BRENTANO'S SOLUTION TO BERTRAND'S PARADOX

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Abstract: Brentano never published on Bertrand's paradox but claimed to have a solution. Adrian Maître has recovered from the *Franz Brentano Archive* Brentano's remarks on his solution. They do not give us a worked demonstration of his solution but only an incomplete and in places obscure justification of it. Here I attempt to identify his solution, to explain what seem to me the clearly discernible parts of his justification and to discuss the extent to which the justification succeeds in the light of current work on Bertrand's paradox.

Keywords: Bertrand's paradox; Brentano; Brentano's correction ratio; principle of indifference; probability.

1. INTRODUCTION

The principle of indifference states that "Events between which we have no epistemic reason to discriminate have equal epistemic probabilities".[1] This is a necessary truth, for consider: if it is false then it is possible for events between which we have *no epistemic reason to discriminate* to have *different epistemic probabilities*, which makes no sense. That would mean, for example, that a known shuffled pack of cards, i.e. one for which you have equal epistemic reason to expect any card, could have different probabilities for you picking the King of Hearts and the Ace of Spades.

Bertrand asks what is the probability of choosing at random a chord of a circle longer than the inscribed equilateral triangle.[2] In the angle case, the tangent angle of the chords[3] that are longer have tangent angle between 60° and 90°,[4] and

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[1] Nicholas Shackel, *Bertrand's paradox and the principle of indifference*, Abingdon, Routledge, 2024, p. iii.

[2] See *ibidem*, pp. 57–59 for illustrations and translation of Bertrand's text. There you may see that he started by asking for the probability of shorter but then himself immediately worked out the probability of longer and the literature has followed him in that ever since.

[3] The angle between a chord and the tangent to the circle at an end point of the chord.

[4] Since then they will lie between the two sides of the inscribed equilateral triangle that has a vertex as the endpoint of the chord.

those that are shorter have tangent angle between 0° and 60°. Applying the principle of indifference to those angles give the probability of longer = 30°/90°=1/3. In the direction case, for each set of parallel chords there is a single diameter that those chords all intersect perpendicularly. Those that are longer intersect it on the inner half of each radius and those that are shorter intersect it on the outer half. Applying the principle of indifference to the points of intersection gives the probability of longer = ½. In the midpoint case, the chords with midpoints inside the circle inscribed within the equilateral triangle (which is, in turn, inscribed in the original circle) are longer and the area of that circle is a quarter of the area of the original circle. Applying the principle of indifference to the areas occupied by midpoints gives the probability of longer = ¼. The probability of an event is unique. Therefore 1/3=1/2=1/4. Hence the principle of indifference implies a contradiction. Whatever implies a contradiction is necessarily false. Therefore Bertrand's chord paradox shows the principle of indifference to be *necessarily false*. And yet, the principle of indifference is *necessarily true*. Whence we have a paradox.

Brentano never published on the paradox but claimed to have a solution. Adrian Maître has now recovered for us from Brentano's nachlass some of Brentano's thoughts on his solution. The manuscripts that appear in the two appendices of Maître (in this issue) do not constitute a fully worked presentation and demonstration of Brentano's solution. They give, rather, an indication of what he claims to be the solution accompanied by incomplete and in some places obscure justifications of that solution. Below I attempt only to identify his solution and explain what seem to me clearly discernible parts of his justification. I neglect passages that I find entirely obscure or not relevant to the chord paradox. The translations into English, and therefore any errors in that translation, are mine alone.

2. MAÎTRE'S APPENDIX 2: BRENTANO 1917 MANUSCRIPT

In this manuscript we can identify Brentano's solution from the initial remark.

> Every point on a circular surface, apart from the centre and circumference, is the centre of one and a single chord; and from this it follows that the set of possible chords is to be set equal to the set of points in the circle.[…] From this arises the justification for determining the probability that the chord of a circle is smaller or larger than the side of the equilateral triangle to be inscribed according to the ratio of the area of the circle with half the radius to the difference between it and the circle with the whole radius, i.e. 1 to 3.[5]

[5] Franz Brentano, "Zum bertrand'schen Problem", Erkenntnislehre und Logik. Graz: Franz Brentano Archive, 1917, p. 2. Strictly speaking, Brentano should be speaking of longer and not-longer chords, but it is convenient to speak of longer and shorter and doesn't change the probabilities. I have already and will continue to follow him in this except in the maths, which is why the '≥' sign will appear there.

Thus, Brentano's claim that the midpoint case is the correct solution. This is supported by his arguments in "Gedankengang beim Beweise für das Dasein Gottes"[6] aimed at eliminating the direction and angle cases and these I leave to the next section.

