contradictory because of "our limited human imaginations" is not really evidence; rather, such a claim is an instance of the fallacy of *the appeal to mystery*.

   So, to disprove my case against infinity, one would need more than dismissive charges or appeals to mystery: it would have to be shown where my reasoning is inconsistent and why the contradictions I show infinity to have are not really contradictions after all. If such can be shown, I will, of course, recant.

2. In *The Cosmic Sphere* (Sewell, 1999, 29-32) I proposed that the traditional understanding of infinity is "meaningless" and "unintelligible." This is technically correct but perhaps a bit misleading: it's not that infinity has no meaning at all; rather, it is just that while the term "infinity" can have multiple meanings, the traditional, literal meaning of infinity is nevertheless self-contradictory. Hence, the traditional notion of infinity can give the impression of being intelligible as long as the self-contradictions in its literal meaning are not noticed or examined. Once the self-contradictions are brought to light, the traditional way of defining infinity is found to be incoherent and so "unintelligible" in that sense.

   I also pointed out in my first book that the lexical or lay definitions for "infinity" have non-literal meanings such as "boundless" and "vast beyond comprehension," and I still maintain that these metaphorical meanings could just as well be attributed to indefinitely large finitudes (Sewell, 1999, 32).

3. I proposed a number of arguments against the traditional notion of infinity in my first book, *The Cosmic Sphere* (Sewell, 1999). I no longer consider all of the arguments I made against infinity in that work to be sound. Consequently, the reader should not assume that I would still defend all of the points I made against infinity in *The Cosmic Sphere*. Rather, this work represents my final thoughts on the subject of infinity and the infinite.

4. Keele University's Dr. Peter Fletcher (2007, 548) asserts that "if you believe that all talk of actual infinity is meaningless then you cannot even pose the question of whether actual infinity exists: if you believe that no actual infinities exist then you evidently believe the question is meaningful and answerable." Fletcher's statement is true enough, but it does not apply to my argument. I am not asserting that "all talk" of infinity is meaningless. I am simply asserting that the traditional mathematical meaning of infinity is logically incoherent, just as the idea of a four-sided triangle is incoherent. And just as one logical absurdity, a four-sided triangle, cannot exist, so too another logical absurdity, (the traditional notion of) infinity, cannot refer to something that exists.

5. Infinity has also been called an "improper finite" or a "potential infinite" (Craig and Sinclair 2009, 104).

6. The infinite is also known as a "true infinite" or "actual infinite" (Craig and Sinclair 2009, 104).

7. See entry for "finite, a. and n." in *The Oxford English Dictionary* (1989, 2nd Edition).

8. See entry for "finite, 1" in *The HarperCollins Dictionary of Mathematics*. (Borowski and Borwein 1991, 221).

9. Maddocks 2010.

10. See entry for "countable" in *The HarperCollins Dictionary of Mathematics* (Borowski and Borwein 1991, 130).

11. Craig and Sinclair 2009, 105.

12. Fleming 1890, 211.

13. See Endnote 16.

14. In *The Cosmic Sphere* (Sewell, 1999, 32-33) I proposed the term "immeasurable" (and "immeasurability") instead of the term "indefinite" (and "indefiniteness") for finite magnitudes that are either so vast or so small they are off any given scale. I now believe the terms immeasurable and immeasurability are misleading: indefinite values are still measurable *in principle* even if not in actual practice, while values that cannot be measured even in principle are traditionally referred to as "infinite." So, the terms indefinite and indefiniteness I now consider more proper to use.

15. Adams 2005 Edition, Footnote p. 5.

16. Kornai 2003, 301-307. See also Sazonov 1995, 30-51.

17. Compare entries for "infinite, 1 and 2" in *The HarperCollins Dictionary of Mathematics* (Borowski and Borwein 1991, 292).

18. See entry for "infinity, 1" in *The HarperCollins Dictionary of Mathematics* (Borowski and Borwein 1991, 294).

19. See entry for "infinity, 4" in *The American Heritage Dictionary*, 4th Edition (Delta 2001). Notice that the mathematical function could just as well indicate a series of steps carried on indefinitely rather than infinitely. The function simply indicates a series of ongoing refinements: it's rather like trying to get your grade point average back up to 4.0 after getting a 3.0 on one assignment—an impossible task no matter how many assignments you ace, unless the original 3.0 is waived. So, the "limit" approached need not be designated as "infinity" since zero or any definite number would do just as well.