It is ambiguous what Brentano means by 'equal'. If he means that the set of chords and the set of points in the circle have the same cardinality, it is true, since both have the cardinality of the continuum. But that equality, as a justification of the midpoint case being the correct solution, is inadequate. It amounts to committing one of what I call the frailties of Bertrand's procedure[7], namely, under-counting the chords. It undercounts them because a single point in the circle, the centre, represents the entire set of diameters. There are therefore continuum many longer chords, the diameters, that are given a probability of zero in this solution (because the area of a point in the circle is zero). For the full explanation of why I think this rules out the midpoint case altogether I refer the reader to §3.13 of my book.[8]

In the passage that I have omitted from the above quotation, Brentano gives what I assume is supposed to be an explanation of why whatever he means when he says "the set of possible chords is to be set equal to the set of points in the circle" suffices for the inference he makes to the correctness of the midpoint case. On its face it appears to be the kind of thinking in terms of infinitesimals that mathematicians now find unreliable and obscure, being tolerable only when they know how to eliminate it (which they learnt how to do from Cauchy and Weierstrass). It might be aimed at the direction case. It might even be aimed at the angle case. It might only be clarifying what he means by 'equal'. I don't think he is necessarily making an error but it is obscure how it is supposed to validate the inference. Consequently, the burden of the justification of his solution falls entirely on his explanation of what is wrong with the angle and direction cases.

3. MAÎTRE FORTHCOMING APPENDIX 3: BRENTANO 1915 MANUSCRIPT

Here we find in page 6 an interesting allusion by Brentano to an opinion of Boltzmann, that Bertrand's paradox proves not the falsity of the principle of indifference but of the *existence of continua*. If held as that there are no *mathematical* continua then Boltzmann would be committed to a radical intuitionism about mathematics,[9] but perhaps he means only no continua in physical reality. This raises the question of whether a result of this kind in probability theory could really

[6] Franz Brentano, "Gedankengang beim Beweise für das Dasein Gottes" (hereafter GBDG), Theologie. Graz, Franz Brentano Archive 1915.

[7] Shackel, *op. cit.*, §3.2.

[8] Shackel, *Bertrand's paradox and the principle of indifference*.

[9] See Charles McCarty, "Continuity in intuitionism", in *The history of continua: Philosophical and mathematical perspectives. Stewart Shapiro and Geoffrey Hellman*, (Eds.), Oxford University Press, 2020, pp. 299–327 for how the intuitionist Brouwer treats continua.

prove any such thing about physical reality! That being said, although interesting from the point of view of understanding Brentano's motivation, this remark is tangential to our interest here.

As it seems to me, page 5 in GBDG gives the crucial material for his claimed solution. Here he starts by faulting the angle and direction cases for concerning themselves with only a subset of the chords. Prima facie the angle case consists in only the chords starting from a single point on the circumference and the direction case considers only the chords perpendicular to a single diameter.

> Since these were completely different subsets [of the chords], no contradiction can initially be found. (GBDG, p. 5)

Thanks to Adrian Maître drawing my attention to his transcription of Brentano's manuscripts I was able at the typesetting stage of my book to add a reference to Brentano as one of the first to raise this worry. He is quite correct about it. It is part of the first of the six significant frailities in Bertrand's original development of his paradox that I have analysed in §3.2 of my book.

Brentano then considers what follows when we use the rotational symmetry of the circle to include all the chords:

> What makes the difference in results between the two selected groups seem paradoxical, however, is that when rotated [vervielfältigt] in exactly the same way, they both seem to lead to the total number of chords of a circle. (GBDG, p. 5)

This comment is not wholly accurate, since rotating the angle case at least double counts the chords whereas the rotating the direction case does not. Setting that aside, Bertrano then brings forward why this use of rotational symmetry may be illegitimate:

> If the process of rotation were really the same for both,…those which are larger than the side of the triangle and those which are smaller than the side of the triangle, in all their parts, or at least if, on average, the chords were rotated the same distance as that assigned to the chords that are smaller than the side of the triangle compared to those corresponding to them, then there would be no objection to the conclusion from the group of chords to the entirety of the chords. [But there would be an objection if the distances are not the same.][10] This is what needs to be examined here. But Bertrand completely overlooked this and we have to make up for what he missed here. (GBDG, p. 5)

Here Brentano introduces the criterion of being the same distance, with the implication that if the distance is not the same then Bertrand's original ratio must

[10] This is my free translation of the very concise sentence in German: "Anders im entgegengesetzten Falle".

be corrected by the ratio of the different distances. In general, distances in mathematics are defined by metrics, which are functions from pairs of objects to the positive real numbers satisfying simple conditions.[11] Brentano has not said by which metric he is defining the distance, but from what we will see shortly he appears to mean a metric defined by how far the midpoint of a chord travels under rotation.

The analysis that then follows is in some ways intriguing but rather loosely developed. One could challenge it on the ground that, for example, formulating an approach based on rotational symmetry would be better pursued in the manner of Jaynes.[12] I am not going to take that line, but attempt to treat Brentano's solution in its own terms.

For simplicity we assume the original circle has radius 1. For the direction case, Brentano says:

> the chords, which are smaller than the side of the triangle, on average are three times as far away from each other at their center as the larger ones. (GBDG, p. 5)

This is ambiguous. On one interpretation it is simply false, since on each radius the centres of the shorter chords are on the outer half of the radius and those of the longer lie on the inner half, so the shorter are not three times as far away from each other at their center as the longer. So I assume it is supposed to be that this greater distance for the shorter arises from considering the distance travelled by the midpoint of a chord under a rotation. Technically, if the midpoint is at distance r from the centre of the circle, the metric is defined by the arc length along the circle radius r travelled by the midpoint under a rotation α,[13] which is αr. It is on this assumption that I offer the following account of Brentano's justification.

Brentano's correction is based on the ratio of the circle of radius ¾ to the circle of radius ¼. The idea is that the point a quarter of the way along the original radius is the average midpoint of the longer chords and the point three quarters of the way along is the average midpoint of the shorter chords.

> if the diameter that the chords intersect perpendicularly is allowed to rotate in a circle, the speed of its movement in the part on which the smaller chords are perpendicular is on average three times as great as that of the part which the larger ones are…. And if you do this, you get exactly the result that Bertrand reached in his third procedure, which does not suffer from a similar oversight. (GBDG, p. 5)

[11] Wilson A. Sutherland, *Introduction to metric and topological spaces*, Oxford, Clarendon, 1998. p. 21.

[12] Edwin T Jaynes, "The well posed problem", *Foundations of Physics*, Vol. 4, nr. 3, 1973, pp. 477–492.

[13] Angle α and angular velocity ω below are given in radians and radians per unit time respectively because this unclutters the formulae.

As the diameter rotates at angular velocity ω the speed of a point ¾ along the radius is ¾ω which is three times the speed of the point ¼ along, which is ¼ω. Consequently, in any period of time, the ratio of distances travelled by the shorter and longer chords is, on average, 3 to 1. He then says that because of this, for reasons of density, the shorter chord proportion should be multiplied by 3 and the longer by 1, which does then give the same result as the midpoint case:

Longer:shorter chords = Bertrand's ratio×Brentano's correction ratio
=1:1×1:3=1×1:1×3=1:3, giving probability of longer = ¼

No further explanation is given for why this is the correct treatment beyond the remark.

> Here you simply stick to the number of center points of the chords, which (since you can ignore the points on the periphery and the central point) correspond to the total number of points. (GBDG, p. 5)

This you cannot do for the reason I gave above.

Prima facie, taking the point ¼ along as the average midpoint for the longer chords cannot be right because it weighs each chord by its distance from the centre and in so doing gives zero weight to the *continuum many chords that are diameters*. It is therefore obscure why Brentano thinks treating all the chords in terms of a speed of rotation and arc length travelled is the correct way to include all the chords. At the very least one would need to see a similar treatment of the other two cases in terms of rotational speed.

Furthermore, as Humpty Dumpty says 'I'd rather see that done on paper'[14], i.e. a fully worked out justification for why this manner of treatment get things right for the third case but not for the other two. After all, by exactly the same reasoning, the ratio of mid-point arc lengths travelled under rotation by shorter and longer chords is the same in the angle case, and therefore the same correction ratio applies, giving:

Longer:shorter chords = Bertrand's ratio×Brentano's correction ratio
= 1:2×1:3=1:6, giving probability of longer = 1/7

And once again, we have two different probabilities, namely ¼ for the midpoint and direction cases and 1/7 for the angle case.

Brentano addresses the angle case on page 7. He says that Bertrand is making the same mistake in appealing to rotation:

> He immediately transferred the result found for the group of chords that have a common starting point in the circumference to the entirety of all chords…without worrying in the least whether, after moving the starting

[14] Lewis Carroll, *Through the looking glass*, London, Folio Society, 1871/1962, p. 74.