20. See entry for "infinity – Science Definition" in *The American Heritage Science Dictionary* (Kleinedler, et al. 2005).

21. Galileo 1914, 31-33.

22. Ibid., 31-33.

23. Cantor 1915, 40-41.

24. Dauben 1990, 50 and 60-62.

25. Cantor 1915, 51-52.

26. Ibid., 47-52.

27. Ibid., 56.

28. Ibid., 109.

29. Cantor 1915.

30. Strobel 2004, 103.

31. Ibid., 103.

32. Morriston 2002, 152.

33. Guminski 2002.

34. Ostler 2006.

35. Oppy 1995, 220.

36. See Oppy (1995) cited in Ostler (2006).

37. Conway 1976.

38. Ibid.

39. Guminski 2002.

40. The term "more perfect" is, literally speaking, nonsense since perfection is that which can't be improved on. (In the Preamble to the United States Constitution, the phrase "more perfect" is used to describe forming the new Union, but is a purely metaphorical phrase intended to signify that the new Federal Government would perpetually solidify the union preexisting between the confederated States.) So too is it nonsensical for the phrase "more infinite" to be taken in any literal sense.

41. In *The Cosmic Sphere* (Sewell, 1999, 2-4; 19-20; 85-86) I stated that points of space are indeed arbitrary divisions of space, which is correct. However, I also proposed that points could be considered "immeasurably small" (indefinite but finite) extents of space that have a negative "structure." I now believe the latter assertion is incorrect. It is more consistent to hold that, as arbitrary divisions of the continuum of space, points have *no extent at all*, rather than an indefinitely or "immeasurably" small finite extent with negative "structure."

42. Johnson 1992, 373.

43. In *The Cosmic Sphere* (Sewell, 1999, 15) I stated that while instants of time are arbitrary divisions of time (which is correct), I also proposed that instants of time can be considered "immeasurably small" (indefinite but finite) durations of time. I now believe the latter assertion is incorrect: as arbitrary divisions of the continuum of time, instants have no duration at all, not an indefinitely or "immeasurably" small finite duration.

44. See, for example, both Bertrand Russell and Rudy Rucker's proposals of the ability to cross an infinite provided *infinite acceleration* is assumed. Russell's illustration is mentioned in James Thomson's article, "Infinity in Mathematics and Logic" (Thomson 1967, 188). Rucker's illustration is in his book *Infinity and the Mind* (Rucker 1983, 69). Both Russell and Rucker's illustrations propose crossing an *actual* infinite set with what amounts to a *potential* infinite (acceleration). However, a potential infinite is impossible to complete. Hence, even the examples of Russell and Rucker do not show that actual infinites can be crossed.

45. See the refutation of this claim offered by Max Black in his article, "Achilles and the Tortoise" (Black 1951, 91-101).

46. Johnson 1992, 370-372. Johnson cites the Tristram Shandy example from Bertrand Russell's book *The Principles of Mathematics* (Russell, 1980, Chapter XLIII, sections 340-341). Be sure to also see Johnson's rebuttals to Raymond Godfrey's remarks in Johnson's article, "More About Infinite Numbers" (Johnson 1994, 369-370).

47. Moreland 1993, 37.

48. Mackie 1982, 93.

49. Moreland 1993, 230.

50. Ibid., 230.

51. Craig 1999, 60.

52. Gardner 1999, 194.

53. See endnote 57.

54. Morris 2003.

55. Aristotle, *Physics*, Book 3, Chapter 6.

56. Barrow 2007, 204. Barrow draws a false analogy between language and mathematics. In Barrow's view, mathematics describes a physical object's intrinsic properties while language does not. Hence, a rose by another name would smell just as sweet, but if the mathematics describing the rose (like say, the number of atoms the rose contains) were to change, the change in math could only describe the rose accurately if indeed the quantitative properties of the rose were not what we previously thought. So, in Barrow's view, math accurately describes nature because nature has intrinsic mathematical properties that are captured in calculation; but language is arbitrary and only describes our reactions to the world.