> point of the individual chords, if one pays attention to all their parts, it is certain that some appear not to be more and others less shifted and moved apart. As soon as you do this, you notice that this was actually the case here too, although not to the same extent as in the version which starts with the group of chords cutting one diameter perpendicularly. Here too, the shifts in the larger chords are smaller than in the smaller ones; (GBDG, p. 7)

Talk of moving points and chords is convenient loose talk but if taken literally it fails to include all the chords, since one is merely talking about the original subset under a change of its orientation. Speaking literally, Bertrand's appeal to symmetry has got nothing to do with *moving* the starting point or *moving* chords, but instead is about claiming that, due to rotational symmetry, each subset of chords starting from each point on the circumference will have the same proportion of longer to shorter chords. Let us name each chord with endpoints having the polar coordinates $(1,\theta)$ and $(1,\phi)$ by the angular components of its endpoints, (θ, ϕ). A rotational symmetry of the chords by angle α maps the chord to the chord $(\theta+\alpha, \phi+\alpha)$, which is a *different* chord from (θ, ϕ) unless $\alpha=0$. So suppose we have the subset of chords all starting from the point with polar coordinate $(1,\theta)$ and we consider a shorter and a longer chord in that subset, (θ, ϕ) and (θ, ψ).[15] When we consider this under a rotation of $\alpha>0$, the correlate chords in the new subset starting at the point $(1, \theta+\alpha)$ are the chords $(\theta+\alpha, \phi+\alpha)$ and $(\theta+\alpha, \psi+\alpha)$. Brentano is saying that the distance between the smaller chords (θ, ϕ) and $(\theta+\alpha, \phi+\alpha)$ is greater than the distance between the larger chords (θ, ψ) and $(\theta+\alpha, \psi+\alpha)$. What does he mean by distance?

If we apply the metric used in the direction case, which is the arc length travelled by the chord midpoint under rotation, this would give the result for the angle case I gave above. But here Brentano says

> the difference in the distances at which the corresponding larger and smaller chords are placed is not as great as in the previous case. If the average ratio was 1:3, here it is 1:3/2, which, when related to the ratio 1:3, is again the ratio of the probability that the chord is larger or smaller than the side of the inscribed equilateral triangle is what 1:3 shows. (GBDG, p. 7)

Calculating the probability again:

Longer:shorter chords = Bertrand's ratio×Brentano's correction ratio
= 1:2×1:3/2=1:3, giving probability of longer = ¼ .

Brentano gives no explanation of where he got the correction ratio of 1:3/2 from. I have various speculations about how he got this but I find none of them satisfactory and for that reason I don't know what metric Brentano is applying to define the

[15] For angles in degrees, to be shorter requires $0<\theta-\phi\leq 120$ or $240\leq\theta-\phi<360$ and to be longer requires $120<\theta-\psi<240$.

distance travelled by longer and shorter chords when they are rotated in the angle case.

For the sake of argument grant Brentano a metric that produces this ratio. Contrary to what he appears to think, that doesn't solve the paradox but just makes things worse. For as I said, the reasoning he applied to the direction case, and therefore the metric of that case, applies just a well to the angle case, and give the probablity of longer = 1/7. So now we have two different probabilites for the angle case. This is a *revenge* Bertrand's paradox using the angle case alone!

Finally, to satisfactorily answer Bertrand's challenge it is not enough to offer different ways of calculating the probability. Brentano must show *why* the intuition to which Bertrand appeals is erroneous due to the distances travelled under rotation. Why exactly is it relevant at all? For consider my justification of the truth of this intuition some time ago:

> In both [angle and direction] cases the set of similar subsets forms a group under the symmetries of a circle,[16] and Bertrand explicitly mentions the symmetry fact. This procedure has intuitive geometrical appeal[…] Bertrand's suggestions for measuring [the chords] in the [angle and direction] cases look like measuring ratios of an abstract cross section of a measure space which has uniform cross section in order to determine ratios in the whole measure space–rather like measuring the ratio of the volume of pink and white candy in seaside rock by measuring the pink and white areas on a slice.[17]

In other words, just as there is no reason to think the proportions of areas on a slice of seaside rock misrepresent the proportion of the volumes, Bertrand thinks there is no reason to think the proportions in the angle case depend on which point of the circle is a common endpoint for a subset of chords. That is a compelling point to which Brentano's talk of distances travelled by chords under rotation has only tangential relevance. Similarly, the 1:1 ratio for the set of parallel chords perpendicular to a diameter applies to every diameter. Yes, Bertrand's appeal to symmetry is too quick, for the reasons I give in my analysis of what I called the 4^{th} and 5^{th} frailties of his procedure.[18] In my book, however, I show how to make both the angle and direction cases completely mathematically rigorous. I then show how the cross section of the chords defined by each point on the circumference (for the angle case) and by each diameter (for the direction case) is indeed definable as a uniform cross section of a measure space on the entire set of chords[19]. The state of play, then, is that unless Brentano could show some way in which distances travelled by chords under rotation has any relevance to that rigorous formulation, his solution fails. I do not think he could do so.

[16] Strictly speaking, the group action of the rotational symmetries of the circle map the similar subsets to one another.

[17] Nicholas Shackel, "Bertrand's paradox and the principle of indifference", *Philosophy of Science*, Vol. 74, 2007, p. 157.

[18] Shackel, *Bertrand's paradox and the principle of indifference*, pp. 62–65.

[19] *Ibidem*, §§3.11 & 3.12.