    Barrow's claim, however, rests on a false distinction between mathematics and language. First, it may be true that a rose by any other name would smell as sweet, but the temperature of the rose would also stay the same regardless of which scale of measurement is used: Fahrenheit or Celsius. So, here we have an arbitrary choice of mathematical scales just as we have an arbitrary choice of labels. Second, language is more than mere labels; it also contains *predicates* that ascribe properties to objects just as mathematics has *quantities* that ascribe properties to objects. To change the predicates ascribed to a physical object would, if accurate, reveal something true about the entity that was previously spoken of falsely. Thus, to discover a chemical "causes disease" instead of "cures disease" would indeed reveal something about the intrinsic properties of the chemical not captured in a purely mathematical description.

So, both mathematics and language are used to describe physical entities and their natures; both math and language can also be either accurate or inaccurate in those descriptions, depending on what concepts are used. But while Barrow's distinction between math and language breaks down, that does not entail that there are Platonic Forms of predicates in the world anymore than there are Platonic Forms of quantities. That is, neither numbers nor predicates need to be somehow "out there" in a Platonic world of Forms, nor do they need to be an intrinsic part of the objects so described by them. Math, like language, it just a tool for describing the natural world—not a metaphysical property of the world so described.

57. The philosopher Ludwig Wittgenstein (1974 Translation, 469-70) also held a neo-Aristotelian position that mathematics is an invention rather than the discovery of pre-existing Platonic forms. Because he saw mathematics as an invention in which number scales are created by following rules and procedures instead of being uncovered already complete, Wittgenstein also believed that infinity does not exist in any actual or complete form (Wittgenstein 1975 Translation, §144).

58. I once thought, as remarked in *The Cosmic Sphere* (Sewell 1999, 33), that Cantor's transfinite mathematics might be revised in the form of an indefinite mathematics, thus saving its basic operations. However, I now believe this to be incorrect for two reasons. First, there is a difference between the infinite as defined in transfinite mathematics and the concept of the indefinite. Transfinite mathematics is based on the idea that an infinite set is equal to an infinite subset (producing contradictions as I've pointed out in this work); but it is not at all clear that an *indefinite* set can be "equal" to an indefinite subset. Second, there is no need to attempt such a revision of transfinite mathematics; Dr. Shaughan Lavine's (1995) system of "finite mathematics" makes a revision of transfinite mathematics superfluous for revealing the genuine nature of indefinite sets.

59. In *The Cosmic Sphere* (Sewell, 1999, 33) I proposed a different notation for indefiniteness, which I had also labeled "immeasurability." I am now dropping the term "immeasurability" for "indefiniteness" and I am dropping the old notation I had proposed in favor of the qof. (Also, any notations I have proposed in previous versions of this work are being jettisoned in favor of those used herein.)

60. Lavine 1995, 391.

61. Ibid., 396.

62. It is important to note that Lavine himself does not believe that his system of finite mathematics demands a change in mathematical practice. Insofar as mathematics represents an intellectual game of sorts, he is correct. But insofar as mathematics represents a tool for discovering the properties of nature, then practice would need to change wherever infinitudes have been invoked. The case against infinity, and the infinite, I present goes further than Lavine's view by stating that finite mathematics is needed to revise classical mathematics and set theory, and that revision would indeed change mathematical practice to a certain extent.

Also, Lavine does believe in the coherence of infinity and infinite sets; he merely asserts that infinitudes are not *implied* by his finite mathematics and that belief in infinitudes is therefore not logically *necessary*. (See Lavine 1995, 390 and endnote 3.) In contrast, my argument exposes the traditional notions of infinity and the infinite to be logically incoherent and so belief in infinitudes is not rational to maintain.

63. Ibid., 389.

64. As Lavine suggested, "…it might be fruitful to consider physical models that are more accurate in that they use the indefinitely large instead of the infinite…" (Lavine 1995, 393) He went on to state that he knew of no case in which there would be a clear advantage, but all cosmological models of an infinite Multiverse would certainly be ruled out, making finite cosmologies more accurate.

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